On the Effect of Body Capacitance to Ground in Tetrapolar Bioimpedance Measurements

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Abstract—Tetrapolar bioimpedance measurements on subjects have long been suspected of being affected by stray capacitance between the subjects’ body and ground. This paper provides a circuit model to analyze that effect in the frequency range from 100 Hz to 1 MHz in order to identify the relevant parameters when impedance is measured by applying a voltage and measuring both the resulting current and the potential difference between two points on the surface of the volume conductor. The proposed model includes the impedance of each electrode and the input impedance of the differential voltage amplifier. When common values for the circuit parameters are assumed, the simplified model predicts: 1) a frequency-independent gain (scale factor) error; 2) inductive artifacts, that is, the measured impedance increases with increasing frequency and may include positive angle phases; and 3) resonance that can affect well below 1 MHz. In addition to the stray capacitance to ground, relevant parameters that determine those errors are the capacitance of the “low-current” electrode and the input capacitance of the differential voltage amplifier. Experimental results confirm those theoretical predictions and show effects from several additional resonances above 1 MHz that also depend on body capacitance to ground.

Index Terms—Bioimpedance measurement, electrode impedance, four-electrode measurements, parasitic capacitance.

I. INTRODUCTION

TETRAPOLAR bioimpedance measurements theoretically exclude the contribution from electrode polarization impedance to the measurement result [1]. Typically, current is injected into the sample tissue or volume conductor by two electrodes and the potential difference between two points is measured by a separate electrode pair. The unknown impedance is then calculated by dividing this voltage drop by the current injected. Hence, if part of the injected current follows a path other than that between the potential electrodes, the result will be wrong. Stray capacitance from the body to (earth) ground offers such a possible separate path, particularly when measuring large volume conductors at frequencies higher than about 100 kHz.

Gersing et al. [2] experimentally found that the capacitance from the body to earth lead to positive phase angles when measuring resistive–capacitive impedances. Scharfetter et al. [3] demonstrated through circuit simulation that body capacitance to earth and capacitance between signal ground and earth can lead to positive phase angles, and experimentally found that significant phase angle errors can occur at 100 kHz, but did not measure any positive phase angle in the range from 5 kHz to 1 MHz. Experiments by Grimnes and Martinsen [1] demonstrated that positive phase angles in resistive–capacitive ionic conductors can also result from tissue electrical properties and electrode positions, even at 1 kHz. In summary, stray capacitance from the body to (earth) ground can yield positive phase angles in bioimpedance measurements, but positive phase angles should not always be attributed to stray capacitance, neither that capacitance always results in positive phase angles below 1 MHz. Hence, there is a need to identify the relevant parameters that together with stray capacitance to ground affect bioimpedance measurement results.

Tetrapolar bioimpedance measurements can be implemented by two basic approaches. One approach is to apply a known current and measure the drop in voltage, whereas the other approach is to apply a voltage and measure both the resulting current and the drop in voltage. The first method results in large common-mode voltage if a single-ended current source is used. Differential current sources reduce that voltage [4], but in any case it is difficult to design current sources with high-output impedance at high frequency [5]. Hence, the second approach is often preferred and it is the one implemented by some commercial autobalancing bridges. These instruments need a front-end amplifier when using electrodes instead of common ohmic contacts, because electrode impedance is higher [6].

This paper analyzes the effect of the stray capacitance from the body to ground on tetrapolar bioimpedance measurements in the range from 100 Hz to 1 MHz when performed by a commercial impedance analyzer (Agilent 4294 A) and a custom-designed front-end amplifier designed to work with pregelled electrodes. A preliminary version of this study has been reported [7].

II. INSTRUMENT AND IMPEDANCE MODELS

A. Use of an Autobalancing Bridge for Tetrapolar Bioimpedance Measurements

Commercial autobalancing bridges are normally intended for impedance measurements performed with ohmic (i.e., low-impedance) contacts, hence, cannot be directly applied to bioimpedance measurements which need electrodes, as these
have large interface impedance. Appendix A shows the operating mode of the commercial impedance analyzer used in this study, a possible connection to apply it to measurements involving electrodes, and the error that can result from their impedances.

A method to overcome electrode impedance error was proposed by Gersing [8] and relies on using an external differential amplifier and rearranging the input connections of the instrument as shown in Fig. 1. \( Z_x \) is the impedance to be measured; \( Z_t \) and \( Z_s \) are the impedances of the electrodes used to inject current plus those of the body segment between each current-injection electrode and the closer voltage-detection electrode, whose impedances are \( Z_2 \) and \( Z_4 \). HC (high current), LC (low current), HP (high potential), and LP (low potential) are the four input terminals of the impedance analyzer. In common two-terminal measurements (see Appendix A), current through \( Z_x \) is forced by a voltage \( V_{osc} \) applied between HC and signal ground, and the drop in voltage across \( Z_x \) is measured between HP (connected to HC) and LP, which is kept at 0 V (virtual signal ground) by a negative feedback circuit, and is connected to LC. In the modified arrangement in Fig. 1, LC and LP are connected too, but HP is not connected to HC but to the output of an instrumentation amplifier (IA) connected to \( Z_x \) through electrodes #2 (\( Z_2 \)) and #4 (\( Z_4 \)). Therefore, there is not any electrode impedance in the feedback loop of the circuit used to measure the current at the LC terminal, as opposed to the circuit in Fig. A2. However, if stray capacitance from the body to ground is large enough, a significant part of the current that flows in \( Z_x \) will be diverted to ground and will not reach terminal LC.

**B. Circuit Model for Tetrapolar Bioimpedance Measurements Including Stray Capacitances**

The circuit in Fig. 2 models tetrapolar bioimpedance measurements performed according to the method in Fig. 1. The impedances of current electrodes \( Z_t \) and \( Z_s \) have been divided into two parts: electrode-skin interface \( Z_{ch}, Z_{cl} \) and body impedance between each current electrode and the closer potential electrode \( Z_{cp}, Z_{pc} \). By applying the same labeling criterion to potential electrodes, their (skin) interface impedances have been termed \( Z_{ph} \) and \( Z_{pl} \), instead of, respectively, \( Z_2 \) and \( Z_4 \) (as termed in Fig. 1). These two potential electrodes are connected to an instrumentation amplifier (IA) whose input impedance is \( Z_{in} \). Hence, if the current injected into the body is \( I_x \) and the drop in voltage between the potential electrodes is \( V_x \), we will have \( V_x = V_{osc}/I_x \) provided \( Z_{in} \gg Z_x \) and all current injected follows the path between potential electrodes. However, there are stray capacitances between current and potential electrodes \( (C_{ch}, C_{cl}, C_{ph}, C_{pl}) \), between each electrode and (earth) ground \( (C_{sh}, C_{sp}, C_{ph}, C_{pl}) \), and from the body to ground \( C_b \). Shielding and guarding cables can largely reduce the effect of those stray capacitances (\([3], [5]\)) exception made of \( C_b \) that cannot be avoided and will basically depend on body size and its closeness to grounded objects. For standing people, \( C_b \) can be from 111 pF to 3.9 nF (\([9], [10]\)), and these large values can affect bioimpedance even when measured at relatively low frequencies. Therefore, we focus on the effects of \( C_b \) because if part of the current at the HC terminal in Fig. 2 flows to ground via \( C_b \) that current will not appear at the LC terminal and \( I_x \) will be smaller than the current actually flowing into (part of) \( Z_x \). Current through \( C_b \) will increase with increasing frequency, \( I_x \) will decrease, and \( Z_x \) will apparently increase the same as if it had an inductive component.

Contrarily to what Fig. 2 may suggest, \( C_b \) is a distributed capacitance. However, at frequencies below 1 MHz, models with lumped capacitances are commonly accepted [11]. A method to model \( C_b \) as a lumped capacitance is shown in Fig. 3 where \( Z_{ph} \) and \( Z_{pl} \) are the impedances of the electrodes used to inject current plus those of the body segment between each current-injection electrode and the closer voltage-detection electrode.
\[ \alpha < 1; \alpha = 0.5 \] would imply a symmetrical coupling from each point of the body to ground. The circuit model in Fig. 3 also includes the impedance of each potential electrode \( Z_{ph}, Z_{pl} \), the input impedance of the voltage amplifier \( Z_{in} \), and the two segmental body impedances between each current electrode and the closest potential electrode \( Z_{pc}, Z_{ic} \).

Circuit analysis in Appendix B shows that the transimpedance measured is

\[ Z_{in} = \frac{V_x}{I_x} = \frac{Z_x Z_{in}}{1 + j \omega \alpha C_b [(1 - \alpha) Z_x + Z_{pc} + Z_{ic}]} \]

\[ \frac{Z_{ph} + Z_{in} + Z_{pl} + Z_{eq}}{Z_{in}} \]

where from (B1)

\[ Z_{eq} = Z_x [1 + j \omega \alpha (1 - \alpha) C_b Z_x] \]

Equation (1) shows that a finite \( Z_{in} \) results in a nonlinear dependence of \( Z_{in} \) on \( Z_x \) (because \( Z_{eq} \) in the denominator includes \( Z_x \)), and that \( C_b \) yields an additional nonlinear effect in the numerator, and also a gain (or scale factor) error that depends on \( Z_{pc} \) (body impedance between the low-potential and low-current electrodes) and \( Z_{ic} \) (impedance of the low-current electrode).

The target impedance \( Z_x \) can be modeled by a resistance \( R_x \) in series with a resistance \( R \) shunted by a capacitance \( C \) [12]

\[ Z_x = R_x + \frac{R}{1 + j \omega RC} \]

If \( R_x = 500 \Omega, R = 1500 \Omega, \) and \( C = 220 \text{nF} \) [13], or \( R_x = 415 \Omega, R = 125 \Omega, \) and \( C = 24 \text{nF} \) [11], at low frequencies we have \( Z_x \approx R_x + R \), whereas at high frequencies where \( \omega RC \gg 1 \), \( Z_x \approx R_x \). This last condition is fulfilled above 500 Hz for the values obtained from [13] and above 50 kHz for the values obtained from [11]. This discrepancy arises from the different model values: those in [11] are intended to better approximate the impedance measured at very high frequencies, whereas those in [13] better model the body impedance at lower frequencies. In both cases, hand-to-foot impedances were estimated from two-terminal measurements and (hand) touch contact, which results in larger \( Z_x \) values than those expected from four-terminal measurements with pregelled electrodes.

The same type of three-component model can be used to describe electrode-skin impedance [14]. Potential electrodes \( Z_{ph}, Z_{pl} \) can be modeled by \( R_e, R_i, \) and \( C_p, C_c \), and the “low-current” electrode \( Z_{ic} \) impedance can be modeled by \( R_x, R_i, \) and \( C_x \). For gelled electrode-skin impedance [14], the series resistance \( R_e, R_i \) is about 100 \( \Omega \), the capacitance \( C_p, C_c \) is from 10 to 40 \text{nF}, and the resistance in parallel with that capacitance \( R_p, R_c \) is from 10 kHz to 1 MHz. If the wires that connect the potential electrodes to the amplifier are very short, \( C_x \) can be less than 5 \text{pF}, and therefore \( C_p \gg C_{in} \), but \( C_{in} \) still determines the input impedance of the amplifier even at frequencies as low as 100 Hz because the input resistance of instrumentation amplifiers can be as high as 1 T\( \Omega \), higher in fact than the surface resistance of some printed circuit boards. Then, if \( Z_x \approx R_x \) in (2) we can approximate

\[ j \omega \alpha (1 - \alpha) C_i Z_x \approx j \omega \alpha (1 - \alpha) R_x C_i \]

Furthermore, whenever \( \omega R_e C_p \gg 1 \), that is, when the measurement frequency is higher than the cutoff frequency of the electrodes, in the numerator of (1) we can also approximate

\[ j \omega \alpha C_b Z_{ic} = j \omega \alpha C_b \left( R_i + \frac{R_e}{1 + j \omega RC_i} \right) \]

\[ \approx j \omega \alpha R_i C_b + \frac{a C_b}{C_i} \]

If \( C_p \gg 2C_{in} \), replacing these two factors in (1) and operating yields

\[ Z_x \approx \frac{1 + \alpha C_b/C_e + j \omega \alpha C_b [(1 - \alpha) R_x + R_i + Z_{pc}]}{1 - \omega^2 \alpha (1 - \alpha) R_e^2 C_b C_i + j \omega (R_x + 2R_e) C_{in}} \]

Therefore, a nonzero capacitance to ground \( C_b \) implies: 1) a frequency-independent gain error that depends on \( \alpha C_b/C_e \); 2) a gain error that increases with increasing frequency (hence and inductive artifact) and with \( R_i + Z_{pc} \) (“low-current” electrode resistance plus body impedance between this electrode and the “low-potential” electrode); 3) a nonlinearity effect that increases with frequency and with the measured resistance \( R_x \); and 4) a resonance at

\[ f_{res} = \frac{1}{2\pi R_x \sqrt{\alpha (1 - \alpha) C_x C_{in}}} \]

If \( C_b \) were negligible, only \( C_{in} \) would affect, by introducing a first-order low-pass response whose cutoff frequency \( f_{in} \) in hertz would be \( 2\pi (R_x + 2R_e) C_{in} \)^{-1}; for \( C_{in} = 3 \text{pF} \) and \( R_x + 2R_e = 1 \text{k\Omega} \, f_{in} \approx 53 \text{MHz} \) hence above the measurement range.

Equation (6) shows that current and potential electrodes play different roles in the undesired effects that arise because of \( C_b \) (and \( C_{in} \)) and therefore they cannot be interchanged, as one would ideally expect from the reciprocity theorem applied to \( Z_x \) when considered alone. Furthermore, current electrodes cannot be interchanged between themselves either, as for this impedance measurement method the high-current electrode \( Z_{hc} \) has no effect whereas the low-current electrode does. On the other hand, when \( C_b \) is negligible, the effect of \( C_{in} \) depends on the resistance of potential electrodes but not on that of current electrodes.

III. MATERIALS AND METHODS

We measured arm-to-arm impedance from 100 Hz to 10 MHz for seven subjects using pregelled electrodes (SkinTact RT-34, Aqua-Tac gel). Although we were only interested on bioimpedance measurements below 1 MHz, we measured up to 10 MHz because any parasitic resonance between 1 and 10 MHz can affect results even below 1 MHz. A current-injection electrode was attached to each wrist and potential electrodes were placed on the forearm at 10 cm from them. The impedance meter was an Agilent 4294 A impedance analyzer preceded by a custom-built front-end amplifier connected as shown in Fig. 1. The respective gain, phase shift, and CMRR for the differential front-end amplifier at 100 Hz, 1 MHz, and 10 MHz were 1.0000/0.720 dB, 1.000/0.88/84 dB, and 1.03/0.8/66 dB. These parameters were obtained by measuring a reference resistor.
about 3 pF, hence the assumption close to ground, the measured impedance increases at all frequencies.

The large CMRR and the small voltage (250 mV) applied by the impedance analyzer minimized errors from common mode signals, such as those described in [1]. The estimated \( C_{in} \) was about 3 pF, hence the assumption \( C_p \gg C_{in} \) that leads to the simplified (6) is right. Wires connecting electrodes to instrument terminals were 10 to 15 cm long, which required the two hands to be very close to the (grounded) instrument enclosure. These wires were untwisted and unshielded to minimize the capacitance from each electrode to ground.

Subjects were seated on an office chair mostly built from reinforced plastic and their arms rested on the wooden bench where the impedance analyzer was placed. Two measurement series were performed for each subject: first with their feet (with shoes) resting on the floor, and then with their feet raised about 10 cm from ground, which should reduce \( C_b \) and hence reveal a change in the measured \( Z_x \).

IV. EXPERIMENTAL RESULTS AND DISCUSSION

Figs. 4 and 5 show the modulus and phase angle of the arm-to-arm impedance measured for subject #1 (1.90 m, 130 kg). The broken straight lines in Fig. 4 are the asymptotes of the magnitude (modulus) of the target impedance \( Z_x \). The corner frequencies defined by the intersections of the asymptotes do not correspond to a −3 dB attenuation of the modulus, and the roll-off slope is far less than 20 dB/decade. This means that the pole \( \omega_{px} = (RC)^{-1} \) and the zero \( \omega_{zx} = (R_x || R || C)^{-1} \) defined by \( Z_x \) are quite close, and that because \( \omega_{px} \) is always smaller than \( \omega_{zx} \), in (3) we have \( R < R_x \). The closeness between \( \omega_{px} \) and \( \omega_{zx} \) yields the small phase angle change (less than 8°) observed in Fig. 5, which confirms that \( Z_x \) is mostly resistive, as assumed to obtain (4).

In Fig. 4, the effect of resting the feet on ground is barely perceptible below 100 kHz when wearing shoes. This can be because of both the relative low body resistance \( R_x \) for that subject and the small \( C_b \) change that makes \( \alpha C_b/C_e \) in (6) to remain about the same. To induce a larger change in \( C_b \), the subject was asked to place his bare feet on a plastic film on ground. The change in the measured impedance modulus was then perceptible at all frequencies.

For the three \( C_b \) values, several resonances appear between 1 and 10 MHz that have a noticeable effect on the modulus and on the phase angle below 1 MHz. For \( R_x = 215 \, \Omega \), \( \alpha = 0.5 \), \( C_b = 120 \, \text{pF} \), and \( C_{in} = 3 \, \text{pF} \), from (7), \( f_{res} \approx 78 \, \text{MHz} \), hence outside the measurement bandwidth. If \( R_x \) and \( C_b \) were much larger, that resonance could happen below 10 MHz, but in any case the other resonances would remain unexplained. These resonances may result from stray capacitive coupling to ground from the segmental impedances \( Z_{c1} \) and \( Z_{p1} \), that exist between current and potential electrodes (see Figs. 2 and 3) because, although these segments were short (10 cm), they were close to ground (instrument enclosure). Nevertheless, connecting an isolated metal electrode between any of these two segments and ground did not bring up any other resonances, neither did their frequencies significantly change. The most noticeable change was the increase of the inductive component due to the larger \( C_b \) in the numerator of (6). Other sources for those resonances could be some of the stray capacitances in Fig. 2 that have been excluded from the model. Other authors have also pointed to circuit self-inductance [1] and transmission line effects [3]. If this were the case, whole-body impedance measurements performed with cables much longer than those used here to connect the forearms to the analog front end to the impedance analyzer would probably be more affected by those stray resonances.

The effects of a larger \( C_b \) on the phase angle are shown in Fig. 5. From (3), the phase angle should decrease from 0° at low frequencies and then increase again toward 0° at higher frequencies. However, positive phase angle values are reached at a frequency that decreases with increasing \( C_b \). This can be the result of the increase of the apparent inductive component predicted by (6).

Measurement results for the other six subjects, five men and one woman (subject #7), are summarized in Table I. They were qualitatively the same obtained for subject #1, but in subjects with large impedance (\( R_x \) compared to \( R \) in Fig. 4) no asymptotical value could be reached at high frequencies. For all subjects, when their feet rested on ground, the magnitude of the measured impedance increased at all frequencies and the frequency where the phase angle of the impedance became positive \( f_z \) decreased with respect to that \( f_1 \) when the impedance was measured with
feet 10 cm above ground. For a given subject, the impedance increase was about the same at low and medium frequencies (20–50 kHz) and was larger at higher frequencies (inductive effect). The magnitude of the impedance for men was inversely proportional to the body mass index (Weight/(Height^2)). The increase of the measured impedance with \( C_b \) at low and medium frequencies and the additional increase at high frequencies depended on the subject’s height and weight, and were probably affected by the thickness of the soles of their shoes. According to (6), the resistance \( R_i \) and the capacitance \( C_c \) of the low-current electrode-skin interface should affect too. Four resonances were observed for all subjects at slightly different frequencies between 1 and 10 MHz, whose relative amplitudes were the same (peak amplitudes increase with frequency).

From (6), the input capacitance \( C_{in} \) of the amplifier should affect the measured impedance. To assess this effect, 4.6 pF were connected between the amplifier input terminals so that the estimated input capacitance was 7.6 pF instead of 3 pF, and subject #1 was measured with his bare feet resting on a plastic sheet on ground. The magnitude of the impedance did not perceptibly change but its phase angle decreased because it started to be positive at 0.75 MHz instead of 0.42 MHz. Contrarily to what it could be expected from (6), the larger \( C_{in} \) did not affect any resonance between 1 and 10 MHz. However, increasing \( C_b \) shifted all those resonances to lower frequencies, but did not bring inside the measured frequency band the resonance expected from (6).

A common test in tetrapolar bioimpedance measurements is to interchange current injection and voltage detection electrodes. If electrode impedances are assumed not to affect, the reciprocity theorem applies and no effect should be observed. However, according to (1) the impedance \( Z_{al} \) of the “low-current” electrode should affect the measured impedance when \( C_b \) is not zero. Applying this test when measuring on subject #1 with feet about 10 cm above ground (small \( C_b \)), did not reveal any noticeable change in the magnitude of the impedance but the phase angle of the impedance increased: the frequency where it started to be positive decreased from 1.12 to 0.94 MHz.

Finally, bioimpedance measurements in tissue samples or organs can probably be affected by the capacitance to ground too, but the effects may be different from those here observed because of stray interelectrode capacitances, shown as \( C_{sh}, C_{al}, \) and \( C_{sc} \) in Fig. 2 but neglected in the subsequent analysis here, as the distance between electrodes on whole body measurements are larger.

V. Conclusion

Stray capacitance form the body (to earth) ground had been pointed out by different authors as a possible source of error in tetrapolar bioimpedance measurements above 100 kHz. By considering only the capacitive coupling from the body segment between potential electrodes to ground \( (C_b) \) in Figs. 2 and 3, and assuming common values for electrode and body impedances, we have obtained a simplified circuit model (6) that predicts gain and nonlinearity errors at any frequency band, and resonance at frequencies above 10 MHz. There is a frequency-dependent gain error that increases with increasing frequency, hence it looks like an “inductive effect” that is proportional to \( C_b \) and to the sum of the measured resistance \( R_i \), the resistance of the “low-current” electrode \( R_I \), and the body impedance between this electrode and the “low-potential” electrode. There is also a frequency-independent gain error that increases according to the ratio \( C_j/C_c \), where \( C_j \) is the capacitance of the low-current electrode, and that shows as an offset of the magnitude of the impedance.

The input capacitance \( C_{in} \) of the differential amplifier used to measure the potential difference introduces a low-pass frequency response and, together with \( C_b \), resonance. However, by keeping \( C_{in} < 10 \text{ pF} \), the low pass frequency is above 10 MHz and having resonance below 10 MHz when \( R_i \approx 230 \Omega \) would require \( C_b > 2 \text{ nF} \).

Experimental results confirm the negligible effects from \( C_{in} \), and both the frequency-independent gain error and the inductive effect that are proportional to \( C_b \), but instead of a single resonance, several resonances are observed between 1 and 10 MHz that for larger \( C_b \) values affect the magnitude and phase angle of the impedance below 1 MHz.

APPENDIX A: TETRAPOLAR IMPEDANCE MEASUREMENTS USING AN AUTOBALANCING PSEUDODRIDGE

Fig. A1 shows the equivalent circuit for the Agilent 4294 A impedance analyzer [15]. The null detector (ND) controls the amplitude and phase angle of voltage source \( V_r \) until \( I_d = 0 \). Then, \( I_e = I_f = -V_r/R_e \), and \( Z_e = V_r/I_e = -R_eV_r/I_f \). Stray capacitance from HC to ground has no effect if \( Z_{osc} \) is small enough, neither has the capacitance from HP to ground because of the large input impedance of the voltmeter that measures \( V_r \). Stray capacitances from LC and LP to ground have no effect because both terminals are at 0 V.

Replacing the null detector in Fig. A1 by an ideal operational amplifier leads to the equivalent circuit in Fig. A2. This circuit is easier to understand because of its similitude to a transimpedance amplifier, and clearly shows that \( Z_e \) can be calculated from two single-ended voltages \( (V_r \) and \( V_f \)).

If the impedances of the connections to \( Z_e \) are not negligible, as happens when they are electrodes rather than ohmic contacts, the equivalent circuit is that in Fig. A3 [15]. LC potential is
no longer 0 V, and  

\[ V_r = -I_r (R_r + Z_3) \]

hence resulting in a measurement error [6]. Because in measurements on an animal body,  

\[ Z_3 \]

and  

\[ Z_4 \]

include the impedance of the body segments between each current electrode and the closest voltage electrode that error will depend not only on electrode impedance but also on electrode placement. Furthermore, because  

\[ Z_3 \]

is undetermined, its presence in a feedback loop could lead to unstable measurement.

APPENDIX B: TRANSIMPEDANCE MEASURED BY AN AUTOBALANCING PSEUDOBridge WITH ACTIVE FRONT END

The circuit to be analyzed is that in Fig. 3. Delta-wye impedance transformation between nodes #1 and #2 leads to the circuit in Fig. B1, where  

\[ Z_{eq} = Z_x + \frac{\alpha Z_x (1 - \alpha) Z_x}{Z_b} = Z_x [1 + j \omega \alpha (1 - \alpha) C_b Z_x] \]

(B1)

\[ Z_{1b} = \alpha Z_x + \frac{Z_b}{1 - \alpha} \]  

(B2)

\[ Z_{2b} = (1 - \alpha) Z_x + \frac{Z_b}{\alpha} \]  

(B3)

The aim is now to calculate \( (V_1 - V_2)/I_x \). First, the direct impedance between nodes #1 and #2 in Fig. B2 is calculated

\[ Z_{12} = Z_{eq} || (Z_{ph} + Z_{in} + Z_{pl}) . \]  

(B4)

Next, the Thevenin equivalent circuits to the left of node #1 and to the right of node #2 are obtained as shown in Fig. B3. This leads to

\[ Z_1 = Z_{2b} || (Z_{pc} + Z_{cl}) \]  

(B5)

\[ V_2 = I_{12} Z_1 = I_{12} [Z_{2b} || (Z_{pc} + Z_{cl})] = I_x (Z_{pc} + Z_{cl}) . \]  

(B6)

\[ I_{12} = I_x \frac{Z_{2b} + Z_{pc} + Z_{cl}}{Z_{2b}} \]  

(B7)

From this last equation

\[ V_1 - V_2 = I_{12} Z_{12} = I_x \frac{Z_{12} Z_{2b}}{Z_{2b}} (Z_{2b} + Z_{pc} + Z_{cl}) . \]  

(B8)

Finally, from Fig. B2

\[ V_1 - V_2 = I_p (Z_{ph} + Z_{in} + Z_{pl}) = \frac{V_p}{Z_{in}} (Z_{ph} + Z_{in} + Z_{pl}) . \]  

(B9)
and from (B8) and (B9) we obtain

\[
\frac{V_x}{I_x} = \frac{Z_{in} (Z_{2b} + Z_{pc} + Z_{cl}) Z_{12}}{(Z_{ph} + Z_{in} + Z_{pl}) Z_{2b}}. \tag{B10}
\]

Replacing now the expression for \( Z_{12} \) in (B4) yields

\[
\frac{V_x}{I_x} = \frac{Z_{in} (Z_{2b} + Z_{pc} + Z_{cl}) Z_{eq}}{(Z_{ph} + Z_{in} + Z_{pl} + Z_{eq}) Z_{2b}}. \tag{B11}
\]

But from (B1) and (B3)

\[
\frac{Z_{eq}}{Z_{2b}} = \frac{\alpha Z_x}{Z_b} \tag{B12}
\]

Hence, (B11) can be rewritten as

\[
\frac{V_x}{I_x} = \frac{Z_{in} (Z_{2b} + Z_{pc} + Z_{cl}) Z_x \alpha}{Z_{ph} + Z_{in} + Z_{pl} + Z_{eq} Z_b} \tag{B13}
\]

and from (B3) and the expression for \( Z_b \), we obtain

\[
Z_{in} = \frac{V_x}{I_x} Z_x Z_{in} \left[ 1 + j \omega \alpha C_b \left( \frac{1 - \alpha}{Z_x + Z_{pc} + Z_{cl}} \right) \right] \frac{Z_{ph} + Z_{in} + Z_{pl} + Z_{eq}}{Z_{ph} + Z_{in} + Z_{eq}} \tag{B14}
\]

which is the same as (1) in the main text.

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