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Stability Criteria of the State Probability Vector of Circulant Markov Chains - Kolmogorov Differential Equations

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Abstract

In this paper, first we find out the eigenvalues of the stochastic matrix, stochastic rate matrix. We derive the bounds of the state probability vector of the first order linear Komogorov differential equations at an instant of the time t. Analysis shows that the state probability vector of the pertinent system away from the steady state vector, and approaches the steady state vector as $t \to \infty$.

Keywords: (Self-Similarity, Circulant stochastic matrices, State probability vector, Eigenvalues, Convergence analysis.)

1. Introduction

Seminal studies reveal that IP packet traffic in both Ethernet and Wide Area Network (WAN) tends to be bursty over many time-scales. This bursty traffic can be characterized mathematically as self-similar or long range dependent (LRD). This type of traffic exhibits statistical similarity over different time scales and is highly correlated. The self-similarity in the network traffic has considerable impact on queueing performance. Characterizing the statistical behavior of traffic is crucial to proper design of routers to provide the quality of service (QoS) [1-3]. The design and control of networks canbe carried out using a set of parameters of traffic. Therefore, it is important to capture packet arrival flows and describe them with suitable stochastic matrices. Hence, the use of stochastic rate matrix is of interest in wide range of in the performance analysis of queuing systems [4-7] internet switch/router of pertaining to communication systems. In most cases, when the network nodes are modeled as queueing systems, the problem of computation of performance measures is reduced to that of steady state probability distribution vector of transition rate matrix or transition probability matrix. Therefore, it is the key importance to investigate pertinent linear system or first order linear differential equations, known as Kolmogorov equations. In this direction, many researchers proposed the methods and their convergence and stability criteria [8-9].

In general, Internet routers can be modeled as a queueing system with MAP and these models are computationally tractable. Markovian models such as MAP have been proposed to emulate the self-similar traffic over the desired time scales [5-8]. A class of MAP, Markov-modulated Poisson process (MMPP) is a generalization of the Poisson process and is widely used in the models of communication systems [4-6]. The Circulant Markov modulated Poisson process (CMMPP) is a restricted version of the Markov modulated Poisson process (MMPP), and is parsimonious when compared to MMPP [6]. Hence the, researchers is now focusing on CMMPP to model the arrival pattern.

In general, the transient analysis and behaviour of the analytical solution of the queueing theory in stochastic models are not widely available as steady state results. Vast literature and behaviour of the solution is available only for the steady state probability of the stochastic models [8-12]. But, in many applications of queueing theory, it is important to know how the system will operate up to some instant time t and how the solution behaves as $t \to \infty$. Moreover, the investigation of the transient analysis of queueing processes is highly essential from the point of view of the theory and its applications. In this paper, we examine the behaviour of the state probability vector of the Kolmogorov first order differential equation. In this direction, first we find out the eigenvalues of the circulant stochastic matrix, and stochastic rate matrix. Then we find out the behaviour of the solution as $t \to \infty$.

The rest of the paper is organized as follows: In section 2, circulant stochastic matrix and characteristics of the eigenvalues are given. In section 3, the stability criteria of the state probability vector of first order differential equation is discussed. Finally, conclusions are drawn in section 4.

2. Circulant Stochastic Matrices and Characteristics of the Eigen Values

In this section, first we overview some definitions and prove some results on eigenvalues of circulant stochastic matrix and stochastic rate matrix. A circulant matrix is a special kind of Toeplitz matrix where each row vector is

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rotated one element to the right relative to the preceding row vector. Consider the circulant matrix

$$\begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_n \\ c_n & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \dots & c_{n-2} \\ \dots & \dots & \dots & \ddots & \vdots \\ c_2 & c_3 & c_4 & \dots & c_1 \end{bmatrix} = (c_1, c_2, \dots, c_n)(\text{say}). \qquad (1)$$

The matrix $(c_1, c_2, \dots, c_n)$ is said to be circulant stochastic matrix if $\sum_{i=1}^{n} c_i = 1$ is for $c_i \geq 0$, and in this case it is denoted by P.

The $(c_1, c_2, \dots, c_n)$ matrix is said to be circulant stochastic rate matrix if $\sum_{i=1}^{n} c_i = 0$ is for $c_i \geq 0, c_i \leq 0, 2 \leq i \leq n.$ In this case it is denoted by Q. Since each row and column sum are zero, the matrix Q is a doubly stochastic matrix.

Consider a finite-state continuous-time Markov chain (CTMC) $\{X(t), t \geq 0\}$ with state space $\Omega = \{1,2,3,\dots,n\}$. Let $Q = (c_1, c_2, \dots, c_n)$ be the circulant stochastic rate matrix. Let $\pi_i(t) = \Pr\{X(t) = i\}$. The state probability at time t is $\pi(t) = [\pi_1, \pi_2, \dots, \pi_n]$. The state probability vector $\pi(t)$ is computed by solving a system of first order linear differential equations, known as Kolmogorov equation

$$\frac{d\pi(t)}{dt} = \pi Q. \qquad (2)$$

The initial condition is specified by the initial state probability vector $\pi(0)$. Before finding the steady state probability vector and the state probability vector, we prove some results of the eigenvalues and stability criteria of the solutions of the system.

Property 1. ([11] Page no.28) Every stochastic matrix has an eigenvalue equal to unity.

Property 2. ([11] Page no.29) The eigenvalues of a stochastic matrix must have modulus less than or equal to 1.

Lemma 1. Let $P = (c_1, c_2, \dots, c_n)$ be the circulant stochastic matrix of odd order and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues, then all the eigenvalues $\lambda_k \neq 1$ for $k = 1,2,3,\dots,n,$ are complex, if $\lambda_1 + \lambda_2 + \dots + \lambda_n \neq 1$ and $c_{k+1} \neq c_{n-k+1}$.

Proof. Let $\omega_k = \exp\left(2\pi i k / n\right)$ are the n^{th} roots of unity.

The eigenvalues λ_k for $k = 1,2,3,\dots,n$ of the circulant matrix are given by [13]

$$\lambda_k = c_1 + c_2\omega_k + c_3\omega_k^2 + \dots + c_n\omega_k^{n-1} \qquad (3)$$

and the corresponding orthogonal eigenvectors are $\vartheta_k = (1, \omega_k, \omega_k^2, \dots, \omega_k^{n-1})$ for $k = 1,2,3,\dots,n$. For $k = 1,2,3,\dots,n,\ \omega_k = \exp\left(2\pi i k / n\right)$, the eigenvalues of the circulant matrix become

$$\lambda_k = c_1 + c_2\left(\cos\left(2\pi k / n\right) + i\sin\left(2\pi k / n\right)\right) + c_3\left(\cos\left(4\pi k / n\right) + i\sin\left(4\pi k / n\right)\right) + \dots + c_n\left(\cos\left(2(n-1)\pi k / n\right) + i\sin\left(2(n-1)\pi k / n\right)\right).$$

Since P is stochastic matrix (i.e., $\sum_{i=1}^{n} c_i = 1$), then the above equation becomes

$$\lambda_k = \frac{1}{(n-1)}(nc_1 - 1) + \alpha + i\beta, \qquad (4)$$

where $\beta = \pm \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sin\theta_k (c_{k+1} - c_{n-k+1})$.

Since $Trace(P) = \lambda_1 + \lambda_2 + \dots + \lambda_n$,

then $nc_1 = \lambda_1 + \lambda_2 + \dots + \lambda_n$,

$$\Rightarrow c_1 = \frac{1}{n}(\lambda_1 + \lambda_2 + \dots + \lambda_n).$$

Therefore,

$$\lambda_k = \frac{1}{(n-1)}((\lambda_1 + \lambda_2 + \dots + \lambda_n) - 1) + \alpha + i\beta.$$

Assume that $\alpha \neq 0$. If $\lambda_1 + \lambda_2 + \dots + \lambda_n \neq 1$ and $c_{k+1} \neq c_{n-k+1}$ for $k = 1,2,3,\dots,\left\lfloor \frac{n}{2} \right\rfloor$ then all the eigenvalues $\lambda_k \neq 1$ of the circulant stochastic matrix are complex.

Corollary 1. Let $P = (c_1, c_2, \dots, c_n)$ be the circulant stochastic matrix of odd order and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues then all the real part of the eigenvalues $\lambda_k \neq 1$ for $k = 1,2,3,\dots,n$ are negative, if $\lambda_1 + \lambda_2 + \dots + \lambda_n < 1$, $\alpha \neq 0$, and $c_{k+1} \neq c_{n-k+1}$.

Property 3. ([11] Page no.29) Every infinitesimal generator matrix Q has at least one zero eigenvalue.

Lemma 2. Let $Q = (c_1, c_2, \dots, c_n)$ be the circulant stochastic rate matrix of odd order and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues, then all the eigenvalues $\lambda_k \neq 0$ for $k = 1,2,3,\dots,n,$ are complex, if $\lambda_1 + \lambda_2 + \dots + \lambda_n \neq 0$ $\alpha \neq 0$, and

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$c_{k+1} \neq c_{n-k+1}$ for $k = 1,2,3,\dots,\left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let $\omega_k = \exp\left(2\pi ik/n\right)$ are the n^{th} roots of unity.

The eigenvalues λ_k for $k = 1,2,3,\dots,n,$ of the circulant matrix are given by [13]

$$\lambda_k = c_1 + c_2\omega_k + c_3\omega_k^2 + \dots + c_n\omega_k^{n-1}$$

and the corresponding orthogonal eigenvectors are $\vartheta_k = \left(1, \omega_k, \omega_k^2, \dots, \omega_k^{n-1}\right)$ for $k = 1,2,3,\dots,n$. For $k = 1,2,3,\dots,n,\ \omega_k = \exp\left(2\pi ik/n\right)$, the eigen values of the circulant matrix becomes

$$\lambda_k = c_1 + c_2\left(\cos\left(2\pi k/n\right) + i\sin\left(2\pi k/n\right)\right) + c_3\left(\cos\left(4\pi k/n\right) + i\sin\left(4\pi k/n\right)\right) + \dots + c_n\left(\cos\left(2(n-1)\pi k/n\right) + i\sin\left(2(n-1)\pi k/n\right)\right).$$

Since Q is stochastic matrix (i.e., $\sum_{i=1}^{n} c_i = 0$), then the above equation becomes

$$\lambda_k = \frac{1}{(n-1)}(nc_1) + \alpha + i\beta, \tag{5}$$

where $\beta = \pm\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \sin\theta_k\,(c_{k+1} - c_{n-k+1})$, and α is a constant. Since $Trace(P) = \lambda_1 + \lambda_2 + \dots + \lambda_n$,

$$\Rightarrow c_1 = \frac{1}{n}\left(\lambda_1 + \lambda_2 + \dots + \lambda_n\right).$$

Therefore,

$$\lambda_k = \frac{1}{(n-1)}(\lambda_1 + \lambda_2 + \dots + \lambda_n) + \alpha + i\beta.$$

Assume that $\alpha \neq 0$. If $\lambda_1 + \lambda_2 + \dots + \lambda_n \neq 0$ and $c_{k+1} \neq c_{n-k+1}$ for $k = 1,2,3,\dots,\left\lfloor \frac{n}{2} \right\rfloor$, then all the eigenvalues $\lambda_k \neq 1$ of the circulant stochastic matrix are complex.

Corollary 2. Let $Q = (c_1, c_2, \dots, c_n)$ be the circulant stochastic rate matrix of odd order and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues then all the real part of the eigenvalues $\lambda_k \neq 0$ for $k =,1,2,3,\dots,n$ are negative, if $\alpha \neq 0$, $\lambda_1 + \lambda_2 + \dots + \lambda_n < 0$, and $c_{k+1} \neq c_{n-k+1}$ for $k = 1,2,3,\dots,\left\lfloor \frac{n}{2} \right\rfloor$.

3. Stability Criteria of the State Probability Vector of First Order Kolmogorov Differential Equation

In this section, we find out the solution of first order Kolmogorov differential and classify the stability criteria in the case of circulant stochastic matrix and circulant stochastic rate matrix.

Theorem 1. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ are the distinct eigenvalues of the circulant stochastic matrix P of odd order. If $\lambda_1 + \lambda_2 + \dots + \lambda_n < 1$, and $c_{k+1} \neq c_{n-k+1}$.for $k = 1,2,3,\dots,\left\lfloor \frac{n}{2} \right\rfloor$, then the solution of the system $\frac{d\pi(t)}{dt} = \pi P$ are unbounded as $t \to \infty$.

Proof. Consider the first order linear Kolmogorov differential equations,

$$\frac{d\pi(t)}{dt} = \pi P. \tag{6}$$

Let $\lambda_1 + \lambda_2 + \dots + \lambda_n \neq 1$, and $c_{k+1} \neq c_{n-k+1}$.for $k = 1,2,3,\dots,\left\lfloor \frac{n}{2} \right\rfloor$, then from the Lemma 3, and corollary 4, all the distinct eigenvalues $\lambda_k \neq 1$ are complex and real part of the eigen values are negative.

Let $\lambda_j = \alpha_j + i\beta_j\ (i = \sqrt{-1},\quad j = 2,3,\dots,n)$ be the eigen value of the matrix P such that $|\lambda_1| = 1 > |\lambda_2| > |\lambda_3| > \dots > |\lambda_n|$ and $\vartheta_k = \left(1, \omega_k, \omega_k^2, \dots, \omega_k^{n-1}\right)$ for $k = \{1,2,3,\dots,n\}$ are the corresponding orthogonal vectors, then the solution of the given system is

$$\pi(t) = \left|e^{Pt}\right| = \left|\sum_{k=1}^{n} \Im_k e^{(\lambda_k)t}\vartheta_k\right|$$

$$= \left|\Im_1 e^{(\lambda_1 t)}\vartheta_1 + \sum_{k=2}^{n} \Im_k e^{(\lambda_k)t}\vartheta_k\right|$$

$$\pi(t) = \left|\Im_1 e^{(\lambda_1 t)}\vartheta_1 + \sum_{k=2}^{n} \Im_k e^{(\alpha_k + i\beta_k)t}\vartheta_k\right|$$

$$\pi(t) \leq \left|\Im_1 e^{(\lambda_1 t)}\vartheta_1\right| + \left|\sum_{k=2}^{n} \Im_k e^{(\alpha_k + i\beta_k)t}\vartheta_k\right|$$

$$\pi(t) \leq \left|\Im_1 e^{t}\vartheta_1\right| + \left|\sum_{k=2}^{n} \Im_k e^{(\alpha_k)t}\vartheta_k\right| \tag{7}$$

where $\Im_k$ is the arbitrary constant.

Let $\xi > \max\{\alpha_k / k = 2,3,\dots,n\}$, then

$$\pi(t) \leq \left|\Im_1 e^{t}\vartheta_1\right| + \left|\sum_{k=2}^{n} \Im_k e^{\xi t}\vartheta_k\right|.$$

Therefore, the solution of the given system (6) is tends to ∞ as $t \to \infty$. The state probability vector is away from the steady state vector as $t \to \infty$.

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Note: If $\Im_1 = 0$ then the solution of the given system tends to 0 as $t \to \infty$.

Theorem 2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the distinct eigenvalues of the circulant stochastic rate matrix Q of odd order. If $\lambda_1 + \lambda_2 + \ldots + \lambda_n < 0$, and $c_{k+1} \neq c_{n-k+1}$.for $k = 1,2,3,\ldots,\left\lfloor \frac{n}{2} \right\rfloor$, then the solution of the system $\frac{d\pi(t)}{dt} = \pi Q$ are bounded as $t \to \infty$.

Proof. Consider the first order linear Kolmogorov differential equations, $\frac{d\pi(t)}{dt} = \pi Q$ (8)

Let $\lambda_1 + \lambda_2 + \ldots + \lambda_n < 0$, and $c_{k+1} \neq c_{n-k+1}$.for $k = \left\{1,2,3,\ldots,\left\lfloor \frac{n}{2} \right\rfloor\right\}$then from the Lemma 6, and corollary 7, all the distinct eigenvalues $\lambda_k \neq 0$ are complex and real part of the eigen values are negative.

Let $\lambda_j = \alpha_j + i\beta_j \; (i = \sqrt{-1}, \quad j = 2,3,\ldots,n)$ be the eigen value of the matrix Q, and $\vartheta_k = \left(1, \omega_k, \omega_k^2, \ldots, \omega_k^{n-1}\right)$for $k = 1,2,3,\ldots,n$ are the corresponding orthogonal vectors then the solution of the given system is

$$\pi(t) = \left|e^{Qt}\right| = \left|\sum_{k=1}^{n} \Im_k e^{(\lambda_k)t} \vartheta_k\right|$$

$$\pi(t) \leq \left|\Im_1 e^{(\lambda_1 t)} \vartheta_1\right| + \left|\sum_{k=2}^{n} \Im_k e^{(\alpha_k + i\beta_k)t} \vartheta_k\right|$$

$$\pi(t) \leq \left|\Im_1 e^{(\lambda_1 t)} \vartheta_1\right| + \left|\sum_{k=2}^{n} \Im_k e^{(\alpha_k)t} \vartheta_k\right| \tag{9}$$

where $\Im_k$ is the arbitrary constant.

Let $\xi > \max\{\alpha_k / k = 2,3,\ldots,n\}$, and $\lambda_1 = 0$, then

$$\pi(t) \leq \left|\Im_1 \vartheta_1\right| + \left|\sum_{k=2}^{n} \Im_k e^{\xi t} \vartheta_k\right|$$

$\pi(t) \leq \left|\Im_1 \vartheta_1\right| = M$ as $t \to \infty$.

Therefore, the solution of the given system (6) is bounded as $t \to \infty$. The state probability vector approaches the steady state vector as $t \to \infty$.

Note: If $\Im_1 = 0$ then the solution of the given system tends to 0 as $t \to \infty$.

4. Conclusions

In general, the transient analysis and behaviour of the analytical solution of the queueing theory in stochastic models are not widely available as steady state results. Hence, we examine the behaviour of the state probability vector of the Kolmogorov first order differential equation. In this direction, first we find out the eigenvalues of the circulant stochastic matrix, and stochastic rate matrix. Then we find out the behaviour of the solution as $t \to \infty$. This kind of analysis is useful for finding the stability criteria of the pertinent system and performance measures of the network nodes to provide the quality of service (QoS).

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