Walrasian Equilibrium in Two-User Multiple-Input Single-Output Interference Channels

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Abstract—We propose a decentralized resource allocation scheme in the two-user multiple-input single-output interference channel. The mechanism is motivated by economic models which define equilibria in competitive settings. We model the situation between the links as a competitive market where the links are consumers, the transmission strategies are goods. In Walrasian equilibrium, the demand of each good equals the supply which constitutes an efficient operating point. For the two-user case, the Walrasian equilibrium and the corresponding prices can be computed in closed form. An arbitrator with perfect channel knowledge computes and distributes the Walrasian prices to the consumers (transmitters) which calculate in a decentralized manner their optimal demand (beamforming) of each good subject to their budget constraint (initial maximum ratio transmission solution). The Walrasian equilibrium is Pareto optimal and dominates the Nash equilibrium. Moreover, utilizing the conflict representation of the consumers in the Edgeworth box, we provide the closed-form solution to all Pareto optimal points for the two-user case.

I. INTRODUCTION

Two transmitter-receiver pairs utilize the same spectral band simultaneously. Each transmitter is equipped with $N$ antennas and each receiver with a single antenna. This setting corresponds to the multiple-input single-output (MISO) interference channel (IFC) [1]. The systems’ performance in such a setting is degraded by mutual interference, and their noncooperative operation is usually not efficient [2]. Therefore, coordination between the links is needed in order to improve their joint outcome. In [3], a Pareto optimal one-shot coordination algorithm is proposed in the MISO IFC where each transmitter independently maximizes its virtual SINR.

Since the Nash equilibrium is generally not efficient, we attempt to use a competitive economy model to find equilibria that are Pareto optimal. In this economic model, as proposed by L. Walras [4], there exists a population in which each individual possesses an amount of divisible goods. The worth of these goods make up the budget of each individual. Each individual has a utility function which reveals his demand on consuming goods. Every individual would use the revenue from selling all his goods to buy amounts of goods such that his utility is maximized. This economy model is competitive because each consumer seeks to maximize his profit independent of what the other consumers demand. Walras investigated if there exist prices of the goods such that the market has neither shortage nor surplus. The existence of such prices, called Walrasian prices, was later explored by Arrow [5]. It is usually assumed that the market or an auctioneer acts as an arbitrator to determine the Walrasian prices.

In [6], the competitive equilibrium is formulated as a linear complementarity problem for a multi-link multi-carrier setting. A decentralized price-adjustment process, also referred to as tâtonnement, is proposed where the users send their power allocations in each iteration to the spectrum manager which adjusts the prices to achieve the equilibrium. In [7], competitive spectrum market is considered where the users, sharing a common frequency band, can purchase their transmit power subject to budget constraints. An agent, referred to as the market, determines the unit prices of the power spectra. Existence of the equilibrium is proven and conditions for its uniqueness are provided. In [8], the competitive equilibrium is used for simultaneous bitrate allocation for multiple video streams. The Edgeworth box is used to illustrate the conflict between the streams. In the context of cognitive radio, spectrum trading is successfully modeled by economic models and market-equilibrium, and competitive and cooperative pricing schemes are developed in [9].

In this work, we propose a coordination mechanism between two MISO interfering links. The setting is modeled as a competitive market where the consumers are the links and the goods correspond to the parametrization of the transmitters’ beamforming vectors. Each good has a price and the consumers demand amounts of goods given these prices. We formulate the consumers’ demand functions and calculate the Walrasian equilibria which are Pareto optimal. Moreover, utilizing the Edgeworth box representation, we determine the Pareto boundary of the SINR region in closed form.

Notations: Column vectors and matrices are given in lowercase and uppercase boldface letters, respectively. $||a||$ is the Euclidean norm of $a, a \in \mathbb{C}^N$. $|b|$ is the absolute value of $b, b \in \mathbb{C}$. $(\cdot)^H$ denotes the Hermitian transpose. The orthogonal projector onto the column space of $Z$ is $\Pi_Z := Z(Z^H Z)^{-1} Z^H$. The orthogonal projector onto the orthogonal complement of the column space of $Z$ is $\Pi_Z^\perp := I - \Pi_Z$, where $I$ is an identity matrix. Throughout the paper, the subscripts $k, \ell$ are from the set \{1, 2\}.

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II. PRELIMINARIES

A. System and Channel Model

The quasi-static block flat-fading channel vector from transmitter $k$ to receiver $\ell$ is denoted by $h_{k\ell} \in \mathbb{C}^N$. We assume that transmission consists of scalar coding followed by beamforming. The beamforming vector used by transmitter $k$ is $w_k \in \mathbb{C}^N$. The matched-filtered, symbol-sampled complex baseband data received at receiver $k$ is $y_k = h_{k\ell}^H w_k s_k + h_{k\ell} w_k s_\ell + n_k, k \neq \ell$, where $s_k$ is the symbol transmitted by transmitter $k$. The random variables $n_k$ are noise terms which are independent and identically distributed (i.i.d.) complex Gaussian with zero mean and variance $\sigma^2$. Each transmitter has a total power constraint of $P := 1$ such that $\|w_k\|^2 \leq 1$.

The transmitters are assumed to have perfect local channel state information (CSI), i.e., each transmitter has perfect knowledge of the channel vectors only between itself and all receivers. We assume there exists an arbitrator who has perfect knowledge of all channels. The arbitrator could be any central controller that can coordinate the transmissions of the users.

B. SINR Region and Efficient Transmission

The signal to interference plus noise ratio (SINR) at receiver $k$ is

$$\phi_k(w_1, w_2) = \|h_{k\ell}^H w_k\|^2 / (\|h_{k\ell}^H w_2\|^2 + \sigma^2), \ k \neq \ell.$$  

This results in the achievable rate $\log_2(1 + \phi_k(w_1, w_2))$ for link $k$ when single user decoding is performed at the receivers. The SINR region is the set of all achievable SINR tuples:

$$\Phi := \{ (\phi_1(w_1, w_2), \phi_2(w_1, w_2)) : \|w_k\|^2 \leq 1 \}. \quad (2)$$

Definition 1: An operating point $(r_1, r_2) \in \Phi$ is Pareto optimal if there is no other operating point $(r_1', r_2') \in \Phi$ such that $(r_1', r_2') \geq (r_1, r_2)$, where the inequality is componentwise and strict for at least one component.

The set of all Pareto optimal operating points makes up the Pareto boundary of the SINR region. For the two-user MISO IFC, the set of beamforming vectors for each transmitter that are relevant for Pareto optimal operation are parameterized by a single real-valued parameter $\lambda_k \in [0, 1]$ as [10, Corollary 1]

$$w_k(\lambda_k) = \sqrt{\lambda_k} \frac{\Pi_{n_k,h_{kk}} h_{kk}}{\|\Pi_{n_k,h_{kk}} h_{kk}\|} + \sqrt{1 - \lambda_k} \frac{\Pi_{n_k,h_{kk}} h_{kk}}{\|\Pi_{n_k,h_{kk}} h_{kk}\|}, \quad (3)$$

where $k \neq \ell$. This parametrization is valuable for designing efficient low complexity distributed resource allocation schemes [11]. The set of beamforming vectors in (3) includes maximum ratio transmission (MRT) ($\lambda_{\text{MRT}}^k = \|\Pi_{n_k,h_{kk}} h_{kk}\| \|\Pi_{n_k,h_{kk}} h_{kk}\|^{-1}$) and zero forcing transmission (ZF) ($\lambda_{\text{ZF}}^k = 0$). According to [10, Corollary 2], it suffices that the parameters $\lambda_k$ only be from the set $[0, \lambda_{\text{MRT}}^k]$ for Pareto optimal operation.

Lemma 1: The power gains at the receivers are

$$\|h_{k\ell}^H w_k(\lambda_k)\|^2 = (\sqrt{\lambda_k} g_k + \sqrt{(1 - \lambda_k)} g_k')^2, \quad (4)$$

$$\|h_{k\ell}^H w_k(\lambda_k)\|^2 = \lambda_k g_k, \ k \neq \ell, \quad (5)$$

where $\lambda_k \in [0, \lambda_{\text{MRT}}^k]$ and $g_k \triangleq \|\Pi_{n_k,h_{kk}} h_{kk}\|$, $g_k' \triangleq \|\Pi_{n_k,h_{kk}} h_{kk}\|$, where $k \neq \ell$.

Proof: The proof is provided in [12].

C. Noncooperative Game

In [2], the outcome of a strategic game [13, Section 2.1] between the links is studied. The set of players is $\{1, 2\}$ consisting of the two links. The pure strategies of player $k$ are the real-valued parameters $\lambda_k \in [0, \lambda_{\text{MRT}}^k]$ in (3), and his utility function is $\phi_k(w_1(\lambda_1), w_2(\lambda_2))$ in (1). In this game, a player always chooses the MRT strategy independent of the choice of the other player [2], i.e., MRT is a dominant strategy for each player. Hence, the unique Nash equilibrium is $(\lambda_{\text{MRT}}^1, \lambda_{\text{MRT}}^2)$. The outcome in Nash equilibrium is usually not Pareto optimal. In order to achieve Pareto improvements from the Nash equilibrium, coordination between the players is required.

III. COMPETITIVE MARKET MODEL

In a competitive market, the consumers buy quantities of goods and also sell goods they possess such that they maximize their profit. In this section, the mapping from transmitters and beamforming to consumers and goods is described.

A. Interpretation of Goods

The characterization of efficient beamforming vectors in (3) requires each transmitter $k$ to choose beamforming vectors corresponding to the parameters in the set $[0, \lambda_{\text{MRT}}^k]$. We will assume that the links are initially in Nash equilibrium and interpret $\lambda_{\text{MRT}}^k$ as the total amount of good consumer $k$ (referring to link $k$) possesses. Consumer $k$ can sell an amount of his good to consumer $\ell, \ell \neq k$. This amount is represented by $x_k^{(\ell)} \leq \lambda_{\text{MRT}}^k$. Selling $x_k^{(\ell)}$ means that transmitter $k$ uses the beamforming vector in (3) which corresponds to $\lambda_{\text{MRT}}^k - x_k^{(\ell)}$. We represent the possession of consumer $k$ from his good and from the good of consumer $\ell, \ell \neq k$, as

$$x_k^{(k)} = \lambda_k^k, \quad x_k^{(\ell)} = \lambda_{\text{MRT}}^k - \lambda_\ell,$$  

respectively. The utility of consumer 1 (analogously consumer 2) in terms of the amount of his good and the amount of good from consumer 2 is the SINR in (1) rewritten as

$$\phi_1(x_1^{(1)}, x_2^{(1)}) = \left( \sqrt{x_1^{(1)} g_1} + \sqrt{(1 - x_1^{(1)}) g_1'} \right)^2 / (\sigma^2 + \lambda_{\text{MRT}}^{g_2} g_2 - x_1^{(1)} g_2),$$  

where we used the expressions of the power gains in Lemma 1 and substituted $\lambda_1 = x_1^{(1)}$ and $\lambda_2 = \lambda_{\text{MRT}}^2 - x_2^{(1)}$ from (6).

Theorem 1: $\phi_1(x_1^{(1)}, x_2^{(1)})$ in (7) is continuous, strictly increasing, and strictly quasiconcave on $[0, \lambda_{\text{MRT}}^1] \times [0, \lambda_{\text{MRT}}^2]$.

Proof: The proof is provided in [12].

As consumers purchase and sell goods from one another, each good has a price. In competitive markets, every consumer takes prices as given. The prices of the goods are not determined by consumers, but arbitrated by markets. In our case the arbitrator, which has full knowledge of the channel states, determines the prices of the goods. Let $p_k$ denote the unit price of good $k$. In order to be able to buy goods, each consumer $k$ has a budget of $\lambda_k p_k$ which is the worth of his initial amounts of goods (in Nash equilibrium). This is the case in the Arrow-Debreu
In the above problem, each consumer calculates his demands for goods he can afford to possess defined as
\[ B_k := \{ (x_1^{(k)}, x_2^{(k)}) \in \mathbb{R}^2_+ : p_1 x_1^{(k)} + p_2 x_2^{(k)} \leq \lambda_{MRT} k \} \]  
(8)
The budget set of consumer 1 is illustrated by the grey area in Fig. 1. Given the budget set, we determine next the amounts of goods a consumer demands such that his utility is maximized.

**B. Consumer Demand Function**

Given the prices \( p_1 \) and \( p_2 \), consumer 1 demands \( x_1^{(1)} \) and \( x_2^{(1)} \) such that these maximize his utility in (7). Thus, consumer 1 solves the following problem:

\[
\begin{align*}
\text{maximize} & \quad \phi_k(x_1^{(k)}, x_2^{(k)}) \\
\text{subject to} & \quad p_1 x_1^{(k)} + p_2 x_2^{(k)} \leq \lambda_{MRT} k.
\end{align*}
\]  
(9)

In the above problem, each consumer calculates his demands without knowing what the demands of the other consumer are. In order to solve (9) in a decentralized manner, each consumer has to know his utility function. For consumer 1 (analogously consumer 2), whose utility function is given in (7), knows the terms \( g_1 \) and \( g_1 \). In addition, he also knows \( \sigma^2 + \lambda_{MRT}^2 g_{21} \) since this is the noise and interference powers in Nash equilibrium. The term \( g_{21} \) is not known to consumer 1 and has to be additionally provided from the arbitrator with the prices.

**Theorem 2:** The unique solution to the problem in (9) is
\[
\begin{align*}
x_1^{(1)} &= \frac{1}{1 + \frac{g_1}{g_1} \left( 1 + \frac{\lambda_{MRT} p_1 / p_2}{\sigma^2 + \lambda_{MRT}^2 g_{21} - \lambda_{MRT}^2 g_{12} / p_1 / p_2} \right)^2}, \\
x_2^{(1)} &= \frac{p_1}{p_2} \left( \lambda_{MRT} - x_1^{(1)} \right),
\end{align*}
\]  
(10, 11)
for consumer 1, and
\[
\begin{align*}
x_1^{(2)} &= \frac{1}{1 + \frac{g_2}{g_2} \left( 1 + \frac{\lambda_{MRT} p_1 / p_2}{\sigma^2 + \lambda_{MRT}^2 g_{21} - \lambda_{MRT}^2 g_{12} / p_1 / p_2} \right)^2}, \\
x_2^{(2)} &= \frac{p_2}{p_1} \left( \lambda_{MRT}^2 - x_2^{(2)} \right),
\end{align*}
\]  
(12, 13)
for consumer 2, where \( g_k, g_k, g_{k2} \) are defined in Lemma 1. The feasible prices ratio are in the range:
\[
\frac{\lambda_{MRT}^2 g_{12}}{\sigma^2 + \lambda_{MRT}^2 g_{12}} \leq \frac{p_1}{p_2} \leq \frac{\sigma^2 + \lambda_{MRT}^2 g_{21}}{\lambda_{MRT}^2 g_{12}}.
\]  
(14)

**Proof:** The proof is provided in [12].

In Fig. 1, the demand of consumer 1 is illustrated as the point where the corresponding indifference curve is tangent to the boundary of the budget set. The indifference curves represent the pairs \((x_1^{(k)}, x_2^{(k)})\) with consumer \( k \) achieves the same constant payoff. According to the properties of the utility function in Theorem 1, the indifference curves are strictly convex. All points above an indifference curve gives the corresponding consumer a higher payoff than at the indifference curve.

If each consumer is to demand amounts of goods without considering the demands of the other consumer, then it is important that the consumers’ demands equal the consumers’ supply of goods. Prices which fulfill this requirement are called Walrasian and are calculated next.

**C. Walrasian Equilibrium**

In a Walrasian equilibrium, the demand equals the supply of each good [14, Definition 5.5]. According to the properties of the utility function in Theorem 1, there exists at least one Walrasian equilibrium [14, Theorem 5.5]. The Walrasian prices \((p_1^*, p_2^*)\) that lead to a Walrasian equilibrium satisfy
\[
x_1^{(1)} + x_2^{(2)} = \lambda_{MRT}^1, \quad \text{and} \quad x_1^{(1)} + x_2^{(2)} = \lambda_{MRT}^2.
\]  
(15)

According to Walras’ law [14, Chapter 5.2], if the demand equals the supply of one good, then the demand would equal the supply of the other good. Hence, in order to calculate the Walrasian prices, it is sufficient to consider one of the conditions in (15).

**Theorem 3:** Ratio of the Walrasian prices are the roots of
\[
a \left[ \frac{p_1}{p_2} \right]^5 + b \left[ \frac{p_1}{p_2} \right]^4 + c \left[ \frac{p_1}{p_2} \right]^3 + d \left[ \frac{p_1}{p_2} \right]^2 + e \left[ \frac{p_1}{p_2} \right] + f = 0,
\]  
(16)
that satisfy the condition in (14). The constant coefficients are
\[
\begin{align*}
a &= T_1 T_3 T_4, & b &= -2 T_3 T_2 T_2 S_2 + T_3 S_1, \\
c &= 2 T_1 T_2 S_3 + 4 S_2 T_2 T_3 + T_1 S_4 T_3, \\
d &= -2 S_4 T_2 S_3 + 4 T_3 T_2 S_2 S_3 - S_1 T_4 S_3, \\
ee &= 2 S_3 S_2 (T_2 S_2 + T_1 S_3), & f &= -S_1 S_2 S_3 S_3,
\end{align*}
\]  
(17, 18, 19, 20)
where
\[
\begin{align*}
T_1 &= (g_1 - \bar{g}_1) / (g_1 + \bar{g}_1), & S_1 &= (g_2 - \bar{g}_2) / (g_2 + \bar{g}_2), \\
T_2 &= \lambda_{MRT}^1 + \sigma^2 / g_{12}, & S_2 &= \lambda_{MRT}^2 + \sigma^2 / g_{21}, \\
T_3 &= (1 - \lambda_{MRT}^1) \lambda_{MRT}^1, & S_3 &= (1 - \lambda_{MRT}^2) \lambda_{MRT}^2, \\
T_4 &= \frac{\bar{g}_1^2 - g_1 g_1 + \bar{g}_1^2}{(g_1 + \bar{g}_1)^2}, & S_4 &= \frac{\bar{g}_2^2 - g_2 g_2 + \bar{g}_2^2}{(g_2 + \bar{g}_2)^2},
\end{align*}
\]  
(21, 22, 23, 24)
and \( \bar{g}_k, g_k, g_{k2} \) are defined in Lemma 1.

**Proof:** Substituting (10) and (13) in (15) and collecting \( p_1 / p_2 \) we get the expression in (16). The condition in (14)
where $C$ is a function of $x_2^{(2)}$ given on the RHS of (31) in Appendix A. The root of interest of (26) is the one in the range $[0, \lambda_{\text{MRT}}]$, and satisfies $A$ and $B$ in (36) in Appendix A to have the same sign.

Proof: The proof is provided in Appendix A.

In Fig. 3, an Edgeworth box is plotted for a sample channel realization. For the prices calculated from Theorem 3 we obtain the allocation on the contract curve where the corresponding indifference curves are tangent. The line passing through this allocation with slope $-p_1/p_2$ defines the budget sets of the consumers. The contract curve is obtained from Theorem 4. Specifically, we take uniformly spaced samples of $x_2^{(2)} \in (0, \lambda_{\text{MRT}})$ and calculate the corresponding values of $x_1^{(1)}$ from Theorem 4.

In Fig. 4, the SINR region is plotted with the outcome in Walrasian equilibrium. The outcome is Pareto optimal and dominates the Nash equilibrium which corresponds to joint MRT transmission. The Pareto boundary corresponds to the allocations on the contract curve calculated in Theorem 4.

The Walrasian equilibrium is an equilibrium in the sense that the links are not willing to deviate from this state given the Walrasian prices. The role of the arbitrator is only to coordinate the operation of the links. This is in contrast to other Pareto optimal allocations described in Theorem 4 where it is not straightforward how to enforce the rational users to operate in these points.

IV. EDGEWORTH BOX REPRESENTATION

The Edgeworth box [15], [14, Chapter 5], is illustrated in Fig. 2. The two points of origin, $O_1$ and $O_2$, correspond to consumer 1 and 2, respectively, and the total amounts of goods define the size of the box (width is $\lambda_{\text{MRT}}$, height is $\lambda_{\text{MRT}}$). Every point $\mathbf{x} = ((x_1^{(1)}, x_1^{(2)}), (x_2^{(1)}, x_2^{(2)}))$ in the box denotes an allocation, i.e., an assignment of bundles of goods to each consumer. The locus of all Pareto optimal allocations in the Edgeworth box is called the contract curve [15]. On these points, the indifference curves are tangent, and are characterized by the following condition [15, p. 21]:

$$\frac{\partial \phi_1(x_1^{(1)}, x_2^{(1)})}{\partial x_1^{(1)}} = \frac{\partial \phi_2(x_1^{(2)}, x_2^{(2)})}{\partial x_1^{(2)}}.$$  \hfill (25)

Theorem 4: The contract curve, $x_1^{(1)}$ as a function of $x_2^{(2)}$, is the solution of the following cubic equation\footnote{After submitting this work we became aware of an independent, concurrent paper that obtained an equivalent one-parameter description of the Pareto boundary [16].}

$$a \left[x_1^{(1)}\right]^3 + b \left[x_1^{(1)}\right]^2 + c \left[x_1^{(1)}\right] + d = 0,$$  \hfill (26)

where

$$a = -(g_1 + \bar{g}_1)(C - g_{12}), \quad d = g_1 \sigma^4,$$ $$b = (C - g_{12})(2\bar{g}_1(C + \sigma^2) + g_1(2\sigma^2 + C - g_{12})), \quad c = -\bar{g}_1(C + \sigma^2)^2 + \sigma^2 g_1(2g_{12} - 2C - \sigma^2),$$

states the set of feasible prices such that the demands of the consumers are feasible. At least one price pair is in this set since a Walrasian equilibrium always exists in our setting.

According to Theorem 3, there exists at most five Walrasian equilibria since the function in (16) might have five roots that satisfy (14). Note that the roots in (16) can be easily calculated using a Newton method. In case more than one Walrasian equilibrium exist, the arbitrator chooses one in which the links then operate in. In all test cases however, we obtain only one Walrasian equilibrium.
\[
\frac{\sqrt{g_1/x_1} - \sqrt{g_1/(1-x_1)} \left( \frac{\sigma^2}{g_1} + \lambda^\text{MRT}_2 - x_2 \right)}{\left( \sqrt{x_1^2 g_1 + \sqrt{1 - x_1^2}) g_1 \right)} = \frac{\sqrt{g_2/x_2^2} - \sqrt{g_2/(1-x_2^2)} \left( \frac{\sigma^2}{g_2} + \lambda^\text{MRT}_1 - x_2^2 \right)}{\left( \sqrt{x_2^2 g_2 + \sqrt{1 - x_2^2}) g_2 \right)} = C := \frac{\sqrt{x_1^2 g_1 + \sqrt{1 - x_1^2}) g_1}{\left( \sqrt{x_2^2 g_2 + \sqrt{1 - x_2^2}) g_2 \right)}
\]