The generalized quasilinearization technique for a second order differential equation with separated boundary conditions

Rahmat Ali Khan

Centre for Advanced Mathematics and Physics, National University of Sciences and Technology, Campus of College of Electrical and Mechanical Engineering, Peshawar Road, Rawalpindi, Pakistan

Received 23 August 2004; accepted 19 May 2005

Abstract

The method of upper and lower solutions and the method of quasilinearization for a second order nonlinear differential equation

\[-x''(t) = f(t, x, x'), t \in [0, 1]\]

subject to the separated boundary conditions

\[p_0 x(0) - q_0 x'(0) = a,\]
\[p_1 x(1) + q_1 x'(1) = b,\]

is developed. A monotone sequence of solutions of linear problems converging uniformly and quadratically to a solution of the problem is obtained.

© 2005 Elsevier Ltd. All rights reserved.

Keywords: Upper and lower solutions; Quasilinearization; General boundary conditions; Rapid convergence

1. Introduction

We consider the following second order nonlinear differential equation

\[-x''(t) = f(t, x, x'), t \in [0, 1],\]  \hfill (1.1)

subject to the separated boundary conditions

\[p_0 x(0) - q_0 x'(0) = a,\]
\[p_1 x(1) + q_1 x'(1) = b,\]  \hfill (1.2)

where \(f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is a continuous function and \(p_0, q_0, p_1, q_1, a, b \in \mathbb{R}\) are such that \(p_0, p_1, q_0, q_1 > 0\). Let \(D = p_0 q_1 + p_1 q_0 + p_0 p_1\), then \(D \neq 0\).

In the present paper we develop the method of upper and lower solutions for the existence of solutions and the method of quasilinearization to approximate the solution of the boundary value problem (1.1), (1.2). For computational purposes, the linear iteration scheme and the quadratic convergence of the iterates are important. We approximate our problem by a sequence of linear problems to obtain a monotone sequence of approximants. We show that, under suitable conditions, these converge quadratically to a solution of the nonlinear original problem (1.1), (1.2).

E-mail address: rahmat_alipk@yahoo.com.
Recently, Cabada and Lois [1], have studied existence results for the BVP (1.1), (1.2) via the method of lower and upper solutions and the monotone iterative technique in the case when the nonlinearity $f$ is independent of the derivative $x'$.

The method of quasilinearization, being a powerful technique for obtaining approximate solutions of nonlinear differential equations, has recently been studied and extended extensively, for example, [2–4]. The fact that one can obtain a monotone sequence of solutions of linear problems whose convergence to a solution of the original nonlinear problem is quadratic, is one of the reasons for the popularity of this technique. It was developed many years ago by Bellman and Kalaba [5] to solve systems of nonlinear ordinary and partial differential equations. It requires that the nonlinearity is convex. Recently, Lakshmikantham [6,7] generalized the method by relaxing the convexity conditions. Its modern developments and examples of its applications to the different fields of science and engineering are given in a recent monograph [8].

However, the method of quasilinearization is less developed for nonlinear problems, when the nonlinearity $f$ depends also on the derivative $x'$. This is the case we study. We control both the function and its first derivative so we prove results on quadratic convergence in the $C^1$ norm. This is more delicate than the corresponding results when there is no $x'$ dependence in $f$.

The purpose of this paper is to develop the method of generalized quasilinearization for the nonlinear differential equation (1.1) subject to general separated boundary conditions (1.2), by establishing a linear iteration scheme to obtain a monotone sequence of solutions converging uniformly and quadratically with respect to the $C^1$ norm to a solution of the problem.

Recently, Eloe and Zhang [9] and Mohapatra et al. [10] have studied the method of quasilinearization for the differential equation (1.2) subject to the boundary conditions

$$x(0) = a, \quad x(1) = b.$$  \hfill (1.3)

Our result contains the results of [9,10] as special cases.

2. Some basic results

Under the conditions given above, we know that the linear homogeneous problem

$$-x''(t) = 0, \quad t \in [0, 1],$$
$$p_0 x(0) = q_0 x'(0) = 0, \quad p_1 x(1) + q_1 x'(1) = 0$$

has only a trivial solution. Consequently, for any $\sigma(t) \in C[0, 1]$, the corresponding nonhomogeneous linear problem

$$-x''(t) = \sigma(t), \quad t \in [0, 1],$$
$$p_0 x(0) - q_0 x'(0) = a, \quad p_1 x(1) + q_1 x'(1) = b$$

has a unique solution $x(t) \in C^2[0, 1]$,

$$x(t) = \frac{a(p_1 + q_1) + bq_0}{D} + \frac{bp_0 - ap_1}{D} t + \int_0^1 G(t, s) \sigma(s) \, ds,$$

where the Green’s function is defined by

$$G(t, s) = \frac{1}{D} \times \begin{cases} (q_0 + p_0 t)(q_1 + p_1 (1 - s)), & 0 \leq t < s \leq 1 \\ (q_0 + p_0 s)(q_1 + p_1 (1 - t)), & 0 \leq s < t \leq 1, \end{cases}$$

and satisfies $G(t, s) > 0$ on $(0, 1) \times (0, 1)$. If $x(t)$ is a solution of (1.1), (1.2), then $x(t)$ is a solution of the integral equation

$$x(t) = \frac{a(p_1 + q_1) + bq_0}{D} + \frac{bp_0 - ap_1}{D} t + \int_0^1 G(t, s) f(s, x(s), x'(s)) \, ds. \quad (2.1)$$

We recall the concept of upper and lower solutions for the BVP (1.1), (1.2) and Nagumo condition.
Definition 2.1. Let $\alpha, \beta \in C^2[0,1]$. We say that $\alpha$ is a lower solution of the BVP (1.1), (1.2), if
\[
-\alpha''(t) \leq f(t, \alpha(t), \alpha'(t)), \quad t \in [0,1]
\]
\[
p_0\alpha(0) - q_0\alpha'(0) \leq a, \quad p_1\alpha(1) + q_1\alpha'(1) \leq b.
\]
Similarly, $\beta$ is an upper solution of the BVP (1.1), (1.2), if $\beta$ satisfies similar inequalities in the reverse direction.

Definition 2.2. Let $f \in C[[0,1] \times \mathbb{R} \times \mathbb{R}]$ and $\alpha, \beta \in C^2[0,1]$ with $\alpha(t) \leq \beta(t)$ on $[0,1]$. We say that $f$ satisfies a Nagumo condition relative to $\alpha, \beta$, if for $t \in [0,1], x \in [\alpha(t), \beta(t)]$ and $x' \in \mathbb{R}$, there exists a positive continuous function, $\omega$, defined on $[0, \infty)$ such that $|f(t, x, x')| \leq \omega(|x'|)$ and
\[
\int_{\lambda}^{\infty} \frac{s}{\omega(s)} \, ds = \infty \tag{2.2}
\]
where $\lambda = \max\{|\alpha(0) - \beta(1)|, |\alpha(1) - \beta(0)|\}$.

For $u \in C[0,1]$ we write $\|u\| = \max\{|u(t)| : t \in [0,1]\}$ and for $v \in C^1[0,1]$ we write $\|v\|_1 = \|v\| + \|v'|$.

Now, we state and prove the following results which establish the existence and uniqueness of a solution in the presence of upper and lower solutions.

Theorem 2.3. Assume that $\alpha, \beta \in C^2[0,1]$ are respectively lower and upper solutions of (1.1), (1.2) such that $\alpha \leq \beta$ on $[0,1] = J$. Assume that $f \in C[J \times \mathbb{R} \times \mathbb{R}]$ and satisfies a Nagumo condition relative to $\alpha, \beta$. Then there exists a solution $x(t)$ of (1.1), (1.2) such that $\alpha(t) \leq x(t) \leq \beta(t)$ on $J$.

Proof. Let $\max_{t \in J} \beta(t) - \min_{t \in J} \alpha(t) = r$, then in view of (2.2), there exists a constant $N$ depending on $\alpha, \beta$ and $\omega$ such that
\[
\int_{\lambda}^{N} \frac{s}{\omega(s)} > r,
\]
where $\omega$ is a Nagumo function.

Let $C > \max\{N, \|\alpha\|, \|\beta\|\}$ and $q(x') = \max\{-C, \min\{x', C\}\}$. Define modification of $f(t, x, x')$ as follows
\[
F(t, x, x') = \begin{cases} 
  f(t, \beta, q(x')) + \frac{x - \beta(t)}{1 + |x - \beta(t)|}, & \text{if } x > \beta(t), \\
  f(t, x, q(x')), & \text{if } \alpha(t) \leq x \leq \beta(t), \\
  f(t, \alpha, q(x')) + \frac{\alpha(t) - x}{1 + |x - \alpha(t)|}, & \text{if } x < \alpha(t).
\end{cases}
\]
Consider the modified boundary value problem
\[
-x''(t) = F(t, x, x'), \quad t \in [0,1],
\]
\[
p_0x(0) - q_0x'(0) = a, \quad p_1x(1) + q_1x'(1) = b \tag{2.3}
\]
This is equivalent to an integral equation,
\[
x(t) = \frac{a(p_1 + q_1) + bq_0}{D} + \frac{bp_0 - ap_1}{D} t + \int_0^1 G(t, s) F(s, x(s), x'(s)) \, ds.
\]
Since $F(t, x, x')$ and $G(t, s)$ are continuous and bounded, this integral equation has a fixed point by Schauder’s fixed point theorem, so the BVP (2.3) has a solution $x \in C^2(J)$. Further, we note that
\[
-\alpha''(t) \leq f(t, \alpha(t), \alpha'(t)) = F(t, \alpha(t), \alpha'(t)), \quad t \in J,
\]
\[
p_0\alpha(0) - q_0\alpha'(0) \leq a, \quad p_1\alpha(1) + q_1\alpha'(1) \leq b
\]
and
\[
-\beta''(t) \geq f(t, \beta(t), \beta'(t)) = F(t, \beta(t), \beta'(t)), \quad t \in J,
\]
\[
p_0\beta(0) - q_0\beta'(0) \geq a, \quad p_1\beta(1) + q_1\beta'(1) \geq b,
\]
that is, $\alpha, \beta$ are lower and upper solutions of (2.3). We claim that any solution $x$, of (2.3) with $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in J$, is also a solution of (1.1), (1.2). For if $x$ is such a solution of (2.3), then for $t \in J$, we have
\[
|F(t, x, x')| = |f(t, x, q(x'))| \leq \tilde{\omega}(|x'|),
\]
where $\tilde{\omega}(s) = \omega(q(s))$ for $s \geq 0$. We note that $q(s) \geq 0$ for $s \geq 0$ and for $s \leq C$, we have $q(s) = s$. Now
\[
\int_{0}^{\infty} \frac{s}{\tilde{\omega}(s)} \, ds = \int_{0}^{C} \frac{s}{\omega(s)} \, ds + \int_{C}^{\infty} \frac{s}{\omega(C)} \, ds = \infty,
\]
which implies that $\tilde{\omega}$ is a Nagumo function. Further,
\[
\int_{0}^{C} \frac{s}{\tilde{\omega}(s)} \, ds \geq \int_{0}^{N} \frac{s}{\omega(s)} \, ds + \int_{0}^{C} \frac{s}{\tilde{\omega}(s)} \, ds \geq \int_{0}^{N} \frac{s}{\omega(s)} \, ds \geq r,
\]
and hence as in the proof of Theorem (1.4.1) of [11], we conclude that $|x'(t)| < C$ on $J$, which implies that $x$ is a solution of (1.1), (1.2).

Now, we show that any solution $x$ of (2.3) does satisfy $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in J$. Set $v(t) = \alpha(t) - x(t)$, $t \in J$. Then, $v(t) \in C^2(J)$ and the boundary conditions imply that
\[
p_0 v(0) - q_0 v'(0) \leq 0, \quad p_1 v(1) + q_1 v'(1) \geq 0. \tag{2.4}
\]
Assume that $\max\{v(t) : t \in J\} = v(t_0) > 0$. If $t_0 = 0$, then $v(0) > 0$ and $v'(0) \leq 0$. However, the boundary conditions (2.4) imply that $v(0) \leq \frac{q_0}{p_0} v'(0) \leq 0$, a contradiction. If $t_0 = 1$, then $v(1) > 0$, $v'(1) \geq 0$ and the boundary conditions (2.4) imply that
\[
v(1) \leq -\frac{q_1}{p_1} v'(1) \leq 0,
\]
again a contradiction. Thus, $t_0 \in (0, 1)$ and consequently
\[v(t_0) > 0, \quad v'(t_0) = 0 \quad \text{and} \quad v''(t_0) \leq 0.
\]
However,
\[
v''(t_0) = \alpha''(t_0) - x''(t_0)
\]
\[\geq -f(t_0, \alpha(t_0), \alpha'(t_0)) + f(t_0, \alpha(t_0), \alpha'(t_0)) + \frac{\alpha(t_0) - x(t_0)}{1 + |x(t_0) - \alpha(t_0)|} > 0,
\]
a contradiction. Thus $\alpha(t) \leq x(t)$, $t \in J$. Similarly, we can show that $x(t) \leq \beta(t)$, $t \in J$. □

We prove a result from which we can deduce the uniqueness of solution of the BVP (1.1), (1.2).

**Theorem 2.4.** Assume that $\alpha, \beta$ are lower and upper solutions of the boundary value problem (1.1), (1.2). If $f \in C[J \times \mathbb{R} \times \mathbb{R}]$ is decreasing in $x$ for each $(t, x') \in J \times \mathbb{R}$, then $\alpha(t) \leq \beta(t)$, $t \in [0, 1]$. In particular, there is at most one solution of the boundary value problem.

**Proof.** Define $m(t) = \alpha(t) - \beta(t)$, $t \in J$. Then $m \in C^2(J)$ and using the boundary conditions, we obtain
\[
p_0 m(0) - q_0 m'(0) \leq 0, \quad p_1 m(1) + q_1 m'(1) \leq 0. \tag{2.5}
\]
We claim that $m(t) \leq 0$, $t \in J$. If not, then $m(t)$ has a positive maximum at some $t_0 \in J$. If $t_0 = 0$, then $m(0) > 0$ and $m'(0) \leq 0$. But the boundary conditions (2.5) gives
\[0 \geq p_0 m(0) - q_0 m'(0) > 0,
\]
a contradiction. If $t_0 = 1$, then $m(1) > 0$ and $m'(1) \geq 0$. But again the boundary conditions (2.5) gives
\[0 \geq p_1 m(1) + q_1 m'(1) > 0,
\]
another contradiction. It follows that $t_0 \in (0, 1)$ and hence
\[m(t_0) > 0, \quad m'(t_0) = 0, \quad m''(t_0) \leq 0.
\]
However, using the definition of upper and lower solution and the decreasing property of \( f(t, x, x') \) in \( x \), we obtain
\[ -m''(t_0) = -\alpha''(t_0) + \beta''(t_0) \leq f(t_0, \alpha(t_0), \alpha'(t_0)) - f(t_0, \beta(t_0), \alpha'(t_0)) < 0, \]
a contradiction. Thus, \( \alpha(t) \leq \beta(t) \) for every \( t \in J \). \( \Box \)

**Example 2.5 (Class of Examples).** Consider the boundary value problem
\[ -x''(t) = |x'(t)|^{p-1}x'(t) + g(x(t))\phi(t), \; t \in J, \; 0 \leq p \leq 2, \]
\[ p_0x(0) - q_0x'(0) = 0, \quad p_1x(1) + q_1x'(1) = 0, \]
where \( 0 \leq \phi(t) \leq 1 \). Assume that \( g \) is continuous and decreasing and \( g(0) \geq 0 \). Then \( \alpha(t) \equiv 0 \) is a lower solution of the problem. Choose \( b > 0 \) so that \( b^p \geq g(0) \) and take \( \beta(t) = \left( \frac{p_1 + q_1}{p_1} - t \right) b \). Then we have \( \beta'(t) = -b, \, \beta''(t) = 0 \) and
\[ p_0\beta(0) - q_0\beta'(0) = \frac{bD}{p_1} > 0, \quad p_1\beta(1) + q_1\beta'(1) = 0. \]
Moreover, by the decreasing property of \( g \), we have
\[ \beta'' + |\beta'|^{p-1}\beta' + g(\beta)\phi(t) \leq -b^p + g(0)\phi(t) \leq 0. \]
Thus, \( \beta(t) = \left( \frac{p_1 + q_1}{p_1} - t \right) b \) is an upper solution of the problem. Clearly, \( \alpha(t) \leq \beta(t) \) on \( J \). Finally, for \( \alpha(t) \leq x(t) \leq \beta(t) \), we have,
\[ ||x'(t)||^{p-1}x'(t) + g(x(t))\phi(t) \leq |x'|^p + g(0) =: h(|x'|), \]
where \( h(s) = s^p + g(0) \) and \( \int_0^\infty \frac{s^p}{s^p + g(0)} \, ds = \infty \) for \( 0 \leq p \leq 2 \). That is, the Nagumo condition is satisfied. Hence by Theorem 2.3 there is a solution of this problem which lies between \( \alpha \) and \( \beta \). Since zero is not a solution if \( g(0) > 0 \), it follows that the boundary value problem has a positive solution on \( J \) when \( g(0) > 0 \).

**Example 2.6.** In this example, we do not require \( g \) to be decreasing and the term in \( x' \) has the opposite sign to Example 2.5.

We consider a nonlinear boundary value problem of the type
\[ -x'' = -|x'|^{p-1}x' + g(x)\phi(t), \; 0 \leq p \leq 2, \; t \in J \]
\[ p_0x(0) - q_0x'(0) = 0, \quad p_1x(1) + q_1x'(1) = 0, \]
where \( \phi \in C[J], \, g \in C[R] \) and \( 0 \leq \phi(t) \leq 1 \). Assume that \( g \) satisfies
\[ g(0) > 0 \quad \text{and} \quad g(x) \leq \delta x^p \text{, for } x \geq 0, \]
where \( \delta \leq \frac{1}{(1 + \frac{q_0}{p_0})^p} \). Firstly, it is easily shown that \( \alpha(t) \equiv 0 \) is a lower solution of the problem.

Secondly, we show that \( \beta(t) = \frac{q_0}{p_0} B + Bt, \, B \geq 0 \) is an upper solution. In fact
\[ p_0\beta(0) - q_0\beta'(0) = 0, \quad p_1\beta(1) + q_1\beta'(1) = \frac{D}{p_0} B \geq 0. \]
Moreover,
\[ \beta'' + |\beta'|^{p-1}\beta' + g(\beta)\phi(t) \leq -B^p + \delta \left( t + \frac{q_0}{p_0} \right)^p B^p \phi(t) \]
\[ \leq -B^p + \delta \left( 1 + \frac{q_0}{p_0} \right)^p B^p \leq 0. \]
Hence \( \beta(t) = \frac{q_0}{p_0} B + Bt \) is an upper solution of the problem. Clearly, \( \alpha(t) \leq \beta(t) \) on \( J \). Finally, we show that a Nagumo condition is also satisfied. For \( \alpha(t) \leq x(t) \leq \beta(t), \, t \in J \), we have
\[ | -|x'|^{p-1}x' + g(x)\phi(t) | \leq |x'|^p + M =: h(|x'|), \]
where \( h(s) = s^p + M \) and \( M = \max\{|g(x)| : x \in [0, \left(\frac{q_0}{p_0} + 1\right)B]\} \). Now,

\[
\int_0^\infty \frac{s}{s^p + M} \, ds = \infty.
\]

Thus by Theorem 2.3, there is a solution of this problem which lies between 0 and \( \frac{q_0}{p_0} B + Bt \). In particular, since \( g(0) > 0 \) this solution is positive on \( J \).

3. Quasilinearization technique

We now study the approximation of the solution of the BVP (1.1), (1.2) by the method of quasilinearizations. We prove that under suitable conditions on the function \( f \), there exists a monotone sequence of solutions of linear problems which converges quadratically to a solution of the nonlinear original problem.

**Theorem 3.1.** Assume that

(A1) \( \alpha \) and \( \beta \) are lower and upper solutions of (1.1), (1.2) such that \( \alpha \leq \beta \) on \( J \).

(A2) \( xf(t, x, 0) > 0 \) for \( |x| > k \), where \( k = \max\{|\alpha|, |\beta|\} \).

(A3) \( f \in C^2(J \times \mathbb{R} \times \mathbb{R}) \) satisfies a Nagumo condition on \( J \) relative to \(-k, k\) and is such that \( f_1(t, x, x') \leq 0 \) and \( H(f) \geq 0 \) on \( J \times \mathbb{R}^2 \), where

\[
H(f) = (x - y)^2 f_{xx}(t, z_1, z_2) + 2(x - y)(x' - y') f_{xx'}(t, z_1, z_2) + (x' - y')^2 f_{x'x'}(t, z_1, z_2)
\]

is the quadratic form of \( f \) and \( z_1 \) is between \( y, x \) and \( z_2 \) lies between \( x', y' \).

Then, there exists a monotone sequence \( \{w_n\} \) of solutions converging uniformly and quadratically to the unique solution of problem (1.1), (1.2).

**Proof.** The conditions (A2) imply that any solution \( x \) of (1.1), (1.2) satisfies \( |x| \leq k \) on \( J \). The Nagumo condition ensures the existence of a positive continuous function \( \omega : [0, \infty) \rightarrow (0, \infty) \) such that

\[
|f(t, x, x')| \leq \omega(|x'|) \text{ for } t \in J, x \in [-k, k],
\]

and

\[
\int_0^\infty \frac{sds}{\omega(s)} = \infty. \tag{3.1}
\]

Let \( r = \max_{t \in J} \beta(t) - \min_{t \in J} \alpha(t) \), then, there exists a constant \( N > 0 \) such that

\[
\int_0^N \frac{sds}{\omega(s)} \geq r,
\]

and hence as in the proof of Theorem (1.4.1) of [11], any solution \( x \) of (1.1), (1.2) with the property \( |x| \leq k \) satisfies \( |x'| \leq N \) on \( J \). Let \( C = \max\{|\alpha'|, |\beta'|, N\} \). Then every solution \( x \) of (1.1), (1.2) with the property \( |x| \leq k \), must satisfy

\[
|x'(t)| < C \text{ on } J. \tag{3.2}
\]

Define \( q(s) = \max\{-C, \min\{s, C\}\} \) and consider the boundary value problem

\[
\begin{align*}
-x''(t) &= f(t, x, q(x')), t \in J, \\
p_0 x(0) - q_0 x'(0) &= a, \\
p_1 x(1) + q_1 x'(1) &= b. \tag{3.3}
\end{align*}
\]

We note that any solution \( x \) of (3.3) which satisfies \( |x'(t)| < C \), is a solution of (1.1), (1.2). Moreover we note that

\[
x f(t, x, q(0)) = x f(t, x, 0) > 0 \text{ for } |x| > k,
\]

and for \( |x| \leq k \), we have

\[
|f(t, x, q(x'))| \leq \omega(|q(x'|) = \tilde{\omega}(|x'|) \text{ for } t \in J, x' \in \mathbb{R},
\]

where \( \tilde{\omega} \) is a function that satisfies

\[
\int_0^\infty \frac{s}{s + M} \, ds = \infty.
\]
where \(\tilde{\omega}(s) = \omega(q(s))\) for \(s \geq 0\). Since
\[
\int_0^\infty \frac{sd\tilde{\omega}(s)}{\tilde{\omega}(s)} = \int_0^C \frac{sd\omega(s)}{\omega(s)} + \int_C^\infty \frac{sd\omega(s)}{\omega(C)} = \infty.
\]
Hence \(\tilde{\omega}\) is a Nagumo function. Also,
\[
\int_0^C \frac{sd\omega(s)}{\omega(s)} \geq \int_0^N N \frac{sd\omega(s)}{\omega(s)} \geq r.
\]
Thus, every solution \(x\) of (3.3) does satisfy \(|x'(t)| < C\) on \(J\) and hence is a solution of (1.1), (1.2).

Now, in view of (A3) and the mean value theorem, we obtain
\[
f(t, x, q(x')) \geq f(t, y, q(y')) + f_x(t, y, q(y'))(x - y) + f_{x'}(t, y, q(y'))(q(x') - q(y')), \quad t \in J
\]
where \(x, y, x', y' \in \mathbb{R}\). Define
\[
K(t, x, x'; y, y') = f(t, y, q(y')) + f_x(t, y, q(y'))(x - y) + f_{x'}(t, y, q(y'))(q(x') - q(y')).
\]
Then, we have the following relations
\[
\begin{align*}
& f(t, x, q(x')) \geq K(t, x, x'; y, y') \\
& f(t, x, q(x')) = K(t, x, x'; x, x')
\end{align*}
\]
for \(t \in J, x, y, x', y' \in \mathbb{R}\). Moreover, \(K\) is continuous and bounded on \(J \times [-k, k] \times \mathbb{R}\) and therefore satisfies a Nagumo condition on \(J\). Hence there exists \(N_1 > 0\) such that any solution \(x\) of
\[
-x''(t) = K(t, x, x'; y, y'), \quad t \in J,
\]
\[
p_0 x(0) - q_0 x'(0) = a, \quad p_1 x(1) + q_1 x'(1) = b.
\]
with \(|x| \leq k\) satisfies \(|x'(t)| \leq N_1\) on \(J\). Choose \(C_1 \geq \max\{C, N_1\}\).

Now, we set \(w_0 = a\) and consider the linear boundary value problem
\[
-x''(t) = K(t, x, x'; w_0, w_0'), \quad t \in J,
\]
\[
p_0 w(0) - q_0 w'(0) = a, \quad p_1 w(1) + q_1 w'(1) = b.
\]
Using (A1) and (3.6), we obtain
\[
K(t, w_0(t), w_0'(t); w_0(t), w_0'(t)) = f(t, w_0(t), w_0'(t)) \geq -w_0''(t), \quad t \in J,
\]
\[
p_0 w_0(0) - q_0 w_0'(0) \leq a, \quad p_1 w_0(1) + q_1 w_0'(1) \leq b,
\]
and
\[
K(t, \beta(t), \beta'(t); w_0(t), w_0'(t)) \leq f(t, \beta(t), \beta'(t)) \leq -\beta''(t)
\]
\[
p_0 \beta(0) - q_0 \beta'(0) \geq a, \quad p_1 \beta(1) + q_1 \beta'(1) \geq b,
\]
which imply that \(w_0\) and \(\beta\) are lower and upper solutions of (3.7) respectively. Hence, by Theorems 2.3 and 2.4, there exists a unique solution \(w_1\) of (3.7) such that \(w_0 \leq w_1 \leq \beta\) on \(J\). Using (3.6) and the fact that \(w_1\) is a solution of (3.7), we obtain
\[
-w''_1(t) = K(t, w_1(t), w_1'(t); w_0(t), w_0'(t)) \leq f(t, w_1(t), q(w_1'(t))), \quad t \in J
\]
\[
p_0 w_1(0) - q_0 w_1'(0) = a, \quad p_1 w_1(1) + q_1 w_1'(1) = b,
\]
which implies that \(w_1\) is a lower solution of (3.3).

Now, consider the linear boundary value problem
\[
-x''(t) = K(t, x, x'; w_1, w_1'), \quad t \in J
\]
\[
p_0 x(0) - q_0 x'(0) = a, \quad p_1 x(1) + q_1 x'(1) = b.
\]
In view of (A1), (3.8) and (3.6), we have
\[
-w''_1(t) \leq f(t, w_1, q(w_1')) = K(t, w_1, w_1'; w_1, w_1'), \quad t \in J
\]
and
\[K(t, \beta(t), \beta'(t); w_1(t), w'_1(t)) \leq f(t, \beta(t), \beta'(t)) \leq -\beta''(t)\]
which imply that \(w_1\) and \(\beta\) are lower and upper solutions of (3.9). Hence by Theorems 2.3 and 2.4, there exists a unique solution \(w_2\) of (3.9) such that \(w_1 \leq w_2 \leq \beta\) on \(J\).

Continuing this process we obtain a monotone sequence \(\{w_n\}\) of solutions satisfying
\[\alpha(t) = w_0 \leq w_1 \leq w_2 \leq \cdots \leq w_{n-1} \leq w_n \leq \beta, \quad t \in J.\]
That is,
\[\alpha(t) \leq w_n(t) \leq \beta(t) \quad \text{and} \quad |w'_n(t)| < C_1, \quad n \in \mathbb{N}, \quad t \in J\]
where \(w_n\) is a solution of the linear problem
\[-x''(t) = K(t, x, x'; w_{n-1}, w'_{n-1}), \quad t \in J\]
\[p_0 x(0) - q_0 x'(0) = a, \quad p_1 x(1) + q_1 x'(1) = b.\]
Thus,
\[w_n(t) = \frac{a(p_1 + q_1) + bq_0}{D} + \frac{bp_0 - ap_1}{D} t + \int_0^1 G(t, s) K(s, w_n, w'_n; w_{n-1}, w'_{n-1}) \, ds. \tag{3.11}\]
Since \(K(t, w_n, w'_n; w_{n-1}, w'_{n-1})\) is continuous and bounded, there is \(L > 0\) such that
\[|K(t, w_n, w'_n; w_{n-1}, w'_{n-1})| \leq L\]
on \(J\).
Moreover, for \(s, t (s \leq t) \in J\), we have
\[|w'_n(t) - w'_n(s)| \leq \int_s^t |K(u, w_n, w'_n; w_{n-1}, w'_{n-1})| \, du \leq L|t - s|. \tag{3.12}\]
From (3.10)-(3.12), it follows that the sequences \(\{w_n^{(j)}\} (j = 0, 1)\) are uniformly bounded and equicontinuous on \(J\). The Arzelà–Ascoli theorem guarantees the existence of subsequences and a function \(x \in C^1(J)\) such that \(w_n^{(j)} \to x^{(j)} (j = 0, 1)\) uniformly on \(J\) as \(n \to \infty\). It follows that \(K(t, w_n, w'_n; w_{n-1}, w'_{n-1}) \to f(t, x, q(x'))\) as \(n \to \infty\). Passing to the limit in (3.11), we obtain
\[x(t) = \frac{a(p_1 + q_1) + bq_0}{D} + \frac{bp_0 - ap_1}{D} t + \int_0^1 G(t, s) f(s, x, q(x')) \, ds,\]
that is, \(x\) is a solution of (3.3) and hence is a solution of (1.1), (1.2).

Now we show that the convergence of the sequence of solutions is quadratic. For this, set \(v_n(t) = x(t) - w_n(t), \quad t \in J, \quad n \in \mathbb{N}\). We note that
\[v_n \in C^2(J) \quad \text{and} \quad v_n(t) \geq 0, \quad t \in J.\]
Moreover, the boundary conditions imply that
\[p_0 v_n(0) - q_0 v'_n(0) = 0, \quad p_1 v_n(1) + q_1 v'_n(1) = 0. \tag{3.13}\]
Since \(v_n \geq 0\) on \(J\), the boundary conditions (3.13) imply that
\[v'_n(0) \geq 0, \quad v_n'(1) \leq 0. \tag{3.14}\]
Using Taylor’s theorem and the definition (3.5), we obtain
\[-v''_{n+1}(t) = -x''(t) + w''_{n+1}(t) = f(t, x, x') - K(t, w_{n+1}, w'_{n+1}; w_n, w'_n)
= f(t, w_n, q(w'_n)) + f_x(t, w_n, q(w'_n))(x - w_n) + f_{x'}(t, w_n, q(w'_n))(x' - q(w'_n))
+ \frac{1}{2} H(f) - \{f(t, w_n, q(w'_n)) + f_x(t, w_n, q(w'_n))(w_{n+1} - w_n)\}}\]
\[ + f_x'(t, w_n, q(w'_n))(q(w'_{n+1}) - q(w'_n)) \]
\[ = f_x(t, w_n, q(w'_n))v_{n+1} + f_x'(t, w_n, q(w'_n))(x' - q(w'_{n+1})) + \frac{1}{2} H(f), \]

where
\[ H(f) = (x - w_n)^2 f_{xx}(t, \xi_1, \xi_2) + 2(x - w_n)(x' - q(w'_n)) f_{xx'}(t, \xi_1, \xi_2) + (x' - q(w'_n))^2 f_{xx'}(t, \xi_1, \xi_2), \]
where \( w_n \leq \xi_1 \leq x \) and \( \xi_2 \) lies between \( x', q(w'_n) \). Let
\[ M_1 = \max\{|f_{xx}(t, \xi_1, \xi_2)|, |f_{xx'}(t, \xi_1, \xi_2)|, |f_{xx'}(t, \xi_1, \xi_2)| : t \in J, \xi_1 \in [w_0(t), \beta(t)], \xi_2 \in [-C, C]\}, \]
then
\[ |H(f)| \leq M_1 |x - w_n|^2 + 2|x - w_n||x' - q(w'_n)| + |x' - q(w'_n)|^2 \]
\[ \leq M_1 |x - w_n|^2 + 2|x - w_n||x' - w'_n| + |x' - w'_n|^2 \]
\[ = M_1 (|x - w_n| + |x' - w'_n|)^2 \leq M_1 \|v_n\|_1^2. \]

Also, by (A3), \( f_x \leq 0 \) on \( J \times \mathbb{R}^2 \). Thus, we have
\[ -v''_{n+1}(t) \leq f_x'(t, w_n, q(w'_n))(x' - q(w'_{n+1})) + \frac{M_1}{2} \|v_n\|_1^2, \quad t \in J. \tag{3.16} \]

Let
\[ M_2 = \max\{|f_{xx'}(t, d_1, d_2)| : t \in J, d_1 \in [w_0(t), \beta(t)], d_2 \in [-C, C]\}, \]
then
\[ -M_2 \leq f_{xx'}(t, d_1, d_2) \leq M_2, \quad t \in J. \tag{3.17} \]

We discuss different cases.

**Case 1.** If \( |w'_{n+1}| \leq C \), then \( q(w'_{n+1}) = w'_{n+1} \) and (3.16) implies that
\[ v''_{n+1}(t) + f_x'(t, w_n, q(w'_n))v'_{n+1}(t) \geq -\frac{M_1}{2} \|v_n\|_1^2. \tag{3.18} \]

Let \( \rho(t) = e^{\int_0^t f_{xx'}(t, w_n, q(w'_n)) \, ds} \) denote the integrating factor for (3.18), then
\[ (v'_{n+1}(t)\rho(t))' \geq -\rho(t) - \frac{M_1}{2} \|v_n\|_1^2. \tag{3.19} \]

Integrating (3.19) from 0 to \( t \), using (3.14) \((v'_{n+1}(0) \geq 0)\) and the fact that \( \rho(t) > 0 \), we obtain
\[ v'_{n+1}(t) \geq -\frac{M_1}{2\rho(t)} \int_0^t \rho(s) \, ds. \tag{3.20} \]

On the other hand, if we integrate (3.19) from \( t \) to 1, use (3.14) \((v'_{n+1}(1) \leq 0)\) and the fact that \( \rho(t) > 0 \), we obtain
\[ v'_{n+1}(t) \leq \frac{M_1}{2\rho(t)} \int_t^1 \rho(s) \, ds. \tag{3.21} \]

Let
\[ \delta = \max\left\{ \max_{[0,1]} \frac{M_1}{2\rho(t)} \int_0^t \rho(s) \, ds, \max_{[0,1]} \frac{M_1}{2\rho(t)} \int_t^1 \rho(s) \, ds \right\}, \]
then from (3.20) and (3.21), it follows that
\[ \|v'_{n+1}\| \leq \delta \|v_n\|_1^2. \tag{3.22} \]
Since \( \rho(t) > 0 \), it follows from (3.21) that
\[
v'_{n+1}(t) \leq \frac{M_1 \|v_n\|^2}{2\rho(t)} \int_0^1 \rho(s) \, ds.
\]
Integrating from 0 to \( t \), we have
\[
v_{n+1}(t) \leq v_{n+1}(0) + \frac{M_1}{2} \|v_n\|^2 \left(1/\rho(t) \int_0^1 \rho(r) \, dr\right) \, ds.
\]
The boundary conditions (3.13) \( (v_{n+1}(0) = \frac{q_0}{p_0} v'_{n+1}(0)) \) and (3.22), give
\[
v_{n+1}(0) \leq \frac{q_0}{p_0} \delta \|v_n\|^2.
\]
Thus,
\[
\|v_{n+1}\| \leq \eta \|v_n\|^2, \tag{3.23}
\]
where \( \eta = \left\{ \frac{q_0}{p_0} \delta + M_1 \int_0^1 [1/\rho(s) \int_0^1 \rho(r) \, dr] \, ds : t \in J \right\} \). Let \( \sigma = \max\{\eta, \delta\} \), then from (3.22) and (3.23), we obtain
\[
\|v_{n+1}\|_1 \leq \sigma \|v_n\|^2.
\]
**Case 2.** If \( w'_{n+1} > C \), then \( x' - q(w'_{n+1}) > x' - w'_{n+1} = v'_{n+1} \) and \( x' - q(w'_{n+1}) < 0 \). These together with (3.17) implies that
\[
-M_2 v'_{n+1} \geq f_{x'}(t, w_n, q(w_n))(x' - q(w_{n+1})) \geq M_2(x' - q(w_{n+1})) \geq M_2 v'_{n+1}, \tag{3.24}
\]
Using (3.24) in (3.16), we obtain
\[
-v''_{n+1}(t) \leq -M_2 v'_{n+1} + \frac{M_1}{2} \|v_n\|^2, t \in J,
\]
which implies that
\[
(v'_{n+1}(t) \rho_1(t))' \geq -\frac{M_1 \rho_1(t)}{2} \|v_n\|^2, t \in J, \tag{3.25}
\]
where \( \rho_1(t) = e^{-M_2 t} \). We note that (3.25) is the same as (3.19) with \( \rho_1 \) instead of \( \rho \). Hence following the same procedure as in Case 1, we can find \( \sigma_1 > 0 \) such that
\[
\|v_{n+1}\|_1 \leq \sigma_1 \|v_n\|^2.
\]
**Case 3.** If \( w'_{n+1} < -C \), then \( x' - q(w'_{n+1}) < x' - w'_{n+1} = v'_{n+1} \) and \( x' - q(w'_{n+1}) > 0 \). These together with (3.17) implies that
\[
-M_2 v'_{n+1} \leq f_{x'}(t, w_n, q(w_n))(x' - q(w_{n+1})) \leq M_2(x' - q(w_{n+1})) \leq M_2 v'_{n+1}. \tag{3.26}
\]
Using (3.26) in (3.16), we obtain
\[
-v''_{n+1}(t) \leq M_2 v'_{n+1} + \frac{M_1}{2} \|v_n\|^2, t \in J,
\]
which implies that
\[
(v'_{n+1}(t) \rho_2(t))' \geq -\frac{M_1 \rho_2(t)}{2} \|v_n\|^2, t \in J, \tag{3.27}
\]
where \( \rho_2(t) = e^{M_2 t} \). We note that (3.27) is the same as (3.19) with \( \rho_2 \) instead of \( \rho_1 \). Hence there exists \( \sigma_2 > 0 \) such that
\[
\|v_{n+1}\|_1 \leq \sigma_2 \|v_n\|^2. \quad \square
\]
4. Generalized quasilinearization technique

Now we introduce an auxiliary function \( \phi \) to allow a weaker hypothesis on \( f \).

**Theorem 4.1.** Assume that

(B1) \( \alpha \) and \( \beta \) are lower and upper solutions of (1.1), (1.2) such that \( \alpha \leq \beta \) on \( J \).

(B2) \( xf(t, x, 0) > 0 \) for \( |x| > k \), where \( k = \max \{|\alpha|, |\beta|\} \).

(B3) \( f \in C^2[J \times \mathbb{R} \times \mathbb{R}] \) satisfies a Nagumo condition on \( J \) relative to \((-k, k)\) and is such that \( f_x(t, x, x') \leq 0 \) and \( H(f + \phi) \geq 0 \) on \( J \times \mathbb{R}^2 \) for some function \( \phi \in C^2[J \times \mathbb{R} \times \mathbb{R}] \) which satisfies \( H(\phi) \geq 0 \) on \( J \times \mathbb{R}^2 \).

Then, there exists a monotone sequence \( \{w_n\} \) of solutions converging uniformly and quadratically to the unique solution of problem (1.1), (1.2).

**Proof.** Define \( F(t, x, x') = f(t, x, x') + \phi(t, x, x') \), then \( F \in C^2[J \times \mathbb{R}^2] \) and

\[
H(F) \geq 0 \text{ on } J \times \mathbb{R}^2. \tag{4.1}
\]

In view of (4.1) and Taylor’s theorem, we obtain

\[
f(t, x, q(x')) \geq F(t, y, q(y')) + F_x(t, y, q(y'))(x - y) + F_x(t, y, q(y'))(q(x') - q(y')) - \phi(t, x, q(x')) \quad t \in J \tag{4.2}
\]

where \( x, y, x', y' \in \mathbb{R} \). Define

\[
h(t, x, x'; y, y') = F(t, y, q(y')) + F_x(t, y, q(y'))(x - y) + F_x(t, y, q(y'))(q(x') - q(y')) - \phi(t, x, q(x')),
\]

then \( h \) is continuous and satisfies the relations

\[
\begin{cases}
  f(t, x, q(x')) \geq h(t, x, x'; y, y'), \\
  f(t, x, q(x')) = h(t, x, x'; x, x').
\end{cases} \tag{4.4}
\]

Using Taylor’s theorem on \( \phi \), we obtain

\[
\begin{align*}
\phi(t, x, q(x')) &= \phi(t, y, q(y')) + \phi_x(t, y, q(y'))(x - y) + \phi_x(t, y, q(y'))(q(x') - q(y')) + \frac{1}{2} H(\phi),
\end{align*}
\]

where

\[
H(\phi) = (x - y)^2 \phi_{xx}(t, c_1, c_2) + 2(x - y)(q(x') - q(y'))\phi_{xx'}(t, c_1, c_2) + (q(x') - q(y'))^2 \phi_{x'x'}(t, c_1, c_2),
\]

\( c_1 \) lies between \( x, y \) and \( c_2 \) lies between \( q(x'), q(y') \). Let

\[
M = \max\{|\phi_{xx}(t, c_1, c_2)|, |\phi_{xx'}(t, c_1, c_2)|, |\phi_{x'x'}(t, c_1, c_2)| : t \in J, c_1 \in [-k, k], c_2 \in [-C, C]\},
\]

then

\[
|H(\phi)| \leq M(|x - y| + |q(x') - q(y')|^2). \tag{4.5}
\]

In view of the assumption (B3) and (4.5), we have the following relations

\[
\phi(t, x, q(x')) \geq \phi(t, y, q(y')) + \phi_x(t, y, q(y'))(x - y) + \phi_x(t, y, q(y'))(q(x') - q(y')) \quad (4.6)
\]

for \( x, y, x', y' \in \mathbb{R} \), and

\[
\phi(t, x, q(x')) \geq \phi(t, y, q(y')) + \phi_x(t, y, q(y'))(x - y) + \phi_x(t, y, q(y'))(q(x') - q(y')) + M(|x - y| + |q(x') - q(y')|^2), \tag{4.7}
\]

for \( x, y, x', y' \in \mathbb{R} \), and

\[
\begin{align*}
\phi(t, x, q(x')) &\geq \phi(t, y, q(y')) + \phi_x(t, y, q(y'))(x - y) + \phi_x(t, y, q(y'))(q(x') - q(y')) + M(|x - y| + |q(x') - q(y')|^2),
\end{align*}
\]
for \( x, y \in [-k, k] \) and \( x', y' \in \mathbb{R} \). From (4.3) and (4.7), we have
\[
    h(t, x, x'; y, y') \geq f(t, y, q(y')) + f_x(t, y, q(y'))(x - y) + f_{x'}(t, y, q(y'))(q(x') - q(y'))
\]
\[
    - \frac{M}{2}(|x - y| + |q(x') - q(y')|)^2.
\]
(4.8)

for \( x, y \in [-k, k] \) and \( x', y' \in \mathbb{R} \). Define
\[
    h^*(t, x, x'; y, y') = f(t, y, q(y')) + f_x(t, y, q(y'))(x - y) + f_{x'}(t, y, q(y'))(q(x') - q(y'))
\]
\[
    - \frac{M}{2}(|x - y| + |q(x') - q(y')|)^2.
\]
(4.9)

Then,
\[
    \begin{cases}
        f(t, x, q(x')) \geq h^*(t, x, x'; y, y'), \\
        f(t, x, q(x')) = h^*(t, x, x'; x, x'),
    \end{cases}
\]
(4.10)

t \in J, \ x, y \in [-k, k] \) and \( x', y' \in \mathbb{R} \). Moreover, \( h^* \) is continuous and bounded on \( J \times [-k, k]^2 \times \mathbb{R} \) and therefore satisfies a Nagumo condition on \( J \). Hence there exists a constant \( C_2 > 0 \) (a Nagumo constant) such that any solution \( x \) of
\[
    -x''(t) = h^*(t, x, x'; y, y'), \quad t \in J,
\]
\[
    p_0x(0) - q_0x'(0) = a, \quad p_1x(1) + q_1x'(1) = b.
\]
with \(|x| \leq k\) satisfies \(|x'(t)| < C_2 \) on \( J \).

Now, we set \( w_0 = \alpha \) and consider the problem
\[
    -x''(t) = h^*(t, x, x'; w_0, w_0'), \quad t \in J,
\]
\[
    p_0x(0) - q_0x'(0) = a, \quad p_1x(1) + q_1x'(1) = b.
\]
(4.11)

Using (A1) and (4.10), we obtain
\[
    h^*(t, w_0(t), w_0'(t); w_0(t), w_0'(t)) = f(t, w_0(t), w_0'(t)) \geq -w_0''(t), \quad t \in J,
\]
\[
    h^*(t, \beta(t), \beta'(t); w_0(t), w_0'(t)) \leq f(t, \beta(t), \beta'(t)) \leq -\beta''(t),
\]
which imply that \( w_0 \) and \( \beta \) are lower and upper solutions of (4.11). Hence, by Theorem 2.3, there exists a solution \( w_1 \) of (4.11) such that \( w_0 \leq w_1 \leq \beta \) on \( J \). Using (4.10) and the fact that \( w_1 \) is a solution of (4.11), we obtain
\[
    -w_1''(t) = h^*(t, w_1(t), w_1'(t); w_0(t), w_0'(t)) \leq f(t, w_1(t), q(w_1'(t))), \quad t \in J
\]
(4.12)

which implies that \( w_1 \) is a lower solution of (3.3). Similarly, we can show that \( w_1 \) and \( \beta \) are lower and upper solutions of
\[
    -x''(t) = h^*(t, x, x'; w_1, w_1'), \quad t \in J
\]
\[
    p_0x(0) - q_0x'(0) = a, \quad p_1x(1) + q_1x'(1) = b.
\]
(4.13)

Hence by Theorem 2.3, there exists a solution \( w_2 \) of (4.12) such that \( w_1 \leq w_2 \leq \beta \) on \( J \).

Continuing this process we obtain a monotone sequence \( \{w_n\} \) of solutions satisfying
\[
    \alpha(t) = w_0 \leq w_1 \leq w_2 \leq w_3 \leq \cdots \leq w_{n-1} \leq w_n \leq \beta, \quad t \in J.
\]

That is,
\[
    \alpha(t) \leq w_n(t) \leq \beta(t) \quad \text{and} \quad |w_n'| < C_2, \quad n \in \mathbb{N}, \ t \in J
\]
(4.14)

where \( w_n \) is a solution of the problem
\[
    -x''(t) = h^*(t, x, x'; w_{n-1}, w_{n-1}'), \quad t \in J
\]
\[
    p_0x(0) - q_0x'(0) = a, \quad p_1x(1) + q_1x'(1) = b.
\]
Using the standard arguments as in Theorem 3.1, we can show that the sequence \( \{w_n\} \) of solutions converges to a solution \( x \) of (1.1), (1.2).
Now we show that the convergence of the sequence of solutions is quadratic. For this, set \( v_n(t) = x(t) - w_n(t) \), \( t \in J \), \( n \in \mathbb{N} \). We note that \( v_n \in C^2[J] \) and \( v_n(t) \geq 0 \), \( t \in J \). Using Taylor’s theorem and (4.6), we obtain

\[
-v''_{n+1}(t) = -x''(t) + w''_{n+1}(t) = (F(t, x, x') - \phi(t, x, x')) - h^*(t, w_{n+1}, w'_{n+1} ; w_n, w'_n)
\leq F(t, w_n, q(w'_n)) + F_\delta(t, w_n, q(w'_n))(x - w_n) + F_\delta(t, w_n, q(w'_n))(x' - q(w'_n))
+ \frac{1}{2} H(F) - [\phi(t, w_n, q(w'_n)) + \phi(t, w_n, q(w'_n))(x - w_n)
+ \phi(t, w_n, q(w'_n))(x' - q(w'_n)) - h^*(t, w_{n+1}, w'_{n+1} ; w_n, w'_n)]
= f(t, w_n, q(w'_n)) + f_\delta(t, w_n, q(w'_n))v_{n+1} + f_\delta(t, w_n, q(w'_n))(x' - q(w'_n))
+ \frac{1}{2} H(F) - h^*(t, w_{n+1}, w'_{n+1} ; w_n, w'_n),
\]

where

\[
H(F) = (x - w_n)^2 F_{xx}(t, \xi_1, \xi_2) + 2(x - w_n)(x' - q(w'_n)) F_{xx'}(t, \xi_1, \xi_2)
+ (x' - q(w'_n))^2 F_{x'x'}(t, \xi_1, \xi_2),
\]

\( w_n \leq \xi_1 \leq x \) and \( \xi_2 \) lies between \( x' \), \( q(w'_n) \). Let

\[
M_3 = \max \{|F_{xx}(t, \xi_1, \xi_2)|, |F_{xx'}(t, \xi_1, \xi_2)|, |F_{x'x'}(t, \xi_1, \xi_2)|
: t \in J, \xi_1 \in [w_n, x], \xi_2 \in [-C, C]\},
\]

then

\[
|H(F)| \leq M_3(|x - w_n| + |x' - q(w'_n)|)^2
\leq M_3(v_n^2 + |v'_n|^2) \leq M_3\|v_n\|^2_1.
\]

Using (4.16), the definition (4.9) of \( h^* \) and the fact that \( w_{n+1} - w_n \leq x - w_n \), we obtain

\[
-v''_{n+1}(t) \leq f_\delta(t, w_n, q(w'_n))v_{n+1} + f_\delta(t, w_n, q(w'_n))(x' - q(w'_n))
+ \frac{M_3}{2} \|v_n\|^2_1 + \frac{M}{2}(|v_n| + |q(w'_{n+1}) - q(w'_n)|)^2.
\]

We note that \( |x' - q(w'_n)| \leq |x' - w'_n| = |v'_n| \) and

\[
|q(w'_{n+1}) - q(w'_n)| \leq |x' - q(w'_n)| + |x' - q(w'_{n+1})| \leq |x' - w'_n| + |x' - w'_{n+1}| = |v'_n| + |v'_{n+1}|.
\]

These together with the assumption \( f_\delta \leq 0 \) implies that

\[
-v''_{n+1}(t) \leq f_\delta(t, w_n, q(w'_n))(x' - q(w'_n)) + \frac{M_3}{2} \|v_n\|^2_1 + \frac{M}{2}(|v'_n| + |v'_{n+1}|)^2.
\]

Again using the mean value theorem and (4.10), we have

\[
-v''_{n+1}(t) = f(t, x, x') - h^*(t, w_{n+1}, w'_{n+1} ; w_n, w'_n) \geq f(t, x, x') - f(t, w_{n+1}, q(w'_{n+1}))
= f_\delta(t, d_1, d_2)v_{n+1} + f_\delta(t, d_1, d_2)(x' - q(w'_{n+1})),
\]

where \( w_{n+1} \leq d_1 \leq x \) and \( d_2 \) lies between \( x' \), \( q(w'_{n+1}) \).

We discuss several cases.

**Case 4.** If \( |w'_{n+1}| \leq C \), then \( q(w'_{n+1}) = w'_{n+1} = \) and (4.18) implies that

\[
-v''_{n+1}(t) \geq -\lambda v_{n+1} + f_\delta(t, d_1, d_2)v'_{n+1}, \ t \in J,
\]

where \( \lambda = \max \{|f_\delta(t, d_1, d_2)| : t \in J, d_1 \in [w_0(t), \beta(t)], d_1 \in [-C, C]\} \geq 0 \). Let \( \rho_3(t) = \int_0^t f_\delta(s, d_1, d_2)ds \) denotes the integrating factor, then

\[
(v'_n(t) \rho_3(t))' \leq \lambda \rho_3(t) v_{n+1}(t) = \lambda \rho_3(t) v_n(t) \leq \lambda \rho_3(t) \|v_n\|.
\]
Integrating from 0 to \(t\), using the boundary conditions (3.13), we obtain
\[
v'_{n+1}(t) \leq \frac{1}{\rho_3(t)} \left( v'_{n+1}(0) + \lambda \int_0^t \rho_3(s) \, ds \| v_n(t) \| \right) \leq \frac{1}{\rho_3(t)} \left( \frac{p_0}{q_0} + \lambda \int_0^1 \rho_3(s) \, ds \right) \| v_n(t) \|. \tag{4.21}
\]
If we integrate from \(t\) to 1, using the boundary conditions (3.13), we obtain
\[
v'_{n+1}(t) \geq -\frac{1}{\rho_3(t)} \left( \rho_3(1) \frac{p_1}{q_1} + \lambda \int_0^1 \rho_3(s) \, ds \right) \| v_n \|. \tag{4.22}
\]
From (4.21) and (4.22), it follows that
\[
|v'_{n+1}(t)| \leq D \| v_n \| \leq D \| v_n \|_1, \tag{4.23}
\]
where \(D = \max \left\{ \frac{1}{\rho_3(t)} (\rho_3(1) \frac{p_1}{q_1} + \lambda \int_0^1 \rho_3(s) \, ds), \frac{1}{\rho_3(t)} (\frac{p_0}{q_0} + \lambda \int_0^1 \rho_3(s) \, ds) : t \in J \right\} \). Using (4.23) in (4.17), we obtain
\[
-v''_{n+1}(t) \leq f \xi'(t, w_n, q(w_n'))v'_{n+1} + R_1 \| v_n \|_1^2, \quad t \in J,
\]
which implies that
\[
(v'_{n+1}(t) \rho(t))' \geq -R_1 \rho(t) \| v_n \|_1^2, \quad t \in J \tag{4.24}
\]
where \(R_1 = \frac{M_1}{2} + \frac{M_2}{2}(1 + D)\) and \(\rho(t) = e^{\int_0^t f \xi'(s, w_n, q(w_n')) \, ds}\). Integrating (4.24) from 0 to \(t\), using the boundary conditions (3.14) \((v'_{n+1}(0) \geq 0)\), we obtain
\[
v'_{n+1}(t) \geq - \frac{R_1 \| v_n \|_1^2}{\rho(t)} \int_0^t \rho(s) \, ds. \tag{4.25}
\]
On the other hand, if we integrate (4.24) from \(t\) to 1, use (3.14) \((v'_{n+1}(1) \leq 0)\), we obtain
\[
v'_{n+1}(t) \leq \frac{R_1 \| v_n \|_1^2}{2 \rho(t)} \int_{[0,1]} \rho(s) \, ds. \tag{4.26}
\]
Let
\[
\delta = \max \left\{ \max_{[0,1]} \frac{R_1}{2 \rho(t)} \int_0^t \rho(s) \, ds, \max_{[0,1]} \frac{R_1}{2 \rho(t)} \int_0^1 \rho(s) \, ds \right\},
\]
then from (4.25) and (4.26), it follows that
\[
\| v'_{n+1} \| \leq \delta \| v_n \|_1^2. \tag{4.27}
\]
Again from (4.26), we have
\[
v'_{n+1}(t) \leq \frac{R_1 \| v_n \|_1^2}{2 \rho(t)} \int_0^1 \rho(s) \, ds.
\]
Integrating from 0 to \(t\), we have
\[
v_{n+1}(t) \leq v_{n+1}(0) + \frac{R_1}{2} \| v_n \|_1^2 \int_0^t \left( 1/\rho(s) \int_0^1 \rho(r) \, dr \right) \, ds.
\]
The boundary conditions (3.13) \((v_{n+1}(0) = \frac{q_0}{p_0} v'_{n+1}(0))\) and (4.27) gives
\[
v_{n+1}(0) \leq \frac{q_0}{p_0} \delta \| v_n \|_1^2.
\]
Thus,
\[
\| v_{n+1} \| \leq \eta \| v_n \|_1^2. \tag{4.28}
\]
where \( \eta = \max \left\{ \frac{q_0}{p_0} \delta + R_1 \int_0^1 \left[ 1 / \rho(s) \int_0^1 \rho(r) \, dr \right] ds : t \in J \right\} \). Let \( \sigma = \max \{ \eta, \delta \} \), then from (4.27) and (4.28), we obtain

\[
\| v_{n+1} \|_1 \leq \sigma \| v_n \|_1^2.
\]

**Case 5.** If \( v''_{n+1} > C \), then using (3.24) in (4.18), we obtain

\[
-v''_{n+1}(t) \geq -\lambda v_{n+1}(t) + M_2 v'_{n+1}(t), \quad t \in J,
\]

which implies that

\[
(v'_{n+1}(t) \rho_2(t))^\prime \leq \lambda \rho_2(t) v_{n+1}(t) \leq \lambda \rho_2(t) \| v_n \|.
\]

where \( \rho_2(t) = e^{M_2t} \). Integrating from 0 to \( t \), using the boundary conditions (3.13), we obtain

\[
v'_{n+1}(t) \leq e^{-M_2t} \left( \frac{p_0}{q_0} + \frac{\lambda}{M_2} (e^{M_2} - 1) \right) \| v_n \|.
\]

If we integrate (4.31) from \( t \) to 1, using the boundary conditions (3.13), we obtain

\[
v'_{n+1}(t) \geq -e^{-M_2t} \left( \frac{e^{M_2p_1}{q_1}}{q_1} + \frac{\lambda}{M_2} (e^{M_2} - 1) \right) \| v_n \|.
\]

From (4.31) and (4.32), it follows that

\[
|v'_{n+1}(t)| \leq D_1 \| v_n \| \leq D_1 \| v_n \|_1,
\]

where \( D_1 = \max \{ e^{-M_2t} \left( \frac{p_0}{q_0} + \frac{\lambda}{M_2} (e^{M_2} - 1) \right), e^{-M_2t} \left( \frac{e^{M_2p_1}{q_1}}{q_1} + \frac{\lambda}{M_2} (e^{M_2} - 1) \right) : t \in J \} \). Using (4.33) and (3.24) in (4.17), we obtain

\[
v''_{n+1}(t) \leq -M_2 v'_{n+1} + R_2 \| v_n \|_1^2, \quad t \in J
\]

which implies that

\[
(v'_{n+1}(t) \rho_2(t))' \geq -R_2 \rho_1(t) \| v_n \|_1^2, \quad t \in J
\]

where \( R_2 = \frac{M_1}{2} + \frac{M_2}{2} (1 + D_2)^2 \). Integrating (4.34) from 0 to \( t \), using the boundary conditions (3.14) \( (v'_{n+1}(0) \geq 0) \), we obtain

\[
v'_{n+1}(t) \geq -\frac{R_2}{M_2} (e^{M_2t} - 1) \| v_n \|_1^2.
\]

On the other hand, if we integrate (4.34) from \( t \) to 1, using (3.14) \( (v'_{n+1}(1) \leq 0) \), we obtain

\[
v'_{n+1}(t) \leq \frac{R_2}{M_2} (1 - e^{-M_2(1-t)}) \| v_n \|_1^2,
\]

\[
v'_{n+1}(0) \leq \frac{R_2}{M_2} (1 - e^{-M_2}) \| v_n \|_1^2.
\]

From (4.35) and (4.36), it follows that

\[
\| v'_{n+1} \| \leq \delta \| v_n \|_1^2,
\]

where

\[
\delta = \max \left\{ \frac{R_2}{M_2} (e^{M_2t} - 1), \frac{R_2}{M_2} (1 - e^{-M_2(1-t)}) : t \in I \right\}.
\]

Integrating (4.36) from 0 to \( t \), using the boundary conditions \( v_{n+1}(0) = \frac{q_0}{p_0} v'_{n+1}(0) \) and (4.37), we have

\[
v_{n+1}(t) \leq E \| v_n \|_1^2.
\]
where $E = \max\left\{ \frac{q_0}{p_0} \left( \frac{R_2}{M_2} (1 - e^{-M_2}) \right) + \frac{R_2}{M_2} \left( t + \frac{e^{-M_2}}{M_2} - \frac{e^{-M_2(1-t)}}{M_2} \right) : t \in I \right\}$. Let $\sigma_3 = \max\{\delta, E\}$, then from (4.38) and (4.39), we obtain

$$\|v_{n+1}\|_1 \leq \sigma_3 \|v_n\|_1^2.$$  

Similarly, we can show that if $w_{n+1}' < -C$, then there exists $\sigma_4 > 0$, such that

$$\|v_{n+1}\|_1 \leq \sigma_4 \|v_n\|_1^2. \quad \Box$$

**References**