ON ALMOST 2-ABSORBING SUBMODULES

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Abstract. Let R be a commutative ring with identity and let M be a unitary R-module. A proper submodule N of an R-module M will be called almost 2-absorbing submodule if $a, b \in R$ and $m \in M$ with $abm \in N - (N : M)N$ implies that $ab \in (N : M)$ or $am \in N$, or $bm \in N$. Also a proper ideal I of R will be called almost 2-absorbing ideal if $a, b, c \in R$ with $abc \in I - I^2$ implies that $ab \in I$ or $ac \in I$, or $bc \in I$. These concepts are generalizations of the notions of 2-absorbing submodules and ideals respectively, which have been studied. In this paper we give several results concerning almost 2-absorbing submodules.

Keywords: 2-absorbing ideal, 2-absorbing submodule, Almost 2-absorbing ideal, Almost 2-absorbing submodule.

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1. Introduction

Through out this paper all rings are commutative with identity and all modules are unitary. Prime ideals play a central rule in commutative ring theory. A proper ideal P of R is said to be prime ideal if $ab \in P$ implies that $a \in P$ or $b \in P$ where $a, b \in R$. Various generalizations of prime ideals have been studied as well as extending ideals to submodules. For example, a proper ideal I of R is said to be almost prime provided that $a, b \in R$ with $ab \in I - I^2$ imply that $a \in I$ or $b \in I$(see [4], [2]). A number of generalizations of prime ideals in commutative

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rings can be found in [5], [6]. Also a proper ideal I of R is said to be 2-absorbing if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$ (see [1]). A proper ideal I of R is said to be almost 2-absorbing ideal if whenever $a, b, c \in R$ with $abc \in I - I^2$, then either $ab \in I$ or $ac \in I$, or $bc \in I$ (see [8]). A proper submodule N of M is called a 2-absorbing (resp. weakly 2-absorbing) submodule of M if whenever $a, b \in R$, $m \in M$ and $abm \in N$ (resp. $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$, or $bm \in N$ (see [8]).

Clearly, if N is a prime R-submodule of M, then N is weakly prime and then it is weakly 2-absorbing and hence it is an almost 2-absorbing R-submodule of M. If N is an almost 2-absorbing R-submodule of M, then N need not to be weakly 2-absorbing; for example $N = \langle 8 \rangle$ in the $\mathbf{Z}$-module $M = \mathbf{Z}_{40}$. So, almost 2-absorbing submodules are generalization of weakly 2-absorbing submodules. We show that if A, B are two ideals of R, N is an almost 2-absorbing R-submodule of M and L is an R-submodule of M such that $ABL \subseteq N - (N : M)N$ and $8(AB + (A + B)(N : M))(L + N) \nsubseteq (N : M)N$, then $AB \subseteq (N : M)$ or $AL \subseteq N$ or $BL \subseteq N$.

The purpose of this paper is to introduce the notion of almost 2-absorbing submodules as a new generalization of both 2-absorbing and almost prime ideals.

2. Almost 2-absorbing submodules

Following [3], a proper submodule N of an R-module M is weakly 2-absorbing if $a, b \in R$, $m \in M$ and $0 \neq abm \in N$ then $ab \in (N : M)$ or $am \in N$, or $bm \in N$. Now we introduce the concept of almost 2-absorbing submodule and generalize the concept of almost prime submodules (see [5]) to an almost 2-absorbing submodules.

Definition 2.1 Let R be an integral domain, M be an R-module and N a proper submodule of M. N is called an almost 2-absorbing submodule of M if $a, b \in R$ and $m \in M$ with $abm \in N - (N : M)N$ then either $ab \in (N : M)$ or $am \in N$, or $bm \in N$.

Let R be a commutative ring and M an R-module, then every weakly 2-absorbing submodule N of M is an almost 2-absorbing submodule of M. But the converse need not to be true. For example, consider $R = Z$, $M = Z_{40}$ and $N = \langle \bar{8} \rangle$ then $(N : M)N = N$. So N is almost prime submodule of M and hence N is an almost 2-absorbing submodule of M. On the other hand, $0 \neq 2.2(\bar{2}) = \bar{8} \in N$ while $2.2 \notin (N : M)$ and $2.\bar{2} \notin N$, so N is not weakly 2-absorbing submodule of M. So almost 2-absorbing submodules are generalization of both almost prime submodules and weakly 2-absorbing submodules.

Theorem 2.2 *Let M be an R-module and N, K be a proper submodules of M with $K \subseteq N$. Then N is an almost 2-absorbing submodule of M if and only if N/K is an almost 2-absorbing submodule of M/K.*

Proof. ($\Rightarrow$) Let N be an almost 2-absorbing submodule of M. Now, for any $a, b \in R$ and $m \in M$ assume that $ab(m+K) \in N/K - (N/K : M/K)N/K$. It follows that $ab(m+K) \in N/K - (N : M)N/K$ and so $abm \in N - (N : M)N$. Since N is an almost 2-absorbing then $ab \in (N : M)$ or $am \in N$, or $bm \in N$. Hence $ab \in (N/K : M/K)$ or $a(m+K) \in N/K$, or $b(m+K) \in N/K$. Thus N/K is an almost 2-absorbing submodule of M/K.

($\Leftarrow$) Let N/K be an almost 2-absorbing submodule of M/K, $a, b \in R$ and $m \in M$ with $abm \in N - (N : M)N$. It follows that $ab(m+K) \in N/K - (N/K : M/K)N/K$. Since N/K is an almost 2-absorbing submodule of M/K then $ab \in (N/K : M/K)$ or $a(m+K) \in N/K$, or $b(m+K) \in N/K$. So $ab \in (N : M)$ or $am \in N$, or $bm \in N$. ∎

Proposition 2.3 *Let M be an R-module and N be a submodule of M. Then the following are equivalent.*

(i) *N is an almost 2-absorbing submodule of M.*

(ii) *For $r_1r_2 \in R - (N :_R M)$,*
$(N :_M \langle r_1r_2\rangle) = (N :_M \langle r_1\rangle) \cup (N :_M \langle r_2\rangle) \cup ((N : M)N :_M \langle r_1r_2\rangle)$.

(iii) *For $r_1r_2 \in R - (N : M)$,*
$(N :_M \langle r_1r_2\rangle) = (N :_M \langle r_1\rangle)$ *or* $(N :_M \langle r_1r_2\rangle) = (N :_M \langle r_2\rangle)$, *or*
$(N :_M \langle r_1r_2\rangle) = ((N : M)N :_M \langle r_1r_2\rangle)$.

Proof. $(i) \Rightarrow (ii)$ If $r_1r_2 \in R-(N :_R M)$ and $m \in (N :_M \langle r_1r_2\rangle)$ then $r_1r_2m \in N$. But if $r_1r_2 \notin R - (N :_R M)N$ then $r_1r_2m \in N - (N :_R M)N$. So $r_1m \in N$ or $r_2m \in N$. Hence $m \in (N :_M \langle r_1\rangle)$ or $m \in (N :_M \langle r_2\rangle)$.

(ii) $\Rightarrow$ (iii) It is well known that if a module equals to the union of two modules then it is one of them.

(iii) $\Rightarrow$ (i) Let $a, b \in R$ and $m \in M$ with $abm \in N - (N : M)N$. Now assume that $ab \notin (N : M)$, we show that $am \in N$ or $bm \in N$. By (ii) $(N :_M \langle ab\rangle) = (N :_M \langle a\rangle)$ or $(N :_M \langle ab\rangle) = (N :_M \langle b\rangle)$, or $(N :_M \langle ab\rangle) = ((N : M)N :_M \langle ab\rangle)$. Since $abm \notin (N : M)N$ then $m \notin ((N : M)N :_M \langle ab\rangle)$. Therefore $m \in (N :_M \langle a\rangle)$ or $m \in (N :_M \langle b\rangle)$. Hence $am \in N$ or $bm \in N$. ∎

Lemma 2.4 *Let M be an R-module, N be an almost 2-absorbing R-submodule of M and $r, s \in R$. If L is an R-submodule of M such that $rsL \subseteq (N : M)N$ and $2rsL \nsubseteq (N : M)$, then $rs \in (N : M)$ or $rL \subseteq N$ or $sL \subseteq N$.*

Proof. Assume $K = (N : M)$ and suppose $rs \notin K$. It is enough to prove $L \subseteq (N :_M r) \cup (N :_M s)$. Let $x \in L$. If $rsx \notin (N : M)N$, then since N is almost 2-absorbing and $rs \notin K$, either $rx \in N$ or $sx \in N$ and then $x \in (N :_M r) \cup (N :_M s)$. Suppose $rsx \in (N : M)N$. Since $2rsL \nsubseteq (N : M)$, there exists $z \in L$ such that $2rsz \notin (N : M)N$ and then $2rsz \in N - (N : M)N$. Since N is almost 2-absorbing and $rs \notin K$, either $rz \in N$ or $sz \in N$. Let $m = z + x$. Then $rsm \in N - (N : M)N$ and since $rs \notin K$, either $rm \in N$ or $sm \in N$. We study three cases:

Case 1. $rz \in N$ and $sz \in N$. Clearly, $rm \in N$ or $sm \in N$ and then either $rx \in N$ or $sx \in N$.

Case 2. $rz \in N$ and $sz \notin N$. Suppose $rx \notin N$. Then $rm \in N$ and then $sm \in N$ which implies that $r(m+z) \notin N$ and $s(m+z) \notin N$. Since N is almost 2-absorbing and $rs \notin K$, $rs(m+z) \in (N:M)N$ which is a contradiction. Hence, $rx \in N$.

Case 3. Similarly, as Case 2. ∎

Lemma 2.5 *Let M be an R-module, N be an almost 2-absorbing R-submodule of M and $r \in R$. If I is an ideal of R and L is an R-submodule of M such that $rIL \subseteq N$ and $4rIL \nsubseteq (N:M)N$, then $rI \subseteq (N:M)$ or $rL \subseteq N$ or $IL \subseteq N$.*

Proof. Assume $K = (N:M)$ and suppose $rI \nsubseteq K$. The there exists $s \in I$ such that $rs \notin K$. We show that there exists $t \in I$ such that $4rtL \nsubseteq (N:M)N$ and $rt \notin K$. Since $4rIL \nsubseteq (N:M)N$, there exists $\alpha \in I$ such that $4r\alpha L \nsubseteq (N:M)N$. If $r\alpha \notin K$ or $4rsL \nsubseteq (N:M)N$, then by taking $t = \alpha$ or $t = s$, we obtain our result. Suppose $r\alpha \in K$ and $4rsL \subseteq (N:M)N$. Then $4r(s+\alpha)L \subseteq N - (N:M)N$ and $r(s+\alpha) \notin K$ and then by taking $t = s+\alpha$, we obtain our result. Hence, $2rtL \nsubseteq (N:M)N$ and then by Lemma 2.4, $L \subseteq (N :_M r) \bigcup (N :_M t)$. If $rL \subseteq N$, we are done. Suppose $rL \nsubseteq N$. Then $tL \subseteq N$. We show that $I \subseteq (K:r) \cup (N:L)$. Let $\beta \in I$. If $2r\beta L \nsubseteq (N:M)N$, then by Lemma 2.4, $r\beta \in K$ or $rL \subseteq N$ or $\beta L \subseteq N$. Since $rL \nsubseteq N$, $\beta \in (K:r) \cup (N:L)$. Suppose $2r\beta L \subseteq (N:M)N$. Then $2r(t+\beta)L \subseteq N - (N:M)N$ and then Lemma 2.4, $r(t+\beta) \in K$ or $rL \subseteq N$ or $(t+\beta)L \subseteq N$. Since $rL \nsubseteq N$, $(t+\beta) \in (K:r) \cup (N:L)$. If $t+\beta \in (N:L)$, then since $t \in (N:L)$, $\beta \in (N:L)$. Suppose $t+\beta \in (K:r) - (N:L)$. Now, $2r(2t+\beta)L \nsubseteq (N:M)N$ and $2r(2t+\beta)L \subseteq N$. Since $rt \notin K$ and $r(t+\beta) \in K$, $r(2t+\beta) \notin K$. So, by Lemma 2.4, $L \subseteq (N :_M r) \cup (N :_M 2t+\beta)$. Since $t+\beta \notin (N:L)$ and $t \in (N:L)$, $2t+\beta \notin (N:L)$ and then $L \subseteq (N:r)$ which is a contradiction. So, $t+\beta \in (N:L)$ and since $t \in (N:L)$, $\beta \in (N:L)$. Thus, $I \subseteq (K:r) \cup (N:L)$ and since $rI \nsubseteq K$, $IL \subseteq N$. ∎

Theorem 2.6 *Let A, B be two ideals of R. Suppose M is an R-module and N be an almost 2-absorbing R-submodule of M. If L is an R-submodule of M such that $ABL \subseteq N - (N:M)N$ and $8(AB + (A+B)(N:M))(L+N) \nsubseteq (N:M)N$, then $AB \subseteq (N:M)$ or $AL \subseteq N$ or $BL \subseteq N$.*

Proof. We study three cases:

Case 1. $8ABL \nsubseteq (N:M)N$. Then there exists $r \in B$ such that $8rAL \nsubseteq (N:M)N$ and then $4rAL \nsubseteq (N:M)N$. By Lemma 2.4, $rA \subseteq (N:M) = K$ or $rL \subseteq N$ or $AL \subseteq N$. If $AL \subseteq N$, then we are done. Suppose $AL \nsubseteq N$. Then $r \in (K:A) \cup (N:L)$. We show that $B \subseteq (K:A) \cup (N:L)$. Let $s \in B$. If $4sAL \nsubseteq (N:M)N$, then by Lemma 2.4, $s \in (K:A) \cup (N:L)$ as $AL \nsubseteq N$. Suppose $4sAL \subseteq (N:M)N$. Then $4(r+s)AL \subseteq N - (N:M)N$ and then by Lemma 2.4, $r+s \in (K:A) \bigcup (N:L)$ as $AL \nsubseteq N$. We study four subcases:

Subcase 1. $r+s \in (K:A)$ and $r \in (K:A)$. Then $s \in (K:A)$.

Subcase 2. $r+s \in (N:L)$ and $r \in (N:L)$. Then $s \in (N:L)$.

Subcase 3. $r \in (K:A)-(N:L)$ and $r+s \in (N:L)-(K:A)$. Then $2r+s \notin (K:A)$ and $2r+s \notin (N:L)$ and then $2r+s \notin (K:A)\cup(N:L)$. Now, $4(2r+s)AL \nsubseteq (N:M)N$ and hence by Lemma 2.4, $2r+s \in (K:A)\cup(N:L)$ as $AL \nsubseteq N$, which is a contradiction. As $r \in (K:A)\cup(N:L)$ and $r+s \in (K:A)$ $\cup(N:L)$, we have two situations:

(1) $r \in (N:L)$ and $r+s \in (N:L)-(K:A)$, so $s \in (N:L)$.

(2) $r \in (K:A)-(N:L)$ and $r+s \in (K:A)$, so $s \in (K:A)$.

Subcase 4. $r+s \in (K:A)-(N:L)$ and $r \in (N:L)-(K:A)$. Then, as Subcase 3, $s \in (K:A)\cup(N:L)$. Thus, $B \subseteq (K:A)\cup(N:L)$.

Case 2. $8ABN \nsubseteq (N:M)N$ and $8ABL \subseteq (N:M)N$. Then $8AB(L+N) \subseteq N-(N:M)N$ and then, by Case 1, $BA \subseteq (N:M)$ or $B(L+N) \subseteq N$ or $A(L+N) \subseteq N$ and then $BA \subseteq (N:M)$ or $BL \subseteq N$ or $AL \subseteq N$.

Case 3. $8B(N:M)L \nsubseteq (N:M)N$ and $8ABL \subseteq (N:M)N$. Then, $8B(A+(N:M))L \nsubseteq (N:M)N$ and then, by Case 1, $8(A+(N:M)) \subseteq (N:M)$ or $BL \subseteq N$ or $(A+(N:M))L \subseteq N$ which implies that $BA \subseteq (N:M)$ or $BL \subseteq N$ or $AL \subseteq N$. Similarly, if $8A(N:M)L \nsubseteq (N:M)N$. ∎

Theorem 2.7 *Let M be an R-module and N be an almost 2-absorbing R-submodule of M. Suppose $r,s \in R$ and $m \in M$ such that $rsm \in (N:M)N$, $rs \notin (N:M)$, $rm \notin N$ and $sm \notin N$. Then $rsN \subseteq (N:M)N$.*

Proof. Suppose $rsN \nsubseteq (N:M)N$. Then there exists $n \in N$ such that $rsn \notin (N:M)N$ and then $rs(m+n) \in N-(N:M)N$. Since N is almost 2-absorbing, $rs \in (N:M)$ or $r(m+n) \in N$ or $s(m+n) \in N$ and then $rs \in (N:M)$ or $rm \in N$ or $sm \in N$ which are contradiction. Hence, $rsN \subseteq (N:M)N$. ∎

Proposition 2.8 *Let (R,M) be a quasi-local ring. Then every proper ideal of R is almost 2-absorbing ideal if $M^3=0$.*

Proof. Let $a,b,c \in M$ with $abc \neq 0$. Since $abc \neq 0$ then $abc \in M-M^2$. So $ab \in M$ or $ac \in M$, or $bc \in M$ then a or b, or c is not in M, which is a contradiction. Hence $abc=0$. ∎

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