Synchronization of chaotic Liouvillian systems: An application to Chua’s oscillator

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A B S T R A C T

In this paper we deal with the synchronization of chaotic oscillators with Liouvillian properties (chaotic Liouvillian system) based on nonlinear observer design. The strategy consists of proposing a polynomial observer (slave system) which tends to follow exponentially the chaotic oscillator (master system). The proposed technique is applied in the synchronization of Chua’s circuit using Matlab-Simulink® and Matlab®-LMI programs. Simulation results are used to visualize and illustrate the effectiveness of Chua’s oscillator in synchronization.

1. Introduction

Since Pecora and Carroll’s observation on the possibility of synchronizing two chaotic systems [1] (so-called drive-response configuration), several synchronization schemes have been developed [2–5]. Synchronization can be classified into mutual synchronization (or bidirectional coupling) [6] and master–slave synchronization (or unidirectional coupling) [1,7].

In the past years, synchronization of chaotic systems problem has received a great deal of attention among scientists in many fields [8–11]. As it is well known, the study of the synchronization problem for nonlinear systems has been very important from the nonlinear science point of view, in particular, the applications to biology, medicine, cryptography, secure data transmission and so on [12,13]. In general, synchronization research has been focused on the following areas: nonlinear observers [2,14–17], nonlinear control [18], feedback controllers [5], nonlinear backstepping control [19], time delayed systems [12,20], directional and bidirectional linear coupling [21], adaptive control [9], adaptive observers [22,23], sliding mode observers [13,24], active control [25], among others.

This work considers the master–slave synchronization problem via an exponential polynomial observer (EPO) based on differential algebraic techniques [26–28]. Differential and algebraic concepts allow us to establish an algebraic observability condition, and therefore they provide a first step for the construction of an algebraic observer. An observable system in this sense can be regarded as a system whose state variables can be expressed in terms of the input and output variables and a finite number of their time derivatives. Thus, chaos synchronization problem can be posed as an observer design procedure, where the coupling signal is viewed as output and the slave system is regarded as observer. The main characteristic is that the coupling signal is unidirectional, that is, the signal is transmitted from the master system (Chua’s circuit) to the slave system (EPO), the slave is requested to recover the unknown state trajectories of the master. The strategy consists of proposing an EPO which exponentially reconstructs the unknown states of Chua’s system.

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Chua’s circuit is a nonlinear electronic chaotic oscillator. This circuit is easily constructed [29] and has been employed in a variety of applications [30], e.g. communication systems [31]. The chaotic associative memory architecture proposed in [32] uses a network of Chua’s oscillators coupled via piecewise linear conductances.

In this paper the Chua’s oscillator is viewed as a chaotic system with some Liouvillian properties [26,28], referred as Chaotic Liouvillian Oscillator (CLO). The Liouvillian character of the system (if a variable can be obtained by the adjunction of integrals or exponentials of integrals) is exploited as an observability criterion, that is to say, by this property we can know whether a variable can be reconstructed with the measurable output.

This paper is organized as follows: In Section 2 we give some definitions about differential-algebraic approach and Liouvillian systems. In Section 3 we treated the synchronization problem and its solution by means of an exponential polynomial observer. In Section 4 we presented the synchronization of the Chua’s circuit [33] and we show some numerical simulations. Finally, in Section 5 we close the paper with some concluding remarks.

2. Definitions

We start with some basic definitions for the understanding of Liouvillian systems, the following definitions are presented.

Definition 1 (Algebraic Observability Condition – AOC). Let us consider a nonlinear dynamical system with input $u$, output $y$, and state vector $x = (x_1, x_2, \ldots, x_n)^T$. A state variable $x_i \in \mathbb{R}$ is said to be algebraically observable if it is algebraic over $\mathbb{R}(u, y)$ that is to say, $x_i$ satisfies a differential algebraic polynomial in terms of $\{u, y\}$ and some of their time derivatives, i.e.,

$$P_i(x, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0, \quad i \in \{1, 2, \ldots, n\},$$

with coefficients in $\mathbb{R}(u, y)$.

Example 1. Consider the following nonlinear system

\begin{align*}
\dot{x}_1 &= x_2 + x_3^2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= u, \\
\end{align*}

If we define $y = x_2$, then

\begin{align*}
\dot{x}_2 &= y, \\
\dot{x}_3 &= 0, \\
\dot{x}_1 &= y + y^2, \\
\end{align*}

The above system is not algebraically observable since $x_1$ cannot be expressed as a differential algebraic polynomial in terms of $\{u, y\}$.

Motivated by this fact, we present the next definition.

Definition 2 (Liouvillian system). A dynamical system is said to be Liouvillian if the elements (for example, state variables or parameters) can be obtained by an adjunction of integrals or exponentials of integrals of elements of $\mathbb{R}$.

Example 2. We consider the nonlinear system as in Example 1. From (3) we can observe that, although $x_1$ does not satisfy the AOC we can obtain it by means of the integral

$$x_1 = \int (y + y^2).$$

Therefore the nonlinear system (2) is Liouvillian.

Example 3. Consider the following nonlinear second order system that models a prey–predator situation,

\begin{align*}
\dot{x}_1 &= x_1 x_2 - x_1 + k, \\
\dot{x}_2 &= -x_1 x_2 - x_2, \\
y &= x_1, \\
\end{align*}

so, we have:

\footnote{$\mathbb{R}(u, y)$ denotes the differential field generated by the field $\mathbb{R}$, the input $u$, the measurable output $y$, and the time derivatives of $u$ and $y$.}
we can solve directly (5), and obtain:

\[
\begin{align*}
\dot{y} &= yx_2 - y + k, \\
\dot{x}_2 &= -yx_2 - x_2,
\end{align*}
\]

we obtain the parameter \(k\) to be:

\[
k = \dot{y} - \exp\left(-\int (y + 1) dt\right) + y
\]

and we say that system (4) is Liouvillian.

For further information we recommend to see [26,28].

3. Problem formulation and main result

Let us consider the following chaotic Liouvillian system,

\[
\begin{align*}
\dot{x} &= Ax + \psi(x) + \varphi(x) + \zeta(u), \\
y &= Cx,
\end{align*}
\]

where \(x \in \mathbb{R}^n\) is the state vector; \(u \in \mathbb{R}^l\) is the input vector; \(l \leq n; y \in \mathbb{R}\) is the measured output; \(\zeta(\cdot) : \mathbb{R}^l \to \mathbb{R}^n\) is an input dependent vector function; \(A \in \mathbb{R}^{n \times n}\) and \(C \in \mathbb{R}^{1 \times n}\) are constants; and \(\psi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n, \varphi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n\) are state dependent nonlinear vector functions.

We restrict each \(\psi_i(\cdot)\) to be nondecreasing, that is, for all \(a, b \in \mathbb{R}, a > b\), it satisfies the following monotone sector condition

\[
0 \leq \frac{\psi_i(a) - \psi_i(b)}{a - b}, \quad i = 1, \ldots, n.
\]

In the same manner, we restrict each \(\varphi_i(\cdot)\) to be nonincreasing, that is, for all \(a, b \in \mathbb{R}, a > b\), it satisfies

\[
\frac{\varphi_i(a) - \varphi_i(b)}{a - b} \leq 0, \quad i = 1, \ldots, n.
\]

To show the relation between the observers for nonlinear systems and chaos synchronization we give the observer’s definition.

**Definition 3 (Exponential Observer).** An exponential observer for (6) is a system with state \(\hat{x}\) such that

\[
\|x - \hat{x}\| \leq \kappa \exp(-\zeta t),
\]

where \(\kappa\) and \(\zeta\) are positive constants.

In the context of master–slave synchronization, \(x\) can be considered as the state variable of the master system, and \(\hat{x}\) can be viewed as the state variable of the slave system. Hence, the master–slave synchronization problem can be solved by designing an observer for (6).

In what follows, we will solve the synchronization problem by using an exponential polynomial observer based upon the Lyapunov method [34]. To this end, we first compute the dynamics of the synchronization error (difference between the master and the slave systems). Next, by means of a simple quadratic Lyapunov function, we prove the exponential convergence.

System (6) is assumed to be a chaotic Liouvillian system, then by Definition 2 all states of (6) can be reconstructed. In this sense, we will propose an observer scheme.

The observer structure. The observer for system (6) has the next form

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + \psi(\bar{x}) + \varphi(\bar{x}) + \zeta(\bar{u}) + \sum_{i=1}^{m} K_i(y - C\bar{x})^{2i-1},
\end{align*}
\]

where \(\bar{x} \in \mathbb{R}^n\), and \(K_i \in \mathbb{R}^n\), for \(1 \leq i \leq m\).

---

2 Mathematically, chaotic systems are characterized by local instability and global boundedness of the trajectories, i.e. \(\|x(t)\|\) is bounded for all \(t > 0\).
Remark 1. The meaning of $m$ can be understood as follows. As it is well known, an Extended Luenberger observer can be seen as a first order Taylor series around the observed state, therefore to improve the estimation performance high order terms are included in the observer structure. In other words, the rate of convergence can be increased by injecting additional terms with increasing powers of the output error.

Observer convergence analysis. In order to prove the observer convergence, we analyze the observer error which is defined as $e = x - \hat{x}$. From Eqs. (6) and (9), the dynamics of the state estimation error is given by

$$\dot{e} = (A - K_i C)e + (A - K_i C)e + \rho(e) - \sum_{i=2}^{m} K_i(e)^{2i-1},$$

where $\phi(e) := \psi(x) - \psi(\hat{x})$ and $\rho(e) := \phi(x) - \phi(\hat{x})$.

It follows from (7) that each component of $\phi(e)$ satisfies

$$0 \leq \frac{\phi_i(e_i)}{e_i}, \quad \forall e_i \neq 0,$$

which implies a relationship between $\phi(e)$ and $e$ as follows,

$$e^T \phi(e) = \sum_{i=1}^{n} e_i \phi_i(e_i) = \sum_{i=1}^{n} e_i \frac{\phi_i(e_i)}{e_i},$$

by using (11) we have the following condition

$$0 \leq e^T \phi(e).$$

By a similar analysis, from (8) we have

$$\rho(e) \leq 0.$$  \hfill (13)

Properties (12) and (13) will allow us prove that the state estimation error $e(t)$ decays exponentially. We have the main result.

Proposition 1. Consider the chaotic Liouvillian system (6) and the observer (9). If there exists a matrix $P = P^T > 0$, and scalars $\varepsilon > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$ satisfying the linear matrix inequality (LMI)

$$\begin{bmatrix} (A - K_i C)^T P + P(A - K_i C) + \varepsilon I & P + \epsilon_1 I & P - \epsilon_2 I \\ P + \epsilon_1 I & 0 & 0 \\ P - \epsilon_2 I & 0 & 0 \end{bmatrix} \leq 0$$

and

$$\lambda_{\min}(M_i + M_i^T) \geq 0, \quad i = 2, \ldots, m,$$

with $M_i := PK_i C$. Then, there exist positive constants $\kappa$ and $\xi$ such that, for all $t \geq 0$,

$$\|e(t)\| \leq \kappa \exp(-\xi t),$$

where $\kappa = \sqrt{\frac{2}{\varepsilon}}\|e(0)\|$, $\xi = \frac{\beta}{\kappa^2} = \lambda_{\min}(P)$, and $\beta = \lambda_{\max}(P)$.

Proof. We use the following Lyapunov function candidate $V = e^T Pe$. From (10), the time derivative of $V$ is

$$\dot{V} = e^T \left[(A - K_i C)^T P + P(A - K_i C) + \varepsilon I \right] e + 2e^T P \phi(e) + 2e^T P \rho(e) - \sum_{i=2}^{m} (e^T)^{2i-2} e^T M_i e$$

and, in view of (14) and (15),

$$\dot{V} \leq -\varepsilon e^T e - 2\epsilon_1 e^T \phi(e) + 2\epsilon_2 e^T \rho(e).$$

By properties (12) and (13) we have

$$\dot{V} \leq -\varepsilon \|e\|^2.$$  \hfill (16)

We write the Lyapunov function as $V = \|e\|^2_p$, then by Rayleigh–Ritz inequality we have that

$$\alpha \|e\|^2 \leq \|e\|_p^2 \leq \beta \|e\|^2,$$  \hfill (17)

where $\alpha := \lambda_{\min}(P)$, and $\beta := \lambda_{\max}(P) \in \mathbb{R}^+$ (because $P$ is positive definite).
By using (17) we obtain the following upper bound of (16)
\[ \dot{V} \leq -\frac{c}{\beta} \|e\|_p^p. \] (18)

Taking the time derivative of \( V = \|e\|_p^2 \) and replacing in inequality (18), we obtain
\[ \frac{d}{dt} \|e\|_p^p \leq -\frac{c}{2\beta} \|e\|_p^p. \]

Finally, the result follows with
\[ \|e(t)\| \leq \kappa \exp(-\zeta t), \] (19)
where \( \kappa = \sqrt{\|e(0)\|} \), and \( \zeta = \frac{c}{2\beta} \).

**Corollary 1.** Let us consider \( \psi(\cdot) \equiv 0 \). Then system (9) is an exponential observer of system (6) if there exists a matrix \( P = P^T > 0 \), and scalars \( \varepsilon > 0 \), \( \varepsilon_2 > 0 \) satisfying
\[ \begin{bmatrix} (A - K_1 C)^T P + P(A - K_1 C) + \varepsilon_1 I & P - \varepsilon_2 I \\ P - \varepsilon_2 I & 0 \end{bmatrix} \leq 0 \] (20)
and
\[ \lambda_{\min}(M_i + M_i^T) \geq 0, \quad i = 2, \ldots, m. \] (21)
With \( \kappa \) and \( \zeta \) defined as in Proposition 1.

**Proof.** The result is proven as in Proposition 1. \( \Box \)

**Corollary 2.** Let us consider \( \varphi(\cdot) \equiv 0 \). Then system (9) is an exponential observer of system (6) if there exists a matrix \( P = P^T > 0 \), and scalars \( \varepsilon > 0 \), \( \varepsilon_1 > 0 \) satisfying the LMI
\[ \begin{bmatrix} (A - K_1 C)^T P + P(A - K_1 C) + \varepsilon_1 I & P + \varepsilon_2 I \\ P + \varepsilon_2 I & 0 \end{bmatrix} \leq 0 \] (22)
and
\[ \lambda_{\min}(M_i + M_i^T) \geq 0, \quad i = 2, \ldots, m. \] (23)
With \( \kappa \) and \( \zeta \) defined as in Proposition 1.

**Proof.** It follows directly from the procedure in Proposition 1. \( \Box \)

4. Numerical results

In this section we consider the synchronization of a Chua’s system considered as a chaotic Liouvillian oscillator. Chua’s circuit [29], shown in Fig. 1, is a simple oscillator circuit which exhibits a variety of bifurcations and chaos. The circuit contains three linear energy-storage elements (an inductor, and two capacitors), a linear resistor, and a single nonlinear resistor \( N_R \).

The state equations for the Chua’s circuit are as follows:
\[ \begin{align*}
C_1 \frac{dv_{C_1}}{dt} &= G(v_{C_2} - v_{C_1}) - g(v_{C_1}), \\
C_2 \frac{dv_{C_2}}{dt} &= G(v_{C_1} - v_{C_2}) + i_l, \\
L \frac{di_l}{dt} &= -v_{C_2},
\end{align*} \] (24)
where \( G = \frac{1}{R} \) and \( g(\cdot) \) is a nonincreasing function defined by:
\[ g(v_R) = m_0 v_R + \frac{1}{2} (m_1 - m_0) (\|v_R + B_p\| - |v_R - B_p|). \] (25)
This relation is shown graphically in Fig. 2, the slopes in the inner and outer regions are $m_0$ and $m_1$ respectively, with $m_1 < m_0 < 0$; $\pm B_p$ denote the breakpoints. The nonlinear resistor $NR$ is termed voltage-controlled because the current in the element is a function of the voltage across its terminals.

In the first reported study of this circuit, Matsumoto [29] showed by computer simulation that the system possesses a strange attractor called the Double Scroll. Experimental confirmation of the presence of this attractor was made shortly afterwards in [35]. Since then, the system has been studied extensively; a variety of bifurcation phenomena and chaotic attractors in the circuit have been discovered experimentally and confirmed mathematically [36].

In what follows we consider the measured output $y = v_{c_2}$. From equations of (24) we obtain:

\[
\begin{align*}
v_{c_1} &= \frac{C_2}{C} y + y + \frac{1}{LC} \int ydt, \\
v_{c_2} &= y, \\
i_L &= -\frac{1}{L} \int ydt.\end{align*}
\]
From (26), the Chua’s system (24) is Liouvillian. This implies that unknown variables $v_{C1}$ and $i_L$ can be reconstructed with the selected output $y = v_{C2}$.

Chua’s system (24) is of the form (6) with $\zeta(u) = 0$, $\varphi(\cdot) = 0$,

$$A = \begin{bmatrix} -\frac{C_1}{C_2} & 0 & -\frac{C_1}{C_2} & 0 \\ \frac{C_2}{C_1} & 0 & -\frac{1}{C_2} & 0 \\ 0 & -\frac{1}{C_2} & 0 & 0 \\ -\frac{1}{C_2} & 0 & -\frac{1}{C_2} & 0 \end{bmatrix}, \quad \psi(x) = \begin{bmatrix} -\frac{g(x_1)}{C_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \ 1 \ 0], \quad x = [v_{C1} \ v_{C2} \ i_L]^T.$$

Since $g(x_1)$ is nonincreasing and $C_1$ is a positive constant, then $\psi_1(x) = \psi_2(x_1) = -g(x_1)/C_1$ is nondecreasing as in (7). Indeed, $\psi_2(x) = \psi_3(x) = 0$ also satisfy property (7), and Chua’s system (24) is Liouvillian, so that, we proceed with the observer design.

Taking into account that $\varphi(\cdot) = 0$, we will use conditions in Corollary 2 to obtain the observer gains. Using LMI software, observer gains are computed to drive the estimation error to zero.

Applying (9), we have the observer for Chua’s system (24).
\[ \dot{x} = \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} -\frac{e(t)}{C_1} \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} k_{1,i} \\ k_{2,i} \\ k_{3,i} \end{bmatrix} (0 \ 1 \ 0) e^{2i-1}. \]

Hence, the state observer is rewritten as,

\[ \begin{align*}
\dot{x}_1 &= \frac{G}{C_1} \dot{x}_1 + \frac{G}{C_1} \dot{x}_2 - \frac{g(x_1)}{C_1} + k_{1,1} e_{1,2} + k_{1,2} (e_{1,2})^3 + \cdots + k_{1,m} (e_{1,2})^{2m-1}, \\
\dot{x}_2 &= \frac{G}{C_2} \dot{x}_1 + \frac{G}{C_2} \dot{x}_2 + \frac{1}{C_2} \dot{x}_3 + k_{2,1} e_{1,2} + k_{2,2} (e_{1,2})^3 + \cdots + k_{2,m} (e_{1,2})^{2m-1}, \\
\dot{x}_3 &= -\frac{1}{L} \dot{x}_3 + k_{3,1} e_{1,2} + k_{3,2} (e_{1,2})^3 + \cdots + k_{3,m} (e_{1,2})^{2m-1}.
\end{align*} \tag{27} \]
Fig. 3 shows the general diagram of the synchronization of Chua's circuit (24) and the exponential observer (27) in master–slave configuration.

Numerical simulations for the synchronization of Chua’s system are carried out in order to show the performance of the exponential observer. The parameter values considered in the numerical simulations correspond to chaotic behavior [33] and these are: \( C_1 = 10 \text{ nF}, \ C_2 = 100 \text{ nF}, R = 1.8 \text{ k}\Omega, \ L = 18 \text{ mH}, m_0 = -0.409 \text{ mS}, m_1 = -0.756 \text{ mS} \) \) and \( B_p = 1.08 \text{ V}. \) The Matlab-Simulink program uses the Dormand–Prince integration algorithm, with the integration step set to \( 1 \times 10^{-5}. \)

We fix \( m = 2 \) in the observer (27). The LMI (22) is feasible with \( \varepsilon = 0.001 \) and \( \varepsilon_1 = 0.001, \) a solution is

\[
P = \begin{bmatrix}
0.0008 & -0.0006 & 0.1021 \\
-0.0006 & 0.0005 & -0.0805 \\
0.1021 & -0.0805 & 15.0959
\end{bmatrix}, \quad K_1 = \begin{bmatrix}
k_{1,1} \\
k_{2,1} \\
k_{3,1}
\end{bmatrix} = \begin{bmatrix}
1.5 \\
0.5 \\
45
\end{bmatrix}
\]

and \( K_2 \) is chosen such that (23) is satisfied, then we obtain
The synchronization results achieved with the polynomial observer are good. The effectiveness of the suggested methodology was shown by means of numerical simulations. A reduced set of measurable state variables were needed to achieve the synchronization with a chaotic system (Chua's circuit). A chaotic masking scheme by using synchronized chaotic systems, Phys. Rev. E 54 (1996) 4803–4811.

5. Concluding remarks

The synchronization problem of chaotic Liouvillian systems has been treated by using differential and algebraic techniques. We proposed a polynomial observer, and by means of properties of nondecreasing and nonincreasing functions, linear matrix inequalities and with the help of the Lyapunov method we proved that the estimation error exponentially converges to zero. This observer has been used as a slave system whose states are exponentially synchronized with the chaotic system (Chua's circuit). A reduced set of measurable state variables were needed to achieve the synchronization with this approach. The effectiveness of the suggested methodology was shown by means of numerical simulations.

References


Figs. 4–6 show the obtained results for the initial conditions $x_1 = x_2 = x_3 = 0$, $\dot{x}_1 = 1$, $\dot{x}_2 = 0.5$ and $\dot{x}_3 = 0.002$, the synchronization results achieved with the polynomial observer are good.

The performance index (quadratic synchronization error) of the corresponding synchronization process is calculated as

$$J(t) = \frac{1}{t + 0.0001} \int_0^t |e(t)|^2 \, dt, \quad Q_0 = I.$$

Figs. 7 and 8 illustrate the performance index, which has a tendency to decrease. Finally, Fig. 9 presents the synchronization in a phase diagram, where clearly is observed the chaotic behavior of the Chua’s circuit.