PSEUDOMATROIDS

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Received 11 March 1986
Revised 29 September 1987

In this paper we suggest a generalization of the well known concept of matroid. This includes as special cases generalized matroids due to A. Frank and some examples of 'Linking of matroids by linking systems' by A. Schrijver in addition to matroids. We do not see any direct relationship between this concept and that of greedoids introduced by Korte and Lovász. The concept generalizes many aspects of matroids. These include combinatorial structures as well as optimization problems. We hope it will provide a unified framework for many of these problems.

1. Introduction

In the last three decades, matroid theory has played an important role in the field of combinatorial optimization [8]. Several equivalent characterizations of matroid have been developed. These include, among others, the characterization in terms of its submodular rank function, in terms of its independent sets and in terms of the optimality of a certain greedy algorithm for a certain combinatorial optimization problem [2]. Each of these characterizations has proved effective in providing deeper insight into several combinatorial structures and optimization problems.

In this paper we suggest a proper, generalization of the concept of matroid. The new system, called pseudomatroid, provides simultaneous generalizations of several properties of matroid such as the submodularity of the rank function, independence axioms and the greedy algorithm. It includes generalized matroid introduced in [3] and various interesting examples of linking systems as defined in [6] as special cases.

In Section 2 we introduce the concept of pseudomatroid in terms of the optimality of an algorithm (called generalized greedy algorithm) when applied to a combinatorial optimization problem. In Section 3 we develop its characterization in terms of its independent sets. A polyhedral characterization of a pseudomatroid is given in Section 4. In Section 5, concept of minor is developed.

* This research was partly supported by a N.S.E.R.C. (Canada) research grant #A8085.
Section 6 contains some interesting special cases and finally in Section 7 some optimization problems associated with a pseudomatroid are discussed.

2. Definition of a pseudomatroid

In all of what follows, $E$ is a finite set and $F$ is a nonempty family of subsets of $E$. The pair $[E, F]$ is called a discrete system.

Now, consider the following combinatorial optimization problem and what we term as the generalized greedy algorithm to solve this problem:

**Problem.** Given a discrete system $[E, F]$ and a real number $w(e)$ for each element $e$ of $E$, find an $X$ in $F$ such that $w[X] = \max\{w[Y] : Y \text{ in } F\}$, where for any $A \subseteq E$

$$w[A] = \sum_{i \in A} w(i).$$

**Generalized greedy algorithm (GGA)**

Let $E = \{1, 2, \ldots, n\}$ with $|w(i)| \geq |w(j)|$ whenever $i \leq j$. Let $A(0) = \emptyset$. For $j = 1, \ldots, n$, $A(j) = A(j - 1) \cup \{j\}$ if $w(j) > 0$ and there is a $B$ in $F$ satisfying $A(j - 1) \cup \{j\} \subseteq B \subseteq A(j - 1) \cup \{j, \ldots, n\}$ or $w(j) \leq 0$ and there is no $B$ in $F$ satisfying $A(j - 1) \subseteq B \subseteq A(j - 1) \cup \{j + 1, \ldots, n\}$; $A(j) = A(j - 1)$ otherwise.

**Output:** $A(n)$.

**Definition 1.** A discrete system $[E, F]$ is a pseudomatroid if the generalized greedy algorithm solves the above problem for all $w$.

The members of $F$ are called independent sets of the pseudomatroid.

3. Independence axioms for pseudomatroid

Just as in the case of a matroid, a pseudomatroid can also be defined by axioms on its independent sets and this is what we do next.

**Lemma 1.** Let $[E, F]$ be a pseudomatroid. Then $\{A \subseteq B ; A, B \in F; j \in B - A\} \Rightarrow \{B - \{j, k\} \in F \text{ for some } k \in B - A\}$

**Proof.** If $|B - A| \leq 2$, the result follows trivially. Else, consider the following
weights with $M$ being a large positive number and $a \in (0, 1/2)$:

$$w(i) = \begin{cases} 
1 + a & \text{if } i \in A \\
1 & \text{if } i \in B - (A \cup j) \\
-1 - 2a & \text{if } i = j \\
-M & \text{if } i \in E - B.
\end{cases}$$

The greedy algorithm gives the desired result. □

**Lemma 2.** Let $[E, F]$ be a pseudomatroid; let $A, B \in F$ and let $j \in A - B$. Then either (i) $B \cup \{j, k\} \in F$ for some $k \in A - B$ or (ii) $(B \cup j) - k \in F$ for some $k \in B - A$.

**Proof.** With the same meaning for $M$ and $a$ as in Lemma 1 let:

$$w(i) = \begin{cases} 
1 + 2a & \text{if } i \in A \cap B \\
1 & \text{if } i \in B - A \\
-1 & \text{if } i \in A - B - j \\
1 + a & \text{if } i = j \\
-M & \text{otherwise}.
\end{cases}$$

One application of the greedy algorithm yields the result. In fact, Lemma 1 is a special case of Lemma 2, but has been given separately to highlight the fact that a pseudomatroid may not be an independence system but may have holes of a specific nature.

**Definition 2.** Let $[E, F]$ be a pseudomatroid. $[E, F']$ with $F' = \{x: E - X \in F\}$ is called the dual of $[E, F]$ and is denoted by $[E, F]^*$.

The following lemma follows easily from the definition of pseudomatroid.

**Lemma 3.** The dual of a pseudomatroid is a pseudomatroid and the dual of the dual is the original pseudomatroid.

**Lemma 4.** Let $[E, F]$ be a pseudomatroid, let $A, B \in F$ and $j \in A - B$. Then either (i) $A - \{j, k\} \in F$ for some $k \in A - B$ or (ii) $(A \cup \{k\}) - \{j\} \in F$ for some $k \in B - A$.

**Proof.** This follows easily from Lemma 3 and application of Lemma 2 to $[E, F]^*$. □

**Theorem 1.** Let $[E, F]$ be a discrete system. Then the following statements are
equivalent:

(i) \([E, F]\) is a pseudomatroid.

(ii) \([E, F]\) satisfies the conditions in Lemmas 2 and 4.

**Proof.** (i) \(\Rightarrow\) (ii) is shown in Lemmas 2 and 4. (ii) \(\Rightarrow\) (i): We do this by showing that (ii) implies the optimality of the generalized greedy solution to the optimization problem on \([E, F]\). Suppose not; let \([E, F]\) be the corresponding system and let \(Y\) be the solution produced by the algorithm when \(X\) is an optimal solution. Let the pair \(X, Y\) be chosen so that \(|X \cap Y|\) is maximum among such choices; if there are ties these are broken by selecting the one with minimum value of \(|X \cup Y|\). By supposition, \(X\) is not \(Y\). Consider the evolution of the greedy algorithm and let the first wrong choice made by it be \(e\); let the set of elements selected by the algorithm so far be denoted by the set \(A\). Then \(A \subseteq X \cap Y\). Now we consider two cases:

**Case a.** The algorithm selects \(e \in Y - X\). This implies that \(w(e) > 0\) and \(|W(e)| \geq |w(j)|\) for all \(j \in (X \cup Y) - A\). Hence either \(X \cup \{e, f\} \in F\) for some \(f \in Y - X\) or \((X \cup e) - d \in F\) for some \(d \in X - Y\). In any case one of these sets that is the applicable case contradicts the manner in which \(X\) was selected.

**Case b.** The algorithm rejects \(e \in X - Y\) and hence \(w(e) \leq 0\) and \(|w(e)| \geq |w(j)|\) for all \(j \in (X \cup Y) - A\). But now either \(X - \{e, f\} \in F\) for some \(f \in X - Y\) or \((X \cup Y) - e \in F\) for some \(j \in Y - X\). In any case these sets contradict the selection of \(X\). Hence the theorem.

We now state some interesting properties of independent sets of a pseudomatroid:

**Definition 3.** \(\text{FMIN} = \{X : X \in F; \text{there is no } A \in F \text{ with } A \subset X\}\).

**Definition 4.** \(\text{FMAX} = \{X : X \in F; \text{there is no } A \in F \text{ with } A \supset X\}\).

**Lemma 5.** \(\text{FMIN forms the set of bases of a matroid on } E\).

**Proof.** It can be easily verified, using the greedy algorithm, that all members of \(\text{FMIN}\) have the same cardinality. Now, for any pair \(A, B \in \text{FMIN}\) and any \(x \in A - B\), Lemma 4 above states that there exists a \(y \in B - A\) such that \((A \cup \{y\}) - \{x\} \in F\) and since all the members of \(\text{FMIN}\) have the same cardinality, this implies that \((A \cup \{y\}) - \{x\}\) is a member of \(\text{FMIN}\). The elements of \(\text{FMIN}\) thus satisfy the exchange axiom of matroids. \(\square\)

**Lemma 6.** \(\text{FMAX forms the set of bases of a matroid defined on } E\).

**Proof.** This follows from Lemma 3 and application of Lemma 5 to \([E, F]^*\). \(\square\)
Lemma 7. Let \([E, F]\) be a pseudomatroid and let \(A\) and \(B\) be members of \(F\) with \(|A| > |B|\). Then there exists an \(x \in A - B\) such that \(B \cup x \in F\) or there is a pair, \(x, y \in A - B\) such that \(B \cup \{x, y\} \in F\).

Proof. Suppose not; choose a counter example with maximum value of \(|A \cap B|\). By Lemma 2, we get the desired result if \(B \subseteq A\). Hence, there exists an \(x \in B - A\). Then an application of Lemma 2 yields the relation: either \(A \cup \{x, y\} \in F\) for some \(y \in B - A\) or \((A \cup x) - y \in F\) for some \(y \in A - B\). Call this set \(A'\). Then in either case \(|A' \cap B| > |A \cap B|\). Hence there exists \(r \in A' - B\) such that \(B \cup r \in F\) or there exist \(r, s \in A' - B\) such that \(B \cup \{r, s\} \in F\). But then \(r, s \in A - B\); this provides a contradiction to the selection of \(A\) and \(B\). \(\square\)

Lemma 8. Let \(A\) and \(B\) be members of a pseudomatroid \([E, F]\) with \(|A| < |B|\). Then there exists an \(x \in B - A\) such that \(B - x \in F\) or there exists a pair \(x, y \in B - A\) such that \(B - \{x, y\} \in F\).

Proof. This follows from application of Lemma 7 to \([E, F]^*\). \(\square\)

It should be noted that given an initial \(A \in F\) and a membership testing oracle, GGA can be applied to a pseudomatroid in \(O(|E|^3)\) time.

4. Polyhedral characterization

In this section, we shall give a polyhedral characterization of pseudomatroids. This will yield a generalization of the similar results for matroids [2]. But first we shall have to consider some definitions:

Let \(E\) be a finite set and let \(T = \{(X, Y) : X, Y \text{ in } E; X \cap Y = \emptyset\}\) be the set of all ordered pairs of disjoint subsets of \(E\).

Definition 5. A function \(r : T \rightarrow R\) is called a generalized submodular function on \(E\) if it satisfies the following relation for all pairs of elements \((A, B)\) and \((C, D)\) of \(T\):

\[
r(A, B) + r(C, D) \geq r(A \cap C, B \cap D) + r((A - D) \cup (C - B), (B - C) \cup (D - A))
\]

Now, consider the following linear program:

\[
\max \{wx : x[A, B] \leq b(A, B) \text{ for all } (A, B) \text{ in } T\}
\]

where \(w\) and \(x\) are vectors in \(R^{|E|}\), \(b : T \rightarrow R\) and

\[
x[A, B] = \sum_{i \in A} x(i) - \sum_{i \in B} x(i) \text{ for } (A, B) \text{ in } T.
\]
The dual of (2) is given by:

\[
\begin{align*}
\min & \quad \sum_{(A, B) \in T} b(A, B)y(A, B) \\
\text{s.t.} & \quad \sum_{A : i \in A} y(A, B) - \sum_{B : i \in B} y(A, B) = w(i) \quad \text{for all } i \in E; \\
& \quad y(A, B) \geq 0 \quad \text{for all } (A, B) \in T.
\end{align*}
\]

Definition 6. A matrix $B$ with elements 0, 1 or $-1$ is said to be signed nested if

(i) each column of $B$ is either a 0/1 or a 0/$-1$ vector;

(ii) $B$ can be transformed, by column permutations, into a matrix $D$ satisfying the property:

\[
D(i, j) = 0 \Rightarrow D(i, k) = 0 \quad \text{for all } i, k \text{ and } j; \quad k \geq j. \tag{4}
\]

Definition 7. Linear program (2) is said to have signed nested property if for any $w$ in $R^{|E|}$ such that (3) has an optimal solution, there exists an optimal solution $y^*$ to (3) such that the row submatrix of the coefficient matrix of (3) corresponding to the positive components of $y^*$ is signed nested.

Let us now consider the following continuous version of the generalized greedy algorithm for (2):

Continuous generalized greedy algorithm (CGGA)

Let the elements of $E = \{1, 2, \ldots, n\}$ be arranged so that $|w(i)| \geq |w(j)|$ whenever $i < j$. Let $A(0) = B(0) = \emptyset$; for $j = 1, \ldots, n$ define $A(j) = A(j - 1)$ and $B(j) = B(j - 1) \cup \{j\}$ if $w(j) \leq 0$; $A(j) = A(j - 1) \cup \{j\}$ and $B(j) = B(j - 1)$ otherwise; $x(j) = b(A(j))$. The resulting vector $x$ is called the continuous generalized greedy solution (CGGS).

The implication of these definitions should be clear from the following theorem:

Theorem 2. The following statements are equivalent:

(i) (2) has signed nested property.

(ii) CGGS is an optimal solution to (2) for all vectors $w$.

(iii) The function $b(\cdot, \cdot)$ in (2) is a generalized submodular function.

Proof. We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Without loss of generality, we assume that absolute values of the components $w$ are distinct. By (i), there exists a signed nested row submatrix $B$ of the coefficient matrix of (3) such that $zB = w$ for some $z \geq 0$. Rearrange the variables of (3) such that $B$ satisfies property (4). We shall then have $|w(i)| \geq |w(j)|$ whenever $i < j$. Furthermore, since the absolute values of the
components of \( w \) are distinct, \( B \) has full row rank and thus \( x^* = B^{-1}b \) is the unique optimal solution to (2). It is easy to check that \( x^* \) is CGGS.

(ii) \( \Rightarrow \) (iii): Given \((A, B)\) and \((C, D)\) in \( T \), define \( w \) as follows:

\[
w(i) = \begin{cases} 
-2 & \text{if } i \in A \cap C \\
-2 & \text{if } i \in B \cap D \\
1 & \text{if } i \in ((A - D) \cup (C - B)) - (A \cap C) \\
-1 & \text{if } i \in ((B - C) \cup (D - A)) - (B \cap D) \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( x^* \) be a CGGS to (2) for the above \( w \). Then,

\[
x^*[A \cap C, B \cap D] = b(A \cap C, B \cap D)
\]

and

\[
x^*[(A - D) \cup (C - B), (B - C) \cup (D - A)] = b((A - D) \cup (C - B), (B - C) \cup (D - A)).
\]

Since \( x^* \) is feasible to (2), we also have:

\[
b(A, B) + b(C, D) \geq x^*[A, B] + x^*[C, D]
\]

\[
= x^*[A \cap C, B \cap D] + x^*[(A - D) \cup (C - B), (B - C) \cup (D - A)]
\]

\[
= b(A \cap C, B \cap D) + b((A - D) \cup (C - B), (B - C) \cup (D - A)).
\]

This proves the result.

(iii) \( \Rightarrow \) (i): For a fixed \( w \) such that (3) has an optimal solution, let \( y^* \) be an optimal solution to (3) such that

\[
\sum_{(A, B) \in T} |A \cup B| y^*_{(A, B)} = \min \left\{ \sum_{(A, B) \in T} |A \cup B| y_{(A, B)} : y \text{ optimal for (3)} \right\} = K \quad (5)
\]

and

\[
\sum_{(A, B) \in T} (|A \cup B|)^2 y^*_{(A, B)} = \max \left\{ \sum_{(A, B) \in T} (|A \cup B|)^2 y_{(A, B)} : \sum_{(A, B) \in T} |A \cup B| y_{(A, B)} = K; y \text{ optimal for (3)} \right\}. \quad (6)
\]

Let \( G \) be the row submatrix of the coefficient matrix of (3) corresponding to the positive components of \( y^* \).

Claim. \( G \) is signed nested.

Proof. If not, there exist \((A, B)\) and \((C, D)\) in \( T \) such that \( y^*_{(A, B)} \) and \( y^*_{(C, D)} \) are
positive and \(((A \not\in C \text{ or } B \not\in D) \text{ and } (C \not\in A \text{ or } D \not\in B))\). Define \(y'\) as follows:

\[
y'(x, y) = \begin{cases} 
y^*(x, y) - a & \text{if } (X, Y) = (A, B) \text{ or } (C, D) \\
y^*(x, y) + a & \text{if } (X, Y) = ((A \cap C, (B \cap D)) \text{ or } ((A - D) \cup (C - B), (B - C) \cup (D - A)) \\
y^*(x, y) & \text{otherwise.}
\end{cases}
\]

It is straightforward to check that \(y'\) is an optimal solution to (3) and it violates either (5) or (6). This proves the claim and hence the theorem. \(\Box\)

We shall now state and prove the main results of this section.

**Theorem 3.** Let \(b : T \to R\) be an integer valued, generalized submodular function on \(E\) satisfying the relations:

(a) \(b(\emptyset, \emptyset) = 0\);

(b) \(b(i, \emptyset) = 0\) or 1 for all \(i \in E\);

(c) \(\{A \subseteq B; C \supseteq D; (A, C), (B, D) \in T\} \leq b(A, C) \leq b(B, D)\).

Then, the extreme points of the polytope \(\mathcal{P}\) in (2) are precisely the characteristic vectors of the independent sets of a pseudomatroid on \(E\).

**Proof.** Using Theorem 2, the following facts are easy to verify:

(i) for each \(w\) in \(R^{(E)}\) \(CGGS\) is an extreme point of \(\mathcal{P}\).

(ii) if \(CGGS\) is a 0/1 valued vector for each \(w\) in \(R^{(E)}\) then \(\mathcal{P}\) has 0/1 valued extreme points and \(CGGA\) is equivalent to \(GGA\) applied to the discrete system \([E, F]\) where the elements of \(F\) are precisely the characteristic sets of the extreme points of \(\mathcal{P}\).

It is therefore sufficient to prove that under the stated conditions, \(CGGS\) is 0/1 valued for all \(w\) in \(R^{(E)}\) and this follows from the following relations:

\[
0 \leq b(X \cup i, Y) - b(X, Y) \leq b(i, \emptyset) \leq 1
\]

and

\[
0 \leq b(X, Y) - b(X, Y \cup i) \leq b(X \cup i, Y) - b(X, Y) \leq 1
\]

for all \((X, Y) \in T\) and \(i \in T - (X \cup Y)\).

This proves the theorem. \(\Box\)

**Theorem 4.** Let \([E, F]\) be a pseudomatroid and for any \((A, B)\) in \(T\) let \(r(A, B)\) be the optimum value of the problem: \(\max \{w[X] : X \in F\}\) where \(w\) is defined as:

\[
w(i) = \begin{cases} 
1 & \text{if } i \in A \\
-1 & \text{if } i \in B \\
0 & \text{otherwise.}
\end{cases}
\]

Then \(r\) is an integer valued generalized submodular function that satisfies
conditions \((a), (b)\) and \((c)\) of Theorem 3 and \(\{x: x[A, B] \leq r(A, B) \text{ for all } (A, B) \in T\}\) is the convex hull of the characteristic vectors of the elements of \(F\).

**Proof.** The fact that \(r(\cdot, \cdot)\) is an integer valued function satisfying conditions \((a), (b)\) and \((c)\) of Theorem 3 is easy to verify. To show that \(r(\cdot, \cdot)\) is a generalized submodular function consider any \((A, B)\) and \((C, D)\) in \(T\) and define a weight function \(w\) on \(E\) as follows:

\[
w(i) = \begin{cases} 
1 & \text{if } i \in (A - D) \cup (C - B) \\
-1 & \text{if } i \in (B - C) \cup (D - A) \\
0 & \text{otherwise.}
\end{cases}
\]

Find GGS \(X\) using the weight function \(w\) on \([E, F]\) by first considering the elements of \((A \cap C) \cup (B \cap D)\) followed by the remaining elements of \((A - D) \cup (B - C)\), then the remaining elements of \((C - B) \cup (D - A)\) and finally the remaining elements of \(E\). Then,

\[
r(A, B) + r(C, D) \geq |X \cap A| - |X \cap B| + |X \cap C| - |X \cap D|
\]

\[
= 2(|X \cap A \cap C| - |X \cap B \cap D|)
\]

\[
+ |X \cap (A - (C \cup D))| + |X \cap (C - (A \cup B))|
\]

\[
- |X \cap (D - (A \cup B))| - |X \cap (B - (C \cup D))|
\]

\[
= r(A \cap C, B \cap D) + r((A - D) \cup (C - B),
(B - C) \cup (D - A)).
\]

This proves the theorem. \(\square\)

We shall call the function \(r(\cdot, \cdot)\) the *rank function* of the pseudomatroid. This generalizes the notion of submodular rank function of a matroid.

**Lemma 9.** The rank function \(r^*(\cdot, \cdot)\) of the dual pseudomatroid \([E, F]^*\) is given by the relation:

\[
r^*(A, B) = |A| - |B| + r(B, A) \text{ for all } (A, B) \text{ in } T.
\]

**5. Minors**

In this section, we shall define the operations of contraction and deletion and through these, the notion of minors.

**Definition 8.** Given a discrete system \([E, F]\) and an element \(i \in E\), the system
\( [E', F'] \) is called a deletion of \( [E, F] \) with respect to \( i \) where \( E' = E - i \) and

\[
F' = \begin{cases} 
\{A : A \in F; i \notin A\} & \text{if } \exists A \in F; \ i \notin A \\
\{A - i : A \in F\} & \text{otherwise}
\end{cases}
\]

and is denoted by \( [E, F]'i \).

**Definition 9.** The system \( [[E, F]^*\setminus i]^* \) is called a contraction of \( [E, F] \) with respect to the element \( i \in E \) and is denoted by the symbol \( [E, F]/i \).

**Definition 10.** \( [E', F'] \) is a minor of \( [E, F] \) if it can be obtained from \( [E, F] \) by contracting and deleting some elements of \( E \). It is easy to verify that the order in which these operations are carried out is irrelevant.

**Lemma 10.** A minor of a pseudomatroid is a pseudomatroid.

**Proof.** Straightforward application of the greedy algorithm. \( \square \)

**Definition 11.** Given \( [E, F] \), the system \( [E, F'] \) with \( F' = \{X : X \in F \text{ and } k \leq |X| \leq k'\} \) for any fixed \( k \) and \( k' \) is called a truncation of \( [E, F] \).

Unlike generalized matroids (where the truncation operation yields a generalized matroid) this operation on a pseudomatroid may not always yield a pseudomatroid. Some well known special cases, however, do have this property as we shall show later.

6. Examples of pseudomatroids

We give some well known special cases first. Certainly matroids are special cases of pseudomatroids; indeed any pseudomatroid that is also an independence system is a matroid. There are simple generalizations of special matroids to corresponding cases of pseudomatroids such as those of uniform and free matroids. We now give some other examples:

1. **Matching Pseudomatroid.** Let \( G = [V, A] \) be an undirected graph. Let \( F = \{S : S \subseteq V; G \setminus S \text{ has a perfect matching}\} \). Then \( [V, F] \) is a pseudomatroid.

2. Let \( E = S \cup T \) where \( S \) and \( T \) are disjoint. Let members of \( F \) be subsets of \( E \) satisfying the relations; (i) \( [T, F(A)] \) is a matroid for all \( A \subseteq S \) where \( F(A) = \{X \cap T : X \in F; X \cap S \subseteq A\} \) and (ii) \( [S, F(B)] \) is a matroid for all \( B \subseteq T \) where \( F(B) = \{X \cap S : X \in F; X \cap T \subseteq B\} \). Then \( [E, F] \) is a pseudomatroid.

We now give more interesting special cases of (2) above:

3. Let \( [E, F] \) be a matroid; let \( B \) be a basis of this matroid. Let \( F' = \{S : S = S' \cup S'' \text{ where } S' = B - B' \text{ and } S'' = B' - B \text{ for some basis } B' \text{ of the matroid}\} \). Then \( [E, F'] \) is a pseudomatroid.
4. Representable Pseudomatroid. Let \( R \) and \( C \) denote the rows and columns respectively of a matrix defined over any given field. Let \( E = R \cup C \) and let \( F = \{A \cup B: \text{the submatrix defined by the rows and columns corresponding to } A \text{ and } B \text{ respectively is nonsingular}\} \). Then, \([E, F]\) is a pseudomatroid. In fact, this is a special case of (3) above.

A useful subclass of representable pseudomatroid is the one corresponding to the identity matrix. We call this parity pseudomutroid.

Recall that \( \text{FMIN} \) and \( \text{FMAX} \) form bases of matroids if \([E, F]\) is a pseudomatroid. One question that arises immediately is: "Is the pseudomatroid \([E, F]\) uniquely defined by \( \text{FMIN} \) and \( \text{FMAX} \)?" The answer is no. However, there is a special class of pseudomatroids in which this is true. These are called generalized matroids and were introduced by A. Frank [3, 7].

**Definition 12.** \([E, F]\) is a generalized matroid if:

(i) \( A, B \in F; \ x \in A - B \implies \text{either } A - x \in F \text{ or } (A \cup y) - x \in F \) for some \( y \in B - A \).

(ii) \( A, B \in F; \ x \in A - B \implies \text{either } B \cup x \in F \text{ or } (B \cup x) - y \notin F \) for some \( y \in B - A \).

**Lemma 11.** Generalized matroids are pseudomatroids; further they are uniquely defined by \( \text{FMIN} \) and \( \text{FMAX} \); i.e. if \([E, F]\) and \([E, F']\) are two generalized matroids with \( \text{FMIN} = \text{FMIN}' \) and \( \text{FMAX} = \text{FMAX}' \) then they are identical.

7. Intersection, partition and parity problems

Now we turn our attention to some traditional types of optimization problems in matroids and their generalizations to pseudomatroids. In the case of matroids the first two are well solved [1] but the third is in general difficult [5]. Since matroids are special cases of pseudomatroids it follows that the parity problem will be in general difficult. However, there are some relationships between these problems when dealing with pseudomatroids that are not present in matroids and we discuss them first.

**Intersection problem**

Let \([E, F]\) and \([E, F']\) be two discrete systems defined on the same set \( E \). Let \( w(e) \) be a number associated with the element \( e \) of \( E \). The intersection problem is to find a set \( S \in F \cap F' \) with maximum value for \( w[S] \).

**Partition problem**

Let \([E, F(i)]\) be discrete systems for \( i = 1, 2, \ldots, n \). A set \( S \subseteq E \) is said to be partitionable if there are sets \( S(i) \in F(i) \) such that \( S(i) \cap S(j) = \emptyset, \forall i \neq j \) and
Let \( S = \bigcup_i S(i) \). Let \( w(e) \) be the weight associated with element \( e \) of \( E \). The partition problem is to find a partitionable set \( S \) with maximum \( w[S] \). If we remove the restriction that \( S(i)'s \) should be disjoint then we have a covering problem.

**Parity problem**

Let \( E \) consist of \( n \) distinct pairs of elements of the form \((i, i')\). Let \([E, F]\) be a discrete system with weight \( w(e) \) for each element \( e \) of \( E \). The parity problem is to select a subset \( S \) in \( F \) which includes either both or none of each pair of elements and among all such sets has the maximum value of \( w[S] \).

When \([E, F]\) is a pseudomatroid and we are interested in the covering problem, since members of \( S(i) \) can without loss be assumed to be from \( \text{FMAX} \) which are bases of a matroid this becomes a covering (and hence a partition) problem on a matroid which is well solved [1].

**Lemma 12.** Parity problems on pseudomatroids can be formulated as intersection problems.

**Proof.** Let \([E, F]\) be a pseudomatroid with paired elements. In the parity problem we want sets that are members of \( F \) with the property that either both or none of a pair of elements be selected. Define \([E, F']\) to be the parity pseudomatroid with members of \( F' \) being precisely those sets that have both or none from each pair of elements. The parity problem is then equivalent to the intersection problem on \([E, F]\) and \([E, F']\).

Since parity problem on a pseudomatroid includes that on a matroid it should be clear that this is a difficult problem. This together with the above lemma implies that the intersection problem is also difficult in general on pseudomatroids. However, there is a class of pseudomatroids for which the intersection problem is nicely solvable. This includes generalized matroids, for which an algorithm has been given in [7]. We discuss this class of pseudomatroids below. □

**Theorem 5.** If the pseudomatroids \([E, F]\) and \([E, F']\) are such that their truncations are pseudomatroids then the intersection and partition problems on these pseudomatroids can be solved in polynomial time.

**Proof.** We shall assume that we are given an oracle for each pseudomatroid which, given an \( X \subseteq E \) checks if \( X \) is independent in that pseudomatroid. In addition, suppose we have an independent set of each possible cardinality for each of the pseudomatroids. For a given instance of pseudomatroid intersection problem with weight function \( w \), let \( S \subseteq E \) solve \( \max\{w(A) : A \in F \cap F'\} \) and let \(|S| = k\). Let us define new weights \( w' \) by \( w'(e) = w(e) + M \) for each \( e \) in \( E \) (here \( M \) is a sufficiently large integer). Then \( S \) solves the matroid intersection problem...
on the matroids \([E, F^\wedge]\) and \([E, F^{\wedge\wedge}]\) with weight function \(w'\), where \(F^\wedge = \{X: X \subseteq Y \text{ for some } Y \in F \text{ and } |Y| = k\}\) and \(F^{\wedge\wedge} = \{X: X \subseteq Y \text{ for some } Y \in F' \text{ and } |Y| = k\}\). Thus we can solve the pseudomatroid intersection problem by solving \([E]\) matroid intersection problems, one for each \(k\), each of which can be solved in polynomial time using the information provided.

Similarly, we can solve the partition problem by converting it to an intersection problem as follows: Make as many copies as necessary of elements that are repeated in different sets \(E(i)\). Let \([E, F]\) be the union pseudomatroid with these disjoint sets \(E(i)\); and let \([E, F\leftarrow]\) be the partition matroid in which members of \(F\leftarrow\) have no more than one of the copies of any element. Now solve the intersection problem with these two pseudomatroids in order to solve the original partition problem. □

References