Comparison between Adomian Decomposition Method and Variational Iteration Method with Exact Solution for Solving Volterra Integral Equations of the Second Kind

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Abstract

In this paper, we used the Adomian Decomposition Method and Variational Iteration Method for solving a generalized to construct the numerical solutions. The solutions in the form of a rapidly convergent power series with easily computable components, the accuracy of the purpose numerical scheme is examined by comparison with other analytical and numerical results.

Keywords:
- Adomian Decomposition Method (ADM)
- Variational Iteration Method (VIM)
- Generalized the methods
- Numerical solutions
- Exact solutions.

Introduction:

In this paper, we implemented Adomian Decomposition Method and Variational Iteration Method for finding the exact and approximate solutions of a generalized Volterra Integral Equations of the Second Kind.

Volterra Integral Equations of the Second Kind

We will first study Volterra integral equations of the second kind given by

\[ u(x) = f(x) + \lambda \int_{0}^{x} k(x, t) \cdot u(t) \, dt \quad \text{equation (1)} \]

The unknown function \( u(x) \), that will be determined, occurs inside and outside the integral sign. The kernel \( k(x, t) \) and the function \( f(x) \) are given real-valued functions, and \( \lambda \) is a parameter. In what follows we will present the methods, new and traditional, that will be used.

The Adomian Decomposition Method

The Adomian Decomposition Method (ADM) was introduced and developed by George Adomian in [5–7] and is well addressed in many references. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral equations as well.

The Adomian Decomposition Method consists of decomposing the unknown function \( u(x) \) of any equation into a sum of an infinite number of components defined by the decomposition series.
\[ u(x) = \sum_{n=0}^{\infty} u_n(x) \]  \hspace{1cm} (2)

or equivalently

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \ldots \]  \hspace{1cm} (3)

where the components
\[ u_n(x), \ n \geq 0 \] are to be determined in a recursive manner.

The decomposition method concerns itself with finding the components

We substitute the Volterra Integral Equations to obtain

\[ \sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t). \sum_{n=0}^{\infty} u_n(t) dt \]  \hspace{1cm} (4)

or equivalently

\[ u_0(x) + u_1(x) + u_2(x) + \ldots = f(x) + \lambda \int_0^x k(x,t). [u_0(t) + u_1(t) + u_2(t) + \ldots ] dt \]  \hspace{1cm} (5)

The zeroth component \( u_0(x) \) is identified by all terms that are not included under the integral sign. Consequently, the components \( u_n(x), \ n \geq 0 \) of the unknown function \( u(x) \) are completely determined by setting the recurrence relation:

\[ u_0(x) = f(x) \]
\[ u_n+1(x) = \lambda \int_0^x k(x,t). u_n(t) dt \quad n \geq 0 \]  \hspace{1cm} (6)

That is equivalent to

\[ u_0(x) = f(x) \]
\[ u_1(x) = \lambda \int_0^x k(x,t). u_0(t) dt \]
\[ u_2(x) = \lambda \int_0^x k(x,t). u_1(t) dt \]
\[ u_3(x) = \lambda \int_0^x k(x,t). u_2(t) dt \]
\[ \vdots \]

\[ u_{n+1}(x) = \lambda \int_0^x k(x,t). u_n(t) dt \quad n \geq 0 \]  \hspace{1cm} (7)

and so on for other components.

The components \( u_0(x), \ u_1(x), \ u_2(x) + \ldots \) are completely determined. As a result, the solution \( u(x) \) of the Volterra integral equation (1) in a series form is readily obtained by using the series assumption in (2).

It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series.

However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical
purposes.
The more components we use the higher accuracy we obtain.

The Variational Iteration Method

The Variational Iteration Method that proved to be effective and reliable for analytic and numerical purposes.
The Variational Iteration Method (VIM) established by Ji-Huan He [11–12] is now used to handle a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in Adomian Decomposition Method (ADM). The Variational Iteration Method (VIM) handles linear and nonlinear problems in the same Manner without any need to specific restrictions such as the so called Adomian polynomials that we need for nonlinear problems. Moreover, the method gives the solution in a series form that converges to the closed form solution if an exact solution exists. The obtained series can be employed for numerical purposes if exact solution is not obtainable. In what follows, we present the main steps of the method.
Consider the differential equation:

\[ Lu + Nu = g(t) \]  

\[ \text{(9)} \]

Where L and N are linear and nonlinear operators respectively, and 
g(t) is the source inhomogeneous term.
The variational iteration method presents a correction functional for equation (3.86) in the form:

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \left( Lu_n(\xi) + Nu_n(\xi) - g(\xi) \right) d\xi \]

\[ \text{(10)} \]

Where \( \lambda \) is a general Lagrange's multiplier, noting that in this method \( \lambda \) may be a constant or a function, and \( \Delta_n^\lambda \) is a restricted value means it behaves as a constant, hence \( \delta \Delta_n^\lambda = 0 \), where \( \delta \) is the variational derivative.
The Lagrange multiplier \( \lambda \) can be identified optimally via the Variational theory as will be seen later.
For a complete use of the variational iteration method, we should Follow two steps, namely:
1. the determination of the Lagrange multiplier \( \lambda(\xi) \) that will be identified optimally, and
2. with \( \lambda \) determined, we substitute the result into (3.87) where the restrictions should be omitted.
Taking the variation of (3.87) with respect to the independent Variable \( u_n(x) \) we find

\[ \frac{\delta u_{n+1}(x)}{\delta u_n(x)} = 1 + \frac{\delta}{\delta u_n(x)} \left( \int_0^x \lambda(\xi) \left( Lu_n(\xi) + Nu_n(\xi) - g(\xi) \right) d\xi \right) \]

\[ \text{....(11)} \]

or equivalently
Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\xi)$. In other words we can use

\[ \int_0^x \lambda(\xi) \left( u''(\xi) \right) d\xi = \lambda(\xi).u_n(\xi) - \int_0^x \lambda'\xi(\xi) u_n(\xi)d\xi \]  

\[ \int_0^x \lambda(\xi) \left( u'''(\xi) \right) d\xi = \lambda(\xi).u_n(\xi) - \lambda(\xi).u_n'(\xi) + \lambda''(\xi).u_n(\xi) - \int_0^x \lambda''(\xi) u_n(\xi)d\xi \]  

and so on. These identities are obtained by integrating by parts.

For example, if $L u_n(\xi) = u''(\xi)$ in eq: (12), then eq: (12) becomes

\[ \delta u_{n+1}(x) = \delta u_n(x) + \delta \left( \int_0^x \lambda(\xi) \left( L u_n(\xi) \right) d\xi \right) \]  

Integrating the integral of (12) by parts using (13) we obtain

\[ \delta u_{n+1} = \delta u_n + \delta \lambda(\xi).u_n(\xi) - \int_0^x \lambda'(\xi) u_n(\xi)d\xi \]  

as a second example, if $L u_n(\xi) = u''(\xi)$ in eq: (12), then eq: (12) becomes

\[ \delta u_{n+1} = \delta u_n + \delta (\lambda(\xi).u_n(\xi))' - \delta (\lambda(\xi).u_n(\xi)) + \int_0^x \lambda''(\xi) u_n(\xi)d\xi \]  

or equivalently

\[ \delta u_{n+1} = \delta u_n + \delta \lambda(\xi).u_n(\xi) - \int_0^x \lambda'(\xi) u_n(\xi)d\xi \]  

The extremum condition of $u_{n+1}$ requires that $u_{n+1} = 0$. This means that the left hand side of (18) is zero, and as a result the right hand side should be 0 as well. This yields the stationary conditions:

\[ 1 + \lambda|_{\xi=x} = 0, \quad \lambda'|_{\xi=x} = 0 \]  

This in turn gives

\[ \lambda = -1 \]  

The extremum condition of $u_{n+1}$ requires that $u_{n+1} = 0$. This means that the left hand side of (22) is zero, and as a result the right hand side should be 0 as well. This yields the stationary conditions:

\[ 1 - \lambda'|_{\xi=x} = 0, \quad \lambda |_{\xi=x} = 0, \quad \lambda''|_{\xi=x} = 0 \quad \lambda = -1 \]
This in turn gives
\[ \lambda = \xi - x \] ....(24)

As a second example, if

\[ L u_n(\xi) = u'''(\xi) \] in eq: (12), then eq: (12) becomes

\[ \delta u_{n+1}(x) = \delta u_n(x) + \delta \left[ \int_0^x \lambda(\xi) \left( L u_n(\xi) \right) d\xi \right] \] ....(25)

Integrating the integral of (25) by parts using (15) we obtain

\[ \delta u_{n+1} = \delta u_n + \delta \left( \lambda(\xi) u''(\xi) \right)_0^x - \delta \left( \lambda'(\xi) u'(\xi) \right)_0^x + \delta \left( \lambda''(\xi) u_n(\xi) \right)_0^x \]

\[ + \int_0^x \lambda'''(\xi) u_n(\xi) d\xi \] ....(26)

or equivalently

\[ \delta u_{n+1}(x) = \delta u_n(x) \left( 1 + \lambda''|_{\xi=x} \right) - \delta \lambda'(u_n)'|_{\xi=x} + \delta \lambda(u_n)'|_{\xi=x} \]

\[ - \int_0^x \lambda'''(\xi) u_n(\xi) d\xi \] ....(27)

The extremum condition of \( u_{n+1} \) requires that \( u_{n+1} = 0 \). This means

That the left hand side of (22) is zero, and as a result the right hand side should be 0 as well. This yields the stationary conditions:

\[ 1 + \lambda''|_{\xi=x} = 0, \quad - \lambda'|_{\xi=x} = 0, \quad \lambda|_{\xi=x} = 0 \] ....(28)

This in turn gives

\[ \lambda = -\frac{(\xi-x)^2}{2} \] ....(29)

Having determined the Lagrange multiplier \( \lambda(\xi) \), the successive approximations \( u_{n+1}, \quad n \geq 0 \), of the solution \( u(x) \) will be readily obtained upon using selective function \( u_0(x) \). However, for fast convergence, the function \( u_0(x) \) should be selected by using the initial conditions as follows:

\[ u_0(x) = u(0), \] for first order \( u'_n \)

\[ u_0(x) = u(0) + xu'(0), \] for second order \( u''_n \)

\[ u_0(x) = u(0) + xu'(0) + \frac{x^2}{2} u''(0), \] for third order \( u'''_n \) ....(30)

and so on. Consequently, the solution

\[ u(x) = \lim_{n \to \infty} u_n(x) \]

In other words, the correction functional (10) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

The determination of the Lagrange multiplier plays a major role in the determination of the solution of the problem. In what follows, we summarize some iteration formulae that show ODE, its corresponding Lagrange multipliers, and its correction functional respectively:

(i)

\[ \begin{align*}
\{ u' + f(u(\xi),u''(\xi)) \} &= 0, \quad \lambda = -1, \\
u_{n+1}(x) &= u_n - \int_0^x [u'_n + f(u_n,u'_n)] d\xi
\end{align*} \]
To use the variational iteration method for solving Volterra integral equations, it is necessary to convert the integral equation to an equivalent initial value problem or to an equivalent integro-differential equation. As defined before, integro-differential equation is an equation that contains differential and integral operators in the same equation. The integro-differential equations will be studied in details in Chapter 5. The conversion process is presented in Section 2.5.1. However, for comparison reasons, we will examine the obtained initial value problem by two methods, namely, standard methods used for solving ODEs, and by using the variational iteration method as will be seen by the following examples.

Example 1
Solve the following Volterra Integro-Differential Equations

\[ u(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{2} \int_0^x (x - t) u(t) \, dt = 0 \]
\[ u(0) = 1 \quad , \quad u'(0) = 1 \quad \text{by} \]

The Adomian Decomposition Method

\[ u(x) = \sum_{n=0}^{\infty} u_n(x) \] \[ \text{.........(33)} \]

or equivalently

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \ldots \] \[ \text{.........(34)} \]

where the components
\[ u_n(x) \quad , \quad n \geq 0 \] are to be determined in a recursive manner.

The decomposition method concerns itself with finding the components

We substitute the Volterra Integral Equations to obtain

\[ \sum_{n=0}^{\infty} u_n(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) \ldots \] \[ \text{.........(35)} \]

\[ \sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t) \sum_{n=0}^{\infty} u_n(t) \, dt \] \[ \text{.........(36)} \]

\[ \sum_{n=0}^{\infty} u_n(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{2} \int_0^x (x - t) \sum_{n=0}^{\infty} u_n(t) \, dt \] \[ \text{.........(37)} \]

\[ f(x) = 1 + x + \frac{1}{2} x^2 \]
\( u_0 = f'(x) \) so
\( u_0(x) = 1 + x + \frac{1}{2} x^2 \) ....(38)

\( u_{n+1} = \frac{1}{2} \int_0^x (x - t) \sum_{n=0}^{\infty} u_n(t) dt \) ....(39)

\( u_1(x) = \frac{1}{2} \int_0^x (x - t) u_0(t) dt \)

\( u_1(x) = \frac{1}{2} \int_0^x (x - t)(1 + t + \frac{1}{2} t^2) dt \)

\( u_1 = \frac{x^3}{4} + \frac{x^5}{4} + \frac{x^4}{8} \) ....(40)

\( u_2(x) = \frac{1}{2} \int_0^x (x - t) u_1(t) dt \)

\( u_2(x) = \frac{1}{2} \int_0^x (x - t) \left( \frac{t^2}{4} + \frac{t^3}{4} + \frac{t^4}{8} \right) dt \)

\( u_2(x) = \frac{x^4}{16} + \frac{x^5}{16} + \frac{x^6}{96} \) ....(41)

\( u_3(x) = \frac{1}{2} \int_0^x (x - t) u_2(t) dt \)

\( u_3(x) = \frac{1}{2} \int_0^x (x - t) \left( \frac{t^3}{16} + \frac{t^4}{16} + \frac{t^5}{32} \right) dt \)

\( u_3(x) = \frac{x^5}{64} + \frac{x^6}{64} + \frac{x^7}{128} \) ....(42)

\( u_4(x) = \frac{1}{2} \int_0^x (x - t) u_3(t) dt \)

\( u_4(x) = \frac{1}{2} \int_0^x (x - t) \left( \frac{t^4}{64} + \frac{t^5}{64} + \frac{t^6}{128} \right) dt \)

\( u_4(x) = \frac{x^6}{256} + \frac{x^7}{256} + \frac{x^8}{512} \) ....(43)

And so on.

or equivalently

\( u(x) = \sum_{n=0}^{\infty} u_n(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \ldots \) ....(44)

\( u(x) = 1 + x + \frac{3x^2}{4} + \frac{x^3}{4} + \frac{3x^4}{16} + \frac{x^5}{16} + \frac{3x^6}{64} + \frac{x^7}{64} + \frac{3x^8}{256} + \frac{x^9}{256} + \frac{x^{10}}{512} + \ldots \) ....(45)

**Graph of (ADM) method in 2D**

![Graph of (ADM) method in 2D](image)

**Graph of (ADM) method in 3D**

![Graph of (ADM) method in 3D](image)
Solve the following Volterra Integro-Differential Equations

\[ u(x) = 1 + x + \frac{1}{4} x^2 + \frac{1}{2} \int_0^x (x - t)u(t) \, dt, \quad u(0) = 1, \quad u'(0) = 1 \]

By

The Variational Iteration Method (VIM)

\[ u(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{2} \int_0^x (x - t)u(t) \, dt, \quad u(0) = 1 \]
\[ u'(x) = 1 + x + \frac{1}{2} \int_0^x (x - t)u(t) \, dt, \quad u'(0) = 1 \]
\[ u''(x) = 1 + \frac{1}{2} u(x), \quad u''(0) - \frac{1}{2} u(x) - 1 = 0, \quad \ldots \ldots \ldots (46) \]

\[ Lu + Nu = g(t) \quad \ldots \ldots \ldots (47) \]

Where \( L \) and \( N \) are linear and nonlinear operators respectively, and \( g(t) \) is the source inhomogeneous term.

The variational iteration method presents a correction functional for equation (3.86) in the form:

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \left( Lu_n(\xi) + Nu_n(\xi) - g(\xi) \right) d\xi \quad \ldots \ldots (48) \]
\[ u_{n+1}(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{2} \int_0^x \lambda(\xi) \left( u''(\xi) - \frac{1}{2} u(\xi) - 1 \right) d\xi \quad \ldots \ldots (49) \]

Where \( L = u''(\xi) \) second order derivative so

\[ \lambda(\xi) = (\xi - x) \]
\[ u_0(x) = 1 + x + \frac{1}{2} x^2 \quad \ldots \ldots \ldots (50) \]
\[ u_1(x) = u_0(x) + \frac{1}{2} \int_0^x (\xi - x) \left( u''_0(\xi) - \frac{1}{2} u_0(\xi) - 1 \right) d\xi \]
\[ u_1(x) = 1 + x + \frac{3}{2} x^2 + \frac{1}{2} \int_0^x (\xi - x) \left( \frac{3}{4} \xi^2 u_0(\xi) - \frac{1}{2} u_0(\xi) - 1 \right) d\xi \]
\[ u_1(x) = 1 + x + \frac{3}{4} x^2 + \frac{3}{4} \int_0^x (1 + \xi + \frac{5}{2} x^2) - \frac{1}{2} \left( 1 + \xi + \frac{1}{2} \xi^2 \right) - 1 \right) d\xi \]
\[ u_1(x) = 1 + x + \frac{3x^2}{4} + \frac{x^3}{4} + \frac{x^4}{3} \quad \ldots \ldots \ldots (51) \]
\[ u_2(x) = u_1(x) + \frac{1}{2} \int_0^x (\xi - x) \left( \frac{d^2}{d\xi^2} u_1(\xi) - \frac{1}{2} u_1(\xi) - 1 \right) d\xi \]
\[ u_2(x) = 1 + x + \frac{3x^5}{4} + \frac{x^6}{8} + \frac{1}{2} \int_0^x (\xi - x) \left( \frac{d^2}{d\xi^2} u_1(\xi) - \frac{1}{2} u_1(\xi) - 1 \right) d\xi \]
\[ u_2(x) = 1 + x + \frac{3x^5}{4} + \frac{x^6}{8} + \frac{1}{2} \int_0^x (\xi - x) \left( \frac{d^2}{d\xi^2} (1 + \xi + \frac{x^2}{4} + \frac{x^3}{4} + \frac{x^4}{8}) - \frac{1}{2} (1 + \xi + \frac{3x^2}{4} + \frac{x^3}{4} + \frac{x^4}{8}) - 1 \right) d\xi \]
\[ u_2(x) = 1 + x + x^2 - \frac{x^5}{4} - \frac{x^6}{16} + \frac{3x^5}{16} + \frac{x^6}{32} + \frac{1}{2} \int_0^x (\xi - x) \left( \frac{d^2}{d\xi^2} u_2(\xi) - \frac{1}{2} u_2(\xi) - 1 \right) d\xi \]
\[ u_3(x) = 1 + x + x^2 - \frac{x^5}{4} - \frac{x^6}{16} + \frac{3x^5}{16} + \frac{x^6}{32} + \frac{1}{2} \int_0^x (\xi - x) \left( \frac{d^2}{d\xi^2} u_2(\xi) - \frac{1}{2} u_2(\xi) - 1 \right) d\xi \]
\[ u_4(x) = 1 + x + \frac{5x^7}{4} - x^3 + \frac{3x^5}{8} + \frac{31x^4}{8} - \frac{9x^3}{64} - \frac{9x^7}{64} + \frac{x^2}{128} + \frac{1}{2} \int_0^x (\xi - x) \left( \frac{d^2}{d\xi^2} u_3(\xi) - \frac{1}{2} u_3(\xi) - 1 \right) d\xi \]
\[ u_{n+1}(x) = u_4 = 1 + x + \frac{3x^7}{2} - \frac{2x^3}{3} + \frac{19x^5}{8} + \frac{3x^6}{16} + \frac{15x^7}{8} + \frac{12x^2}{256} + \frac{15x^3}{256} - \frac{17x^9}{512} + \frac{x^7}{512} \]

Graph of (VIM) method in 2D
Graph of (VIM) method in 3D

Graph of Exact Solution in 3D

Graph of Exact Solution in 2D

\[ u(x) = \frac{1}{2} e^{\frac{x}{\sqrt{2}}} (3 - \sqrt{2} + 4e^{\frac{x}{\sqrt{2}}} + 3e^{\sqrt{2}x} + \sqrt{2}e^{\sqrt{2}x}) \]

Comparison of (ADM) and (VIM) and Exact Solution
### Values From \( x = -0.1 \) \( t_0 = 0.1 \)

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### 4. Conclusions

In this paper we have studied the variational iteration method (VIM) and Adomian Decomposition Method (ADM) generalize and Solve the problem of Volterra Integral Equation with both methods, which does not require small parameter in an equation as the perturbation techniques do. The results show that

1. A correction functional can be easily constructed by a general Lagrange multiplier, and the multiplier can be optimally identified by variational theory. The application of restricted variations in correction functional makes it much easier to determine the multiplier.
2. The initial approximation can be freely selected with unknown constants, which can be determined via various methods.
3. The approximations obtained by this method are valid not only for small parameter, but also for very large parameter, furthermore their first-order approximations are of extreme accuracy.
4. Comparison of Adomian’s and Variational’s with Exact Solution reveals that the approximations obtained by the proposed method converge to its exact solution faster than those of Variational’s.

### References:


