On $J$-normal matrices and symplectic diagonalizability

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Abstract

Define $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \in M_{2n}$ and let $\phi_J(A) = J^{-1}A^TJ$ for all $A \in M_{2n}$. A nonsingular matrix $P \in M_{2n}$ is called symplectic if $P\phi_J(P) = I$. A matrix $P \in M_{2n}$ is called $J$-normal if $P\phi_J(P) = \phi_J(P)P$. We show that a diagonalizable matrix $A$ is symplectically diagonalizable if and only if $A$ is $J$-normal. We characterize matrices which are symplectically equivalent or symplectically congruent to a diagonal matrix. These results are the symplectic analogues of the Takagi factorization and the singular value decomposition. We also show that a matrix $A$ is both $J$-normal and normal if and only if $A$ is diagonalizable by a matrix which is both symplectic and unitary.

1 Introduction

Denote by $M_{m,n}$ the set $m$-by-$n$ complex matrices and we write $M_{m,m} \equiv M_m$ for brevity. Let $(\cdot, \cdot) : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$. Then $(\cdot, \cdot)$ is called sesquilinear if it is conjugate linear in the first component and linear in the second and $(\cdot, \cdot)$ is called bilinear if it is linear in both components. Let $\star$ be conjugate transposition if the form is sesquilinear, or let $\times$ be transposition if the form is bilinear. It is known that if $(\cdot, \cdot)$ is sesquilinear or bilinear, then there exists a unique matrix $S \in M_n$ such that for for all $x, y \in \mathbb{C}^n$, $(x, y) = x^*Sy$.

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In this case, we say that the form is defined by $S$. A bilinear or sesquilinear form $(\cdot,\cdot)$ is degenerate if there exists a nonzero $x$ such that $(x,y) = 0$ for all $y \in \mathbb{C}^n$, otherwise, it is called nondegenerate. It is known that the bilinear form or sesquilinear form defined by a matrix $S$ is nondegenerate if and only if $S$ is nonsingular. From here on, we only consider nondegenerate forms. For $A \in M_n$, define $\phi_S(A) : M_n \to M_n$ by

$$\phi_S(A) = S^{-1}A^*S.$$  

Note that $\phi_S$ acts as the adjoint with respect to the form defined by $S$, in the sense that for every $A \in M_n$, $\phi_S(A)$ is the unique matrix satisfying $(Ax,y) = (x,\phi_S(A)y)$ for all $x, y \in \mathbb{C}^n$. The operator $\phi_S$ generalizes the idea of transpose and conjugate transpose and it also has properties analogous to those of these operators: $\phi_S$ is additive ($\phi_S(A + B) = \phi_S(A) + \phi_S(B)$) and is an antihomomorphism ($\phi_S(AB) = \phi_S(B)\phi_S(A)$). Given a form $(\cdot,\cdot)$ defined by a matrix $S$, there are three sets of special matrices associated to $(\cdot,\cdot)$ which we define as follows. The automorphism group $G_S$ is the set

$$G_S = \{ M \in M_n : M \text{ is nonsingular and } M^{-1} = \phi_S(M) \} = \{ M \in M_n : (x,y) = (Mx,My) \text{ for all } x, y \in \mathbb{C}^n \} ,$$

the Jordan algebra $J_S$ is the set

$$J_S = \{ M \in M_n : M = \phi_S(M) \} = \{ M \in M_n : (Mx,y) = (x,My) \text{ for all } x, y \in \mathbb{C}^n \} ,$$

and the Lie algebra $L_S$ is the set

$$L_S = \{ M \in M_n : M = -\phi_S(M) \} = \{ M \in M_n : (Mx,y) = -(x,My) \text{ for all } x, y \in \mathbb{C}^n \} .$$

A matrix $A$ is called $S$-normal if it commutes with $\phi_S(A)$. Note that $G_S$, $J_S$, and $L_S$ are subsets of the set of $S$-normal matrices. We also have that when $S = I$, the set $G_I$ is the set of unitary (orthogonal) matrices, the set $J_I$ is the set of Hermitian (symmetric) matrices and the set $L_I$ is the set of skew-Hermitian (skew-symmetric) matrices, if $(\cdot,\cdot)$ is sesquilinear (respectively, if $(\cdot,\cdot)$ is bilinear). The elements of $G_S$ are called $S$-orthogonal matrices, the elements of $J_S$ are also called $S$-symmetric matrices, and the elements of $L_S$ are also called $S$-skew symmetric. We note that $G_S$ is a group under multiplication and the sets $J_S$ and $L_S$ are additive groups.

We consider the following problem.
**Problem 1** Let $S \in M_n$ be nonsingular and $A \in M_n$.

1. Is $A = PDP^{-1}$ for some diagonal $D$ and $S$-orthogonal $P$?
2. Is $A = PDP^T$ for some diagonal $D$ and $S$-orthogonal $P$?
3. Is $A = PDQ$ for some diagonal $D$ and $S$-orthogonal matrices $P$ and $Q$?

Note that when the form is sesquilinear and when $S = I$, the answer to Problem 1 (1) is yes if and only if $A$ is normal; the answer to Problem 1 (2) is yes if and only if $A$ is symmetric (this is more known as the Takagi factorization of symmetric matrices); and the answer to Problem 1 (3) is yes, and this is the well known singular value decomposition (SVD). A similar problem to Problem 1 (3) is in [5], where the authors considered the ‘structured’ SVD of matrices in $J_J$, where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Recall that a ‘structured’ SVD of a matrix $A$ in $J_J$ is the usual SVD of $A$, where the factors are also in $J_J$.

We are also interested in the case when the form is bilinear and $S = J$. Note that $J$-orthogonal matrices are called symplectic matrices, and the real $J$-symmetric and $J$-skew symmetric matrices are called real skew-Hamiltonian and Hamiltonian matrices, respectively. It is known that the 2-by-2 symplectic matrices are exactly the 2-by-2 matrices with determinant one.

The following is vital for our results, a proof of which is in [3].

**Proposition 2** Let $S = G_J, L_J$ or $J_J$ and let $M, N \in S$. Then $M$ and $N$ are similar if and only if $M$ and $N$ are symplectically similar.

Let $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in M_{2m}$ and $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \in M_{2n}$ such that $A_i \in M_m$ and $B_i \in M_n$ for each $i$. As in [7], the expanding sum of $A$ and $B$ is defined to be

$$A \boxplus B = \begin{bmatrix} A_1 \oplus B_1 & A_2 \oplus B_2 \\ A_3 \oplus B_3 & A_4 \oplus B_4 \end{bmatrix} \in M_{2(m+n)}.$$

Note that $A \boxplus B$ is $J$-symmetric, $J$-skew symmetric, or symplectic if and only if both $A$ and $B$ are $J$-symmetric, $J$-skew symmetric, or symplectic, respectively. We also have that $A \boxplus B$ is similar to $A \oplus B$. 


A matrix is similar to a $J$-symmetric matrix if and only if it is similar to $A \oplus A$ for some $A \in M_n$ [7]. A matrix is similar to a $J$-skew symmetric matrix if and only if it is similar to the direct sum of matrices of the form $A \oplus -A$ for some nonsingular matrix $A \in M_n$ or a nilpotent matrix whose odd sized Jordan blocks come in pairs ([4], Theorem 26). The following is a consequence of Proposition 2 and the preceding. We denote by $J_k(\lambda)$ the $k$-by-$k$ upper triangular Jordan block corresponding to $\lambda$.

**Theorem 3** Let $M \in M_{2n}$.

1. If $M$ is $J$-symmetric, then $M$ is symplectically similar to $A \oplus A^T$ for some $A \in M_n$.

2. If $M$ is $J$-skew symmetric, then $M$ is symplectically similar to $(A \oplus -A^T) \boxplus B$ for some nonsingular $A$ such that $\sigma(A) \cap \sigma(-A) = \phi$ and $B$ is nilpotent and $J$-skew symmetric.

Theorem 3 implies that if $M$ is $J$-symmetric and diagonalizable, then $M$ is symplectically similar to $A \oplus A^T$, where $A$ is diagonalizable. Let $X$ be nonsingular such that $XAX^{-1}$ is diagonal. Observe that $P = X \oplus X^{-T}$ is symplectic such that $P(A \oplus A^T)P^{-1}$ is diagonal, and so $M$ is symplectically diagonalizable.

## 2 Main Results

### 2.1 Symplectic Diagonalizability

Note that if $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2n}$, where $A, B, C, D \in M_n$, then

$$
\phi_J(M) = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}.
$$

Let $M$ be diagonal. Observe that $\phi_J(M)$ is also diagonal and so $M$ commutes with $\phi_J(M)$, that is, $M$ is $J$-normal. Now if $P$ is symplectic and $M$ is diagonal, we have that $PMP^{-1}$ is $J$-normal. This gives us the following.

**Lemma 4** Let $M$ be symplectically diagonalizable. Then $M$ is $J$-normal.
Let $M \in M_{2n}$. If $M - \phi_J(M)$ is nilpotent, we immediately see that the converse of Lemma 4 is true. Recall that two diagonalizable matrices $A$ and $B$ are simultaneously diagonalizable, that is, there exists a nonsingular matrix $X$ such that $XAX^{-1}$ and $XBX^{-1}$ are both diagonal, if and only if $AB = BA$.

**Lemma 5** Let $M \in M_{2n}$ be diagonalizable and $J$-normal such that $M - \phi_J(M)$ is nilpotent. Then $M$ is $J$-symmetric. Moreover, $M$ is symplectically diagonalizable.

**Proof.** Let $M \in M_{2n}$ be diagonalizable and $J$-normal such that $M - \phi_J(M)$ is nilpotent. Since $\phi_J(M)$ is similar to $M$, we have that $\phi_J(M)$ is diagonalizable. Moreover, $M\phi_J(M) = \phi_J(M)M$ implies that there exists a nonsingular $X$ such that $XMX^{-1} = D_1$ and $X\phi_J(M)X^{-1} = D_2$ are diagonal. We have that

$$X(M - \phi_J(M))X^{-1} = D_1 - D_2,$$

and since $M - \phi_J(M)$ is nilpotent, it follows that $D_1 - D_2 = 0$. We conclude that $M = \phi_J(M)$ and since $M$ is diagonalizable and $J$-symmetric, Theorem 3 implies that $M$ is symplectically diagonalizable. 

(Email the author for the complete preprint)

**References**


