NONLINEAR HYPERBOLIC CONSERVATION LAWS

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SYNOPSIS

We are concerned with nonlinear conservation laws and we are aiming for a realistic multi-space dimensional framework. The main issue we are interested in is the following “when can we use hyperbolic (simplified) models?” and to answer it, we study zero diffusion-dispersion limits. While our proofs establish integrity, the major emphasis is on reliability and failure. In particular, we look to all known physically relevant solutions, both classical and nonclassical. The techniques we use depend upon an analytical setting on measure-valued function theory and are energy based methods.

INTRODUCTION

Nonlinear hyperbolic conservation laws arise in the modeling of many problems from continuum mechanics, physics, chemistry, biology, environment, etc., in such diverse areas as gas dynamics, nonlinear elasticity, shallow water theory, geometric optics, magneto-fluid dynamics, kinetic theory, chromatography, combustion theory, chemostasis, petroleum engineering, traffic flow, …

If most of the applications are modeled by vector equations, for which analytical theory is essentially open, even there, arguing with applied perturbation theory, scalar equations provide quite often quantitative special solutions for these complex systems: scalar equations are not merely “theoretical laboratory” equations. When small scale mechanisms of diffusion or dispersion are taken into account, heat conduction or capillarity in fluids, the equations become “parabolic”. From a general standpoint, solutions of hyperbolic equations are discontinuous (have “shocks”), while parabolic equations have smooth solutions. But, while diffusion has a dissipative effect, dispersion has an oscillatory one.

For example, if we are concerned with optimal design and active control in structures, wind towers, the Thames barrier, bridges, airplanes or the orbital spatial station, shocks and oscillations are fundamental issues. Let us mention for airplane optimal shape design the importance of the shocks strength magnitude at the transonic regime and the control of vibrations (or the consequences for the rotor blades of helicopters…). And, as E. Zuazua recently pointed out, for the engineer working in the wind tunnel, these design and control problems are much harder as they become inverse problems.

We go back to Poincaré’s ideas: physical processes are described by natural balance laws, then the ability to formulate the proper constitutive state laws appearing in these balance laws
is the main task with which we must be confronted. In practice we are, all the time, also concerned with simplifying approximation procedures. The relationship between these issues is strongly sensitive and imbricate, giving a vigorous emphasis to the questions “when can we replace the given balance laws by simpler (hyperbolic) models?” and the umbilical one “what about accurate realistic models for the nonlinear functions describing the occurring dissipations and dispersions in these balance laws?”, which we will motivate below and, we claim, urge the attention of the scientific community.

To be specific, we consider here two such balance laws. Regarding the first question, we bypass the usual continuum mechanics deduction of balance laws or their divergence-form (e.g., Landau-Lifschitz, 1971), leading to the (hyperbolic) conservation laws. For the sake of simplicity, we do not consider source terms.

We consider the, multi-space dimensional, generalized Benjamin-Bona-Mahony-Burgers (BBMB) equation

\[ \partial_t u + \text{div} f(u) = \epsilon \text{div} b(\nabla u) + \delta \text{div} \partial_t c(\nabla u), \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, \]

and the generalized Korteweg-de Vries-Burgers (KdVB) equation obtained from the previous one by changing the right-hand side time-derivative with a derivative in one of the space variables.

Then for both equations, the associated nonlinear hyperbolic conservation law obtained as the formal \( \epsilon, \delta \to 0 \) limit is

\[ \partial_t u + \text{div} f(u) = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+. \]

Hyperbolic equations, as is well known, usually develop discontinuities (“shocks”) in finite time. Thus, we must consider new notions of global-time solutions, weak-solutions, which are not unique. This lack of uniqueness and shock handling is the crucial theoretical challenge: “how can we select the physically relevant solution?” see, e.g., (Dafermos, 2005) or (Serre, 1999).

To answer this question we study the convergence (compactness) of \( \epsilon, \delta \)-solutions of the BBMB and KdVB equations, as the \( \epsilon, \delta \)-parameters tend to zero. This is a singular limit: we can obtain different limits (the so called classical and nonclassical weak-solutions), or have no limit at all, according to the balance of strengths of \( \epsilon \)-diffusion and \( \delta \)-dispersion, and also on the \( b \)-diffusion and \( c \)-dispersion constitutive laws.

So we are walking in circles and the way to deal with it is to fix, in each application, the pertinent form of the dissipation and dispersion functions: if we do not have convergence (failure), then (1.2) fails to replace (1.1); when we have convergence (reliability), we must then characterize which of (1.2) solutions we obtain. We will explain below a little more. Meanwhile observe that a numerical approach is a priori of no help, e.g., “what does apparent numerical divergence mean if we can have many solutions?” Otherwise, if a dissipative device, like the nonlinear diffusion term in (1.1), provides a good approximation to (1.2), then knowing the qualitative behavior of solutions is welcome. As dissipative devices regularize equations, there numerics must be successful. Hsiao (Hsiao, 1997) considers some of such pure dissipative mechanisms (viscosity, heat diffusion, frictional damping, relaxation) and reports that the qualitative behavior of solutions for damping and nonlinear diffusion are related—we need to find accurate models for these nonlinear functions in each application.
In some applications in solid mechanics (material science) or magnetohydrodynamics (Hall effect) the pure dissipative limits do not select the physically relevant solutions. Thus, we must also consider the dispersive mechanisms in order to capture these nonclassical entropy solutions, see (LeFloch, 2002) and references therein—until recently people looked at 2nd order derivative terms as dissipative and 3rd order terms as dispersive, Brenier-Levy (Brenier-Levy, 2000) noticed that 3rd order terms, depending on their form, can also be dissipative (still maintaining the oscillation profile). So we need to be even more careful in their modelization.

We note that the uncomfortable need of different kinds of entropy conditions is quite natural from the mathematical point of view: while discontinuities in data are solved by a transmission principle implied by the hyperbolic equations, the so called Rankine-Hugoniot conditions, it remains to know more information on data rates of change. In the dissipative case it is solely a qualitative, sign, information (the entropy inequality, which select the good discontinuities, called ‘shocks’), in the balanced diffusion-dispersion case more information seems to be needed (new entropy conditions).

An overview:

(Failure) when $\varepsilon = 0$: (1.1) become a generalized version of the Korteweg-de Vries (KdV) and of the Benjamin-Bona-Mahony (BBM) equations. If dispersion is modeled by a linear function, we expect both equations behave similarly to the KdV equation, the solutions become more and more oscillatory as $\delta$ tend to zero and the approximate solutions do not converge (Lax-Levermore, 1983). The use of the nonlinear hyperbolic conservation law (1.2) to make previsions about the pure-dispersive regime is hopeless.

(Integrity) classical weak-solutions, $\delta = 0$: it is the pure dissipative case like for the Burgers equation (Burgers, 1948; Hopf, 1950) or for the pseudo-viscosity approximation of von Neumann and Richtmyer (Neumann-Richtmyer, 1950). We take the limit as the single parameter $\varepsilon$ tend to zero, the so called “vanishing viscosity method”. We have convergence: for the linear diffusion (Kruzhkov, 1970), for the pseudo-viscosity (Marcati-Natalini, 1994), for the case of viscosity and damping, which is a lower order dissipation see (Hsiao, 1997), for “general” diffusion (Correia-LeFloch, 1998) or also as a sub product of the present work. In fact the physical limit solutions of (1.2) become directly characterized by some (entropy) inequalities (Serre, 1999), see Appendix B. Such weak-solutions are called (classical) entropy solutions. This is reminiscent of gas dynamics where the entropy is given by the 2nd law of thermodynamics.

(Reliability) in order to ensure ‘integrity’ (convergence of the zero diffusion-dispersion approximation to the classical entropy solutions) a dominant diffusion regime will be natural. Henceforth, ‘reliability’ given by these diffusive-dispersive approximations, instead of the “classical” solely diffusive approximations, is no more the same: we can now reach nonclassical (new) solutions, relevant in models of solid mechanics or magnetohydrodynamics. In particular, the ‘integrity’ domain must be larger, see LeFloch (LeFloch, 2002). We assure here ‘integrity’ by two main ways: one concerns $\delta-\varepsilon$ balance, the other concerns the growth competition between the diffusion and the dispersion. The approach (started in Correia-LeFloch, 1998) is based on energy methods: a priori, careful analysis of higher order Lebesgue measures must be carried out.

Finally, we note that in the case of the BBMB model also the flux growth is a variable and the ‘integrity’ frontier domain must be differently located for BBMB and KdVB equations. This puts in evidence major differences between both models. In particular, such unequal
convergence behavior shows that Whitham's change between time and space derivatives (Whitham, 1974) must be of non-trivial use in nonlinear models.

**Remark** Correia-LeFloch (Correia-LeFloch, 1999) and Brenier-Levy (Brenier-Levy, 2000) regard some KdV-type equations where the 3rd order term depend on functions of the Hessian matrix of the second order partial derivatives, such models are out of consideration here, but the striking fact in (Brenier-Levy, 2000) is the numerical evidence of the dissipative behaviour, still maintaining the oscillation profile, of the dispersive equations. At last, we have in conclusion a work which generalize the results here, in even less restrictive assumptions on the KdVB equation

$$
\partial_t u + \text{div} f(u) = \text{div} \left( \varepsilon b_j(u, \nabla u) + \delta \sum_k \partial_{x_k} c_{j,k}(\nabla u) \right)_{\leq j \leq d}.
$$

**ASSUMPTIONS**

After the vicious circle we have described before in the introduction, our strategy here is to look at the integrity region (existence of the classical entropy solution) of the hyperbolic initial value problem

(2.1a) \hspace{1cm} \partial_t u + \text{div} f(u) = 0, \hspace{1cm} (x,t) \in \mathbb{R}^d \times [0, +\infty[,

(2.1b) \hspace{1cm} u(x,0) = u_0(x), \hspace{1cm} x \in \mathbb{R}^d,

performed as the \(\varepsilon, \delta\) zero-limits of the \(u^{\varepsilon, \delta} : \mathbb{R}^d \times [0,T] \to \mathbb{R}\), defined on an uniform (independent of \(\varepsilon, \delta\)) interval of time \([0,T]\), smooth solutions to one of the initial value approximate problems for the KdVB or the BBMB equations (according the partial derivative \(\partial_x\) is, respectively, in one space variable \(\partial_{x_k}\) or in time \(\partial_t\))

(2.2a) \hspace{1cm} \partial_t u + \text{div} f(u) = \varepsilon \text{div} b(u, \nabla u) + \delta \text{div} \partial_x c(\nabla u), \hspace{1cm} (x,t) \in \mathbb{R}^d \times [0, +\infty[,

(2.2b) \hspace{1cm} u(x,0) = u_0^{\varepsilon, \delta}(x), \hspace{1cm} x \in \mathbb{R}^d,

where \(u_0^{\varepsilon, \delta}\) is a convenient regularized approximation of the initial data \(u_0 : \mathbb{R}^d \to \mathbb{R}\) in (2.1b).

And then, hoping that our results are optimal (it is proved for some and work is in progress), identify the boundary of the integrity region where nonclassical solutions can be located, their union forming the reliability region and the outside the failure domain (we note the difficulty to obtain directly both the divergence, failure region, and the convergence to nonclassical entropy solutions).

**Remark 1.** Throughout, it is assumed \(u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)\) and that the \(u_0^{\varepsilon, \delta}\) are smooth functions with compact support, uniformly bounded in \(L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)\) for some \(q \geq 2\). In fact we assume (boundary conditions) that the solutions \(u^{\varepsilon, \delta}\) decay rapidly at infinity (after all this introduces another dissipative mechanism, even if a natural one).

As we restrict our attention to the diffusion-dominant regime we regard \(\delta = \delta(\varepsilon)\), and we suppose that \(u_0^{\varepsilon, \delta}\) approaches the initial condition \(u_0\) in the sense that
\[
\lim_{\varepsilon \to 0^+} u_0^{\varepsilon, \delta} = u_0 \quad \text{in} \quad L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),
\]

\[
\|u_0^{\varepsilon, \delta}\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)},
\]

\[
\delta \left\| \nabla u_0^{\varepsilon, \delta}\right\|_{L^{p^*}(\mathbb{R}^d)} = o(\varepsilon),
\]

(only for the BBMB eq.)

where \( \gamma \) is to be appropriately fixed, later, and \( \rho \) is the dispersion growth exponent defined in the next few lines.

\[
\lim_{\varepsilon \to 0^+} u_0^{\varepsilon, \delta} = u_0 \quad \text{in} \quad L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),
\]

\[
\|u_0^{\varepsilon, \delta}\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)},
\]

\[
\delta \left\| \nabla u_0^{\varepsilon, \delta}\right\|_{L^{p^*}(\mathbb{R}^d)} = o(\varepsilon),
\]

(only for the BBMB eq.)

About the \( f \)-flux, the \( b \)-diffusion and the \( c \)-dispersion functions we make the following assumptions. Growth hypothesis:

\[(G_f) \quad \exists m \geq 1, \ \exists k_f \geq 1 : \ |f'(u)| \leq k_f |u|^{m-1}, \ \forall u \in \mathbb{R};\]

\[(G_b) \quad \exists r \geq 1, \ \exists k_b > 0 : \ |b(\lambda)| \leq k_b |\lambda|^r, \ \forall \lambda \in \mathbb{R}^d;\]

\[(G_c) \quad \exists \rho \geq 1, \ \exists k_c > 0 : \ |c(\lambda)| \leq k_c |\lambda|^\rho, \ \forall \lambda \in \mathbb{R}^d.\]

In fact the last two are, respectively, consequences of the next assumptions, if we define the constants \( k_b := \max_{|\lambda|=1} |b(\lambda)| \) and \( k_c := \max_{|\lambda|=1} |c(\lambda)| \). Homogeneous potential:

\[(HP_b) \quad b = \nabla B, \quad B : \mathbb{R}^d \to \mathbb{R} \text{ a positively homogeneous potential of degree } r + 1 \geq 2;\]

\[(HP_c) \quad c = \nabla C, \quad C : \mathbb{R}^d \to \mathbb{R} \text{ a positively homogeneous potential of degree } \rho + 1 \geq 2.\]

**Remark 2.** We think these growth conditions are quite natural and non-restrictive. Anyway with respect to the flux function and according to the \( L^p \)-Young measure setting we use (see Appendix B; which is, to our knowledge, the unique analytic mathematical tool we dispose to treat the multidimensional equation), it is a basilar assumption. In applications the simpler and least exponent of such flux functions is that of the KdV and the BBM equations \( f(u) = u^2 / 2 \), but, as we are interested also in nonclassical entropy solutions and the case of convex flux functions excludes them (LeFloch, 2000), such generality seems to be appropriate. With respect to the diffusion and dispersion functions, we collect in the literature, out of the linear cases, the example of the \( \nabla u \nabla u \) pseudo-viscosity or the \( p \)-Laplacian \( |\nabla u|^{p-2} \nabla u \), which verify each of our assumptions. We expect to keep a realistic setting, although with some generality. In this context we must assume only formal solutions of the equations (2.2).

\[
\exists d_b > 0 : \ \lambda \cdot b(\lambda) \geq (r + 1)d_b |\lambda|^{r+1}, \ \forall \lambda \in \mathbb{R}^d; \quad \text{(usual 'diffusion hypothesis')}\]

\[
\exists d_c > 0 : \quad C(\lambda) \geq d_c |\lambda|^{\rho+1}, \ \forall \lambda \in \mathbb{R}^d.
\]

Coercivity hypothesis: define the constants \( d_b := \min_{|\lambda|=1} B(\lambda) \) and \( d_c := \min_{|\lambda|=1} C(\lambda) \) and assume \( d_b, d_c \geq 0 \), then
\[(C_C) \quad \exists d_C > 0: \forall v \in \mathbb{R}^d \quad \forall \lambda \in \mathbb{R}^d, \left| D^2 C(\lambda) v \right| \geq d_C \left| \lambda \right|^{-1} \left| v \right|^2, \quad \forall \lambda \in \mathbb{R}^d. \quad \text{(Strict convexity)}\]

**RESULTS and COMMENTS**

As we will regard to integrity, but keeping a *true* dispersion (one conducting to failure when we take the pure zero-dispersion limit), we must force a diffusion-dominant regime \( r \geq \rho + 1 \) for KdVB and \( r \geq 2\rho + 1 \) for BBMB). It relies direct and naturally on the energy estimates we do in Appendix A.

In the following theorems we will consider the KdVB or the BBMB problems (2.1), we explicit, in each case, the hypothesis on the flux, diffusion and dispersion functions. For all cases considered, the conclusion is *integrity*, which stands for: “as \( \varepsilon \) tends to zero, the sequence of solutions \( u^{\varepsilon, \delta} \) of the approximate problems (2.2) converges in \( L^s((0,T); L^{r,\text{loc}}(\mathbb{R}^d)) \) for all \( s < \infty \) and \( p < \alpha + 1 \), to a function \( u \in L^s((0,T); L^r(\mathbb{R}^d) \cap L^{p+1}(\mathbb{R}^d)) \), which is the unique classical entropy solution to (2.1).”

**Theorem 1. (KdVB)** We suppose satisfied \((G_f), (G_b), (G_c)\) and \((C_b)\) such that \( r \geq \rho + 1 \) and \( m < q \) if \( u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \), which is always possible if \( q \) is large enough. Then the sufficient condition of integrity is \( \delta = o\left(\varepsilon^{\rho + 1} r \right) \). Moreover \( \alpha + 1 = q \) (can be any one of the exponents \( q \)).

In particular, with a linear dispersion \( (\rho = 1) \) the diffusion should be at least quadratic \( (r \geq \rho + 1 = 2) \) and the situation is worse in that concern the BBMB problem \( (r \geq 2\rho + 1) \) in the next results. Thus, if the dispersions we consider can be linear or nonlinear, the diffusion will be nonlinear.

The following results regard the BBMB problem. For conciseness, in all the remaining statements we use the definitions:

\[
M_{\rho,m} := 2 \frac{\rho + 1}{\rho - 1}(m - 1); \quad M_{r,m} := 2 \frac{r + 1}{r - 1}(m - 1); \\
M_{r,\rho,\rho} := 6\frac{(r + 1)}{r + 3 + 2\rho}; \quad M_{r,\rho} := 4 + 2 \frac{r - (2\rho + 1)}{r + 1}. 
\]

With respect to the BBMB problem, we need to distinguish between four different regimes that we present separately. We remark that in all instances above, the assumptions we make imply we have the necessary conditions \( \alpha + 1 > m \) and \( \alpha + 1 > 2 \).

**Theorem 2. (BBMB)** We assume \((G_f), (G_b), (G_c), (C_b), (C_c)\) and \((C_C)\) for \( 1 \leq \rho \leq 3 \) and \( r \geq 2\rho + 1 \). Then the sufficient conditions of integrity are, for \( u_0 \in L^1(\mathbb{R}^d) \cap L^r(\mathbb{R}^d) \).
\( q \geq \alpha + 1 \) and \( \delta = O \left( \varepsilon^{\frac{\rho+1}{2}} \right) \), in fact \( \delta = o(\varepsilon) \) in the extreme case of \( \rho = 1 \) and \( r = 3 \).

Furthermore

\[
\alpha + 1 = \begin{cases} 
M_{\rho,m}, & \text{if } \frac{5 \rho - 1}{2 (\rho + 1)} \leq m < \frac{2 r \rho - \rho^2 - r + 2 \rho}{(\rho + 1)(r + 1)}; \\
M_{\rho', r}, & \text{if } \frac{2 r \rho - \rho^2 - r + 2 \rho}{(\rho + 1)(r + 1)} \leq m < \frac{5 r - 1}{2 (r + 1)}; 
\end{cases}
\]

If we want to compare the results on the KdVB and the BBMB problems, we note that not only the diffusion growth domination for the BBMB problem is more severe \( (r \geq 2 \rho + 1) \) than for the KdVB problem \( (r \geq \rho + 1) \) as also it involves the growth of the flux function. In particular, this determines, fixes, the integrity Lebesgue space (mandatory \( \alpha + 1 \)) for the BBMB problem which, remarkably and in deep contrast, remain a “free” choice for the KdVB problem (any \( \alpha + 1 \); take \( q = \infty \)).

The relevant technicality is that the flux-growth in the case of the KdVB problem “do not” interfere in the energy estimates: the diffusion is sufficiently dominant to cover all “space”-dispersive effects, while in the BBMB problem the “time”-dispersion interact strongly with the flux.

**Theorem 3.(BBMB)** We assume \((G_j), (G_b), (G_c), (C_b), (C_c)\) and \((C_C)\) for \( \rho > 3 \). Then the sufficient conditions of integrity are, for \( u_0 \in L^q \left( \mathbb{R}^d \right) \cap L^q \left( \mathbb{R}^d \right) \), \( q \geq \alpha + 1 \) and \( \delta = O \left( \varepsilon^{\frac{\rho+1}{2}} \right) \).

Furthermore

\[
\alpha + 1 = \begin{cases} 
M_{\rho,m,}, & \text{if } 2 r \rho + 1 \leq m < \frac{4 \rho^2 - 9 \rho - 1}{3 (\rho - 3)} \text{ and } \frac{5 \rho - 1}{2 (\rho + 1)} \leq m < \frac{2 r \rho - \rho^2 - r + 2 \rho}{(\rho + 1)(r + 1)}; \\
M_{\rho', r}, & \text{if } r \geq \frac{4 \rho^2 - 9 \rho - 1}{3 (\rho - 3)} \text{ and } \frac{5 \rho - 1}{2 (\rho + 1)} \leq m < \frac{5 r - 1}{2 (r + 1)}; \\
M_{r, r}, & \text{if } 2 r \rho + 1 \leq m < \frac{4 \rho^2 - 9 \rho - 1}{3 (\rho - 3)} \text{ and } \frac{2 r \rho - \rho^2 - r + 2 \rho}{(\rho + 1)(r + 1)} \leq m < \frac{5 r - 1}{2 (r + 1)}.
\end{cases}
\]

**Theorem 4.(BBMB)** We assume \((G_j), (G_b), (G_c), (C_b), (C_c)\) and \((C_C)\) for \( \rho \geq 1 \), \( r \geq 2 \rho + 1 \) and \( \frac{5 r - 1}{2 (r + 1)} \leq m < \frac{2 r + \rho}{r + 3 + 2 \rho} \). Then the sufficient conditions of integrity are, for

\[
u_0 \in L^q \left( \mathbb{R}^d \right) \cap L^q \left( \mathbb{R}^d \right), \ q \geq \alpha + 1 = M_{r,m} \text{ and } \delta = O \left( \varepsilon^{\frac{\rho+1}{2}} \right) \]
We remark here the extra assumptions we need in the BBMB problem, in contrast to the KdVB problem (which we notice we can even relax, without changing the crucial $\delta - \varepsilon$ relation). Moreover, we note that in the KdVB case the integrity region does not include the boundary, but in the BBMB case it usually does (see the little ‘$o$’ and the big ‘$O$’). The few examples working with nonclassical entropy solutions seem to indicate that such solutions do not exist in the case of big ‘$O$’ results and lead us to guess the non-existence of such solutions. In any case, we must have located the frontier where these solutions can be formed.

Comparing again the KdVB and BBMB problems, this configures one more dissimilitude, as their frontiers must be differently located. Of course this supposes the results are optimal, so our prior, technical, concern is the $\delta - \varepsilon$ balance. (It is already proved for some. And, work is in progress concerning both issues.)

In the case we mentioned of a ‘false’ dispersion, we must be looking to the boundary between say “pure” diffusive behavior and a, new, dispersion-diffusive one (as the failure domain is empty).

**Theorem 5.** **(BBMB)** We assume $(G_f)$, $(G_b)$, $(G_c)$, $(C_b)$, $(C_c)$ and $(C_C)$ for $\rho \geq 1$ and $r \geq 2 \rho + 1$. Then the sufficient conditions of integrity are, for $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, $q \geq \alpha + 1$ and $\delta = o\left(\varepsilon^{\frac{1}{\rho+1}}\right)$. Furthermore

$$\alpha + 1 = \begin{cases} M_{r,m}, & \text{if } \frac{2r + \rho}{r + 3 + 2\rho} \leq m < \frac{2r^2 - r\rho + \rho}{(r + 1)^2}; \\ M_{r,\rho}, & \text{if } \frac{2r^2 - r\rho + \rho}{(r + 1)^2} \leq m \leq 2 + \frac{r - \rho - 1}{r + 1}. \end{cases}$$

Ending with a final remark, we remember the usual “Whitham’s” handling, changing between time and space derivatives: in view of such unequal convergence behavior for the BBMB and KdVB equations (consequence of nonlinear phenomena: fluxes) we must emphasize the limitations of such a procedure.

In view of the intricate correlations between (non)convergence of the approximate solutions and the nonuniqueness of solutions of the approximated equation, we need, first, accurate constitutive laws for the possibly various pertinent dissipation and dispersion functions, then select a specific zero dissipation-dispersion limit which must be validated by (e.g., numerical evaluation of the qualitative behavior of solutions over) testing cases. Finally, try to characterize the, possibly new, entropy conditions. Of course, both new, entropy solutions and approximating devices are open possibilities.

**APPENDICES**

**A. Proofs**

**A priori generalized energy estimates**
While the equations, the methods and the results (proofs) are space multidimensional, here we simplify by considering the one-dimensional case. Also, we explain only the simplest and shortest case of the KdVB problem (a full presentation will be done elsewhere).

Superscripts $\varepsilon$ and $\delta$ are omitted, except if emphasis is necessary, and we use right subscripts to denote partial derivatives. There is the approximate equation:

$$
(A.1) \quad u_t + f(u)_x = \varepsilon b(u_x)_x + \delta c(u_x)_{xx}.
$$

Multiply by a function $\eta'(u)$ and let $q' = \eta' f'$ be the new flux function, $B' = b$ the homogeneous diffusion of degree $r + 1$ and $C' = c$ the homogeneous dispersion of degree $\rho + 1$:

$$
\eta(u)_t + q(u)_x = \varepsilon \left( (\eta'(u) b(u_x))_x - \varepsilon (r+1) \eta''(u) B(u_x) \right) + \delta \left( (\eta'(u) c(u_x))_x - \delta \rho \eta''(u) C(u_x)_x \right).
$$

Then, integrate over $\mathbb{R} \times [0,1]$ with $\eta(u) = \frac{b''}{\varepsilon (r+1)}$. The conservative terms vanish (we suppose $u(\infty, \cdot) = 0$, $u_x(\infty, \cdot) = 0$), Remark 1) and we obtain the

**Lemma.** Let $\alpha \geq 1$ and $B, C : \mathbb{R} \to \mathbb{R}$ be diffusion and dispersion homogeneous potentials of degree $r + 1$ and $\rho + 1$. Each solution of (A.1) satisfies, for $t \in [0,T]$,

$$
(A.3) \quad \int_{\mathbb{R}} |u(t)|^{\alpha+1} \, dx + (\alpha + 1) \alpha (r + 1) \varepsilon \int_{0}^{T} \int_{\mathbb{R}} |u|^{|\alpha-1|} B(u_x) \, dx \, ds = \int_{\mathbb{R}} |u_0|^{\alpha+1} \, dx.
$$

For $\alpha \geq 2$ we have also

$$
(A.4) \quad \int_{\mathbb{R}} |u(t)|^{\alpha+1} \, dx + (\alpha + 1) \alpha (r + 1) \varepsilon \int_{0}^{T} \int_{\mathbb{R}} |u|^{|\alpha-1|} B(u_x) \, dx \, ds = \int_{\mathbb{R}} |u_0|^{\alpha+1} \, dx.
$$

We have deduced the KdVB first energy estimates ($\alpha = 1$ in Lemma; from now on $k$’s stands for constants):

**Proposition 1.** For any solution of (A.1), with diffusion verifying $(\text{C}_k)$, we have for $t \in [0,T]$

$$
(A.5) \quad \int_{\mathbb{R}} u(t)^2 \, dx + 2d_b (r + 1) \varepsilon \int_{0}^{T} \int_{\mathbb{R}} u_x^{r+1} \, dx \, ds \leq \|u_0\|_2^2.
$$
Then, we ask for higher than the $L^2$ a priori estimate. We use (A.4) in the Lemma which switch derivatives-order in gradient-degree, thus we explore diffusion and dispersion growths. Bound (A.4) at left using $(C_b)$ and at right by $(G_c)$:

$$
\int_{\mathbb{R}} |u(t)|^{\alpha + 1} \, dx + (\alpha + 1)\alpha (r+1) d_t \varepsilon \int_{\mathbb{R}} |u|^{\rho - 1} |u_x|^{\alpha + 1} \, dxds
$$

(A.6) integrate over $[0, t]$,

$$
\int_{\mathbb{R}} |u(t)|^{\alpha + 1} \, dx + (\alpha + 1)\alpha (r+1) d_t \varepsilon \int_{\mathbb{R}} |u|^{\rho - 1} |u_x|^{\alpha + 1} \, dxds
$$

\leq \|u_0\|_{L^{\alpha + 1}}^{\alpha + 1} + (\alpha + 1)\alpha (\alpha - 1) k_c \rho \delta \int_{\mathbb{R}} |u|^{\rho - 1} |u_x|^{\alpha + 1} \, dxds.
$$

Apply Young’s inequality to the last term within the Proposition 1 (remember, we want optimal $\delta-\varepsilon$ relations, in particular we do not use dimensional arguments):

$$
(\alpha + 1)\alpha (\alpha - 1) k_c \rho \delta \int_{\mathbb{R}} |u|^{\rho - 1} |u_x|^{\alpha + 1} \, dxds
$$

$$
= \int_{\mathbb{R}} \left[ p_2 |u|^{\alpha + 1} \right]^{\frac{1}{p_2}} \left[ p_3 \left( (\alpha + 1)\alpha (r+1) d_t \varepsilon \int_{\mathbb{R}} |u|^{\rho - 1} |u_x|^{\alpha + 1} \, dxds \right) \right]^{\frac{1}{p_3}} \left[ \frac{2^{n-2} ((\alpha + 1)\alpha t)^{\frac{1}{p_2}} \left( k_c \rho (\alpha - 1) p_2 \right)^{\frac{1}{p_2}} \left( \frac{1}{p_3} \right)^{\frac{1}{p_3}} \left( \frac{1}{p_4} \right)^{\frac{1}{p_4}} \delta \varepsilon \right]^{\frac{1}{p_4}} 2^{d_s(r+1)} \varepsilon \|u_x\|_{L^{\alpha + 1}}^{\alpha + 1} \, dxds
$$

$$
\leq \frac{1}{2} \left( \int_{\mathbb{R}} |u|^{\alpha + 1} \, dxds + (\alpha + 1)\alpha (r+1) d_t \varepsilon \int_{\mathbb{R}} |u|^{\rho - 1} |u_x|^{\alpha + 1} \, dxds \right)
$$

$$
+ \frac{1}{p_4} 2^{n-2} ((\alpha + 1)\alpha t)^{\frac{1}{p_2}} \left( k_c \rho (\alpha - 1) p_2 \right)^{\frac{1}{p_2}} \left( \frac{1}{p_3} \right)^{\frac{1}{p_3}} \left( \frac{1}{p_4} \right)^{\frac{1}{p_4}} \delta \varepsilon \|u_x\|_{L^{\alpha + 1}}^{\alpha + 1}
$$

where

$$
\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad \rho + 2 = \frac{r+1}{p_1} + \frac{r+1}{p_3}, \quad \alpha - 2 = \frac{\alpha + 1}{p_2} + \frac{\alpha - 1}{p_3},
$$

so that, we must have $r \geq \rho + 1$ and

$$
\frac{1}{p_2} = 1 - \frac{\rho + 2}{r + 1}, \quad \frac{1}{p_3} = \frac{1}{\alpha - 1} \left( \frac{\alpha + 1}{\alpha + 2} \right).
$$

Define

$$
\frac{1}{p_1} = \frac{1}{\alpha - 1} \left( 3 - 2 \frac{\rho + 2}{r + 1} \right), \quad \frac{1}{p_1} + \frac{1}{p_3} = \frac{\rho + 2}{r + 1}.
$$
We, conclude, both
\[
\left\{ \begin{array}{l}
\|u(0,.)\|_{L^p}^p \\
n_p \{ k, \rho (\alpha - 1) p_2, \frac{1}{p_1 - 1} \}
\end{array} \right.
\]
and
\[
(\alpha + 1) \left[ \alpha (\alpha - 1) + \alpha + 1 \right] \left\{ \begin{array}{l}
\|u(0,.)\|_{L^p}^p \\
n_p \{ k, \rho (\alpha - 1) p_2, \frac{1}{p_1 - 1} \}
\end{array} \right.
\]
So, we can finally come back to (A.6): we have then proved the following proposition which gives rise to an arbitrarily large \( L^q \) bound.

**Proposition 2.** Assume that \((G_b), (G_c), (C_b)\) holds with \( \rho \geq 1, \ r \geq \rho + 1 \) and \( u_0 \in L^2(\mathbb{R}) \cap L^r(\mathbb{R}) \) for some \( q \geq \alpha + 1 > 3 \frac{r + 1}{\rho + 2} \). For \( t \in [0,T] \) we have
\[
\frac{1}{2} \int_0^t \left\{ \begin{array}{l}
\|u(t,.)\|_{L^p}^p \\
n_p \{ k, \rho (\alpha - 1) p_2, \frac{1}{p_1 - 1} \}
\end{array} \right.
\]
Moreover, if \( \delta = 0 \left\{ \begin{array}{l}
\|u(t,.)\|_{L^p}^p \\
n_p \{ k, \rho (\alpha - 1) p_2, \frac{1}{p_1 - 1} \}
\end{array} \right. \), then \( H_a \left\{ \begin{array}{l}
\|u(t,.)\|_{L^p}^p \\
n_p \{ k, \rho (\alpha - 1) p_2, \frac{1}{p_1 - 1} \}
\end{array} \right. \leq \text{Const.} \)

**Convergence Proof**

Let reconsider the equation (A.2), with an arbitrary convex function \( \eta \) (where we assume \( \eta', \ \eta'', \ \eta''' \) bounded functions on \( \mathbb{R} \)),
\[
\eta(u(t,.) + q(u(t,.)x = \varepsilon (\eta'(u)b(u(t,.)x - \varepsilon (r + 1) \eta''(u)B(u(t,.)x + \delta \eta''(u)C(u(t,.)x.
\]
We prove (B.3). As sufficient condition, we claim that there exists bounded measure \( \mu \leq 0 \) such that
\[
\eta(u(t,.) + q(u(t,.)x \rightarrow \mu, \quad \text{in } D'(\mathbb{R} \times (0,T))
\]
We use the notation:
\[ \mu_1 = \varepsilon (\eta'(u)b(u_\ast) ) ; \]
\[ \mu_2 = -\varepsilon(r+1)\eta''(u)B(u_\ast) ; \]
\[ \mu_3 = \delta \eta'(u)c(u_\ast) ; \]
\[ \mu_4 = -\delta \rho \eta''(u)C(u_\ast) ; \]

and, for each positive \( \theta \in C_0^\infty (\mathbb{R} \times (0,T)) \) we evaluate \( \langle \mu_i, \theta \rangle \) for \( i = 1,2,3,4 \):

\[
\begin{align*}
\langle \mu_i, \theta \rangle &\leq \varepsilon \int_0^T \int_{\mathbb{R}} \theta \eta'(u)b(u_\ast) x dt \\
&\leq \text{Const} \varepsilon \int_0^T \int_{\mathbb{R}} |\theta| |u_\ast|^{r+1} x dt
\end{align*}
\]

in view of the growth hypothesis \((G_\ast)\). Use Hölder’s inequality within Proposition 1 and assumption \((R_1b)\). We get

\[
\langle \mu_i, \theta \rangle \leq \text{Const} \varepsilon \int_0^T \int_{\mathbb{R}} |\theta| |u_\ast|^{r+1} x dt
\]

For \( \mu_2 \), because \( B(u_\ast) \geq 0 \) and \( \eta \) is convex,

\[
\langle \mu_2, \theta \rangle = -(r+1)\varepsilon \int_0^T \int_{\mathbb{R}} \theta \eta''(u)B(u_\ast) x dt \leq 0,
\]

with, by Proposition 1 and assumption \((R_1b)\),

\[
\langle \mu_2, \theta \rangle \leq \text{Const} \varepsilon \int_0^T \int_{\mathbb{R}} |\theta| |u_\ast|^{r+1} x dt
\]

By Hölder’s inequalities

\[
\langle \mu_2, \theta \rangle \leq \text{Const} \varepsilon \int_0^T \int_{\mathbb{R}} |\theta| |u_\ast|^{r+1} x dt
\]

Finally, for \( \mu_4 \),

\[
\langle \mu_4, \theta \rangle \leq \rho \delta \int_0^T \int_{\mathbb{R}} \left[ |\theta| \eta''(u)C(u_\ast) x dt + \rho \delta \int_0^T \int_{\mathbb{R}} |\theta| |u_\ast|^{r+1} x dt
\]

Then

\[
\begin{align*}
\langle \mu_2, \theta \rangle &\leq \text{Const} \varepsilon \int_0^T \int_{\mathbb{R}} |\theta| |u_\ast|^{r+1} x dt
\end{align*}
\]

For \( \mu_3 \), we have by hypothesis \((G_\ast)\)

\[
\langle \mu_3, \theta \rangle \leq \delta \int_0^T \int_{\mathbb{R}} |\theta| |u_\ast|^{r+1} x dt + \delta \int_0^T \int_{\mathbb{R}} \left[ |\theta| \eta''(u)C(u_\ast) x dt + \delta \int_0^T \int_{\mathbb{R}} |\theta| |u_\ast|^{r+1} x dt
\]

therefore, by Proposition 1 and assumption \((R_1b)\),

\[
\langle \mu_3, \theta \rangle \leq \text{Const} \delta \int_0^T \int_{\mathbb{R}} |\theta| |u_\ast|^{r+1} x dt
\]

Finally, for \( \mu_4 \),

\[
\begin{align*}
\langle \mu_4, \theta \rangle &\leq \rho \delta \int_0^T \int_{\mathbb{R}} \left[ |\theta| \eta''(u)C(u_\ast) x dt + \rho \delta \int_0^T \int_{\mathbb{R}} |\theta| |u_\ast|^{r+1} x dt
\end{align*}
\]
\[ \left\langle \mu_i, \theta \right\rangle \leq \text{Const} \delta e^{-\frac{\rho+1}{r+1} \left\| \theta \right\|_{r+1} \left( \varepsilon \int |u_\varepsilon|^2 \right)^{\frac{r+1}{r-1}} } \]

and, by Proposition 1 and assumption (R1b),

\[ \left\langle \mu_i, \theta \right\rangle \leq C \delta e^{-\frac{\rho+2}{r+1} \left( \left\| \theta \right\|_{r+1} + \left\| \theta \right\|_{r+1} \right)} , \]

now, the condition \( \delta = o \left( \frac{\rho}{\varepsilon^{r+1}} \right) \) is sufficient for the conclusion.

Using a standard regularization of \( \text{sgn}(u) \) and \( |u-k| \) (for \( k \in \mathbb{R} \)), which fulfills the growth condition (B.1) in the Young measure representation theorem, Lemma B.1, we apply the limit representation (B.2) and conclude that \( \nu \) satisfies (B.3).

To show (B.4) we follow DiPerna (DiPerna, 1985) and Szepessy (Szepessy, 1989) arguments. We have to check that, for each compact \( K \subseteq \mathbb{R} \),

\[ \lim_{t \to 0^+} \frac{1}{t} \int_0^t \int_K \left( u^\varepsilon, \delta(x,s) - u_0(x) \right) dx ds = 0. \]

By Jensen’s inequality, where \( m(K) \) stands for Lebesgue measure of \( K \),

\[ \frac{1}{t} \int_0^t \int_K u^\varepsilon, \delta(x,s) - u_0(x) \right| dx ds \leq m(K)^{1/2} \left\{ \frac{1}{t} \int_0^t \int_K (u^\varepsilon, \delta(x,s) - u_0(x))^2 dx ds \right\}^{1/2} . \]

We will establish that

\[ \lim_{t \to 0^+} \frac{1}{t} \int_0^t \int_K (u^\varepsilon, \delta(x,s) - u_0(x))^2 dx ds = 0. \]

Let \( K_i \subset K_{i+1} \) (\( i = 0, 1, \ldots \)) be an increasing sequence of compact sets such that \( K_0 = K \) and \( \bigcup_{i \geq 0} K_i = \mathbb{R} \), use the identity \( u^2 - u_0^2 = 2u_0(u-u_0) = (u-u_0) \) :

\[ \frac{1}{t} \int_0^t \int_K (u^\varepsilon, \delta(x,s) - u_0(x))^2 dx ds \]

\[ \leq \frac{1}{t} \int_0^t \left( \int_K |u^\varepsilon, \delta(x,s)|^2 dx - \int_K u_0^2 dx - 2 \int_K u_0 (u^\varepsilon, \delta(x,s) - u_0(x) dx \right) ds \]

\[ \leq \int_{\mathbb{R} \setminus K_i} u_0^2 dx + \frac{2}{t} \int_0^t \left| \int_K u_0 (u^\varepsilon, \delta(x,s) - u_0(x) dx \right| ds \]

for all \( i = 0, 1, \ldots \), using Proposition 1 and assumption (R1b).

Since

\[ \lim_{t \to 0^+} \int_{\mathbb{R} \setminus K_i} u_0^2 dx = 0, \]

We need to consider only the second term.
Take \( \{\theta_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}) \) such that \( \lim_{n \to \infty} \theta_n = u_0 \) in \( L^2(\mathbb{R}) \). Cauchy-Schwartz inequality gives

\[
\left| \int_{K_i} u_0^e(\tau,s) - u_0 \right| \leq \int_{K_i} |u_0 - \theta_n| \cdot |u_0^e(\tau,s) - u_0| \, dx \\
+ \left| \int_{K_i} \theta_n^e(\tau,s) - u_0 \right| \, dx + \int_{K_i} \theta_n(\tau,s) \cdot (u_0^e(\tau,s) - u_0) \, dx
\]

\[
\leq \|u_0 - \theta_n\|_{L^2(\mathbb{R})} \left( \|u_0^e(\tau,s)\|_{L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})} \right)
\]

\[
+ \|\theta_n\|_{L^2(\mathbb{R})} \|u_0^e - u_0\|_{L^2(\mathbb{R})} + \int_0^T \int_{K_i} \theta_n \cdot u_0^e \, dx \, d\tau.
\]

In view of Proposition 1 and (R1b),

\[
\|u_0 - \theta_n\|_{L^2(\mathbb{R})} \left( \|u_0^e(\tau,s)\|_{L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})} \right) \leq \text{Const} \|u_0 - \theta_n\|_{L^2(\mathbb{R})},
\]

which tends to zero when \( n \to \infty \) and since \( \lim_{\varepsilon \to 0^+} \|u_0^e - u_0\|_{L^2(\mathbb{R})} = 0 \), it remains to see that

\[
\lim_{\varepsilon \to 0^+} \|u_0^e - u_0\|_{L^2(\mathbb{R})} = 0.
\]

We have, by (A.1),

\[
\left| \int_0^T \int_{K_i} \theta_n^e \, dx \, d\tau \right| = \left| \int_0^T \int_{K_i} \theta_n\left( -f(u_0^e) + \varepsilon \cdot b(u_0^e) \right) + \delta \cdot c(u_0^e) \, dx \, d\tau \right|
\]

\[
\leq \int_0^T \int_{K_i} \left| \theta_n\right| \, f(u_0^e) \, dx \, d\tau + \varepsilon \int_0^T \int_{K_i} \left| \theta_n\right| \cdot b(u_0^e) \, dx \, d\tau + \delta \int_0^T \int_{K_i} \left| \theta_n\right| \cdot c(u_0^e) \, dx \, d\tau
\]

\[
\leq \mu_1 + \mu_2 + \mu_3.
\]

To deal with \( \mu_1 \), we use (G_f), Hölder’s inequality, Proposition 2 and (R1b):

\[
\int_0^T \int_{K_i} \left| \theta_n\right| \, f(u_0^e) \, dx \, d\tau \leq k_f \left[ \int_0^T \int_{K_i} \left| \theta_n\right|^a \right]^\mu_1 \left[ \int_0^T \int_{K_i} \left| u_0^e \right|^\mu_1 \right]^\mu_2 \cdot \left[ \int_0^T \int_{K_i} \left| u_0^e \right|^\mu_1 \right]^\mu_3
\]

\[
\leq k_f \left( \|\theta_n\|_{L^{\mu_1}(\mathbb{R})} \right)^\mu_1.
\]

For \( \mu_2 \), using (G_b) and once more Hölder’s inequality with Proposition 1 and (R1b), we get

\[
\varepsilon \int_0^T \int_{K_i} \left| \theta_n\right| \, b(u_0^e) \, dx \, d\tau \leq k_b \varepsilon \int_0^T \int_{K_i} \left| \theta_n\right| \, u_0^e \, dx \, d\tau
\]

\[
\leq k_b \varepsilon \left( \frac{r}{r+1} \right) \left( \|\theta_n\|_{L^{r+1}(\mathbb{R})} \right)^{\frac{1}{r+1}} \left[ \int_0^T \int_{K_i} \left| u_0^e \right|^r \, dx \, d\tau \right]^{\frac{1}{r+1}}
\]

\[
\leq k_b \varepsilon \left( \frac{r}{r+1} \right) \left( \|\theta_n\|_{L^{r+1}(\mathbb{R})} \right)^{\frac{1}{r+1}}.
\]
Finally, for \( \mu_3 \) with \((G_c)\), Hölder’s inequality, Proposition 1 and \((R1b)\), we have

\[
\delta \int_0^s \int_{K_i} \left| \langle \theta, \phi \rangle \right| dx d\tau \leq k_i \delta \int_0^s \int_{K_i} \left| \langle \theta, \phi \rangle \right| dx d\tau
\]

\[
\leq k_i \delta \left[ \int_0^s \int_{K_i} \left| \langle \theta, \phi \rangle \right| dx d\tau \right]^{\rho \frac{r+1}{r+1-\rho}}
\]

\[
\leq k \delta \left( \int_0^s \int_{K_i} \left| \langle \theta, \phi \rangle \right| dx d\tau \right)
\]

Thus, since \( \delta = \alpha \left( \frac{\nu + 2}{\nu + 1} \right) \),

\[
\lim_{\epsilon \to 0^+} \frac{1}{t} \int_0^t \int_{0}^{s} \int_{K_i} \theta_n \ u_\tau \ dx \ d\tau \ ds
\]

\[
\leq \lim_{\epsilon \to 0^+} \frac{1}{t} \int_0^t \left( k_n \left\| \langle \theta_n \rangle \right\|_{L^{t+1}}^{\frac{\nu+1}{\nu+1}} + k_n \left\| \langle \theta_n \rangle \right\|_{L^{t+1}}^{\frac{\nu+1}{\nu+1}} + k \delta \left( \int_0^s \int_{K_i} \left| \langle \theta, \phi \rangle \right| dx d\tau \right)^{\rho \frac{r+1}{r+1-\rho}} \right)
\]

and the desired conclusion follows as \( t \to 0^+ \).

**B Entropy Measure-Valued Solutions**

Here, we review basic material on Young measures and entropy measure-valued (e.m.-v.) solutions for conservation laws.

Beginning with Schonbek’s representation theorem (Schonbek, 1982) for the Young measures associated with a sequence uniformly bounded in \( L^q \), generalization of the \( L^\infty \) setting first established by Tartar (Tartar, 1983).

Along this appendix, we suppose \( 1 < q < \infty \) and \( T \leq \infty \) are fixed, \( \text{Prob}(\mathfrak{R}) \) is the space of probability measures (non-negative measures with unit total mass).

**Lemma B.1.** Let \( \{u_n\} \) be a bounded sequence in \( L^\infty \left( [0,T); L^q \left( \mathfrak{R}^d \right) \right) \). Then there exists a subsequence denoted by \( \{u_{n_k}\} \) and a weakly-* measurable mapping \( \nu: \mathfrak{R}^d \times (0,T) \to \text{Prob}(\mathfrak{R}) \) such that, for all functions \( g \in C(\mathfrak{R}) \) satisfying

\[
\left( v_{(x,t)}, g \right) \]

belongs to \( L^\infty \left( (0,T); L^{\frac{q}{q+m}} \left( \mathfrak{R}^d \right) \right) \) and the following limit representation holds

\[
\left( v_{(x,t)}, g \right) \]

\[
\lim_{n \to \infty} \int_{\mathfrak{R}^d \times (0,T)} g \left( \theta_{n_k} \phi \right) \phi(x,t) \ dx dt = \int_{\mathfrak{R}^d \times (0,T)} \left( v_{(x,t)}, g \right) \phi(x,t) \ dx dt
\]
for all \( \phi \in L^1 \left( \mathbb{R}^d \times (0, T) \right) \cap L^\infty \left( \mathbb{R}^d \times (0, T) \right) \).

Conversely, given \( \nu \), there exists a sequence \( \{u_n\} \) satisfying the same conditions as above and such that (B.2) holds for any \( g \) verifying (B.1).

We use the notation \( \left\langle v, g \right\rangle = \int_{\mathbb{R}^d} g(x) \, d\nu(x) \). Then, ‘weakly-* measurable’ means that the real-valued functions \( \left\langle v, g \right\rangle \) is measurable with respect to \( (x,t) \) for each continuous \( g \) satisfying (B.1). The measure-valued function \( \nu \) is called a Young measure associated with the sequence \( \{u_n\} \). As simple example we have the Dirac mass \( \delta_{u(x,t)} \) defined by

\[
\left\langle \delta_{u(x,t)}, g \right\rangle = g(u(x,t)), \quad \text{for all } g \in C(\mathbb{R}) \text{ satisfying (B.1)}.
\]

The following result reveals the connection between the structure of \( \nu \) and the strong convergence of the subsequence.

**Lemma B.2.** Suppose that \( \nu \) is a Young measure associated with a sequence \( \{u_n\} \), bounded in \( L^\infty \left( \mathbb{R}^d \right) \). For \( u \in L^1 \left( \mathbb{R}^d \right) \cap L^q \left( \mathbb{R}^d \right) \), the following statements are equivalent:

(i) \( \lim_{n \to \infty} u_n = u \) in \( L^s \left( \mathbb{R}^d \right) \) for all \( s < \infty \) and \( p \in [1, q) \);

(ii) \( \nu \) is equal to \( \delta_{u(x,t)} \) a.e. \( (x,t) \in \mathbb{R}^d \times (0,T) \).

Following DiPerna (DiPerna, 1985) and Szepessy (Szepessy, 1989), we define a very weak notion of entropy solution to the hyperbolic first order Cauchy problem (2.1).

**Definition B.1.** Assume that \( f \in C(\mathbb{R}) \) satisfies the growth condition (B.1) and \( u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \). A Young measure \( \nu \) associated with a bounded sequence \( \{u_n\} \) in \( L^\infty \left( \mathbb{R}^d \right) \) is called an entropy measure-valued (e.m.-v.) solution to (2.1) if

\[
\partial_t \left\langle v, \left| u - k \right| \right\rangle + \text{div} \left\langle v, \text{sgn}(u-k)(f(u)-f(k)) \right\rangle \leq 0, \quad \text{for all } k \in \mathbb{R},
\]

in the sense of distributions on \( \mathbb{R}^d \times (0,T) \);

\[
\lim_{x \to 0^+} \int_0^t \int_K \left\langle v, |u - u_0(x)| \right\rangle \, dx \, ds = 0,
\]

for all compact set \( K \subseteq \mathbb{R}^d \).

A function \( u \in L^\infty \left( \mathbb{R}^d \right) \cap L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \) is an (classical) entropy weak solution to (2.1) in the sense of Kruzkov (Kruzkov, 1970) and Volpert (Volpert, 1967) if and only if the Dirac
measure $\delta_{u(t)}$ is an e.m.-v. solution. In the case $q = +\infty$, existence and uniqueness of such solutions were proved in (Kruzkov, 1970). The following results on e.m.-v. solutions were proved in (Szepessy, 1989): Proposition B.1 states that e.m.-v. solutions are actually Kruzkov solutions. Proposition B.2 states that the problem has a unique solution in $L^q$.

**Proposition B.1.** Assume that $f$ satisfies (B.1) and $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Suppose that $\nu$ is an e.m.-v. solution to (2.1). Then there exists a function $u \in L^\infty((0,T);L^q(\mathbb{R}^d) \cap L^p(\mathbb{R}^d))$ such that

$$\nu_{x,t} = \delta_{u(x,t)}, \quad \text{for a.e. } (x,t) \in \mathbb{R}^d \times (0,T).$$

**Proposition B.2.** Assume that $f$ satisfies (B.1) and $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for $q > 1$. Then there exists a unique entropy solution $u \in L^\infty((0,T);L^q(\mathbb{R}^d) \cap L^p(\mathbb{R}^d))$ to (2.1) which, moreover, satisfies

$$\left\| u(t) \right\|_{L^p(\mathbb{R}^d)} \leq \left\| u_0 \right\|_{L^p(\mathbb{R}^d)}, \quad \text{for a.e. } t \in (0,T) \text{ and all } p \in [1,q].$$

The measure-valued mapping $\nu_{x,t} = \delta_{u(x,t)}$ is the unique e.m.-v. solution of the same problem.

Combining Propositions B.1 and B.2 and Lemma B.2, we obtain the main convergence tool:

**Corollary B.1.** Assume that $f$ satisfies (B.1) and $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for $q > 1$. Let be $\{u_n\}$ a bounded sequence in $L^\infty((0,T);L^p(\mathbb{R}^d))$ with associated Young measure $\nu$. If $\nu$ is an e.m.-v. solution to (2.1), then

$$\lim_{n \to \infty} u_n = u \quad \text{in } L^s((0,T);L^p_{\text{loc}}(\mathbb{R}^d)), \quad \forall s < \infty, \quad p \in [1,q).$$

$u \in L^\infty((0,T);L^q(\mathbb{R}^d) \cap L^p(\mathbb{R}^d))$ is the unique entropy solution to (2.1).

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