Abstract

Continuous Galerkin formulations are appealing for their low computational cost, whereas discontinuous Galerkin formulation facilitate adaptive mesh refinement and are more accurate in regions with jumps of physical parameters. Since many electromagnetic problems involve materials with different physical properties, this last point is quite important. For this reason, in this article we have developed a combined cG-dG formulation for Maxwell’s problem that allows arbitrary finite element spaces with functions continuous in patches of finite elements and discontinuous on the interfaces of these patches. In particular, the second formulation we propose comes from a novel continuous Galerkin formulation that reduces the amount of stabilization introduced in the numerical system. In all cases, we have performed stability and convergence analyses of the methods. The outcome of this work is a new approach that keeps the low CPU cost of the recent nodal continuous formulations with the ability to deal with coefficient jumps and adaptivity of discontinuous ones. All these methods have been tested using a problem with singular solution, in order to prove that in fact these formulations can properly deal with these problems.

Keywords: Maxwell’s problem, electromagnetism, finite elements, stabilization techniques, discontinuous Galerkin methods, singular solutions

1 Introduction

The simulation of electromagnetic phenomena increasingly demands more complex geometries and larger scale problems. As an example, Maxwell’s equations are in the core of plasma physics and magnetohydrodynamics simulations, coupled with the Vlasov and Navier-Stokes equations respectively. Finite element (FE) methods are a popular approach for the numerical simulation of Maxwell’s systems, since realistic geometries can be handled via unstructured grids and can be straightforwardly applied to multi-physics. Furthermore, they posses a strong mathematical foundation that allows one to analyze stability and convergence properties of the algorithms.
Maxwell’s equations can be casted in a saddle-point mathematical structure, with the particular feature that the multiplier is zero. With a crude Galerkin formulation, we are enforced to use inf-sup stable FE spaces; for this particular problem, this type of elements are the so-called Nédélec’s FEs for the magnetic field and continuous Lagrangian interpolations for the multiplier. Unfortunately, exact penalty formulations that eliminate the Lagrange multiplier from the equations and allow FE spaces that do not satisfy the inf-sup condition lead to spurious solutions (see [13]). In order to rehabilitate the exact penalty formulation, some techniques have been proposed, e.g. the weighted penalty formulation in [13] or the decomposition of singular and smooth parts of the solution in [2, 18]. Unfortunately, these approaches require to know where the singularities will appear (re-entrant corners) and so, are hard to use in an automatic and general way. Very recently, the authors have considered an alternative way to circumvent the inf-sup condition in [3]. We still consider a saddle-point formulation formulation, but now using an equivalent augmented formulation and a stabilized FE discretization of the problem. This way, the resulting method allows one to capture singular solutions for continuous nodal (Lagrangian) FE spaces that do not satisfy the discrete inf-sup condition of the original system. Further, the method is very easy to implement, essentially general and automatically applicable to any problem, without the need to know where the singularities will take place.

Nodal FE approximations are very appealing in terms of CPU cost versus accuracy and by the simplicity of data bases (both the solution and Lagrange multiplier can be approximated using the same FE space). Furthermore, they are very effective when considering multi-physics problems, since all the unknowns of the different problems can be solved using the same finite dimensional spaces. For instance, in the magnetohydrodynamics (MHD) problem, the Navier-Stokes equations are often solved using nodal FE stabilization techniques. Analogously, the Vlasov equations in plasma physics can be solved this way. It is clear that this kind of approach greatly simplifies multi-physics codes and monolithic solvers.

However, continuous Lagrangian formulations, onwards denoted as cG, have some drawbacks that can be solved using discontinuous nodal formulations, indicated as dG formulations, that enforce continuity weakly. Examples of dG formulations for Maxwell’s problem can be found in [22, 19]. dG approaches are expensive but allow local mesh adaptation and are more accurate in regions in which the solution exhibits jumps, i.e. regions with jumps of physical parameters. Since many electromagnetic problems involve materials with different physical properties, this last point is quite important. For this reason, in this article we have developed a combined cG-dG formulation that allows arbitrary FE spaces with functions continuous in patches of FEs and discontinuous on the interfaces of these patches; this is the sense in which the term “combined” is used in this article. This way, we are able to restrict weak continuity to a very reduced number of nodes (e.g., material interfaces or refined regions). As a result, the cG-dG approach shares the low CPU cost of cG formulations with the ability to deal with adaptivity and different materials.

The outline of the paper is as follows. In Section 2 we state the continuous problem and consider different functional settings, as well as the cG formulation in [3] and the dG one in [19]. The combined cG-dG formulation is introduced in Section 3. We have considered two alternative formulations, the second one coming from a new cG formulation based on projections that introduces less numerical dissipation. In all cases, we have performed stability and convergence analyses. Finally, in Section 4 we present some numerical experiments for a problem with singular solution.
2 Problem statement

2.1 Notation

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, with $d = 2, 3$ the space dimension. Given a Banach space $X$, we denote its associated norm by $\| \cdot \|_X$; for the sake of conciseness, we will omit the subscript for the $L^2(\Omega)$ space of square integrable functions. The space of vector-valued functions with components in $X$ is denoted by $X^d$. The dimension superscript will be omitted in the norm, i.e., we will simply denote its norm by $\| \cdot \|_X$ instead of $\| \cdot \|_{X^d}$. The dual space of $X$ is denoted as $X'$. The inner product between two scalar or vector functions $f_1, f_2 \in L^2(\Omega)$ is denoted by $(f_1, f_2)$, whereas $\langle f_1, f_2 \rangle$ is used for a duality pairing in $X \times X'$ based on the integral.

$W^{s,m}(\Omega)$ is used for the standard Sobolev space, with real coefficients $s \geq 0$ and $m \geq 1$. Hilbert spaces $W^{s,2}(\Omega)$ are denoted by $H^s(\Omega)$. We write $H^1_0(\Omega)$ for the space of functions in $H^1(\Omega)$ with null trace on $\partial \Omega$. We will make use of the following spaces of vector fields:

$H(\text{div}; \Omega) := \{ v \in L^2(\Omega)^d \text{ such that } \nabla \cdot v \in L^2(\Omega) \}$,

$H(\text{curl}; \Omega) := \{ v \in L^2(\Omega)^d \text{ such that } \nabla \times v \in L^2(\Omega)^d \}$,

and the subspaces

$H(\text{div } 0; \Omega) := \{ v \in H(\text{div}; \Omega) \text{ such that } \nabla \cdot v = 0 \}$,

$H_0(\text{curl}; \Omega) := \{ v \in H(\text{curl}; \Omega) \text{ such that } n \times v = 0 \text{ on } \partial \Omega \}$.

We use the notation $A \lesssim B$ to indicate that $A \leq CB$, where $A$ and $B$ are expressions depending on functions that in the discrete case may depend on the discretization as well, and $C$ is a positive constant.

2.2 The continuous problem

We consider the Maxwell problem, which physically describes magnetostatics in a bounded domain $\Omega$ surrounded by a perfect conductor. Let us consider $\Omega \subset \mathbb{R}^d$ to be a simply connected in general non-convex polyhedral domain with a connected Lipschitz continuous boundary $\partial \Omega$. Besides its range of applicability, this system of partial differential equations exhibits the mathematical complications encountered in more involved model problems (see, e.g., [14, 21]). Maxwell’s problem can be stated as a saddle-point problem by enforcing the divergence constraint with a Lagrange multiplier $p$. The Euler-Lagrange equations read as follows: seek a pair $(u, p)$ solution of

\[
\nabla \times (\lambda \nabla \times u) - \varepsilon \nabla p = f, \\
\n\nabla \cdot (\varepsilon u) = 0,
\]

with $n \times u = 0$ and $p = 0$ on $\partial \Omega$, for some divergence-free datum $f$; $\lambda(x)$ and $\varepsilon(x)$ are positive coefficient functions such that

\[
0 < \lambda_- \leq \lambda(x) \leq \lambda_+ < \infty, \quad 0 < \varepsilon_- \leq \varepsilon(x) \leq \varepsilon_+ < \infty.
\]
Taking the divergence of (1a) and invoking the boundary condition for \( p \) one easily proves that \( p = 0 \).

The variational interpretation of the mixed problem (1) admits two functional settings. The so-called curl formulation reads as: find \( u \in H_0(\text{curl}; \Omega) \) and \( p \in H_0^1(\Omega) \) such that

\[
(\lambda \nabla \times u, \nabla \times v) - (\varepsilon \nabla p, v) = (f, v), \quad \forall v \in H_0(\text{curl}; \Omega),
\]

\[
(\varepsilon \nabla q, u) = 0, \quad \forall q \in H_0^1(\Omega),
\]

where \( f \in H(\text{div}; \Omega) \) is assumed. However, this is not the only functional setting in which the problem is well-posed; the \( H^1(\Omega) \) regularity for \( p \) can be “transferred” to \( u \), leading to a curl-div variational formulation: find \( u \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) and \( p \in L^2(\Omega)/\mathbb{R} \) such that

\[
(\lambda \nabla \times u, \nabla \times v) + (p, \nabla \cdot (\varepsilon v)) = (f, v), \quad \forall v \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega),
\]

\[
-(q, \nabla \cdot (\varepsilon u)) = 0, \quad \forall q \in L^2(\Omega).
\]

On the other hand, since \( p = 0 \), the curl-div formulation can be equivalently written as an exact penalty method: seek \( u \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) such that

\[
(\lambda \nabla \times u, \nabla \times v) + \left( \frac{\lambda}{\varepsilon^2} \nabla \cdot (\varepsilon u), \nabla \cdot (\varepsilon v) \right) = (f, v), \quad \forall v \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega).
\]

The equivalence between these three formulation can be found e.g. in [17]. At the discrete level, however, these formulations lead to methods with quite different properties.

In the curl formulation (3), let us denote by \( V \) the space \( H(\text{curl}; \Omega) \) for the solution \( u \) and by \( Q \) the space \( H^1_0(\Omega) \) for the Lagrange multiplier \( p \). Furthermore, let us write (3) in compact form as

\[
c(u, p, v, q) = (f, v) \quad \text{where} \quad c(u, p, v, q) = (\lambda \nabla \times u, \nabla \times v) - (\varepsilon \nabla p, v) + (\varepsilon \nabla q, u).
\]

In [3], the authors have proposed a novel numerical approximation of the Maxwell problem whose starting point is a different augmented formulation of the continuous problem. Since we are interested in a curl formulation for reasons that will be explained later on, we add the term \( \frac{\ell^2}{\lambda_0} \nabla \cdot (\varepsilon^2 \nabla p) \) to (1b); \( \ell > 0 \) has dimensions of length whereas \( \lambda_0 \) is an average value of \( \lambda(x) \), e.g. \( \lambda_0 := |\Omega|^{-1} \int_{\Omega} \lambda \).

\( \ell \) and \( \lambda_0 \) are introduced in order to end up with a dimensionally consistent method. A length scale is inherent to the problem, since it is needed to define dimensionally consistent norms for these spaces, which are given by

\[
\|v\|_V := \|v\|_{H(\text{curl}; \Omega)} = \frac{1}{\ell} \|v\| + \|\nabla \times v\|,
\]

\[
\|q\|_Q := \|q\|_{H^1_0(\Omega)} = \frac{1}{\ell} \|q\| + \|\nabla q\|,
\]

where \( \ell = \ell(\Omega) \) makes the norms dimensionally consistent. Theoretically, this length scale comes from the Poincaré-Friedrichs inequality of the problem at hand. Let us define \( \ell_\lambda := \frac{\ell}{\lambda_0^{1/2}} \). The augmented formulation in strong form consists of finding \( u \) and \( p \) such that

\[
\nabla \times (\lambda \nabla \times u) - \varepsilon \nabla p = f,
\]

\[
-\nabla \cdot (\varepsilon u) - \ell_\lambda^2 \nabla \cdot (\varepsilon^2 \nabla p) = 0,
\]

in \( \Omega \), satisfying \( u \times u = 0 \) and \( p = 0 \) on \( \partial \Omega \). Since \( p \in Q \) is identically zero, the penalty is exact. The equivalence of the new formulation (8) and (3) can be found in [3].
2.3 Numerical approximation

Although all the formulations introduced above are equivalent, stable and consistent, numerical approximations of the curl-div formulations (4) and (5) lead to spurious solutions for nonconvex domains, e.g. domains with re-entrant corners. Costabel provided in [11] a mathematical justification to this surprising observation: since Ω is not convex, $V \cap H^1(\Omega)^d$ is a closed proper subspace of $V \cap H(\text{div}; \Omega)$. One could think that a $H^1$-conforming FE space cannot be used, since the FE space is a closed proper subspace of $V \cap H(\text{div}; \Omega)$ and the FE solution $u_h$ can be shown to be bounded in $H^1(\Omega)^d$ (see e.g. [17] or [15, Corollary 2.30]). However, this reasoning is wrong in general. The approximability lost only happens when the sequence of solutions $\{u_h\}_{h>0}$ is uniformly bounded in $H^1(\Omega)$, which is only true for curl-div formulations (see [3, Corollary 1]). Therefore, approximations based on (4) and (5) cannot capture solutions $u \not\in V \cap H^1(\Omega)^d$ of the Maxwell problem (3), so, are not suitable for numerical purposes. This kind of solutions are called nonsmooth or singular solutions. Note that the key for this negative result is the spurious control on the divergence of the approximations based on (4) and (5), which implies that the whole gradient is uniformly bounded in $L^2(\Omega)$, since $u_h$ is a $H^1(\Omega)^d$ function for all $h$. It is not an approximability problem of FEs spaces $V_h \subset H^1(\Omega)^d$ for $h$ fixed, which may well approximate $H(\text{curl}; \Omega)$, as we will see later. These FE spaces are dense not only in $H(\text{curl}; \Omega)$, but also in $L^2(\Omega)^d$.

On the other hand, problem (3) in the good functional setting, with a crude Galerkin discretization can only be used with inf-sup stable elements satisfying:

$$\inf_{(u_h,p_h)\in V_h \times Q_h \setminus \{0,0\}} \sup_{(v_h,q_h)\in V_h \times Q_h \setminus \{0,0\}} \frac{c(u_h,p_h;v_h,q_h)}{\|u_h\|_V \|v_h\|_Q \|p_h\|_Q} \geq \beta_d > 0,$$  \hspace{1cm} (9)

for $\beta_d > 0$ uniform with respect to $h$ (see e.g. [7]). This is satisfied by $V_h$ built by the celebrated Nedelec’s (or edge) elements; those elements are only conforming in $H(\text{curl}; \Omega)$, since they do not satisfy normal continuity over the element faces. A nodal FE space can be used for $Q_h$ (see, e.g., [23]).

In the next, we consider an alternative nodal (Lagrangian) FE approach to this problem, that can circumvent (9). It is motivated by the augmented formulation (8). For simplicity, we assume that $\lambda$ and $\varepsilon$ are elementwise constant.

2.3.1 Finite element notation

Let $T_h$ be a partition of $\Omega$ into a set of FEs $\{K\}$. For every element $K$, we denote by $h_K$ its diameter, and set the characteristic mesh size as $h = \max_{K \in T_h} h_K$. We denote by $h_f$ the piecewise constant function that takes the value $h_K$ at every element $K \in T_h$. For simplicity, we consider a regular and quasi-uniform family $\{T_h\}_{h>0}$ of FE partitions. The space of polynomials of degree less or equal to $k > 0$ in a FE $K$ is denoted by $P_k(K)$. Summation over all the elements will be indicated by $\sum_K$ and the broken integral $\sum_K \int_K$ will be denoted by $\int_{T_h}$.

Suppose now that elements $K_1$ and $K_2$ share an edge $E$, and let $n_1$ and $n_2$ be the normals to $E$ exterior to $K_1$ and $K_2$, respectively. For a scalar function $f$, possibly discontinuous across $E$, we
define its jump and average as

\[ [f] := n_1 f_{\partial K_1 \cap E} + n_2 f_{\partial K_2 \cap E}, \]
\[ \{f\} := \frac{1}{2} (f_{\partial K_1 \cap E} + f_{\partial K_2 \cap E}), \]

whereas for vectorial quantities we will use

\[ [v] := n_1 \cdot v_{\partial K_1 \cap E} + n_2 \cdot v_{\partial K_2 \cap E}, \quad [v]|_t := n_1 \times v_{\partial K_1 \cap E} + n_2 \times v_{\partial K_2 \cap E}, \]
\[ \{v\} := \frac{1}{2} (v_{\partial K_1 \cap E} + v_{\partial K_2 \cap E}). \]

We extend these definitions on \( \partial \Omega \) as \( [f] := n f \) and \( \{f\} := f \) and similarly for vector functions. Further, let us define the mesh size \( h_{\perp} \) (defined over the set of edges of the mesh) that on \( E \) takes the value \( h_{\perp} := (x_{b_2} - x_{b_1}) \cdot n_1 > 0 \), where \( x_{b_2} \) and \( x_{b_1} \) are the barycentric coordinates of elements \( K_1 \) and \( K_2 \).

The space of discontinuous piecewise polynomials of order \( k > 0 \) is defined as

\[ \mathcal{H}_k(\Omega) := \left\{ v_h \in L^2(\Omega) \text{ such that } v_h|_K \in P_k(K) \forall K \in \mathcal{T}_h \right\}. \]

(10)

Any function in \( \mathcal{H}_k(\Omega) \) can be uniquely determined by its values on a set of points (nodes) in \( \Omega \) (see [6, 15]), and so this is a nodal FE approximation. Continuous FE spaces are denoted as \( \mathcal{N}_k := \mathcal{H}_k \cap C^0(\Omega) \), where \( C^0(\Omega) \) is the space of continuous functions in \( \Omega \). These approximations are usually called \( H^1 \)-conforming approximations since, because of the inter-element continuity, functions in \( \mathcal{N}_k \) are \( H^1 \) for each fixed \( h \).

For quasi-uniform partitions, there is a constant \( C_{\text{inv}} \), independent of the mesh size \( h \) (the maximum of all the element diameters), such that

\[ \|\nabla v_h\|_{L^2(K)} \leq C_{\text{inv}} h^{-1} \|v_h\|_{L^2(K)}, \]
\[ \|\Delta v_h\|_{L^2(K)} \leq C_{\text{inv}} h^{-1} \|\nabla v_h\|_{L^2(K)}, \]

(11)

(12)

for all FE functions \( v_h \) defined on \( K \in \mathcal{T}_h \). This inequality can be used for scalars, vectors or tensors. Similarly, the trace inequality

\[ \|v\|^2_{L^2(\partial K)} \leq C_{\text{tr}} \left( h^{-1} \|v\|^2_{L^2(K)} + h \|\nabla v\|^2_{L^2(K)} \right) \]

(13)

is assumed to hold for functions \( v \in H^1(K), K \in \mathcal{T}_h \). If \( \psi_h \) is a piecewise (continuous or discontinuous) polynomial, the last term in the previous inequality can be dropped using an inverse inequality, getting \( \|\psi_h\|^2_{L^2(\partial K)} \lesssim h^{-1} \|\psi_h\|^2_{L^2(K)} \).

### 2.3.2 A cG nodal solver

For the reasons explained above, nodal elements have always been related to curl-div conforming formulations, whereas edge elements have always been related to curl formulations. In [3] the authors claim that this correspondence is false. So, they propose a stabilized FE formulation based on the
augmented formulation (8), which is stated in the curl setting and does not require to satisfy (9) for stability purposes. In fact, the resulting method is stable for any pair of Lagrangian FE spaces.

The cG method in [3] can be motivated as a residual-based stabilized discretization of the exact augmented formulation (8). Let us consider nodal FE approximations \( V_h = \mathcal{N}_k^h \cap V \) and \( Q_h = \mathcal{N}_l^h / \mathbb{R} \) of the continuous spaces \( V \) and \( Q \), respectively; \( k, l > 0 \) are the orders of approximation for \( u \) and \( p \), respectively; there is no restriction between \( k \) and \( l \), and equal-order approximations are allowed. The method consists of seeking \( u_h \in V_h \) and \( p_h \in Q_h \) solution of

\[
(\lambda \nabla \times u_h, \nabla \times v_h) - (\varepsilon \nabla p_h, v_h) + s_u(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{14a}
\]

\[
(\varepsilon \nabla q_h, u_h) + s_p(p_h, q_h) = 0, \quad \forall q_h \in Q_h, \tag{14b}
\]

where the stabilization terms read:

\[
s_u(u_h, v_h) = \frac{c_u}{\ell^2} \int_{T_h} h^2 \nabla \cdot u_h \nabla \cdot v_h, \quad s_p(p_h, q_h) = \ell^2 \int_{T_h} \varepsilon^2 \nabla p_h \cdot \nabla q_h, \tag{15}
\]

c, being an algorithmic constant; the subscript \( K \) denotes elementwise functions. The reason why the term \( s_u \) is needed becomes evident from both theoretical analysis and numerical experimentation. Obviously, as \( h \to 0 \) this term vanishes, and the method is not a curl-div conforming algorithm.

We can easily see that (14) is a residual-based FE approximation of the augmented formulation (8) (see e.g. [20, 9]). However, it has been observed in [4] that also the multiplier stabilization term can be motivated as a residual-based stabilization term, and so, it does not require to pass through the augmented formulation (see Remark 3.1). This interpretation is appealing for multi-physics problems, since residual-based stabilized formulation are available for the Navier-Stokes and Darcy equations, and a unified framework for all these problems can be considered.

### 2.3.3 A dG nodal solver

Nodal dG formulations for the Maxwell problem have been proposed e.g. in [22, 19]. The method in [22] included some terms that were not essential and were eliminated in the more recent formulation [19]. This last formulation is based on an interior penalty dG approximation of the Maxwell problem in the curl setting (3). For the sake of brevity, we just indicate in the next section the terms in the combined cG-dG formulation that have to be eliminated to recover the dG formulation in [19]. Further details in the motivation of the method and its numerical analysis can be found in the original paper. Let us remark that the method proposed in [19] is only suitable for equal-order approximations of \( u \) and \( p \).

### 3 The combined cG-dG formulation

#### 3.1 First formulation of the method

We consider an agglomeration of the triangulation \( \mathcal{T}_h \) into a set of \( n_p \) patches. Let \( S_h \) be the partition of \( \Omega \) into patches. We assume that the agglomeration is such that every patch \( S \in S_h \) is connected.
At this point, we can define the set of FE functions in the combined cG-dG formulation as follows: we denote the cG-dG nodal FE space of order \( k \) as

\[
\mathcal{M}_k(\mathcal{S}_h, \mathcal{T}_h) := \{ v_h \in L^2(\Omega) \text{ such that } v_h|_K \in \mathcal{P}_h(K), \forall K \in \mathcal{T}_h, \\
\text{and } v_h|_S \in C^0(S), \forall S \in \mathcal{S}_h \}.
\]

We simply denote this finite dimensional space as \( \mathcal{M}_k \) for the sake of conciseness. Let us indicate by \( \mathcal{F}_h := \cup_{S \in \mathcal{S}_h} \partial S \) the set of patch interfaces (edges), whereas the interior interfaces are denoted by \( \mathcal{F}_h^0 := \mathcal{F}_h \cap \partial \Omega \). The \( L^2 \) norm over the manifold \( \mathcal{F}_h \) is denoted by \( \| \cdot \|_{\mathcal{F}_h} \), and analogously for \( \mathcal{F}_h^0 \). We also denote the broken integral over the patches as \( \int_{\mathcal{S}_h}(\cdot) := \sum_{S \in \mathcal{S}_h} \int_S(\cdot) \). The broken norm over \( \mathcal{S}_h \) is denoted by \( \| \cdot \|_{\mathcal{S}_h} \). Limit cases in this approach are the dG case in which there is no agglomeration, i.e. every FE space becomes a patch (\( \mathcal{S}_h = \mathcal{T}_h \)), and the cG case in which there is only one patch (\( \mathcal{S}_h = \{ \Omega \} \)).

We can easily check that the proposed nodal cG approach cannot be straightforwardly used with \( \mathcal{M}_k \), since \( C^0 \) continuity has been assumed in its design. On the other hand, a dG formulation is not stable when the unknowns are continuous in patches, because it assumes inclusions between the FE pairs for \( u_h \) and \( p_h \) that are not true in general. So, a stable cG-dG formulation requires to combine the stable nodal formulation in [3] with an effective dG formulation (e.g., the one in [19]) in a suitable way.

The cG-dG method we propose is designed starting with the cG nodal formulation and adding the corresponding boundary terms on \( \mathcal{F}_h \). It is well-known that it is not enough with the additional terms that appear after integration-by-parts, and stabilizing jump terms have to be added; we refer to [1] for a detailed exposition of different approaches. In particular, we use an interior penalty stabilization, since this is the approach used in [19]. Let us pick \( V_h = (\mathcal{M}_k)^d \) and \( Q_h = \mathcal{M}_\ell \); we consider the general case in which different approximation orders can be used for the different unknowns.

Let us motivate the jump terms, for the sake of completeness. Integrating by parts \( \nabla \times (\lambda \nabla \times u) \) for \( u \) the exact solution, against a function \( v_h \in V_h \), we have:

\[
\begin{align*}
\int_{\Omega} v_h \cdot \nabla \times (\lambda \nabla \times u) &= \int_{\mathcal{S}_h} \nabla \times v_h \cdot \lambda \nabla \times u - \sum_S \int_{\partial S} n \times v_h \cdot \lambda \nabla \times u \\
&= \int_{\mathcal{S}_h} \nabla \times v_h \cdot \lambda \nabla \times u - \int_{\mathcal{F}_h} [v_h]_t \cdot \{ \lambda \nabla \times u \} - \sum_S \int_{\partial S} n \times \{ v_h \} \cdot \lambda \nabla \times u \\
&= \int_{\mathcal{S}_h} \nabla \times v_h \cdot \lambda \nabla \times u - \int_{\mathcal{F}_h} [v_h]_t \cdot \{ \lambda \nabla \times u \} - \int_{\mathcal{F}_h} [u]_t \cdot \{ \lambda \nabla \times v_h \},
\end{align*}
\]

(16)

after invoking the continuity of the solution and its flux. Using a typical dG formulation, the gradient term may be written as

\[
\int_{\Omega} \varepsilon \nabla p \cdot v_h = \int_{\mathcal{S}_h} \varepsilon \nabla p \cdot v_h - \int_{\mathcal{F}_h} [p] \cdot \{ \varepsilon v_h \},
\]

(17)

whereas the divergence term reads as:

\[
\int_{\Omega} q_h \nabla \cdot (\varepsilon u) = - \int_{\mathcal{S}_h} \varepsilon \nabla q_h \cdot u + \int_{\mathcal{F}_h} [q_h] \cdot \{ \varepsilon u \}.
\]

(18)
A new term that comes from the proposed augmented formulation and does not appear in [19] is the multiplier Laplacian. We have:

\[
- \int_{\Omega} q_h \nabla \cdot (\varepsilon^2 \nabla p) = \int_{S_h} \varepsilon^2 \nabla q_h \cdot \nabla p - \int_{F_h} [q_h] \cdot \{ \varepsilon^2 \nabla p \} - \int_{F_h} [\varepsilon^2 \nabla p] \{ q_h \} = \int_{S_h} \varepsilon^2 \nabla q_h \cdot \nabla p, \tag{19}
\]

where we have neglected the boundary terms, since \( p = 0 \) and they are only effective for consistency but not for stability purposes. Finally, in order to prove stability in the appropriate broken norms to be defined, we will introduce to the final bilinear form of the problem the interior penalty stabilizing terms:

\[
c_{tu} \int_{F_h} \left\{ \frac{\lambda}{h} \right\} [u_h]_t \cdot [v_h]_t + c_{mu} \frac{h}{\varepsilon \lambda} \int_{F_h} \left\{ \frac{h}{\varepsilon} \right\} [\varepsilon u_h] [\varepsilon v_h] + c_{np} \varepsilon^2 \int_{F_h} \left\{ \frac{\varepsilon^2}{h} \right\} [p_h] \cdot [q_h],
\]

where \( c_{tu}, c_{mu} \) and \( c_{np} \) are algorithmic constants.

Continuing the cG approach (14a)-(14b) with the dG strategy described, the cG-dG formulation we propose reads as follows: find \( u_h \in V_h \) and \( p_h \in Q_h \) such that

\[
a_h(u_h, v_h) - b_h(p_h, v_h) = (f, v_h), \quad \text{for any } v_h \in V_h, \tag{20a}
\]
\[
b_h(q_h, u_h) + s_h(p_h, q_h) = 0, \quad \text{for any } q_h \in Q_h, \tag{20b}
\]

where the discrete forms have the expressions:

\[
a_h(u_h, v_h) := \int_{S_h} \lambda \nabla \times v_h \cdot \nabla \times u_h - \int_{F_h} [v_h]_t \cdot \{ \lambda \nabla \times u_h \} - \int_{F_h} [u_h]_t \cdot \{ \lambda \nabla \times v_h \}
\]
\[+ c_{tu} \int_{F_h} \left\{ \frac{\lambda}{h} \right\} [u_h]_t \cdot [v_h]_t + c_{mu} \frac{h}{\varepsilon \lambda} \int_{F_h} \left\{ \frac{h}{\varepsilon} \right\} \nabla \cdot u_h \nabla \cdot v_h + c_{mu} \frac{h}{\varepsilon \lambda} \int_{F_h} \left\{ \frac{h}{\varepsilon} \right\} [\varepsilon u_h] \varepsilon v_h,
\]
\[
b_h(p_h, v_h) := \int_{S_h} \varepsilon \nabla p_h \cdot v_h - \int_{F_h} [p_h] \cdot \{ \varepsilon v_h \},
\]
\[
s_h(p_h, q_h) := \frac{\varepsilon^2}{\lambda} \int_{S_h} \varepsilon \nabla q_h \cdot \nabla p_h + c_{np} \varepsilon^2 \int_{F_h} \left\{ \frac{\varepsilon^2}{h} \right\} [p_h] \cdot [q_h]. \tag{21}
\]

**Remark 3.1** For constant \( \varepsilon \) the grad-grad term in (19) can also be motivated as a residual-based stabilization term of streamline-upwind Petrov-Galerkin type (see [4]), since:

\[
(\nabla \times (\lambda \nabla \times u) - \varepsilon \nabla p - f, \varepsilon \nabla q) = (\varepsilon \nabla p, \varepsilon \nabla q).
\]

**Remark 3.2** In the most general case, in which we consider a non-solenoidal forcing term \( f \) and \( u \) such that \( \nabla \cdot (\varepsilon u) = g \), with \( g(x) \) different from zero, we can still use the previous formulation. In this case, the multiplier \( p \) is no longer zero, in general. Instead, it satisfies \( -\nabla \cdot (\varepsilon \nabla p) = \nabla \cdot f \), which is obtained taking the divergence of Eq. (1a). So, Eq. (8b) in the augmented formulation now reads

\[
\nabla \cdot u - \varepsilon^2 \lambda \nabla \cdot (\varepsilon^2 \nabla p) = g + \varepsilon^2 \lambda \nabla \cdot f,
\]
and Eq. (20b) is now

\[ b_h(q_h, u_h) + s_h(p_h, q_h) = (g + \ell_2^A \nabla \cdot f, q_h). \]

Let us remind that in this case, the consistency terms that have been eliminated from (19) must be kept now. This straightforward generalization of our approach makes it suitable for the approximation of the so-called generalized formulations of Maxwell’s equations, which are a key ingredient in the numerical simulation of plasmas via Vlasov-Maxwell approximations. Roughly speaking, these solutions are such that \( p \neq 0 \). Since \( f \) is numerically obtained in some cases, \( \nabla \cdot f \) is going to be close to zero at most. Formulations that essentially assume that \( p = 0 \) in their definition are not appropriate for these problems. We refer to [5] and references therein for a detailed exposition of these concepts.

Let us make some comments about the resulting formulation for the limit cases of cG and dG-type approximation spaces. In the first case, we can easily see that all the interior face (edge) terms cancel out and we recover the original cG formulation in [3]. However, when considering the dG case, we do not recover the interior penalty formulation in [19]. Two additional terms appear, namely the element interior stabilization terms:

\[ c_h \int_{T_h} h_i^2 \nabla \cdot u_h \nabla \cdot v_h + \ell_2^A \int_{S_h} \varepsilon^2 \nabla p_h \cdot \nabla q_h. \]

These terms are required for stability purposes when the inclusions

\[ \nabla \cdot V_h \subseteq Q_h, \quad \nabla Q_h \subseteq V_h \] (22)

are not true. We can easily see that for dG equal order FE approximations these inclusions are satisfied; since the method in [19] was restricted to this particular case, these terms were not needed. However, in the most general case, in which we consider a dG formulation with different approximation order for \( V_h \) and \( Q_h \) or we consider the combined cG-dG approach, these terms cannot be neglected.

In the following, we perform the numerical analysis of the cG-dG formulation. First, let us introduce some norms for the stability analysis of the method:

\[ |||u_h|||_h := \| \lambda^\frac{3}{2} \nabla \times u_h \|_{S_h} + \| \{ \lambda \}^\frac{3}{2} h_1^{-\frac{1}{2}} [\nabla u_h] \|_{F_h} + \ell_\lambda^{-1} |||h_1 \nabla \cdot u_h|||_{S_h}, \]

\[ |||p_h|||_h := \ell_\lambda \| \nabla p_h \|_{S_h} + \ell_\lambda |||h_1^{-\frac{1}{2}} \{ \varepsilon \} p_h |||_{S_h}. \] (23)

The norm in the product space \( V_h \times Q_h \) is denoted by \( |||u_h, p_h|||_h := |||u_h|||_h + |||p_h|||_h \), whereas the norm that also includes \( L^2 \) stability over \( u_h \) is simply denoted by \( |||u_h, p_h||| := |||u_h, p_h||| + \ell_\lambda^{-1} |||u_h||| \).

Let us denote the bilinear form of the problem as

\[ c_h(u_h, p_h, v_h, q_h) := a_h(u_h, v_h) + b_h(p_h, v_h) - b_h(q_h, u_h) + s_h(p_h, q_h). \] (24)

In the next theorem, we prove coercivity of this bilinear form with respect to \( \| \cdot \| \). This analysis is inspired by the results in [3] and [19] for the cG and dG versions, respectively. For the sake of conciseness, we consider \( \lambda = 1 \), \( \varepsilon = 1 \), \( \ell = 1 \) and \( h_1 = h_1 = h \) in the following proofs. Anyway, these proofs apply to the general case, with minor modifications.
Theorem 3.1 The stabilized bilinear form (24) with the expressions in (21), with $c_{nu} > C_{inv}$, satisfies the following coercivity property:

$$c_h(u_h, p_h, u_h, p_h) \geq \alpha \|u_h, p_h\|^2$$

for a constant $\alpha > 0$ uniform with respect to $h$.

Proof The stability of the problem in the mesh-dependent norm is straightforward. We easily obtain:

$$c_h(u_h, p_h, u_h, p_h) = \|\lambda^{\frac{1}{2}} \nabla \times u_h\|_{S_h}^2 + c_{tu} \|\{\lambda\}^{\frac{1}{2}} h^{\frac{1}{2}} [u_h]_F^2 + \frac{c_{nu}}{\ell^2} h^2 \nabla \cdot u_h\|_{S_h}^2$$

$$+ \frac{c_{nu}}{\ell^2} \|h^{\frac{1}{2}} \{\varepsilon\}^{-1} [\varepsilon u_h]\|_{F_h}^2 - 2 \int_{F_h} [u_h]_t \cdot \{\nabla \times u_h\}.$$

Now, using the lifting operator $L$ defined in [22, Sec. 5.1],

$$\int_{\Omega} L(v) \cdot w_h := \int_{F_h} v \cdot \{w_h\}, \quad \forall w_h \in D_h,$

where $D_h$ is a space of possibly discontinuous piecewise polynomials such that $\nabla \times \nabla \nabla \subset D_h$, and the continuity property

$$\|L([v_h]_t)\| \leq C_{inv} h^{-\frac{1}{2}} \|[v_h]_t\|_{F_h},$$

proved in [22], we get

$$\int_{F_h} [u_h]_t \{\nabla \times u_h\} = \int_{\Omega} L([u_h]_t) \cdot \nabla \times u_h \leq C_{inv} h^{-\frac{1}{2}} \|[u_h]_t\|_{F_h} \|\nabla \times u_h\|$$

$$\lesssim \frac{C_{inv}^2}{2\delta} \|\nabla \times u_h\|^2 + \frac{\delta h^{-1}}{2} \|[u_h]_t\|_{F_h}^2. \tag{25}$$

Using the fact that $C_{inv} < c_{nu}$, we take $C_{inv}^2 < \delta < c_{nu}^2$. This way, we can absorb all these terms by $\|u_h, p_h\|^2_h$. At this point, it only remains to prove that $\|u_h\| \lesssim \|u_h, p_h\|$. Let us consider the auxiliary problem:

$$\nabla \times \nabla \times z + \nabla \varphi = u_h, \quad \nabla \cdot z = 0, \tag{26}$$

with $n \times z = 0$ and $\varphi = 0$ on $\partial \Omega$. By the continuous inf-sup condition we have

$$\|\nabla \times z\| + \|\nabla \varphi\| \lesssim \|u_h\|.$$

Given $w := \nabla \times z$, we know from [16] that there exists $w_0 \in H_0^1(\Omega)$ such that

$$\nabla \times w_0 = \nabla \times w \quad \text{in } L^2(\Omega), \quad \|w_0\|_1 \lesssim \|w\|_{H_0(\text{curl}, \Omega)} \lesssim \|u_h\|.$$

Now, we multiply the first equation in the auxiliary problem by $u_h$ and integrate over $\Omega$, getting

$$\int_{\Omega} u_h \cdot \nabla \times \nabla \times z + \int_{\Omega} u_h \cdot \nabla \varphi = \|u_h\|^2. \tag{27}$$
The first term is integrated by parts, obtaining
\[
\int_{\Omega} \mathbf{u}_h \cdot \nabla \times \nabla \times \mathbf{z} = \int_{S_h} \nabla \times \mathbf{u}_h \cdot \mathbf{w}_0 - \int_{F_h} [\mathbf{u}_h]_\ell \cdot \{\mathbf{w}_0\} - \int_{F_h} \mathbf{n} \times \{\mathbf{u}_h\} \cdot \mathbf{w}_0
\]
\[
= \int_{S_h} \nabla \times \mathbf{u}_h \cdot \mathbf{w}_0 - \int_{F_h} [\mathbf{u}_h]_\ell \cdot \{\mathbf{w}_0\} \lesssim \| \nabla \times \mathbf{u}_h \|_{S_h} \| \mathbf{w}_0 \|_1 + \| h^{-\frac{1}{2}} \mathbf{w}_0 \|_{F_h} \| h^{-\frac{1}{2}} \{\mathbf{u}_h\}_\ell \|_{F_h}
\]
\[
\lesssim \| \mathbf{u}_h, p_h \|_h \| \mathbf{w}_0 \|_1. \tag{28}
\]

where we have used (13). For the second term, let us introduce \( \varphi_h \), an optimal projection of \( \varphi \) onto \( Q_h \) with \( H^1 \)-continuity that keeps the homogeneous boundary condition, e.g. the Scott-Zhang projector (see [6]). We get
\[
\int_{\Omega} \mathbf{u}_h \cdot \nabla \varphi = \int_{\Omega} \mathbf{u}_h \cdot \nabla (\varphi - \varphi_h) + \int_{S_h} \nabla \varphi_h \cdot \nabla p_h, \tag{29}
\]
where we have used the divergence constraint discrete equation. Now, we treat the first term as follows:
\[
\int_{\Omega} \mathbf{u}_h \cdot \nabla (\varphi - \varphi_h) = -\int_{\Omega} \nabla \cdot \mathbf{u}_h (\varphi - \varphi_h) + \int_{F_h} [\mathbf{u}_h]_\ell (\varphi - \varphi_h)
\]
\[
\lesssim (h \| \nabla \cdot \mathbf{u}_h \|_{S_h} + h^{-\frac{1}{2}} \| [\mathbf{u}_h]_\ell \|_1) (h^{-1} \| (\varphi - \varphi_h) \| + h^{-\frac{1}{2}} \| (\varphi - \varphi_h) \|_{F_h})
\]
\[
\lesssim (h \| \nabla \cdot \mathbf{u}_h \|_{S_h} + h^{-\frac{1}{2}} \| [\mathbf{u}_h]_\ell \|_1) \| \varphi \|_1 \tag{30}
\]

The treatment of the second term is easy:
\[
\int_{S_h} \nabla \varphi \cdot \nabla p_h \lesssim \| \nabla p_h \|_{S_h} \| \varphi \|_1.
\]
Combining all these results, we show that \( \| \mathbf{u}_h \| \lesssim \| \mathbf{u}_h, p_h \|_h \). It proves the theorem.

The next result states optimal convergence for the combined cG-dG formulation. We define the norms,
\[
e_1(\mathbf{v}, q) := \| \mathbf{v}, q \| + \frac{h^{\frac{1}{2}}}{\ell_\lambda} \| \mathbf{v} \|_{F_h}, \quad e_2(\mathbf{v}) := \frac{1}{\ell_\lambda} \| \mathbf{v} \| + \frac{h^{\frac{1}{2}}}{\ell_\lambda} \| \mathbf{v} \|_{F_h}. \tag{31}
\]
as well as the error function:
\[
E_h(\mathbf{v}, q) := \inf_{\mathbf{v}_h \in \mathbf{V}_h, q_h \in Q_h} e_1(\mathbf{v} - \mathbf{v}_h, q - q_h) + \ell_\lambda \inf_{\mathbf{w}_h \in \mathbf{V}_h} e_2(\nabla \times \mathbf{v} - \mathbf{w}_h).
\]

**Theorem 3.2** The solution \((\mathbf{u}_h, p_h)\) of system (20)-(21) with \( c_{nu} \geq C_{inv} \) satisfies the error estimate
\[
\| \mathbf{u}_h - \mathbf{u}, p_h - p \| \lesssim E_h(\mathbf{u}, p),
\]
where \((\mathbf{u}, p)\) is the solution of the continuous problem.
**Proof** Once again, we consider $\lambda = 1$, $\varepsilon = 1$ and $\ell = 1$ to simplify the proof. Since there is no consistency error, we can easily check that for any $\tilde{u}_h \in V_h$ and $\tilde{p}_h \in Q_h$, we have

$$c_h(\tilde{u}_h - u_h, \tilde{p}_h - p_h, \tilde{u}_h - u_h, \tilde{p}_h - p_h) = c_h(\tilde{u}_h - u, \tilde{p}_h - p, \tilde{u}_h - u_h, \tilde{p}_h - p_h).$$

Stability has already been proved above. The only remaining point is to bound the interpolation error. Let us prove that

$$c_h(\tilde{u}_h - u, \tilde{p}_h - p, v_h, q_h) \lesssim (e_1(u - \tilde{u}_h, p - \tilde{p}_h) + e_2(\nabla \times u - w_h)) \|v_h, q_h\|, \quad (32)$$

for any $v_h, w_h \in V_h$ and $q_h \in Q_h$. The bound for the symmetric jump terms is straightforward. For the sake of conciseness, we only bound the most interesting terms. The element interior stabilization term over for the divergence is treated as follows:

$$h^2 \int_{S_h} \nabla \cdot (\tilde{u}_h - u) \nabla \cdot v_h = -h^2 \int_{S_h} (\tilde{u}_h - u) \nabla \nabla \cdot v_h + h^2 \int_{\mathcal{F}_h} n \cdot (\tilde{u}_h - u) \nabla \cdot v_h$$

$$\lesssim \|\tilde{u}_h - u\|_h \|\nabla \cdot v_h\|_{S_h} + h^\frac{1}{2} \|	ilde{u}_h - u\|_{\mathcal{F}_h} \|\nabla \cdot v_h\|_{\mathcal{F}_h}$$

$$\lesssim (\|\tilde{u}_h - u\| + h^\frac{1}{2} \|	ilde{u}_h - u\|_{\mathcal{F}_h}) \|\nabla \cdot v_h\|_{S_h}, \quad (33)$$

where we have used integration-by-parts and inverse inequalities. For any $w_h \in V_h$, we also have:

$$\int_{\mathcal{F}_h} [v_h]_t \cdot \{\nabla \times (\tilde{u}_h - u)\} = \int_{\mathcal{F}_h} [v_h]_t \cdot \{\nabla \times \tilde{u}_h - w_h\} + \int_{\mathcal{F}_h} [v_h]_t \cdot \{w_h - \nabla \times u\}$$

$$\leq h^{-\frac{1}{2}} \|[v_h]_t\|_{\mathcal{F}_h} (\|\nabla \times (\tilde{u}_h - u)\| + \|w_h - \nabla \times u\| + h^\frac{1}{2} \|w_h - \nabla \times u\|_{\mathcal{F}_h}). \quad (34)$$

where we have used again inverse inequalities. The rest of terms are easily bounded. Finally, using the coercivity of $c_h$ proved in Theorem 3.1 in expression (32), together with the fact that $\|u - v_h\| \lesssim e_1(u - v_h)$ for any $v_h \in V_h$ and the triangle inequality, we prove the theorem. ⊡

For smooth solutions, i.e. $u \in H^1(\Omega)$, the previous error bounds are optimal. The case of nonsingular solutions is not so clear. Under some assumptions over the typology of the mesh, we can prove that $\lim_{h \to 0} E_h(u, p) = 0$ without regularity requirements (see [3] and references therein). Similar situations have been found in [13, 8]. One can show that at least $u \in H^s(\Omega)$ with $s > \frac{1}{2}$, and $\nabla \times u \in H^s(\Omega)$ (see [12] for a detailed exposition). So, the second component of the error function, that measures the distance between $\nabla \times u$ and $V_h$ measured in terms of $e_2$, cannot spoil the convergence towards singular solutions.

### 3.2 An alternative formulation based on projections

As commented above, the previous cG-dG algorithm is not reduced to the dG interior penalty formulation in [19]. We can however consider a modified formulation, in which the element interior stabilization is reduced. The idea consists of stabilizing only the components of $\nabla \cdot u_h$ and $\nabla p_h$ that are orthogonal to $Q_h$ and $V_h$, respectively; the corresponding FE projections are bounded by the
Galerkin terms. Therefore, the new method consists of (20) with the following new expressions for the bilinear forms:

\[
\begin{align*}
a_h(\mathbf{u}_h, \mathbf{v}_h) & := \int_{S_h} \lambda \nabla \times \mathbf{v}_h \cdot \nabla \times \mathbf{u}_h - \int_{S_h} \| \mathbf{v}_h \|_t \cdot \{ \lambda \nabla \times \mathbf{u}_h \} - \int_{S_h} [\mathbf{u}_h]_t \cdot \{ \lambda \nabla \times \mathbf{v}_h \} \\
& + c_{tu} \int_{\mathcal{F}_h} \frac{1}{h} [\mathbf{u}_h]_t \cdot [\mathbf{v}_h]_t + \frac{c_{tu}}{\ell^2} \int_{\mathcal{F}_h} h P_{Q_h}(\nabla \cdot \mathbf{u}_h) \nabla \cdot \mathbf{v}_h + \frac{c_{tu}}{\ell^2} \int_{\mathcal{F}_h} h \nabla^2 [\mathbf{u}_h]_t \parallel \mathbf{v}_h \parallel, \\
s_h(p_h, q_h) & := \ell^2 \int_{S_h} P_{V_h}^1(\varepsilon \nabla p_h) \cdot \varepsilon \nabla q_h + c_{np} \ell^2 \int_{\mathcal{F}_h} \frac{1}{h} [p_h] \cdot [q_h],
\end{align*}
\]

where \( P_{Q_h} \) and \( P_{V_h} \) are the \( L^2 \) projections onto \( Q_h \) and \( V_h \), respectively. The introduction of the orthogonal projection in the divergence term does not introduce any problem in the stability and convergence analysis of the method. However, the orthogonal projection in the gradient term introduces some complications in the following analysis. It motivates the introduction of a norm weaker than \( \| \cdot \| \), in which we prove stability and convergence of the method:

\[
\| \mathbf{u}_h, p_h \|_w := \| \mathbf{u}_h \|_h + \ell \| P_{V_h}(\varepsilon \nabla p_h) \|_{S_h} + \ell \| P_{V_h}(\varepsilon \nabla p_h) \|_{S_h} + \| \varepsilon \|_{h^{-1/2}} \{ \varepsilon \} \| p_h \|_{\mathcal{F}_h},
\]

where \( h \lambda := \frac{h}{\ell^2} \). In any case, the method exhibits good stability and convergence behavior even for the approximation of singular solutions (see Section 4). Stability is proved in the next theorem; we remark the fact that in this case stability is stated in terms of an inf-sup condition, instead of coercivity, as in the orthogonal subgrid scale stabilization (OSS) method analyzed in [10].

**Theorem 3.3** The bilinear form (24) with the expressions in (35) and \( c_{tu} > C_{inv} \) satisfies the inf-sup condition

\[
\inf_{(\mathbf{u}_h, p_h) \in \mathcal{V}_h \times Q_h} \sup_{(\mathbf{v}_h, q_h) \in \mathcal{V}_h \times Q_h} \frac{c_h(\mathbf{u}_h, p_h, \mathbf{v}_h, q_h)}{\| \mathbf{u}_h, p_h \|_w \| \mathbf{v}_h, q_h \|_w} \geq \beta,
\]

for a constant \( \beta > 0 \) uniform with respect to \( h \).

**Proof** Assume for simplicity that \( \lambda = 1, \varepsilon = 1 \) and \( \ell = 1 \), and the mesh quasi-uniform of size \( h \). As above, we initially test the system against \((\mathbf{u}_h, p_h)\). Analogously to Theorem 3.1, we are able to prove stability for all the terms in \( \| \cdot \|_w \) but those associated to the projections \( hP_{Q_h}(\nabla \cdot \mathbf{u}_h) \) and \( hP_{V_h}(\nabla p_h) \). Stability over the first term is obtained taking \((\mathbf{0}, h^2P_{Q_h}(\nabla \cdot \mathbf{u}_h))\) as test functions in (20)-(35), where \( \delta > 0 \) is an appropriate constant to be defined. We obtain:

\[
\begin{align*}
c_h(\mathbf{u}_h, p_h, \mathbf{0}, h^2P_{Q_h}(\nabla \cdot \mathbf{u}_h)) & := \delta h^2 \| P_{Q_h}(\nabla \cdot \mathbf{u}_h) \|_2^2 - \delta h^2 \int_{\mathcal{F}_h} \{ P_{Q_h}(\nabla \cdot \mathbf{u}_h) \} [\mathbf{u}_h] \\
& + \delta h^2 \int_{S_h} \nabla P_{Q_h}(\nabla \cdot \mathbf{u}_h) \cdot \nabla p_h + \delta h \int_{\mathcal{F}_h} [p_h] \cdot [P_{Q_h}(\nabla \cdot \mathbf{u}_h)].
\end{align*}
\]

Using inverse inequalities (11)-(13), we easily obtain the bound:

\[
\begin{align*}
c_h(\mathbf{u}_h, p_h, \mathbf{0}, h^2P_{Q_h}(\nabla \cdot \mathbf{u}_h)) & \geq \delta h^2 \| P_{Q_h}(\nabla \cdot \mathbf{u}_h) \|_2^2 - \delta c_1 h \| \mathbf{u}_h \|_{S_h}^2 \\
& - \delta c_2 \| \nabla p_h \|_{S_h}^2 - \delta c_3 h^{-1} \| p_h \|_{\mathcal{F}_h}.
\end{align*}
\]
for appropriate constants $c_1$, $c_2$ and $c_3$. On the other hand, testing system (20)-(35) against the function $(\varphi h^2 P_{Vh}(\nabla p_h), 0)$, with $\varphi > 0$ to be defined, we get:

$$
c_h(u_h, p_h, \varphi h^2 P_{Vh}(\nabla p_h), 0) = \varphi h^2 \|P_{Vh}(\nabla p_h)\|^2 + \varphi h^2 \int_{S_h} \nabla \times P_{Vh}(\nabla p_h) \cdot \nabla \times u_h$$

$$- \varphi h^2 \int_{F_h} [P_{Vh}(\nabla p_h)]_t \cdot \{\nabla \times u_h\} - \varphi h^2 \int_{F_h} [u_h]_t \cdot \{\nabla \times P_{Vh}(\nabla p_h)\}$$

$$+ \varphi h c_t u_h \int_{F_h} [u_h]_t \cdot \{P_{Vh}(\nabla p_h)\}_t + \varphi h^4 c_u \int_{S_h} P_{Qh}^\perp (\nabla \cdot u_h) \nabla \cdot P_{Vh}(\nabla p_h)$$

$$+ \varphi h^3 c_m h \int_{F_h} [u_h] \{P_{Vh}(\nabla p_h)\} - \varphi h^2 \int_{F_h} [p_h] \{P_{Vh}(\nabla p_h)\}. \quad (36)$$

The conflictive terms that do not allow full control over $P_{Vh}(\nabla p_h)$ are those related to the curl-curl term, i.e. the second to fourth terms in the right-hand side of (36). After some manipulation, using expressions (11)-(13) properly, we obtain:

$$
c_h(u_h, p_h, \varphi h^2 P_{Vh}(\nabla p_h), 0) = \frac{\varphi h^2}{8} \|P_{Vh}(\nabla p_h)\|^2 - \varphi c_4 \|\nabla \times u_h\|^2_{S_h} - \varphi c_5 h^{-1} \|[u_h]_t\|^2_{F_h}$$

$$- \varphi c_6 h^4 \|P_{Qh}^\perp (\nabla \cdot u_h)\|^2_{S_h} - \varphi c_7 h^3 \|[u_h]_t\|^2_{F_h} - \varphi c_8 h \|[p_h]\|_{F_h}. \quad (37)$$

for appropriate constants $c_4$, ..., $c_8$. Combining the previous results with $\delta$ and $\varphi$ small enough, we get:

$$c_h(u_h, p_h, u_h + \varphi h^2 P_{Vh}(\nabla p_h), p_h + \delta h^2 P_{Qh}(\nabla \cdot u_h)) \geq \|u_h, p_h\|^2_w.$$

The remaining point is to prove that $\|\varphi h^2 P_{Vh}(\nabla p_h), \delta h^2 P_{Qh}(\nabla \cdot u_h)\|_w \leq \|u_h, p_h\|_w$, which can be easily checked by using repeatedly (11) and (13).

Again, the introduction of the orthogonal projection in the stabilization term for $\nabla p_h$ introduces some complications in the convergence analysis, due to the weaker stability results. In particular, instead of bounding the error of the numerical solution by the one of the best FE approximation with respect to $E_h(\cdot)$, we are only able to bound the error related to $u$ by the one of the $L^2$-projector. In any case, this projector has optimal interpolation properties for smooth solutions (see, e.g., [6]).

**Theorem 3.4** The numerical solution $(u_h, p_h)$ of system (20)-(35) satisfies

$$\|u_h - u, p_h - p\| \leq \inf_{q_h \in Q_h} e_1(u - P_{Vh}(u), p - q_h) + \inf_{w_h \in V_h} e_2(\nabla \times u - w_h).$$

where $(u, p)$ is the solution of the continuous problem (3).

**Proof** The term that complicates the analysis and forces to consider the $L^2$ projector in the definition of the interpolation error is the following:

$$\int_{\Omega} \nabla q_h \cdot (P_{Vh}(u) - u) = \int_{\Omega} P_{Vh}^\perp (\nabla q_h) \cdot (P_{Vh}(u) - u) \leq \|P_{Vh}^\perp (\nabla q_h)\| \|P_{Vh}(u_h) - u\|.$$

We omit the rest of terms, since their bounds are obtained as in the proof of Theorem 3.2. Combining again stability and the bound for the interpolation error, we prove the theorem.
Remark 3.3 Let us remark that an intermediate problem, in which the projection is only introduced in the divergence term, exhibits the stability result in Theorem 3.1 with respect to the strong norm \( \| \cdot \| \), and the same happens for the convergence result in Theorem 3.2. We omit the details here, since it is straightforward from the previous analysis. As a conclusion, the fact that complicates the problem is the introduction of the projection on \( \nabla p_h \) in the term

\[
\ell_h^2 \int_{S_h} P_{V_h}^\perp (\varepsilon \nabla p_h) \cdot \varepsilon \nabla q_h.
\]

This can hardly be motivated using the augmented formulation motivation, but is perfectly acceptable interpreting this term as an additional stabilization term.

Remark 3.4 Suppose that inclusions (22) hold, i.e. equal discontinuous interpolation for \( u \) and \( p \) is used. Then \( P_{Q_h}^\perp (\nabla \cdot v_h) = 0 \) for any \( v_h \in V_h \) and \( P_{Q_h} (\varepsilon \nabla q_h) = 0 \) for any \( q_h \in Q_h \). The method (20) with the bilinear forms \( a_h \) and \( s_h \) given in (35) and \( b_h \) given in (21) reduces to the method proposed in [19] (up to the scaling), with the only difference that the algorithm in [19] is not dimensionally consistent.

4 Numerical experiments

4.1 The singular problem

In this section, we want to check how the previous approach is suitable for the numerical approximation of singular solution of system (1). With this aim, we consider the datum \( f \) such that the solution in polar coordinates \( (r, \theta) \) is:

\[
u = \nabla \left( r^{2n} \sin \frac{2n\theta}{3} \right)
\]

in the nonconvex domain \( \Omega \equiv [-1, 1]^2 \setminus ([0, 1] \times [0, -1]) \), with one re-entrant corner. We have that \( u \in H^{2n-\epsilon}(\Omega) \), for any \( \epsilon > 0 \). We consider \( \lambda = \varepsilon = 1 \). For \( n = 1 \) we have a singular solution since \( u \notin H^1(\Omega)^2 \). Larger values of \( n \) lead to smooth solutions.

In the following, we have considered all the algorithmic constants equal to one, except \( c_{nu} = 10 \), in order to satisfy \( c_{nu} > C_{inv} \) (see [19]).

4.2 The cG method

It has been proved in [3] that the nodal cG formulation (14) converges to the good solution even in the singular case. In order to show the behavior of the cG formulation, we have plotted the two components of \( u \) in Figures 1-2 for the exact solution (38) with \( n = 1 \), as well as the solution of the stabilized cG formulation in (20)-(21) for a structured uniform mesh of 4096 linear triangular elements. We see that the numerical approximation is close to the appropriate singular solution.
Figure 1: Comparison of the exact and numerical solution for different approximations of the test problem with exact solution (38) for $n = 1$. Component $u_x$.

On the other hand, we have performed this test for the same cG formulation but switching off the divergence stabilization term $s_h$ in (14a); it is clear out of the Figures 1-2 that this term is not only needed for technical purposes but also basic for the good behavior of the algorithm. We remark that this term is required in order to get control over $\|u_h\|$. Finally, we include the results obtained for a stabilized method in the wrong functional setting $H^1(\Omega)^d \times L^2(\Omega)$ (see [3, Section 4.2]). As expected, this method is affected by the approximability property, tending to a spurious solution. The convergence rates of these methods for different values of $n$ can be found in [3].

The formulation with orthogonal projections (20)-(35) has been tested for the cG FE approximation. We can easily check that the weaker stability and convergence results also apply for the cG particular
Figure 2: Comparison of the exact and numerical solution for different approximations of the test problem with exact solution (38) for $n = 1$. Component $u_y$. Case. We have performed the convergence analysis for $n = 1, 2, 4$ and $h = 2^{-i}$ for $i = 3, 4, 5, 6$ (see Figure 3). Despite the stability and convergence theoretical results are slightly weaker than those in the original formulation (20)-(21), the numerical results are almost identical to those for the original cG formulation, that can be found in [3]. The method approximates well both singular and smooth solutions.
Figure 3: Error plots for different quantities in \( L^2(\Omega) \) norm for Formulation (20)-(35) (with orthogonal projections) and the problem with analytical solution (38), for different values of \( n \). Values in parenthesis indicate the experimental slope of the convergence curves.

4.3 The dG method

Similar convergence results to those presented earlier can be found in [19] for the dG formulation. Since the first cG-dG formulation we propose in Section 3.1 includes additional terms in the dG limit, we aim at checking that these additional stabilization terms are harmless. In Figure 4, we plot the numerical error for different norms using linear FEs, for both the original dG method and the over-stabilized one. We can easily check that the results are very similar for these two methods, even though the errors are always slightly lower for the over-stabilized formulation. As a conclusion, the use of the cG-dG formulation (20)-(21) is suitable, and the fact that it involves additional stabilization terms that are not needed in the dG limit, when inclusions (22) are satisfied, does not represent any accuracy reduction. All these experiments agree with the theoretical results.

Figure 4: Error plots for different quantities in \( L^2(\Omega) \) norm for the dG method in [19] and Formulation (20)-(21) (in the dG limit) for the test problem with analytical solution (38) and \( n = 1 \).
4.4 The cG-dG method

Now, let us consider a simple cG-dG problem, in which we have split the domain into two different patches (see Figure 5(a)). As defined above, the continuity between patches is weakly enforced via the interior penalty interface terms whereas the FE functions are continuous inside patches. In Figure 5(b), we show how the error is reduced for the cG-dG formulation, for a refinement with characteristic mesh size $h^{-i}$ with $i = 3, 4, 5, 6$. The convergence rates are excellent for the problem with singular solution.

![cG-dG mesh (2 patches)](image1)

![Error plots for cG-dG Formulation (20)-(21)](image2)

Figure 5: Decomposition of the domain into two patches and error plots for different quantities in $L^2(\Omega)$ norm for the cG-dG Formulation (20)-(21) for the problem with analytical solution (38) and $n = 1$.

Figure 6 compares the error associated to the cG, dG and combined cG-dG formulation. In order to compare the different methods in terms of computational cost, we define an effective mesh size $h_{\text{eff}} = L_0(n_{\text{dof}})^{-\frac{1}{d}}$, where $n_{\text{dof}}$ indicates the number of degrees of freedom.

We have considered both the error for $u$ in terms of the $L^2$ norm and $\|\cdot\|_h$ norm. From these results it is observed that the cG method is the best approach for this particular choice of the patches in terms of $\|u - u_h\|$. However, the cG-dG formulation is more accurate with respect to $\|u - u_h\|_h$. In any case, what becomes quite evident out of these results is the fact that a full dG formulation is less accurate for the singular re-entrant corner problem than a cG formulation for a fixed computational cost. It justifies the combined cG-dG approach in the paper. In applications with discontinuous jumps, in which cG alone is not suitable, a combined formulation allows us to weaken continuity in the interfaces between different materials keeping accuracy for a low computational cost of the nodal cG formulation.
Figure 6: Error evaluated in terms of $\|u - u_h\|$ for different norms in the cG case, dG case and cG-dG case with the patches in Figure 5(a) against the effective mesh size $h_{\text{eff}}$.

5 Conclusions

In this work, we have considered a numerical approximation of the Maxwell operator that allows us to use continuous and discontinuous nodal FE spaces, and so, combined spaces built of patches of continuous FE functions with discontinuities on the patch interfaces. We have denoted these three cases as cG, dG and cG-dG formulations, respectively.

The use of cG nodal FE spaces without any compatibility requirement is very appealing in terms of computational cost. Furthermore, this approach is very well suited for multi-physics problems, e.g. magnetohydrodynamics, since all the unknowns of the different subproblems can be approximated by the same FE spaces using stabilization techniques, a main difference with respect to inf-sup stable FEs (see [4]). This new approach to the Maxwell problem has been recently proposed in [3]. However, in electromagnetic applications that involve materials with different physical properties, cG approaches lead to large errors and wiggles on the material interfaces. For this reason, we have considered in this article a way to consider nodal cG formulations in patches, but allowing jumps on the patch interfaces. This cG-dG approach inherits the low computational cost of the cG method and is able to deal with coefficient jumps as accurately as a full dG formulation.

The combined mixed cG-dG formulation we propose combines the nodal cG formulation in [3] with the nodal dG formulation in [19] (a stabilized formulation via an interior penalty technique). In the dG limit, the first combined formulation we propose does not reduce to the dG formulation in [19]; additional terms are needed, which are not required under some inclusions, that are true for dG FE spaces using equal interpolation for both $u$ and $p$. Alternatively, we have motivated a new method based on orthogonal projections that does reduce the amount of element interior stabilization and recovers the formulation in [19] for the dG limit. For both formulations, stability and convergence analyses have been performed. Numerical experiments show the good behavior of the different formulations, exceeding the theoretical expectations in some cases. The experiments have been performed for a problem with a singular solution $u \not\in H^1(\Omega)^d$. 

References


