Efficiency of Sequential Bandwidth and Power Auctions With Rate Utilities

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Abstract—We study a sequential second-price auction for allocating wireless resources between two non-cooperative users. This mechanism requires relatively little computation and information exchange among agents, but does not always achieve an efficient allocation. This is a continuation of previous work in which the worst-case efficiency is evaluated, assuming each user has full knowledge of the other user’s utility function. Here we assume that the users are randomly placed within a region, and evaluate the associated efficiency via simulation. Sequential auctions for bandwidth (with fixed power) and for power (with fixed bandwidth) are considered, where each user utility is the achievable rate, and interference is treated as background noise. Our results show that the sequential auction typically achieves the efficient (utility-maximizing) allocation. We also relate observed improvements in the worst-case efficiency to constraints on the size of the marginal utilities associated with each resource.

I. INTRODUCTION

In some dynamic spectrum sharing scenarios, a spectrum owner, or licensee may wish to lease spectrum to secondary users (e.g., see [1], which discusses secondary spectrum markets). This leads to the problem of designing an efficient mechanism for allocating the available wireless resources (e.g., power and bandwidth) among non-cooperative agents. The allocation mechanism can be implemented through a spectrum manager, or broker, which also controls the amount of bandwidth or power allocated to the secondary users. Examples of resource allocation mechanisms, which have been recently proposed for this type of application, are presented in [2], [3]. Other distributed spectrum sharing mechanisms, which do not rely on the presence of a spectrum manager to determine the allocation, are considered in [4]–[6].

The spectrum manager can mitigate the effects of externalities (interference) and increase the overall utility of the resource allocation by soliciting information about user utilities and channel conditions. Here we consider a sequential, second-price auction for allocating either bandwidth or power among non-cooperative users in a peer-to-peer network. This is in contrast to the iterative auction mechanism proposed in [2]. Namely, the resource is divided into \( n \) smaller units, and each unit is auctioned off sequentially according to a second-price auction. Sequential auctions have been used in many applications (e.g., see [7], [8]), since they require relatively little computation and information exchange among the agents and the broker, compared with other mechanisms. In addition, sequential auctions easily accommodate scenarios in which agents enter and leave the market at arbitrary times, and allow the broker to allocate resources incrementally. However, it is well known that sequential auctions do not always achieve efficient allocations [9].

This paper is a continuation of previous work in which a worst-case bound on the efficiency loss of the sequential second-price auction is presented [10]. Specifically, the sequential second-price auction is used to allocate the divisible resource (e.g., bandwidth) between two agents, each of which has decreasing marginal valuations, corresponding to a concave increasing utility function. It is assumed in [10] that each user has full knowledge of both users’ utilities, so that the auction can be modeled as an extended-form game [11]. It is then possible to determine a subgame perfect equilibrium in which each user adopts a bidding strategy to maximize her anticipated surplus (utility minus payments) associated with the outcome. It is shown in [10] that the total utility corresponding to this equilibrium is lower bounded by \( 1 - e^{-1} \) of the maximum total utility. Furthermore, it is shown that this bound is tight as the number of goods (sub-divisions) becomes large.

Although the worst-case efficiency loss is significant, it is not clear how often it can be expected to occur in practical situations. In this paper, we therefore study via simulation the efficiency loss of the sequential second-price auction with random utilities. We again assume two users, which corresponds to the important scenario in which a transmitter-receiver pair has one dominant interferer. The users are assumed to be randomly placed within a region, and the attenuation from each transmitter to a receiver is determined by the distance between them. The utility function for each user is the maximum

\[ V(x) = x^{\alpha} \]

\[ \alpha > 0 \]

1Namely, each unit is allocated to the highest bidder, who pays the second-highest bid.

2In a multistage game a subgame perfect equilibrium is a refinement of the concept of Nash equilibrium with the restriction that agents cannot make non-credible threats (see e.g. [11])).
achievable (Shannon) rate, where interference is treated as background noise.

As in [10], we assume that the users have full information. While this may not be true in practice, this assumption can be viewed as pessimistic, since less available information is more likely to lead to an efficient allocation. That is, with full information the agents have an incentive to alter their bidding strategy to maximize their own anticipated surplus (utility minus payment), rather than bid their true marginal values. In contrast, without knowledge of other users’ utilities, there is no incentive for a user to deviate from bidding her marginal utility at each iteration, in which case the sequential auction always achieves the efficient allocation. Furthermore, with two users the corresponding full information game is tractable, and provides substantial insight into advantages and shortcomings of the mechanism.

For the sequential bandwidth auction our results show that except for a small fraction of realizations, the subgame perfect equilibrium achieves the efficient (utility-maximizing) allocation. Furthermore, the observed worst-case efficiency is somewhat higher than the bound presented in [10]. We show that this is due to constraints on the marginal utilities imposed by the rate utility function, and provide a modified worst-case bound for $n = 2$, which depends on the ratio of the marginal utilities.

In addition to the bandwidth auction, corresponding to concave increasing utilities, we also consider a sequential auction for a particular user’s power. In this setting we assume that both users spread over the same bandwidth, and therefore interfere with each other. The maximum available power for the user is divided into $n$ smaller units, and both users bid on each power increment. (The competing user bids to lower interference.) The sequential auction then determines how many units of power are allocated in response to the user’s request. In this case, the rate (utility) function for the competing user is a convex increasing function of the other user’s power.

For the power auction we show that the worst-case efficiency is $1/n$. Simulation results with randomly placed users show that this worst-case efficiency is achieved, although for a relatively small fraction of realizations. The results again indicate that the sequential auction typically achieves the efficient allocation, although with somewhat lower probability than the bandwidth auction.

In the next section, we introduce the spectrum sharing model, and in Section III we specify the sequential second-price auction by an example. Sections IV and V discuss the worst-case efficiency loss, and simulation results with randomly placed users are presented in Section VII.

II. SPECTRUM SHARING MODEL

We consider a model for spectrum sharing between two users, where each user consists of a distinct transmitter-receiver pair. As in [2], [4], we model this as a two-user Gaussian interference channel with frequency flat fading. The channel gain between user $i$’s transmitter and user $j$’s receiver is denoted by $h_{ij}$. Each transmitter has an average power constraint $P$, and the total available bandwidth is $W$ Hz. We further assume that each transmitter uses an optimal (capacity achieving) code, where the received interference is treated as background noise (i.e., no interference cancellation is used).

We focus on two spectrum sharing techniques: frequency division multiplexing (FDM) and spread-spectrum signaling with frequency-flat power allocations across the entire band. With FDM, each user $i$ receives bandwidth $W_i$, where $W_1 + W_2 = W$. User $i$’s resulting achievable rate is then

$$r_i(W_i) = W_i \log(1 + \frac{h_{ii} P_i}{N_0 W_i})$$

(1)

where $N_0$ is the power spectral density of the additive noise. Note that (1) is concave in $W_i$. With full spreading user $i$ chooses a power $P_i \in [0, P]$, resulting in the achievable rate

$$r_i(P_i, P_j) = W \log(1 + \frac{h_{ij} P_i}{N_0 W + h_{jj} P_j})$$

(2)

Note that (2) is concave in $P_i$, but is convex in $P_j$.

Each agent is endowed with a utility function, $U_i(r_i)$, which is increasing and concave. Assume that agent 1 is initially using the spectrum with a given bandwidth or power allocation, and agent 2 wants to share this spectrum. The spectrum manager divides the appropriate resource into $n$ units and re-allocates these units between the two agents. In the FDM case, each allocation unit represents a frequency band of $W/n$ Hz. The manager either re-allocates a unit to agent 2 or lets agent 1 continue to use that unit. Let $u^*_i$ be agent $i$’s marginal valuation for receiving her $s$-th unit, i.e.,

$$u^*_s = U_i(r_i(sW/n)) - U_i(r_i((s-1)W/n)).$$

(3)

From (1), it follows that the marginal valuations of each agent are decreasing, i.e. $u^*_1 \geq \ldots \geq u^*_n$.

In the full-spread case, we assume that the manager allows agent 2 to continue transmitting at its current power, $P_2$, and allocates power to agent 1. Each allotted unit represents a power increment of $P/n$. A unit allocated to agent 1 allows her to increase her transmission power, whereas a unit allocated to agent 2 decreases the power assigned to agent 1, thereby reducing agent 2’s interference. Agent 1’s marginal valuation for the $s$-th unit is

$$u^*_s = U_1(r_1(sP/n, P_2)) - U_1(r_1((s-1)P/n, P_2))$$

(4)

which is again decreasing in $s$. However, agent 2’s marginal valuation for the $t$-th unit of agent 1’s power is

$$u^*_t = U_2(r_2(P_2, (n-t)P/n)) - U_2(r_2(P_2, (n-t+1)P/n)),$$

(5)

which is not necessarily decreasing in $t$. For example, if $U_2(r_2)$ is linear, then $u^*_2$ is increasing in $t$, i.e. $u^*_2 \leq \ldots \leq u^*_n$. 4

3More generally, the users could each pick a power allocation over frequency; our choices represent two specific classes of power allocations. Restricting ourselves to these classes simplifies resource allocation. Furthermore, for many choices of channel gains, the optimal power allocation is in this set [4].

4Indeed user 2’s marginal valuations may be neither increasing nor decreasing for all $t$. In general this depends on the utility, the choice of channel gains and the power levels. A necessary condition for the marginals to be increasing or decreasing can be given in terms of the utility’s coefficient of relative risk aversion, as in [5].
III. SEQUENTIAL SECOND PRICE AUCTION

The manager (auctioneer) runs a sequential second price auction to allocate the \( n \) resource units between the two agents. Each unit is sequentially auctioned off in a sequence of \( n \) rounds. In a given round, each agent submits a bid, and the auctioneer allocates the current unit to the agent with the largest bid and charges the agent the price of the second largest bid. This procedure is repeated until all \( n \) units are allocated. This sequential auction can be viewed as an extended-form game [11]. Assuming each agent has full information, the detailed bidding strategy (sophisticated bidding) and the subgame perfect equilibrium for two agents are discussed in [10].

Instead of presenting the details of the second-price auction, we give an example with \( n = 2 \) units. In Figure 1 (c), the directed graph representing this auction is shown with each node labeled by the allocation \((s, t)\). Assume agent 1’s marginal valuation of each resource unit is 5. Agent 2 values the first unit at 4 and the second unit at 1, i.e., \( u_1^{(1)} = 5 \), \( u_1^{(2)} = 4 \), and \( u_2^{(1)} = 1 \). Since agent 1 values each unit more than agent 2 values any unit, the socially optimal allocation is to give both units to agent 1.

Now let us examine how the agents would bid in the sequential second price auction with full information. Assume that the game reaches the node \( v = (1, 0) \) and the second unit is on the market for the auction (see Figure 1 (a)). At this node two agents are bidding for the one remaining unit. For a single unit, the optimal bids for the agents are their valuation, hence agent 1 bids 5 and agent 2 bids 4. The auctioneer therefore allocates the second unit to agent 1 and charges her a price of 4. Hence, the value of node \( v = (1, 0) \) to agent 1 is \( 5 + (5 - 4) = 6 \) where the first 5 represents the value from winning the first stage and \( (5 - 4) \) is the value minus the payment at the second stage. The value of \( v = (1, 0) \) to agent 2 is 0. Similarly, the value of node \( v = (0, 1) \) to agent 1 is 4 and that to agent 2 is 4. Since each agent knows the values of all the nodes with \( s + t = 1 \) in advance, they can both optimize their bids for the first unit auction. In particular, it is again optimal for each agent to bid her valuation, i.e., agent 1 bids \( 6 - 4 = 2 \) and agent 2 bids \( 4 - 0 = 4 \) for the first unit, respectively. It follows that agent 2 wins the first unit with payment 2. Therefore, the equilibrium path of this example is \((0,0) \to (0,1) \to (1,1)\) as shown in Figure 1 (b) and the payoff after the sequential auction ends is \( 5 - 1 = 4 \) for agent 1 and \( 4 - 2 = 2 \) for agent 2. The equilibrium path of the example does not terminate in an efficient allocation. Namely, the utility corresponding to the equilibrium path is \( u_1^{(1)} + u_2^{(2)} = 5 + 4 = 9 \), whereas the efficient allocation gives \( u_1^{(1)} + u_2^{(2)} = 10 \). Hence the efficiency of the subgame perfect equilibrium is 0.9.

IV. EFFICIENCY BOUNDS

In this section we state lower bounds on the efficiency of the subgame perfect equilibrium corresponding to the sequential second-price auctions discussed in Section II. The bound with decreasing marginal valuations (concave utility functions), along with the proof, is given in [10].

**Definition 1:** Given marginal valuations \( \{u_i^{(k)}\}_{i=1}^n \) and \( \{v_i^{(k)}\}_{i=1}^n \), let \( k \) and \( n - k \) be an efficient allocation, and let \( k' \) and \( n - k' \) be the sequential allocation. (Note that \( k \) and \( k' \) are functions of the marginal valuations.) Given \( n \) goods, the worst-case efficiency is given by

\[
\eta(n) = \min_{\{u_i^{(k)}\}, \{v_i^{(k)}\}} \frac{\sum_{k=1}^{k'} u_i^{(k)} + \sum_{k=n-k'}^{n} v_i^{(k)} - \sum_{i=1}^{n-k} u_i^{(k)} - \sum_{i=1}^{k} v_i}{\sum_{i=1}^{n-k} u_i^{(k)} - \sum_{i=1}^{k} v_i^{(k)}}. \tag{6}
\]

We first state the efficiency bound for the case where both users have decreasing marginal valuations, i.e., \( u_1^{(1)} \geq \ldots \geq u_n^{(1)} \), which corresponds to the bandwidth auction.

**Theorem 1:** If both agents have decreasing marginal valuations, then for all \( n \)

\[
\eta(n) \geq 1 - e^{-1}. \tag{7}
\]

It can be shown that the worst-case efficiency decreases with \( n \), i.e.,

\[
\eta(1) \geq \eta(2) \geq \ldots \geq \eta(n) \geq \ldots \geq 1 - \frac{1}{e}, \tag{8}
\]

and as \( n \to \infty \), the bound in Theorem 1 is asymptotically tight. To illustrate this, consider the sequential auction with \( n = 2 \). If \( u_1^{(1)} = u_2^{(1)} = 1 \) and \( u_1^{(2)} = \frac{1}{2} + \varepsilon_1 \), \( u_2^{(2)} = 0 \), then the efficient allocation is to assign both units to agent 1, giving a utility of 2. However, in the sequential auction agent 1’s payoff for both units would be \( 1 - 2\varepsilon_1 \). Agent 1 would prefer to get only one unit with payoff 1, so that the allocation corresponding to the subgame perfect equilibrium is \((1,1)\). The corresponding efficiency is \( \frac{1 + \varepsilon_1}{1 + 2\varepsilon_1} = \frac{3}{2} \), which is in fact the worst-case efficiency for two users and two goods.

Similarly, we can construct worst-case marginal valuations for all \( n > 2 \). Table 1 shows these marginal valuations, where \( \varepsilon_j \), \( j = 1, 2, \ldots \), are small, positive numbers, along with the worst-case efficiency \( \eta(n) \). Table 1 shows that \( \eta(n) \) decreases monotonically to the lower bound in Theorem 1 as \( n \to \infty \).

Next we consider the case where agent 1 has decreasing marginal values, and agent 2 has increasing marginal values, i.e., \( u_1^{(1)} \geq \ldots \geq u_n^{(1)} \) and \( u_1^{(2)} \leq \ldots \leq u_n^{(2)} \). As discussed
in section II, this corresponds to the power auction with spreading.

Theorem 2: If the marginal values of one agent are decreasing and the marginal values of the other agent are increasing, then \( \eta(n) \leq \frac{1}{n} \).

The proof is based on the following example. Let \( u_1^1 = \frac{1}{n}, u_1^2 = \cdots = u_n^1 = \epsilon \) and \( u_2^1 = \cdots = u_n^2 = 0, u_2^2 = 1 \). The efficient allocation is \((0, n)\); however, it is easy to determine that the allocation associated with the subgame perfect equilibrium is \((n, 0)\). The efficiency therefore approaches \( \frac{1}{n} \) as \( \epsilon \to 0 \).

The worst-case efficiency for the sequential power auction tends to zero as the number of goods \( n \) increases. Comparing Theorems 1 and 2 indicates that the efficiency for the power auction can be much lower than for the bandwidth auction.

V. Efficiency with Constrained Marginal Values

As indicated in the preceding section, the marginal values that achieve the worst-case efficiency in each case are quite special. With additional constraints on the marginal values, we expect the worst-case efficiency to increase. Here we illustrate this for \( n = 2 \).

First, we consider decreasing marginal valuations for both agents, and assume that \( u_1^1 = \lambda_1 u_1^1 \) and \( u_2^1 = \lambda_2 u_2^1 \), where \( \lambda_1 < 1 \) and \( \lambda_2 < 1 \). That is, we restrict the amount the marginal utilities can change for each good. It can be shown that the subgame perfect equilibrium for the sequential auction does not achieve the efficiency allocation if \( u_1^1 > \lambda_1 u_1^1 > u_1^2 > \lambda_2 u_2^2 \). Symmetry, this case is equivalent to \( u_2^2 > \lambda_2 u_2^2 > u_2^1 > \lambda_1 u_1^1 \), and it can be shown that the efficient allocation is achieved for all other orderings. The worst-case efficiency with these constrained marginal valuations is given by

\[
\eta(2; \lambda_1, \lambda_2) = \frac{2 + \lambda_1 - \lambda_2}{(1 + \lambda_1) \cdot (2 - \lambda_2)},
\]

where \( u_1^1 < \lambda_1 < 1 \) and \( 0 < \lambda_2 < 1 \). Note that \( \eta(2; \lambda_1, \lambda_2) \geq 3/4 \), the worst-case efficiency over unconstrained marginal values, and equality holds as \( \lambda_1 \to 1 \) and \( \lambda_2 \to 0 \). Hence the worst-case efficiency improves when constraints are placed on the relative size of the marginal values.

For the case in which one agent has decreasing marginal values and the other has increasing marginal values, we let \( u_1^2 = \lambda_1 u_1^1 \), and \( u_2^2 = \lambda_2 u_2^1 \), where \( \lambda_1 < 1 \) and \( \lambda_2 > 1 \). In this case, any ordering of marginal values can lead to an inefficient allocation. Hence all orderings must be considered to compute the worst-case efficiency. As an example, we assume that \( \lambda_2 u_1^2 > u_2^2 > u_1^1 > \lambda_1 u_1^1 \). Then the worst-case efficiency is given by

\[
\eta(2; \lambda_1, \lambda_2) = \frac{2 - \lambda_1 + \lambda_2}{(2 - \lambda_1) \cdot (1 + \lambda_2)},
\]

where \( 0 < \lambda_1 < 1 \) and \( 1 < \lambda_2 < 2 \). Here we have \( \eta(2; \lambda_1, \lambda_2) \geq 2/3 \), and equality holds as \( \lambda_1 \to 0 \) and \( \lambda_2 \to 2 \). Again, restricting the marginal values increases the worst-case efficiency.

For the numerical results in the next section corresponding to the bandwidth auction, agent \( i \) has the rate utility function \( U_i(n) = n \log_2(1 + \frac{\alpha_i}{n}) \), where \( \beta_1 \) and \( \beta_2 \) depend on the randomly generated channel gains, and satisfy the constraints \( 5 \leq \beta_1 \leq 15 \) and \( 1 \leq \beta_2 \leq 6 \). The marginal values for each agent are decreasing, and are given by \( u_1^1 = \log_2(1 + \beta_1) \), \( u_2^1 = 2 \log_2(1 + \beta_1/2) - \log_2(1 + \beta_1) \), \( u_1^2 = \log_2(1 + \beta_2) \), \( u_2^2 = 2 \log_2(1 + \beta_2/2) - \log_2(1 + \beta_2) \). Therefore, \( 0.398 \leq \lambda_1 \leq 0.544, 0.170 \leq \lambda_2 \leq 0.425 \), and from (9), the worst-case efficiency occurs when \( \lambda_1 = 0.544 \) and \( \lambda_2 = 0.17 \), which gives \( \eta = 0.84 \). This, of course, is larger than the worst-case efficiency 3/4 given in Table I. In contrast, the results for the power auction show that the worst-case efficiency \( 1/n \) is actually achieved for some realizations.

Obtaining the worst-case efficiency with constraints on marginal values for \( n > 2 \) becomes considerably more difficult. However, we observe from numerical examples that the corresponding efficiency of the subgame perfect equilibrium is generally higher than the worst-case efficiency shown in section II.

VI. Simulation Model

To generate the simulation results in this section we randomly place the two transmitters and receivers within a given region, as illustrated in Figure 2. We then determine all (four) scalar channel gains and compute the corresponding rates treating the interference as Gaussian noise. For the bandwidth auction, the two users do not interfere, so that only the distance between the transmitter and receiver for each user is relevant. This distance is assumed to be uniformly distributed between zero and a maximum value \( d_0 \). For the power auction, user 1’s transmitter is randomly placed within a circle of radius \( d_{12} \) centered at user 2’s receiver. This captures the scenario in which a user experiences a single dominant interferer. User 1’s receiver is then randomly placed within a circle of radius \( d_0 \) centered at user 1’s transmitter, and similarly, user 2’s transmitter is randomly placed within a circle of radius \( d_0 \) centered at user 2’s receiver.

The channel gain from transmitter \( i \) to receiver \( j \) is given by \( h_{ij} = l_{ij}^{-\delta} \), where \( l_{ij} \) is the distance between the transmitter \( i \) and the receiver \( j \). In the bandwidth auction the rate, or utility, for user \( i \) is given by (1), where \( W_i \) is the assigned bandwidth, and \( P \) is the (fixed) transmitted power. The total bandwidth \( W_{\text{max}} \) is divided into \( n \) units, so that \( W_i = n_i W_{\text{max}}/n \), where

<table>
<thead>
<tr>
<th>n</th>
<th>Marginals</th>
<th>( j^* )</th>
<th>( \eta(n) )</th>
</tr>
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<tr>
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<td>1</td>
<td>3/4</td>
</tr>
<tr>
<td>3</td>
<td>1, 1, 1 : 2/3 + ( \varepsilon_1 ), 1/2 + ( \varepsilon_2 ), 0</td>
<td>1</td>
<td>13/18</td>
</tr>
<tr>
<td>4</td>
<td>1, 1, 1, 1 : 1/2 + ( \varepsilon_1 ), 1/3 + ( \varepsilon_2 ), 0, 0</td>
<td>2</td>
<td>17/24</td>
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<td>...</td>
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<tr>
<td>( \infty )</td>
<td>...</td>
<td>...</td>
<td>1 - ( \frac{1}{n} )</td>
</tr>
</tbody>
</table>
A. Bandwidth Auction

worst probability distribution function (PDF) for the which may give different total utilities. Figure 3 shows the auctioned off sequentially. There are three possible allocations, i.e., the bandwidth is split into two subbands, and each is allocated to user 2. This generally leads to an equilibrium with user 2’s rate, given that the preceding power units have been allocated to user 2. That is, user 2 bids for units of \( P \) and \( d \) are increasing. Because of the nature of the interference, user 1 are decreasing, whereas the marginal utilities for user 2 are increasing. Because of the nature of the interference, allotting the last unit of \( P \) to user 1 significantly decreases user 2’s rate, given that the preceding power units have been allocated to user 2. This generally leads to an equilibrium with the worst-case efficiency.

VII. SIMULATION RESULTS

A. Bandwidth Auction

We first show results for the bandwidth auction with \( n = 2 \), i.e., the bandwidth is split into two subbands, and each is auctioned off sequentially. There are three possible allocations, which may give different total utilities. Figure 3 shows the probability distribution function (PDF) for the worst possible efficiency for every realization. That is, for each realization the worst possible efficiency is defined as the ratio of minimum sum utility to maximum sum utility over the three possible allocations. For these results, and in what follows, \( P_1 = P_2 = 10^{-6} \) watt and \( d_0 = d_{12} = 50 \) m unless otherwise specified. This figure shows that without an appropriate resource allocation mechanism, the efficiency can be very low.

Figure 4 shows the cumulative distribution function (CDF) of the efficiency of the subgame perfect equilibrium associated with the sequential auction. This figure shows a substantial improvement in efficiency relative to the worst possible results in Figure 3. Although the worst-case efficiency for \( n = 2 \) is lower bounded by \( \frac{3}{4} \), these results show a significantly higher equilibrium efficiency than the worst-case efficiency.

This is due to the nature of the rate utility function, as discussed in Section V. The simulation results show that the lowest efficiency is 0.844, and the auction achieves an efficient allocation for more than 80% of the realizations.

To illustrate the effect of the constraint on marginal values, discussed in Section V, Figure 5 shows the CDF of the equilibrium efficiency for different powers \( P \). As \( P \) increases, so does the observed worst-case efficiency. Moreover, the fraction of realizations for which the equilibrium achieves full efficiency increases from 80.1% to 96.3%. This is because the ratio \( \lambda_k = u^k_2/u^k_1 \) increases with \( P \). (That is, \( \beta_k \) increases with \( P \), which causes \( \lambda_k \) to increase.) According to the discussion in Section V, this leads to an increase in worst-case efficiency. Figure 5 also shows that the entire CDF is shifted to the right, improving average efficiency as well.

Figure 6 shows the CDF of the efficiency of the sequential auction outcome for \( n \geq 2 \). As the number of units \( n \) increases, the worst-case efficiency should decrease according to the analysis in [10]. However, the figure shows that the smallest efficiency observed actually increases from 0.844 when \( n = 2 \) to 0.914 when \( n = 20 \). This is because as
Fig. 5. CDF of the efficiency of the subgame perfect equilibrium for the bandwidth auction with $P_1 = 10^{-6}, 10^{-4}$, and $10^{-2}$, $n = 2$.

Fig. 6. CDF of the efficiency of the subgame perfect equilibrium for the bandwidth auction with different $n$. The transmitted power $P = 10^{-6}$ and $d_0 = d_{12} = 50$ m.

Fig. 7. PDF of the worst possible efficiency for the power auction with $n = 2$.

Fig. 8. CDF of the efficiency of the subgame perfect equilibrium for the power auction with $n = 2$.

Fig. 9. CDF of the efficiency of the subgame perfect equilibrium for the power auction with different $d_{12}$ between 10 and 50 m. As $d_{12}$ decreases, the interference to user 2 increases, so that the ratio $\lambda_2 = u_2^2/u_1^2$ increases. As indicated by the discussion in Section V, this increases the number of equilibria with relatively low efficiencies. In addition, the probability that the sequential auction does not achieve the efficient allocation increases slightly (from 0.114 to 0.153). Results for the sequential power auction with different values of $n$ are shown in Figure 10. The lowest efficiency is $1/n$, as predicted by Theorem 2, and the fraction of realizations for which the full efficiency is achieved decreases as $n$ increases.

Finally, we remark that our results for both the power and bandwidth auctions only indicate efficiency loss, and do not compare the sum utilities achieved with the power and bandwidth auctions. Further results show that in addition to having lower efficiency loss, the bandwidth auction typically achieves a higher sum utility than the power auction.

VIII. CONCLUSIONS

The performance of a sequential second-price auction for allocating power and bandwidth between two users has been illustrated by simulation. This mechanism is relatively simple and requires little information exchange among users, which makes it attractive for dynamic allocation of wireless resources.

$n$ increases, the specific marginal values, which achieve the worst-case efficiency, are much less likely to be observed. However, the fraction of realizations for which the full efficiency is achieved decreases as $n$ increases. In part, this is simply due to the increase in number of possible outcomes (allocations) with $n$.

B. Power Auction

Figure 7 shows the PDF of the worst possible efficiency for the power auction with $n = 2$. Unlike the bandwidth auction, the lowest efficiency observed is 0.575, which is close to the worst-case efficiency given in Theorem 2. Because of the interference, the marginal value of the second unit for agent 2 can be very large relative to the marginal value of the first unit, which leads to the worst-case efficiency. However, the sequential power auction still achieves the efficient allocation for more than 85% of the realizations.

To illustrate the effects of increasing interference, Figure 9 shows the CDF of the equilibrium efficiency for different values of $d_{12}$ between 10 and 50 m. As $d_{12}$ decreases, the interference to user 2 increases, so that the ratio $\lambda_2 = u_2^2/u_1^2$ increases. As indicated by the discussion in Section V, this increases the number of equilibria with relatively low efficiencies. In addition, the probability that the sequential auction does not achieve the efficient allocation increases slightly (from 0.114 to 0.153). Results for the sequential power auction with different values of $n$ are shown in Figure 10. The lowest efficiency is $1/n$, as predicted by Theorem 2, and the fraction of realizations for which the full efficiency is achieved decreases as $n$ increases.

Finally, we remark that our results for both the power and bandwidth auctions only indicate efficiency loss, and do not compare the sum utilities achieved with the power and bandwidth auctions. Further results show that in addition to having lower efficiency loss, the bandwidth auction typically achieves a higher sum utility than the power auction.
among secondary users. A drawback, however, is that it does not always achieve the efficient allocation.

Although previous work has shown that the efficiency loss can be substantial (especially for the power auction), our results show that with the rate utility function, which accounts for random placement of users, the subgame perfect equilibrium with full information typically corresponds to the efficient allocation. Furthermore, when the equilibrium is inefficient, the efficiency loss is typically less than the worst-case efficiency loss. This is due to the rate utility function, which places constraints on the ratios of marginal utilities for the successive units being auctioned.

The results presented here apply to only two users; however, the sequential auction mechanism can clearly be extended to any number of users. The efficiency loss becomes difficult to evaluate with more than two users, since it becomes much harder to determine a subgame perfect equilibrium, if it exists at all. Of course, this computational difficulty would be faced by each agent in practice, which perhaps reduces the chances that an agent would attempt to strategize bidding in this way.

In the absence of full information about other users’ utilities, each user may attempt to strategize bidding by assuming a distribution over those utilities. Computing equilibria and efficiency loss in that case is an open problem, although in general less information seems more likely to encourage bidding according to marginal utilities, which leads to an efficient allocation. Extensions to joint power and bandwidth auctions are also interesting possibilities for future work.

### References


