INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 4, Number 2, Pages 167–178

AN INTERPOLATION METHOD FOR RIGID CONTACT LENSES

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Abstract. Custom designed rigid gas permeable lenses show promise for correcting the vision of subjects with significant higher order aberration beyond defocus (myopia and hyperopia) and astigmatism. Such aberrations could be coma, trefoil, spherical aberration, secondary astigmatism etc. Although, there exist several techniques for designing the central part of the front optic zone for subjects with significant higher order aberration, there does not exist a method for continuously differentiable interpolation of the corrected region and the rest of the front surface of the lens. This paper presents a method for C^1 interpolation based on the solution of the biharmonic equation with appropriate boundary conditions.

Key words. rigid contact lenses, higher order aberration, two dimensional cubic spline interpolation, biharmonic equation.

1. Introduction

In the early 1960's, it was Smirnov, who suggested that higher-order aberrations can be corrected with customized lenses to compensate for aberrations in individual eyes (page 19 in [1]). Aberrations in human vision, and optics, in general, may be characterized by a variety of methods such as the point spread function, the optical transfer function, Seidel polynomial expansion, or Zernike polynomial expansion [2, 3, 4]. Of these, the Zernike polynomial representation is especially useful in the design of contact lenses [2]. Traditionally, the majority of aberration correction methods dealt with a class called lower-order aberrations. This class consists of refractive errors such as Myopia (nearsightedness), Hyperopia (farsightedness), and Astigmatism. Glasses, contact lenses, and laser surgery can correct these types of refractive errors. Unfortunately, the eye can suffer from many more types of aberrations called higher-order aberrations, such as Coma, Trefoil, Spherical Aberration, and Secondary astigmatism, which are characterized by the coefficients in a Zernike expansion of the wavefront measured by an aberrometer. Such aberrations are especially prominent in patients suffering from a disease called keratoconus and significantly affects their vision [5].

In this paper, we present a method for C^1 interpolation of the corrected region on a customized rigid gas permeable contact lens with the rest of the front surface. There exist several techniques for the design of the front surface for the correction of aberrations [2, 6, 7, 8]. All of these methods will result in a modification of a circular region of the trial lens directly in front of the pupil. This leads to an edge dislocation at the boundary of the modified region, which leads to a C^1 interpolation problem. In this paper, we consider an aberration correction method by Guirao et. al., which accounts for decentrations and rotations of the lens. We then interpolate on the contact lens by solving the biharmonic equation with Dirichlet boundary conditions. This procedure completes the process of designing the front surface of the contact lens. We illustrate the process with data from a human subject.

Received by the editors March 10, 2013.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.

2. Wavefront aberration correction using a contact lens

A wavefront is a constant phase solution to the scalar wave equation. The causal solution to the scalar inhomogenous wave equation is given by the Kirchhoff formula [4, 11]. The solution for the particular case of a point source is a spherical wave. Therefore, if a constant (in time) point light source is "placed" at the retina, the solution to wave equation yields a constant (in time) spherical wave solution within the vitreous humor of the eye. After passing through the lens in the eye, the opening in the iris, the anterior chamber, and the cornea, the solution in the space in front of the eye takes the form of a plane wave in the ideal case of no ocular aberrations (see figure 2.1 in [2] for an illustration). An imperfect optical system consisting of the cornea and lens will result in a wave that is not planar.

A device that "places" a point source at the retina and measures the reflected wave coming out of the cornea is a Shack-Hartmann wavefront sensor. The wavefront sensor shines a low-power laser through a patient's cornea and iris onto the retina. As the retina is a rough surface, the reflected light inside the eye is not a ray travelling back along the optical axis but a spherical wave.

The patient wears a lens of known front and back surface radii while being tested on a Sharck-Hartmann aberrometer. The back surface of the lens is designed by an optometrist after a corneal topography measurement. It is desirable for the patient to wear a trial lens during the aberrometry because the tear layer between the cornea and the contact lens is optically active, which is unaccounted for in the absence of the trial lens. Therefore, a lens designer only has to make corrections on the front surface of the trial lens itself to cancel the aberrations after an aberrometry measurement.

We present a simple lens design below based on the idea of equalizing optical path lengths so that the reflected wave coming out of the trial lens is a plane wave. Let $W(x,y)$ be the function of the wavefront reflected from the trial lens, and let $L(x,y)$ be the thickness of the contact lens. Furthermore, let L_{max} be the thickest part of the contact lens, and η_{glass} be the refractive index of the contact lens glass material. The expression

$$L(x,y) = L_{max} - \frac{W(x,y)}{\eta_{glass} - 1} \tag{1}$$

yields a simple method for designing a contact lens [2, 8]. Unfortunately, decentrations and rotations can occur each time the patient blinks (page 126 in [7]). As a result, it is essential that equation (1) take decentrations and rotations into account. Guirao, Cox, and Williams presented a method for optimizing the correction of the aberrations of the eye with decentrations and rotations (pages 126-127 in [7]). The method partially corrects each of the Zernike coefficients such that the average variance of the residual wavefront aberration for each decentration can be minimized. The interpolation method presented in the next section does not depend on the aberration correction method used. Any method including Equation (1) or the method in [7] may be used followed by an application of the interpolation method described in this paper.

2.1. Wavefront representation using Zernike polynomials. In 1934, Fritz Zernike introduced the concept of Zernike Polynomials. Unlike Seidel Aberrations, Zernike Polynomials represents aberrations of any order on the unit disk. Since

Zernike Polynomials are orthogonal and complete over the unit disk, we may represent each L^2 function over the unit disk as a linear combination of Zernike Polynomials (pages 28-29 in [12]). Below is a formal definition of the Zernike Polynomial (as found in [13]).

Definition 1. *A polynomial $Z_n^m(\rho,\theta)$ is called a Zernike polynomial if,*

$$Z_n^m(\rho,\theta) = \begin{cases} N_n^m R_n^{|m|}(\rho)\cos m\theta & \text{for } m \geq 0, \\ -N_n^m R_n^{|m|}(\rho)\sin m\theta & \text{for } m < 0, \end{cases} \tag{2}$$

where (ρ,θ) are in Polar Coordinates ($0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$), n is the order of the polynomial and m is the azimuthal frequency of the sinusoidal component. In addition, N_n^m is the normalization constant given by

$$N_n^m = \sqrt{\frac{2(n+1)}{1+\delta_m}}, \text{ where } \delta_m = \begin{cases} 1 & \text{for } m = 0, \\ 0 & \text{for } m \neq 0, \end{cases}$$

and $R_n^{|m|}$ is the radial component given by

$$R_n^{|m|}(\rho) = \sum_{i=0}^{\frac{n-|m|}{2}} \frac{(-1)^i(n-i)}{i!\left[\frac{1}{2}\left(n+|m|-i\right)\right]!\left[\frac{1}{2}\left(n-|m|-i\right)\right]!}\rho^{n-2i}.$$

Table 1 lists the first 10 Zernike Polynomials based on Definition 1. As the

Table 1. First 10 Zernike polynomials [13]

Order (n)	Frequency (m)	Polynomial $[Z_n^m(\rho,\theta)]$
0	0	1
1	-1	$2\rho\sin\theta$
1	1	$2\rho\cos\theta$
2	-2	$\sqrt{6}\rho^2\sin 2\theta$
2	0	$\sqrt{3}(2\rho^2-1)$
2	2	$\sqrt{6}\rho^2\cos 2\theta$
3	-3	$\sqrt{8}\rho^3\sin 3\theta$
3	-1	$\sqrt{8}(3\rho^3-2\rho)\sin\theta$
3	1	$\sqrt{8}(3\rho^3-2\rho)\cos\theta$
3	3	$\sqrt{8}\rho^3\cos 3\theta$

Zernike polynomials form an orthogonal basis for the space of L^2 functions on a unit circle, we can express the continuous function $W(x,y)$ as a linear combination of the Zernike Polynomials defined over the pupil region. Therefore,

$$W(x,y) = \sum_{n=0}^{N} \sum_{m} \alpha_n^m Z_n^m(x,y), \quad \textit{where} \quad m = -n, -n+2, \ldots, n. \tag{3}$$

Each Zernike coefficient α_n^m is related to a particular type of aberration [2]. The types of aberrations that a person notices depends on the magnitude of each Zernike coefficients. Through Table 2 we can see the types of aberrations corresponding to the first few Zernike coefficients. As coefficients of second order and above affect vision, the contact lens should be designed to minimize the Zernike coefficients of second order and above as much as possible while taking into account possible decentrations and rotations.

Table 2. First 20 aberrations corresponding to their Zernike coefficients (page 35 in [12])

Coefficient	Aberration	Coefficient	Aberration	Coefficient	Aberration
α_0^0	Piston	α_1^{-1}	Tilt	α_2^{-2}	Primary astigmatism
		α_1^1	Tilt	α_2^0	Defocus
				α_2^2	Primary astigmatism
α_3^{-3}	Trefoil	α_4^{-4}	Quadrafoil	α_5^{-5}	Pentafoil
α_3^{-1}	Primary coma	α_4^{-2}	Secondary astigmatism	α_5^{-3}	Secondary trefoil
α_3^1	Primary coma	α_4^0	Spherical aberration	α_5^{-1}	Secondary coma
α_3^3	Trefoil	α_4^2	Secondary astigmatism	α_5^1	Secondary coma
		α_4^4	Quadrafoil	α_5^3	Secondary trefoil
				α_5^5	Pentafoil

3. C^1 interpolation using the biharmonic equation

Equation (1) or the Guirao, Cox, Williams [7] method yield a formula to calculate the front surface of contact lens after correcting for wavefront aberrations. However, the lens will contain corners and discontinuities along its surface after correction. Upon decentration, the corners can cause optical errors (see Figure 3). To account for this, we will smoothly interpolate the corrected and uncorrected regions of the contact lens by solving the biharmonic equation with Dirichlet boundary conditions (see Figure 4).

Consider the Laplacian in Polar Coordinates (page 78 in [14])

$$\triangle = \triangle_r + \frac{1}{r^2}\triangle_\phi \tag{4}$$

where,

$$\triangle_r = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}; \quad \triangle_\phi = \frac{\partial^2}{\partial \phi^2}. \tag{5}$$

The annular domain $D_{a,b}$ on which the biharmonic equation is defined is: $D_{a,b} = \{(r,\phi) : 0 < a < r < b\,; 0 \le \phi \le 2\pi\}$. A function u defined on the annular domain

$D_{a,b}$ is Biharmonic if and only if it satisfies the fourth-order equation:

$$\triangle^2 u(x) = 0 \quad x \in D_{a,b}. \tag{6}$$

Substituting (4) into (6) we obtain,

$$\triangle^2 u(x) = \left(\triangle_r + \frac{1}{r^2}\triangle_\phi\right)\left(\triangle_r + \frac{1}{r^2}\triangle_\phi\right)u(x) = 0. \tag{7}$$

Let $h(r,\phi)$ be a biharmonic function in the annulus where $h(r,\phi) \in C^4(D_{a,b}) \cap C(\bar{D}_{a,b})$ for every r with $0 < a < r < b$. Therefore, we can express $h(r,\phi)$ as a Fourier Series,

$$h(r,\phi) = \frac{u_0(r)}{2} + \sum_{k=1}^{\infty}[u_k(r)\cos(k\phi) + v_k(r)\sin(k\phi)]. \tag{8}$$

Theorem 1 below considers behavior of an arbitrary function $g(r,\phi)$ as a function of ϕ with r fixed, while Theorem 2 considers the case when ϕ is the fixed variable and r is the independent variable. Finally, Theorem 3 applies to the solution of the biharmonic equation.

Theorem 1. *Let $g_r(\phi)$ denote the function $g(r,\phi)$ where $r \in [a,b]$ is fixed. For each $r \in [a,b]$, if g_r is absolutely continuous and periodic on $[0,2\pi]$, and in addition, $\dfrac{\partial g_r}{\partial \phi} \in L^2[0,2\pi]$, then*

(i) *The Fourier Series for $g_r(\phi) \sim \sum_{k=1}^{\infty}[u_k(r)\cos(k\phi) + v_k(r)\sin(k\phi)]$ converges absolutely.*

(ii) $\dfrac{\partial g_r}{\partial \phi}(\phi) \sim \sum_{k=1}^{\infty}[-k\,u_k(r)\,\sin(k\phi) + k\,v_k(r)\,cos(k\phi)]$.

(iii) $\sum_{k=1}^{\infty} k^2(u_k^2(r) + v_k^2(r)) < \infty$ *for almost all $u_k(r)$ and $v_k(r)$.*

(iv) *If $\dfrac{\partial g_r}{\partial \phi}(\phi) \in C^{0,\alpha}[0,2\pi]$, (that is $\exists\ L > 0 \in \mathbb{R} \ni \left|\dfrac{\partial}{\partial r}g_r(\phi_1) - \dfrac{\partial}{\partial r}g_r(\phi_2)\right| \leq L|\phi_1 - \phi_2|^\alpha$ with $\frac{1}{2} < \alpha \leq 1$), then $\sum_{k=1}^{\infty}[-k\,u_k(r)\,\sin(k\phi) + k\,v_k(r)\,cos(k\phi)]$ converges absolutely.*

The proof may be found in Sections 6.35 (page 138), 2.12 (page 15), 5.6 (page 125), and 6.3 (page 135) in Zygmund [15].

Theorem 2. *Suppose $f(r,\phi)$ is continuous on $[a,b]\times[0,2\pi]$ and differentiable with respect to r for each $\phi \in [0,2\pi]$, with either (a) $\dfrac{\partial}{\partial r}f(r,\phi)$ a continuous function on $(a,b)\times[0,2\pi]$ or (b) $\dfrac{\partial}{\partial r}f(r,\phi) \in L^2[0,2\pi]$ and $\left|\dfrac{\partial}{\partial r}f(r,\phi)\right| \leq K(\phi)$, where $K(\phi)$ is an integrable function on $[0,2\pi]$, and suppose $f(r,\phi) \sim \sum_{k=1}^{\infty}[u_k(r)\cos(k\phi) + v_k(r)\sin(k\phi)]$. Then, for $r \in (a,b)$, we have*

(i)

$$\frac{\partial}{\partial r}u_k(r) = \int_0^{2\pi}\frac{\partial}{\partial r}f(r,\phi)\cos(k\phi)d\phi, \tag{9}$$

$$\frac{\partial}{\partial r} v_k(r) = \int_0^{2\pi} \frac{\partial}{\partial r} f(r, \phi) \sin(k\phi) d\phi, \tag{10}$$

$$\frac{\partial}{\partial r} f(r, \phi) \sim \sum_{k=1}^{\infty} \frac{\partial}{\partial r} \Big[u_k(r) \cos(k\phi) + v_k(r) \sin(k\phi) \Big]. \tag{11}$$

(ii) *Furthermore, if* $\frac{\partial}{\partial r} f(r, \phi) \in C^{0,\alpha}[0, 2\pi]$ *with* $\frac{1}{2} < \alpha \leq 1)$*, then* $\frac{\partial}{\partial r} f(r, \phi) \sim \sum_{k=1}^{\infty} \frac{\partial}{\partial r} [u_k(r) \cos(k\phi) + v_k(r) \sin(k\phi)]$*, where the series converges absolutely.*

The proof follows from the Leibniz formula and section 6.3 (page 135) of Zygmund [15].

Theorem 3. *Let* $h(r, \phi)$ *be a biharmonic function in the annulus* $D_{a,b}$ *where* $h(r, \phi)$ *can be represented as in* (8)*. Suppose* $h(r, \phi)$ *is absolutely continuous on* $[a, b] \times [0, 2\pi]$*, periodic in* ϕ *(that is, for almost ever* r*,* $h(r, 0) = h(r, 2\pi)$*), four times differentiable in both* r *and* ϕ*, and* $\triangle_\phi^2 h(r, \phi), \triangle_r^2 h(r, \phi) \in C^{0,\alpha}[0, 2\pi]$ *with* $\frac{1}{2} < \alpha \leq 1)$*, then we have*

(i) $\triangle_\phi^2 \, h(r, \phi) \sim \sum_{k=1}^{\infty} \triangle_\phi^2 [u_k(r) \cos(k\phi) + v_k(r) \sin(k\phi)],$

(ii) $\triangle_r \triangle_\phi \, h(r, \phi) \sim \sum_{k=1}^{\infty} \triangle_r \triangle_\phi [u_k(r) \cos(k\phi) + v_k(r) \sin(k\phi)],$

(iii) $\triangle_r^2 \, h(r, \phi) \sim \triangle_r^2 \left(\frac{u_0(r)}{2} \right) + \sum_{k=1}^{\infty} \triangle_r^2 [u_k(r) \cos(k\phi) + v_k(r) \sin(k\phi)],$

(iv) $\triangle_\phi \triangle_r \, h(r, \phi) \sim \sum_{k=1}^{\infty} \triangle_\phi \triangle_r [u_k(r) \cos(k\phi) + v_k(r) \sin(k\phi)]$*, where all the series converge absolutely.*

In addition, $u_k(r)$ *and* $v_k(r)$ *satisfy the equations*

$$\left(\triangle_r - \frac{k^2}{r^2} \right)^2 u_k(r) = 0 \quad k = 0, 1, 2, 3, \ldots \tag{12}$$

$$\left(\triangle_r - \frac{k^2}{r^2} \right)^2 v_k(r) = 0 \quad k = 1, 2, 3, \ldots \tag{13}$$

for $0 < a < r < b$ *and* $0 \leq \phi \leq 2\pi$

The proof of the first three items follow from Theorems 1 and 2, while the proof of the last item involves straightforward calculations and may be found in [8].

Equations (12) and (13) obtained in Theorem 3 are Ordinary Differential Equations that we can obtain solutions for as seen in the following Proposition (see Proposition 7.12 on page 90 in [14]).

Proposition 4. *Given equations (12) and (13) where* $k \geq 0$*, we have the following solution subspaces:*

(i) *If* $k = 0$*, then*
$H_0 = span\{1, r^2, \ln r, r^2 \ln r\}.$

(ii) *If* $k = 1$*, then*
$H_1 = span\{r^{-1}, r, r^3, r \ln r\}.$

(iii) *Else if, $k \geq 2$, then*
$H_k = span\{r^{-k}, r^{-k+2}, r^k, r^{k+2}\}$.

For example, consider the set $H_1 = span\{r^{-1}, r, r^3, r\ln r\}$ for $k = 1$. One can show that any solution for $k = 1$ is a linear combination of the spanning set for H_1 described in Proposition 4 by showing that: [8]

(1) each function u in the spanning set satisfies the equation $\left(\Delta_r - \frac{1}{r^2}\right)^2 u(r) = 0$,
(2) the functions in the spanning set are linearly independent, and
(3) using the fact from linear ordinary equations theory that a homogeneous fourth order linear ordinary differential equation has at most 4 linearly independent solutions (see page 87 of [9], or Chapter 3 of [10]).

Therefore, the spanning set for each k described in Proposition 4 is a basis for the corresponding solution subspace for Equations (12) and (13), and we may find general solutions to Equations (12) and (13). The boundary conditions to the Dirichlet Problem yields a particular solution to (12) and (13), which is the C^1 interpolation between the corrected and the uncorrected regions of the contact lens.

Let f_a and g_a be continuous functions of the angular variable defined on the interior circle $C_{int}(0;a)$. In addition, let f_b and g_b be continuous functions of the angular variable on the exterior circle $C_{ext}(0;b)$. We require the Dirichlet Problem to satisfy the following conditions (see section 7.3.2 on page 92-93 in [14]):

(i) $\triangle^2 h(r,\phi) = 0 \;\forall\; 0 < a < r < b$ and $0 \leq \phi \leq 2\pi$,
(ii) $h(a,\phi) = f_a(\phi), \quad \dfrac{\partial h(a,\phi)}{\partial r} = g_a(\phi), \quad h(b,\phi) = f_b(\phi), \quad \dfrac{\partial h(b,\phi)}{\partial r} = g_b(\phi).$

As h, $f_a(\phi)$, $f_b(\phi)$, $g_a(\phi)$, and $g_b(\phi)$ are continuous functions, then we can represent them as a Fourier Series. Therefore, we have

$$f_a(\phi) = \frac{f_{0,a}}{2} + \sum_{k=1}^{\infty}[f_{k,a}\cos(k\phi) + \bar{f}_{k,a}\sin(k\phi)], \tag{14}$$

$$g_a(\phi) = \frac{g_{0,a}}{2} + \sum_{k=1}^{\infty}[g_{k,a}\cos(k\phi) + \bar{g}_{k,a}\sin(k\phi)], \tag{15}$$

$$f_b(\phi) = \frac{f_{0,b}}{2} + \sum_{k=1}^{\infty}[f_{k,b}\cos(k\phi) + \bar{f}_{k,b}\sin(k\phi)], \tag{16}$$

$$g_b(\phi) = \frac{g_{0,b}}{2} + \sum_{k=1}^{\infty}[g_{k,b}\cos(k\phi) + \bar{g}_{k,b}\sin(k\phi)]. \tag{17}$$

Substituting equations (14) through (17) into the Dirichlet boundary conditions, we obtain the following four equations (page 94 in [14]):

(i) $\dfrac{u_0(a)}{2} + \displaystyle\sum_{k=1}^{\infty}[u_k(a)\cos(k\phi) + v_k(a)\sin(k\phi)] = \dfrac{f_{0,r}}{2} + \displaystyle\sum_{k=1}^{\infty}[f_{k,a}\cos(k\phi) + \bar{f}_{k,a}\sin(k\phi)].$

(ii) $\dfrac{u_0'(a)}{2} + \displaystyle\sum_{k=1}^{\infty}[u_k'(a)\cos(k\phi) + v_k'(a)\sin(k\phi)] = \dfrac{g_{0,a}}{2} + \displaystyle\sum_{k=1}^{\infty}[g_{k,a}\cos(k\phi) + \bar{g}_{k,a}\sin(k\phi)].$

(iii) $\dfrac{u_0(b)}{2} + \displaystyle\sum_{k=1}^{\infty}[u_k(b)\cos(k\phi) + v_k(b)\sin(k\phi)] = \dfrac{f_{0,b}}{2} + \displaystyle\sum_{k=1}^{\infty}[f_{k,b}\cos(k\phi) + \bar{f}_{k,b}\sin(k\phi)].$

(iv) $\dfrac{u_0^{'}(b)}{2} + \sum_{k=1}^{\infty}[u_k^{'}(b)\cos(k\phi) + v_k^{'}(b)\sin(k\phi)] = \dfrac{g_{0,b}}{2} + \sum_{k=1}^{\infty}[g_{k,b}\cos(k\phi) + \bar{g}_{k,b}\sin(k\phi)].$

Therefore, our desired solution is given by

$$h(r,\phi) = \frac{u_0(r)}{2} + \sum_{k=1}^{\infty}[u_k(r)\cos(k\phi) + v_k(r)\sin(k\phi)],$$

where, for $k = 0$ we have, $u_0(a) = f_{0,a} = c_{0,1} + c_{0,2}a^2 + c_{0,3}\ln a + c_{0,4}a^2\ln a$,
$u_0(b) = f_{0,b} = c_{0,1} + c_{0,2}b^2 + c_{0,3}\ln b + c_{0,4}b^2\ln b$,
$u_0'(a) = g_{0,a} = 2\cdot c_{0,2}a + c_{0,3}\frac{1}{a} + c_{0,4}(a + 2a\ln a)$,
$u_0'(b) = g_{0,b} = 2\cdot c_{0,2}b + c_{0,3}\frac{1}{b} + c_{0,4}(b + 2b\ln b)$.
For $k = 1$ we have,
$u_1(a) = f_{1,a} = c_{1,1}a^{-1} + c_{1,2}a + c_{1,3}a^3 + c_{1,4}a\ln a$,
$u_1(b) = f_{1,b} = c_{1,1}b^{-1} + c_{1,2}b + c_{1,3}b^3 + c_{1,4}b\ln b$,
$u_1'(a) = g_{1,a} = -c_{1,1}a^{-2} + c_{1,2} + 3\cdot c_{1,3}a^2 + c_{1,4}(\ln a + 1)$,
$u_1'(b) = g_{1,b} = -c_{1,1}b^{-2} + c_{1,2} + 3\cdot c_{1,3}b^2 + c_{1,4}(\ln b + 1)$,
$v_1(a) = \bar{f}_{1,a} = d_{1,1}a^{-1} + d_{1,2}a + d_{1,3}a^3 + d_{1,4}a\ln a$,
$v_1(b) = \bar{f}_{1,b} = d_{1,1}b^{-1} + d_{1,2}b + d_{1,3}b^3 + d_{1,4}b\ln b$,
$v_1'(a) = \bar{g}_{1,a} = -d_{1,1}a^{-2} + d_{1,2} + 3\cdot d_{1,3}a^2 + d_{1,4}(\ln a + 1)$,
$v_1'(b) = \bar{g}_{1,b} = -d_{1,1}b^{-2} + d_{1,2} + 3\cdot d_{1,3}b^2 + d_{1,4}(\ln b + 1)$.
For $k \geq 2$ we have,
$u_k(a) = f_{k,a} = c_{k,1}a^{-k} + c_{k,2}a^{-k+2} + c_{k,3}a^k + c_{k,4}a^{k+2}$,
$u_k(b) = f_{k,b} = c_{k,1}b^{-k} + c_{k,2}b^{-k+2} + c_{k,3}b^k + c_{k,4}b^{k+2}$,
$u_k'(a) = g_{k,a} = -k\cdot c_{k,1}a^{-k-1} + (-k+2)\cdot c_{k,2}a^{-k+1} + k\cdot c_{k,3}a^{k-1} + (k+2)\cdot c_{k,4}a^{k+1}$,
$u_k'(b) = g_{k,b} = -k\cdot c_{k,1}b^{-k-1} + (-k+2)\cdot c_{k,2}b^{-k+1} + k\cdot c_{k,3}b^{k-1} + (k+2)\cdot c_{k,4}b^{k+1}$,
$v_k(a) = \bar{f}_{k,a} = d_{k,1}a^{-k} + d_{k,2}a^{-k+2} + d_{k,3}a^k + d_{k,4}a^{k+2}$,
$v_k(b) = \bar{f}_{k,b} = d_{k,1}b^{-k} + d_{k,2}b^{-k+2} + d_{k,3}b^k + d_{k,4}b^{k+2}$,
$v_k'(a) = \bar{g}_{k,a} = -k\cdot d_{k,1}a^{-k-1} + (-k+2)\cdot d_{k,2}a^{-k+1} + k\cdot d_{k,3}a^{k-1} + (k+2)\cdot d_{k,4}a^{k+1}$,
$v_k'(b) = \bar{g}_{k,b} = -k\cdot d_{k,1}b^{-k-1} + (-k+2)\cdot d_{k,2}b^{-k+1} + k\cdot d_{k,3}b^{k-1} + (k+2)\cdot d_{k,4}b^{k+1}$.
Let $\Phi_{k,j}$, where $j = 1,2,3,4$, be the set of four functions spanning the solution subspace H_k. Therefore, we have

$$\begin{bmatrix} \Phi_{k,1}(a) & \Phi_{k,2}(a) & \Phi_{k,3}(a) & \Phi_{k,4}(a) \\ \Phi_{k,1}^{'}(a) & \Phi_{k,2}^{'}(a) & \Phi_{k,3}^{'}(a) & \Phi_{k,4}^{'}(a) \\ \Phi_{k,1}(b) & \Phi_{k,2}(b) & \Phi_{k,3}(b) & \Phi_{k,4}(b) \\ \Phi_{k,1}^{'}(b) & \Phi_{k,2}^{'}(b) & \Phi_{k,3}^{'}(b) & \Phi_{k,4}^{'}(b) \end{bmatrix} \begin{bmatrix} c_{k,1} & d_{k,1} \\ c_{k,2} & d_{k,2} \\ c_{k,3} & d_{k,3} \\ c_{k,4} & d_{k,4} \end{bmatrix} = \begin{bmatrix} f_{k,a} & \bar{f}_{k,a} \\ g_{k,a} & \bar{g}_{k,a} \\ f_{k,b} & \bar{f}_{k,b} \\ g_{k,b} & \bar{g}_{k,b} \end{bmatrix}, \tag{18}$$

where $\Phi_{k,j}$ is the j-th basis function for the subspace H_k defined in Proposition 4. As $\Phi_{k,j}$, $\Phi_{k,j}^{'}$, $f_{k,r}$, $g_{k,r}$, $\bar{f}_{k,r}$, and $\bar{g}_{k,r}$ are known, we can solve for the unknown coefficients $c_{k,j}$ and $d_{k,j}$ in equation (18) for $j = 0,1,2,3$ and $r = a, b$.

As a result, we obtain a particular solution $h(r,\phi)$ to the biharmonic equation with Dirichlet boundary conditions. This particular $h(r,\phi)$ will allow us to smoothly interpolate the corrected and uncorrected regions of the contact lens as desired.

4. RESULTS

In this example, we will apply the smoothing algorithm to data from an actual patient. An aberrometer reading of Patient X showed the first six order Zernike coefficients given in Table 3.

Figure 1 shows a trial lens with a bevel and prism. The prism is for weighting one end of the lens and this will reduce rotational motion. The bevel is a shaping of the corner to look like a cone. This reduces the weight of the lens, which is increased by adding the prism, and also reduces chances of injury to the lower eyelid.

A patient wearing a trial lens had Zernike coefficients as measured by a Shack-Hartmann aberrometer produced and marketed by Wavefront Sciences Inc. The Zernike coefficients for this patient X is shown in Table 3. Due to space limitations only the first 20 coefficients are shown while the design used 27 coefficients, which is the maximum measurable with the

Table 3. The first 20 non-dimensional Zernike coefficients for patient X with pupil radius = 1.7 mm

Order 1 $\times 10^{-4}$	Order 2 $\times 10^{-4}$	Order 3 $\times 10^{-5}$	Order 4 $\times 10^{-5}$	Order 5 $\times 10^{-5}$
$\alpha_1^{-1} = 9$	$\alpha_2^{-2} = 4$	$\alpha_3^{-3} = -10$	$\alpha_4^{-4} = 10$	$\alpha_5^{-5} = -6$
$\alpha_1^{1} = -5$	$\alpha_2^{0} = -8$	$\alpha_3^{-1} = 8$	$\alpha_4^{-2} = -6$	$\alpha_5^{-3} = 4$
	$\alpha_2^{2} = -4$	$\alpha_3^{1} = -9$	$\alpha_4^{0} = 2$	$\alpha_5^{-1} = 3$
		$\alpha_3^{3} = 20$	$\alpha_4^{2} = 5$	$\alpha_5^{1} = 6$
			$\alpha_4^{4} = -0.9$	$\alpha_5^{3} = -5$
				$\alpha_5^{5} = 2$

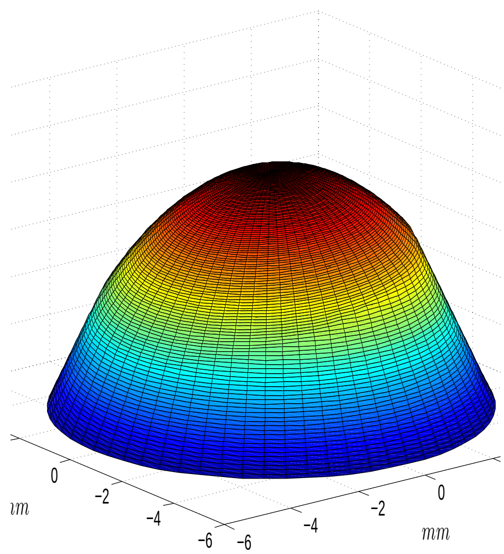

Figure 1. Trial lens for patient X with bevel and prism. The units for the axes are millimeters. The diameter of the lens is 8.7 mm, with front surface radius 7.39 mm. The prism angle is 1° and the width of the bevel from the edge of the lens is 1mm.

aberrometer used. The Guirao, Cox, Williams correction method [7] yields the required correction to be added to the lens as shown in Figure 2.

Figure 3 shows the base lens with the center part corrected. The pupil is concentric with the lens. It may be observed that part of the lens that is sticking out and is recessed in other parts as determined by Figure 2.

Figure 4 shows the base lens with the corrected region in the center and the smoothed interpolation region. To solve for the interpolated region, we need to solve Equation (18). It turns out that this equation is very badly conditioned, with singular values of the order of 10^{15}, 10^{12}, 1 and 10^{-7} for each value of $k \geq 0$. Therefore, a regularized equation was solved using only the two largest singular values for each value of k.

Unlike the contact lens in Figure 3, this contact lens contains no corners or discontinuities along the surface. Therefore, by solving the Biharmonic equation with Dirichlet boundary conditions, our contact lens has a smooth surface.

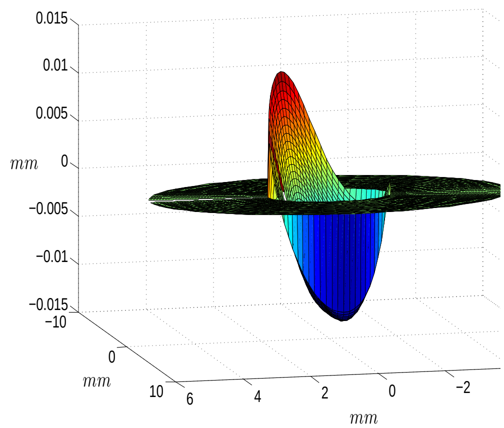

FIGURE 2. Required correction on the trial lens using the Guirao, Cox, and Williams [7] method.

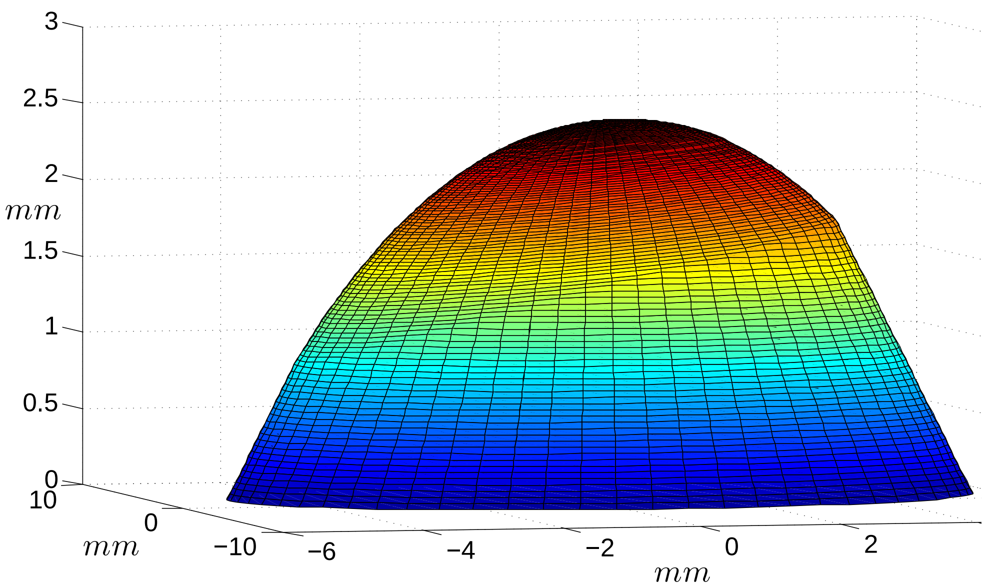

FIGURE 3. Trial lens with aberration correction. The dislocation (discontinuity) in the surface may be seen near the top. This dislocation falls in the functional optical zone of the lens.

5. Conclusion

In this paper, we have presented a method to smooth the regions of a contact lens after a wavefront aberration correction is implemented. The correction method may vary but the result of the correction is that there is a discontinuity in the surface of the lens, which may cause injury to the eyelid in addition to causing a vision deficiency due to lens motion. Our method involves solving the biharmonic equation with Dirichlet boundary conditions. We have presented the results of the design for an actual patient with data about wavefront aberrations obtained from a Shack-Hartmann aberrometer.

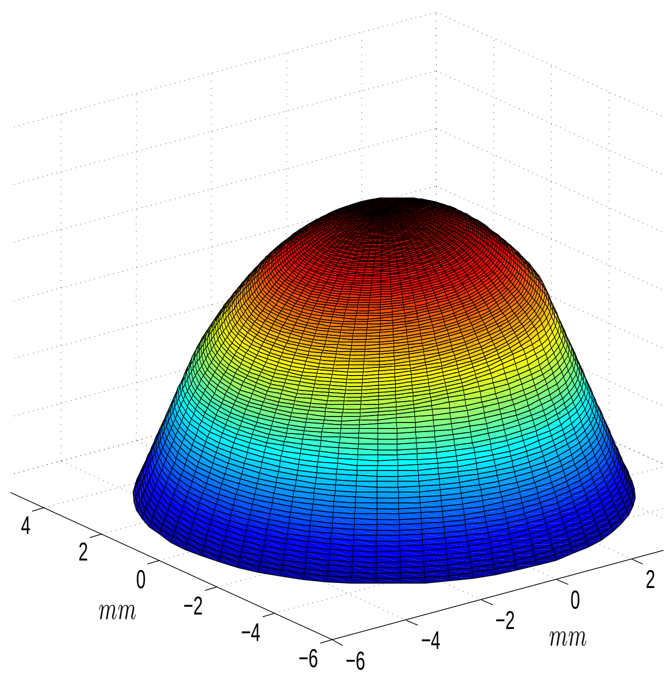

FIGURE 4. Lens with aberration correction and C^1 interpolation between the corrected and uncorrected regions. There is no longer any dislocation of the surface.

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