Control of a flexible joint robot manipulator via a non-linear control-observer scheme

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A non-linear controller-observer scheme for the output tracking of a class of non-linear singularly perturbed systems based on a two-time scale sliding-mode technique and a high gain estimator, is presented. An analysis of stability of the resultant closed-loop system is given. The proposed scheme is applied to the model of a two degrees of freedom flexible joint robot to show the controller-observer methodology proposed.

1. Introduction

During the last few years, considerable research efforts have been directed toward the control problem of flexible joint robots. Various techniques such as feedback linearization, singular perturbations, sliding-mode, passivity and adaptive techniques have been proposed to tackle the problem (see, for example, Marino and Spong 1986, Spong 1990, Kokotovic et al. 1993, Battilotti and Lanari 1995, Brogliato et al. 1995, 1998, Hernandez and Barbot 1996).

The aim of this paper is to design an observer-based controller for tracking the desired reference signal for flexible manipulators combining the advantages of the singular perturbation methods and sliding modes techniques.

It is well known that many physical systems involve dynamic phenomena occurring in different time scales and the flexible joint manipulator is a representative example of them. Typically, these systems can be modelled using a singular perturbation approach which allows us to obtain subsystems of reduced dimension in order to design control laws and analyse their stability (see, for example, Kokotovic et al. 1993).

The sliding-mode control of rigid robotic manipulators as a robust approach has attracted a number of researchers (see Nathan and Singh 1987, De Carlo et al. 1988, Utkin 1992, Stepanenko and Su 1993). Moreover, a sliding controller is characterized as a high-speed switching controller that provides a robust means of controlling non-linear systems by forcing the trajectories to reach a sliding manifold in finite time and stay on the manifold for all time. Due to switching behaviour of the controller some theoretical and practical problems rise.

On the one hand, the controller contains a discontinuous non-linear term for which the existence and uniqueness of solutions should be examined. On the other hand, the observer design for flexible joint manipulators is an important problem in robot control theory and of great practical importance as well. In several works, the control law requires measurement of the joint position and the joint velocity. However, the joint velocity is often contaminated by noise which reduces the performance of the controller. In order to avoid these problems, some interesting results on the observer problem for these robots are reported, for example, in Tomei (1990) and Nicosia et al. (1988). More recently, in Hernandez and Barbot (1996), a sliding observer-based feedback control law, using singular perturbed methods and sliding-modes, has been designed for a class of non-linear systems which admits a singular perturbation representation for tracking desired time-varying trajectories for flexible manipulators. The flexible joint manipulator model can be split into two subsystems: the slow and fast subsystems, where the slow variables are the link positions and velocities and the fast variables are the elastic forces and their time derivatives. In Hernandez and Barbot (1996) the joint positions and the elastic force, which represent the output of the slow subsystem and the output of the fast subsystem, respectively, are measured. For that, the authors propose a sliding observer to estimate the unmeasurable variables.

Assuming that the output variable is the only one that is available for measurement in a non-linear system, in this paper a controller-observer scheme is proposed to make that variable track a given reference signal. The controller is based on a sliding mode technique recently developed in Alvarez and Silva (1997) to stabilize a class of non-linear singular perturbed systems. The main difference of the present work with respect to that of...
Alvarez and Silva (1998) is that, first, a tracking objective is considered and, second, local exponentially stability assumptions are made on the slow and fast sliding mode dynamics, instead of making stability assumptions on their linear approximations. The assumptions considered allow the existence of certain Lyapunov function candidates that are instrumental in investigating the stability properties of the closed-loop system. The observer is a non-linear high gain observer based on the one proposed by Busawon et al. (1998).

For the stability analysis of the closed-loop system, with and without an observer, sufficient conditions are given that allow us to guarantee the ultimate boundedness of the variables of the system. The design methodology is applied to the model of a flexible joint manipulator to make the link position track a reference signal.

This paper is organized as follows. In §2, the design of a controller based on singular perturbation methods is introduced. In order to overcome the problem of estimating the unmeasurable variables, a non-linear observer is also given in that section. An analysis of stability of the closed-loop system is presented in §3. In §4, the observer-based controller scheme obtained is applied to the model of a flexible joint manipulator. Finally, some conclusions are given in §5.

2. Some preliminary results

2.1. Two-time scale sliding-mode control

Let us consider the class of non-linear singular systems described by the so-called standard singularly perturbed form

\[
\dot{x} = f_1(x) + F_1(x)z + g_1(x)u, \quad x(t_0) = x_0 \tag{1}
\]

\[
\varepsilon \dot{z} = f_2(x) + F_2(x)z + g_2(x)u, \quad z(t_0) = z_0 \tag{2}
\]

where \( t_0 \geq 0, x \in B \subset \mathbb{R}^n \) is the slow state, \( z \in B \subset \mathbb{R}^m \) is the fast state, \( u \in \mathbb{R}^r \) is the control input and \( \varepsilon \in [0, 1) \) is the small perturbation parameter. \( f_1, f_2, g_1 \) and \( g_2 \) are assumed to be bounded with their components being smooth functions of \( x \) and \( y \). The columns of the matrices \( F_1, F_2, g_1 \) and \( g_2 \) are assumed to be bounded and centered at the origin. \( F_1(x) \) is assumed to be non-singular for all \( x \in B \). It is assumed that \( f_1(0) = f_2(0) = 0 \) and for \( u = 0 \), the origin \( (x, z) = (0, 0) \) is an isolated equilibrium state.

The slow reduced system is found by making \( \varepsilon = 0 \) in (1) and (2), obtaining the slow order slow system

\[
\dot{x}_s = f(x_s) + g(x_s)u_s, \quad x_s(t_0) = x_0 \tag{3}
\]

\[
z_s = h(x_s) := -F_2^{-1}(x_s)[f_2(x_s) + g_2(x_s)u_s] \tag{4}
\]

where \( x_s \), \( z_s \) and \( h_s \) denote the slow components of the original variables \( x \), \( z \) and \( u \), respectively, and

\[
f(x_s) = f_1(x_s) - F_1(x_s)F_2^{-1}(x_s)f_2(x_s) \tag{5}
\]

\[
g(x_s) = g_1(x_s) - F_1(x_s)F_2^{-1}(x_s)g_2(x_s) \tag{6}
\]

In (3) and (4), \( u_s(x_s) \) denotes the slow state feedback which only depends on \( x_s \). According to Kokotovic et al. (1993), the n-dimensional invariant system of the system (1) and (2) parametrized by \( \varepsilon \) and known as the slow invariant manifold, is defined by (see Khorasani 1989)

\[
M_\varepsilon := \{ z \in B \subset \mathbb{R}^m : z = h_s(x_s, \varepsilon) + \psi(x_s, \varepsilon)u_s(x_s, \varepsilon) \} \tag{7}
\]

where the functions \( \phi(x_s, \varepsilon), \psi(x_s, \varepsilon) \) and \( u_s(x_s, \varepsilon) \) satisfy the so-called manifold condition

\[
f_2 + F_2\phi + F_2\psi u_s + g_2 u_s = \epsilon \left( \frac{\partial \phi}{\partial x_s} + \frac{\partial \psi}{\partial x_s} u_s + \frac{\partial u_s}{\partial x_s} \right)
\]

\[
\times \left[ f_1 + F_1 \phi + F_1 \psi u_s + g_1 u_s \right] \tag{8}
\]

for all \( x_s \in B \), and for \( \varepsilon \) sufficiently small, where the subscript \( 's' \) stands for the exact solution.

The fast dynamics (also known as boundary layer system) is obtained by transforming the (slow) time scale \( t \) to the (fast) time scale \( \tau := (t - t_0) / \varepsilon \) and introducing the deviation of \( z \) from \( M_\varepsilon \), i.e.

\[
\eta := z - h_s(x_s, \varepsilon) \tag{9}
\]

The original system (1) and (2) then becomes

\[
\frac{d\tilde{x}}{dt} = \varepsilon \left[ f_1(\tilde{x}) + F_1(\tilde{x})[\eta + h_s(\tilde{x}, \varepsilon)] + g_1(\tilde{x})u_s \right] \tag{10}
\]

\[
\frac{d\eta}{d\tau} = \frac{\partial h_s(\tilde{x}, \varepsilon)}{\partial \tilde{x}} \frac{d\tilde{x}}{d\tau} - \frac{\partial \eta(\tilde{x}, \varepsilon)}{\partial \tilde{x}} \frac{d\tilde{x}}{d\tau} \tag{11}
\]

where \( \eta(0) = z_0 - h(x_0), \tilde{x}(\tau) := \varepsilon(\tau + t_0), \) with \( \tilde{x}(0) = \tilde{x}_0 \) and \( \tilde{x}(\tau) := x(\varepsilon\tau + t_0), \) with \( \tilde{x}(0) = x_0 \). The so-called composite control for the original system (1) and (2) is defined by

\[
u(x, \eta, \varepsilon) = u_{ss}(x, \varepsilon) + u_{sf}(x, \eta, \varepsilon) \tag{12}
\]

where \( u_{ss} \) and \( u_{sf} \) denote the slow and fast control components, respectively. The component \( u_{ss} \) is used to make \( M_\varepsilon \) attractive and vanishes there, i.e. \( u_{ss}(x, 0, t_0) = 0 \). If \( u_{ss}(\tilde{x}, \varepsilon) \) and \( \partial h_s(\tilde{x}, \varepsilon)/\partial \tilde{x} \) are bounded and \( \tilde{x} \) remains relatively constant with respect to \( \tau \), then the term \( \varepsilon \partial h_s(\tilde{x}, \varepsilon)/\partial \tilde{x} \) can be neglected for \( \varepsilon \) sufficiently small. Since equation (11) defines the fast reduced subsystem, an \( O(\varepsilon) \) approximation can be obtained for
this subsystem using equation (4) and setting \( \varepsilon = 0 \) in (10) and (11), this is
\[
\frac{d \eta_{\text{opx}}}{d \tau} = F_2(\tilde{x}) \eta_{\text{opx}} + g_2(\tilde{x}) u_f
\]  
where \( \eta_{\text{opx}}, h_2(\tilde{x},0) = h(\tilde{x}) \) and \( u_f \) are \( O(\varepsilon) \) approximations for \( \eta_1, h_1(\tilde{x}, \varepsilon) \) and \( u_{f} \) during the initial boundary layer and \( \eta_{\text{opx}}(0) = z_0 - H(x_0,0) \).

2.2. Sliding-mode control design

The sliding-mode control for the system (1) and (2) is designed in two stages. First, the slow control is designed for the slow subsystem (3). Now, consider the \((n - r)\)-dimensional slow non-linear switching surface defined by
\[
\sigma_s(x_s, x_{sd}) = \text{col}(\sigma_{s_1}(x_s, x_{sd}), \ldots, \sigma_{s_n}(x_s, x_{sd})) = 0
\]  
where \( x_{sd} = \text{col}(x_{sd}, \ldots, x_{sd}) \) is a reference vector and each function \( \sigma_{s_i}: B_{x_s} \times B_{x_{sd}} \rightarrow \mathbb{R} \) is a \( C^1 \) function such that \( \sigma_{s_i}(0,0) = 0 \). The equivalent control method (see Nathan and Singh 1987, De Carlo et al. 1988, Utkin 1992) is used to determine the slow control \( u_s \), yielding
\[
u_s = u_{se} + u_{sN}
\]  
where \( u_{se} \) is the slow equivalent control that takes the form
\[
u_{se} = -\left[ \frac{\partial \sigma_s}{\partial x_s} g(x_s) \right]^{-1} \left[ \frac{\partial \sigma_s}{\partial x_s} f(x_s) + \frac{\partial \sigma_s}{\partial x_{sd}} T \right]
\]  
and \( u_{sN} \) is the control that acts when \( \sigma_s(x_s, x_{sd}) \neq 0 \) given by
\[
u_{sN} = -\left[ \frac{\partial \sigma_s}{\partial x_s} g(x_s) \right]^{-1} L_s(x_s) \sigma_s(x_s, x_{sd})
\]  
where the matrix \( \left[ \frac{\partial \sigma_s}{\partial x_s} g(x_s) \right] \) is assumed to be non-singular for all \( x_s, x_{sd} \in B_{x_s} \), \( L_s(x_s) \) is an \( r \times r \) positive definite matrix whose components are \( C^0 \) bounded non-linear functions of \( x_s \), such that
\[
\|L_s(x_s)\| \leq \rho_s
\]  
for all \( x_s \in B_{x_s} \) with a constant \( \rho_s > 0 \). The equation that describes the projection of the slow subsystem motion outside \( \sigma_s(x_s, x_{sd}) = 0 \) is given by
\[
\dot{\sigma}_s(x_s, x_{sd}) = -L_s(x_s) \sigma_s(x_s, x_{sd}) = 0
\]  
The stability properties of \( \sigma_s(x_s, x_{sd}) = 0 \) in (19) can be studied by means of the Lyapunov function candidate
\[
V_s(x_s, x_{sd}) = \frac{1}{2} \sigma_s^T(x_s, x_{sd}) \sigma_s(x_s, x_{sd})
\]  
and \( \dot{V}_s(x_s, x_{sd}) = -\sigma_s^T(x_s, x_{sd}) L_s(x_s) \sigma_s(x_s, x_{sd}) < 0 \), for all \( x_s, x_{sd} \in B_{x_s} \), thus assuring the existence of a slow sliding mode. The system (3) feedback with slow control (15) yields the slow reduced closed-loop system
\[
\dot{x}_s = f_s(x_s, 0) + p_s(x_s, x_{sd}, x_{sd})
\]  
where
\[
p_s(x_s, x_{sd}, x_{sd}) = f_s(x_s, x_{sd}) - f_s(x_s, 0) + g_s(x_s, x_{sd}) \dot{x}_{sd}
\]  
with
\[
f_s(x_s, x_{sd}) = f(x_s) - g(x_s) \left[ \frac{\partial \sigma_s}{\partial x_s} g(x_s) \right]^{-1} \left[ \frac{\partial \sigma_s}{\partial x_{sd}} \right] f(x_s) + L_s(x_s) \sigma_s(x_s, x_{sd})
\]  
Also, from the boundedness off \( \sigma_s \) and the columns of \( g(x_s) \), the non-singularity of the matrix \( \left[ \frac{\partial \sigma_s}{\partial x_s} g(x_s) \right] \) and the continuous differentiability of \( \sigma_s(x_s, x_{sd}) \), it follows that
\[
\|p_s(x_s, x_{sd}, x_{sd})\| \leq l_1 \|x_s\| + l_2 \|x_{sd}\| + l_3 \|\dot{x}_{sd}\|
\]  
for all \( x_s, x_{sd}, \dot{x}_s, \dot{x}_{sd} \in B_{x_s} \), where \( l_1, l_2 \) and \( l_3 \) are positive constants. We now introduce the following assumptions.

Assumption 1: The equilibrium \( x_s = 0 \) of \( \sigma_s = f_s(x_s, 0) \) is locally exponentially stable.

Assumption 2: The reference vector \( x_{sd}(t) \) and its time derivatives \( \dot{x}_{sd} \) and \( \ddot{x}_{sd} \) are uniformly bounded and satisfy
\[
\|x_{sd}\| \leq b_1, \quad \|\dot{x}_{sd}\| \leq b_2, \quad \|\ddot{x}_{sd}\| \leq b_3
\]  
for some positive constants \( b_1, b_2 \) and \( b_3 \).

By a converse theorem of Lyapunov (see Khalil 1996), Assumption 1 assures the existence of a Lyapunov function \( V_s(x_s) \) which satisfies the inequalities
\[
c_1 \|x_s\|^2 \leq V_s(x_s) \leq c_2 \|x_s\|^2
\]  
\[
\frac{\partial V_s(x_s)}{\partial x_s} f_s(x_s, 0) \leq -c_3 \|x_s\| \|x_s\|= 0 (27)
\]  
\[
\left\| \frac{\partial V_s(x_s)}{\partial x_s} \right\| \leq c_4 \|x_s\|
\]  
for some positive constants \( c_1, c_2, c_3 \) and \( c_4 \). Using \( V_s(x_s) \) as a Lyapunov function candidate to investigate the stability of the origin \( x_s = 0 \) as an equilibrium point for the system (21). From Assumptions 1, 2, equation (27) and using the property \( \partial h \leq (k/2) a^2 \ t \ (1/2k) b^2 \), with \( k \in (0,1) \), the time derivative of \( V \), along the trajectories of (21) then satisfies
\[
\dot{x}_2 (x_3) \leq -v_1 \|x_3\|^2 + v_2 
\]
where
\[
v_1 = c_3 - c_4 I_p, \quad v_2 = \frac{c_4}{2 \kappa_1} (l_2 b_1 + l_3 b_2)
\]
with \(\kappa_1 \in (0,1)\) and \(I_p = [I_1 + \frac{1}{2} (l_2 b_1 + l_3 b_2) \kappa_1]\). Then, if \(I_p\) is small enough to satisfy the bound
\[
l_p \leq \frac{c_3}{c_4}
\]
then the reduced slow system (21) is locally ultimately bounded.

On the other hand, the fast control design for the subsystem (13) can be obtained in a similar way to the one used for the slow control. That is, one considers an \((m-r)\)-dimensional fast switching surface defined by
\[
\sigma_f (\eta_{apx}, x_{fd}) = \text{col} (\sigma_{f1} (\eta_{apx}, x_{fd}), \ldots, \sigma_{fr} (\eta_{apx}, x_{fd})) = 0
\]
where \(x_{fd} = \text{col} (x_{fd}, \ldots, x_{fd})\) is the reference vector and each function \(\sigma_{fi} : B_r \times B_r \to R\), \(i = 1, \ldots, r\), is also a \(C^1\) function such that \(\sigma_{fi}(0,0) = 0\). The complete fast control takes the form
\[
u_f = u_{fe} + u_{fn}
\]
(32)
where \(u_{fe}\) is the fast equivalent control given by
\[
u_{fe} (x, \eta_{apx}, x_{fd}) = -\left[ \frac{\partial \sigma_f}{\partial \eta_{apx}} g_2 (x) \right]^{-1} \times \left[ \frac{\partial \sigma_f}{\partial \eta_{apx}} F_2 (x) \eta_{apx} + \frac{\partial \sigma_f}{\partial x_{fd}} \frac{dx_{fd}}{d\tau} \right]
\]
(33)
and
\[
u_{fn} (x, \eta_{apx}, x_{fd}) = -\left[ \frac{\partial \sigma_f}{\partial \eta_{apx}} g_2 (x) \right]^{-1} L_f (\eta_{apx}) \sigma_f (\eta_{apx}, x_{fd})
\]
(34)
where the matrix \(\left[ \frac{\partial \sigma_f}{\partial \eta_{apx}} g_2 (x) \right]^{-1}\) is assumed to be non-singular for all \((x, \eta_{apx}, x_{fd}) \in B_r \times B_r \times B_r\), and \(L_f (\eta_{apx})\) is a positive definite matrix of dimension \(r \times r\), whose components are \(C^0\) bounded non-linear real functions of \(\eta_{apx}\), such that
\[
\|L_f (\eta_{apx})\| \leq \rho_f
\]
(35)
for all \((x, \eta_{apx}, x_{fd}) \in B_r \times B_r \times B_r\), with a constant \(\rho_f\). The projection of the fast subsystem motion outside \(\sigma_f (\eta_{apx}, x_{fd}) = 0\) is described by \(\frac{d \sigma_f}{d\tau} = -L_f (\eta_{apx}) \sigma_f (\eta_{apx}, x_{fd})\), and arguments similar to the ones used for the slow subsystem motion can be applied to this system to conclude the existence of a fast sliding-mode.

When the complete fast control (32) is substituted into (13), the fast reduced closed-loop system takes the form
\[
\frac{d \eta_{apx}}{d\tau} = g_r (x, \eta_{apx}, 0) + p_f \left( \frac{dx_{fd}}{d\tau} \right) - g_e (x, \eta_{apx}, 0)
\]
(36)
for some positive constants \(c_1, c_2, c_3 \) and \(c_4\). Consider \(W_f(\eta_{apx})\) as a Lyapunov function candidate to investigate the stability of the origin \(\eta_{apx} = 0\) as an equilibrium point for the system (36). Using assumptions 3, 4, (42) and the property \(ab \leq (k/2)u^2 + (1/2k)b^2\), with \(k \in (0, 1)\), the time derivative of \(W_f\) along the trajectories of (36) then satisfies
\[
\frac{dW_f(\eta_{apx})}{dt} \leq -\eta_1^2 + \eta_2 \tag{43}
\]
where
\[
\eta_1 = c_3 - c_4F_p, \quad \eta_2 = \frac{c_5}{2\kappa_2}(l_1^2b_1 + l_3b_2) \tag{44}
\]
with \(\kappa_2 \in (0, 1)\) and \(l_p = [l_1 - \frac{1}{2}(l_1^2b_1 + l_3b_2)\kappa_2]\). If \(l_p\) is small enough to satisfy the bound
\[
l_p \leq \frac{c_3}{c_4} \tag{45}
\]
then the reduced fast system (36) is locally ultimately bounded.

Based on the reduced order sliding-mode control described above, the original slow and fast state variables are used to construct the composite control, i.e.
\[
u = u_s + u_f \tag{46}
\]
where
\[
u_s = - \left[ \frac{\partial \sigma_s}{\partial x} g(x) \right]^{-1} \left[ \frac{\partial \sigma_s}{\partial x} F(x) + \frac{\partial \sigma_s}{\partial x} \dot{x}_{apx} + L_s(x) \sigma_s(x, x_{apx}) \right] \tag{47}
\]
\[
u_f = - \left[ \frac{\partial \sigma_f}{\partial \eta} g_2(x) \right]^{-1} \left[ \frac{\partial \sigma_f}{\partial \eta} F_2(x) \eta + \frac{\partial \sigma_f}{\partial \eta} \dot{x}_{apx} + L_f(\eta) \sigma_f(\eta, x_{apx}) \right] \tag{48}
\]
When the composite control (46)-(48) is substituted in (1) and (2), one obtains the closed-loop non-linear singularly perturbed system
\[
\dot{x} = f_c(x, \eta, x_{apx}) + g_c(x, x_{apx}) \dot{x}_{apx} \tag{49}
\]
\[
\dot{\eta} = g_c(x, \eta, x_{apx}) + g_e(x, x_{apx}) \epsilon \dot{x}_{apx}
\]
\[
-\epsilon \frac{\partial \eta}{\partial x} \left[ f_c(x, \eta, x_{apx}) + g_e(x, x_{apx}) \dot{x}_{apx} \right] \tag{50}
\]
where \(\eta = z - h(x), \dot{x}(t_o) = x_{apx}, \dot{z}(t_o) = z_{apx}\) and
\[
\epsilon_c(x, \eta, x_{apx}) = f(x) + F(x) \eta - g(x) \left[ \frac{\partial \sigma_s}{\partial x} g(x) \right] \tag{51}
\]
\[
\times \left[ \frac{\partial \sigma_f}{\partial \eta} f(x) + L_s(x) \sigma_s(x, x_{apx}) \right] \tag{51}
\]

The Lyapunov function candidates \(V_s\) and \(W_f\) are instrumental in investigating the stability properties of the closed-loop system obtained when the composite control \(u = u_s + u_f\) is used. The following proposition can be proved using the above reasoning.

**Proposition 1:** Consider a non-linear singularly perturbed system (1) utzd (2) for which a composite control (46)-(48) is designed such that (30) and (45) are satisfied. Then, the closed-loop non-linear singularly perturbed system (49)-(51) is locally ultimately bounded, for sufficiently small \(\epsilon\).

**Remark 1:** In Alvarez and Silva (1998, pp. 870–871), the stability of the slow and fast closed-loop systems with \(x_d = 0\) is studied, making the assumption that the linear approximations of \(x = f_c(x, \eta, 0)\) and \(d\eta/dt = g_c(x, \eta, 0)\), around \(x = 0\) and \(\eta_{apx} = 0\) respectively, were exponentially stable. In the present work, the Lyapunov function candidates \(V_s\) and \(W_f\) are instrumental to investigate the stability properties of the closed-loop system obtained when the composite control \(u = u_s + u_f\) is used and an observer is introduced to estimate the whole state \((x, \eta)\).

### 2.3. A non-linear observer

The control law developed above requires the knowledge of the whole vector slate to be implemented. Since it is only possible to have information of some components of the state vector by direct measurement, it is necessary to substitute the unknown information by means of an observer. Recently, some state estimators have been proposed for non-linear systems (see, for example, Hammouri and De Leon 1990, Gauthier et al. 1992, De Leon et al. 1996, Hernandez and Barbou 1996, Bornard et al. 1998, Busawon et al. 1998). In this paper a uniform observer, which allows us to estimate the states of a non-linear system and recently reported in Busawon et al. (1998), is used.

Let us consider the singularly perturbed system (1) and (2) together with an output variable
\[
y = \pi(x) \tag{52}
\]
where \(y \in \mathbb{R}\) and \(\pi\) is a smooth function. Suppose that this system can be written, after a (possible) state coordinate transformation \(z = \Phi(x, z)\), in the form
\[
z = F(y)z + G(u, y, z) \tag{53}
\]
where \(z \in B, x \in \mathbb{R}, u \in \mathbb{R}, C = \{1 \quad 0 \quad \ldots \quad 0\}\) and
It is also assumed that the functions $f_i, i=1,\ldots,n+m-1,$ are of class $C^{r}, r \geq 1,$ with respect to $y$ and that the functions $g_i, i=1,\ldots,n+m,$ are globally Lipschitz with respect to $z$ and uniformly with respect to $u.$ In addition, one assumes that there is a class of Lipschitz constants $\alpha, \beta,$ such that for each $u \in U$ and each output $y$ associated to $u$ and the initial condition $z(0) \in B_x \times B_z$

$$0 \leq \alpha \leq |f_i(y)| \leq \beta, \quad i=1,\ldots,n+m-1 \quad (55)$$

These inequalities guarantee that there is no loss of observability because of singularities in the system.

Let us define the system

$$\dot{z} = F(y)z + G(u,y,z) - S^{-1}_\theta(y)C^TCz - y \quad (56)$$

where $S^{-1}_\theta = \Omega(y)S_\theta \Omega(y)$ with

$$\Omega(y) = \text{diag} \left(1, f_1(y), f_2(y), \ldots, \prod_{i=1}^{n+m-1} f_i(y) \right) \quad (57)$$

and assume that the time derivative of $\Omega(y)$ is bounded. Let be $S_\theta = S_\theta^T > 0$ the unique solution of the algebraic Lyapunov equation

$$\theta S_\theta + A^T S_\theta + S_\theta A - C^T C = 0 \quad (58)$$

with $\theta > 0$ and

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

We then have the following result

Theorem 1: Consider the system (53). If there exists a positive constant $\theta$ such that for all $u \in U,$ for all initial conditions $z(0), \quad z(0) \in B_x \times B_z, \quad \theta_0 > 0,$ such that $\theta \geq \theta_0,$ then the system (56) is an exponential observer for the system (53) with arbitrarily decay exponential rate.

Proof: Let $e = \hat{z} - z$ be the estimation error, whose dynamics is given by

$$\dot{e} = \{F(y) - S^{-1}_\theta(y)C^TC\}e + \Gamma(u,y,z) \quad (59)$$

where $\Gamma(u,y,z) = G(u,y,z) - G(u,y,z)$ and $S_\theta$ is a solution of $\theta S_\theta + F(y)S_\theta + S_\theta F(y) - C^T C = 0.$ After some computations, one can see that $F(y) = \Omega^{-1}(y)A\Omega(y), \quad \Omega(y) = C$ and $S_\theta = \{1/\theta\} \Delta_\theta S_\theta \Delta_\theta,$ where $S_\theta$ is the unique solution of (58) with $\theta = 1$ and $\Delta_\theta = \text{diag}\{1,1/\theta,\ldots,1/\theta^{n+m-1}\}$ and rewrite (59) as

$$\dot{e} = \Omega^{-1}(y)\{A - S^{-1}_\theta C^TC\}e + \langle u,y,z \rangle$$

It is clear, under suitable conditions (see also Busawon et al. 1998), that $\|e\| \leq k\|e\|$ where $k > 0.$ To show the convergence of estimation error dynamics (60) to zero, let us make the change of variable $\tilde{e} = \Omega(y)\Omega^{-1}(y)e$

$$\ddot{e} = \theta \{A - S^{-1}_\theta C^TC\}e + \langle u,y,z \rangle$$

Now, considering the Lyapunov candidate function $V_\theta(e) = e^2 S_\theta^2 e$ and taking the time derivative of this function along the trajectories of the estimation error dynamics (60), it follows that $\dot{V}_\theta(e) \leq -\mu \tilde{e}(e),$ where $\mu = \theta - 2k\sqrt{\lambda_{\max}(S_\theta)/\lambda_{\min}(S_\theta)}$. Then

$$\|e(t)\| \leq \tilde{K} \exp(-\mu t) \quad (62)$$

where $\tilde{K}$ is a constant which depends on the initial conditions $\|e(0)\|$. On the other hand, $\dot{e} = \Omega(y)\Delta_\theta e$ and let $N$ be a constant such that for $\tilde{K}_1$ and $\tilde{K}_2$ positive constants we have

$$\sqrt{N}\|e\| = (\tilde{K}_1/\theta^{n+m-1})\|e\| \leq \|\Omega(y)\|\|\Delta_\theta\|_{\min}\|e\|$$

$$\leq \|e\| \leq \|\Omega(y)\|\|\Delta_\theta\|_{\max}\|e\| = \tilde{K}_2\|e\|$$

together with $\dot{V}_\theta(e) \leq -\mu N\|e\|^2.$ Then

$$\|e(t)\| \leq \tilde{K} \exp(-\mu t)$$

where $\tilde{K} = \tilde{K}/\sqrt{N}$. Thus, an arbitrarily exponential decay rate for $\|e(t)\|$ is obtained. $\square$

3. Closed-loop stability

Suppose that a composite control (47) and (48) has been designed such that the non-linear singularly perturbed system (49) and (50) is ultimately bounded. and that an observer (56), with exponential rate of convergence, is designed in order to estimate that the state is not measurable. The fundamental question of knowing if the stability of the closed-loop system is preserved, when the state is replaced by its estimate in the control law, is now addressed. Let us consider the augmented
closed-loop non-linear singularly perturbed system described by

\[
\begin{align*}
\dot{x} &= f(x) + F_1(x)\eta + g(x)u_0(x, x_{sd}, \dot{x}_{sd}) \\
&\quad + g_1(x)u_f\left(x, \dot{x}, x_{sd}, \frac{dx_{sd}}{d\tau}\right), \quad x(t_0) = x_0 \\
\varepsilon\dot{\eta} &= F_2(x)\eta + g_2(x)u_f\left(x, \dot{x}, x_{sd}, \frac{dx_{sd}}{d\tau}\right) \\
&\quad - \varepsilon\left[\left(\frac{\partial h}{\partial \xi_x} + \frac{\partial h}{\partial \xi_x}\right)\dot{x} + \frac{\partial h}{\partial \xi_{sd}}\dot{x}_{sd}\right] \\
&\quad - \varepsilon\left[\left(\frac{\partial h}{\partial \xi_x} + \frac{\partial h}{\partial \xi_x}\right)\dot{x}_{sd} + \frac{\partial h}{\partial \xi_{sd}}\dot{x}_{sd}\right]
\end{align*}
\]

\[
\eta(t_0) = z_0 - h(x_0, x_{sd}, \dot{x}_{sd})
\]
\[
\dot{\varepsilon} = \Omega^{-1}(y) \{A - S_{\varepsilon}^{-1}C^TC\} \Omega(y)\varepsilon + \Gamma(u, \dot{\xi}, \dot{\zeta})
\]
\[
e(0) = e_0
\]

(63)

Noting that the composite control now depends on the estimate \((\hat{x}, \hat{\eta})\) and that \(\frac{d\hat{\xi}}{dt} = \dot{\hat{x}}, \frac{d\xi}{dt} = P_x\dot{\varepsilon} + \left(\frac{\partial h}{\partial \xi_x}\right)\dot{x}\), the system (63) can be rewritten as

\[
\begin{align*}
\dot{x} &= f_c(x, \eta, x_{sd}) + g_c(x, x_{sd})\dot{x}_{sd} \\
&\quad + g_1(x)\Delta u_0(x, x_{sd}, \dot{x}_{sd}) \\
&\quad + g_2(x)\Delta u_f(x, \eta, x_{sd}, \dot{x}_{sd}) \\
&\quad + g_3(x, x_{sd})\dot{x}_{sd} + g_4(x)\Delta u_0(x, x_{sd}, \dot{x}_{sd}) \\
&\quad + g_5(x, x_{sd})\dot{x}_{sd} + g_6(x, x_{sd})\Delta u_0(x, x_{sd}, \dot{x}_{sd}) \\
&\quad - \varepsilon\left[\left(\frac{\partial h}{\partial \xi_x} + \frac{\partial h}{\partial \xi_x}\right)\dot{x} + \frac{\partial h}{\partial \xi_{sd}}\dot{x}_{sd}\right] \\
&\quad - \varepsilon\left[\left(\frac{\partial h}{\partial \xi_x} + \frac{\partial h}{\partial \xi_x}\right)\dot{x}_{sd} + \frac{\partial h}{\partial \xi_{sd}}\dot{x}_{sd}\right]
\end{align*}
\]

\[
\eta(t_0) = z_0 - h(x_0, x_{sd}, \dot{x}_{sd})
\]
\[
\dot{\varepsilon} = \Omega^{-1}(y) \{A - S_{\varepsilon}^{-1}C^TC\} \Omega(y)\varepsilon + \Gamma(u, \dot{\xi}, \dot{\zeta})
\]
\[
e(0) = e_0
\]

(64)

where \(f_c, g_c, g_c\) and \(g_{c_1}\) are defined as in $3$, and

\[
\begin{align*}
\Delta u_0(x, \hat{x}, x_{sd}, \dot{x}_{sd}) &= u_0(\hat{x}, x_{sd}, \dot{x}_{sd}) \\
&\quad - u_0(x, x_{sd}, \dot{x}_{sd}) \\
\Delta u_f(x, \hat{x}, \eta, x_{sd}, \dot{x}_{sd}, \dot{x}_{sd}/d\tau) &= u_f(\hat{x}, \eta, x_{sd}, \dot{x}_{sd}/d\tau) \\
&\quad - u_f(x, \eta, x_{sd}, \dot{x}_{sd}/d\tau)
\end{align*}
\]

(65)

From the properties of the functions involved in \(u_0\) and \(u_f\), one has that \(\Delta u_0\) and \(\Delta u_f\) satisfy the local Lipschitz conditions

\[
\begin{align*}
\|\Delta u_0(x, \hat{x}, x_{sd}, \dot{x}_{sd})\| &\leq m_j\|\varepsilon\| \\
\|\Delta u_f(x, \hat{x}, \eta, x_{sd}, \dot{x}_{sd}, \dot{x}_{sd}/d\tau)\| &\leq m_j\|\varepsilon\|
\end{align*}
\]

(66)

for all \((x, \hat{x}, \eta, \dot{\eta}) \in B_x \times B_x \times B_\eta \times B_\eta\), with \(m_j\) and \(m_j\) being the Lipschitz constants of \(u_0(x, x_{sd}, \dot{x}_{sd})\) and \(u_f(x, \eta, x_{sd}, \dot{x}_{sd}, \dot{x}_{sd}/d\tau)\) with respect to \(x\) and \((x, \eta)\), respectively. From the fact that the columns of \(g(x), g_1(x)\) and \(g_2(x)\) are bounded, one has that

\[
\begin{align*}
\|g(x)\Delta u_0(x, \hat{x}, x_{sd}, \dot{x}_{sd})\| &\leq m_{0m}\|\varepsilon\|, \\
\|g_1(x)u_f(\hat{x}, \eta, x_{sd}, \dot{x}_{sd}, \dot{x}_{sd}/d\tau)\| &\leq m_1(\delta_1\|\varepsilon\| + \delta_2\|\eta\| + \delta_3\|e\|), \\
\|g_2(x)\Delta u_f(x, \hat{x}, \eta, x_{sd}, \dot{x}_{sd}, \dot{x}_{sd}/d\tau)\| &\leq m_2m_j\|\varepsilon\|
\end{align*}
\]

(67)

for all \(x, \hat{x} \in B_x, \eta, \dot{\eta} \in B_\eta, \) and \(e \in B_e, \delta_1, \delta_2, \delta_3, m_0, m_1, \) and \(m_2\) are some positive constants.

In view of the properties of all the functions involved in \(f_c(x, \eta, x_{sd})\) and \(g_c(x, x_{sd})\), these satisfy the local Lipschitz conditions

\[
\|f_c(x, \eta, x_{sd}) - f_c(x, \eta, x_{sd})\| = \|f_1(\dot{\xi})\| \leq m_{f_1}\|\dot{\xi}\|.
\]

\[
\forall (x, \eta) \in B_x \times B_\eta
\]

\[
\|g_c(x, x_{sd}) - g_c(x, x_{sd})\| \leq m_{g_1}\|x\|, \quad \forall x, z \in B_e
\]

(68)

\[
\|g_c(x, x_{sd}) - g_c(0, x_{sd})\| \leq m_{f_2}\|x\|, \quad \forall x, z \in B_e
\]

therefore \(I_{f_1}\) and \(I_{g_1}\) are the Lipschitz constants of \(f_c(x, \eta, x_{sd})\), with respect to the fast variable \(\eta\), and \(g_c(x, x_{sd})\), with respect to the slow variable \(x\). Furthermore, \(f_1(0, 0, 0) = 0\) and \(g_1(0, 0) = 0\), thus

\[
\|f_c(x, 0, x_{sd})\| \leq I_{f_1}\|x\| + I_{f_2}\|x_{sd}\|, \quad \forall x \in B_e
\]

(70)

\[
\|g_c(x, 0, x_{sd})\| \leq I_{f_1}\|x_{sd}\|, \quad \forall x \in B_e
\]

(71)

where \(I_{f_1}\) and \(I_{f_2}\) are positive constants and \(I_{f_1}\) denotes the Lipschitz constant of \(g_c(0, x_{sd})\) with respect to \(x_{sd}\). Also, from the continuous differentiability of \(h\) with respect to its arguments and that of the mapping \(\Phi\) it follows that
for all $x, \dot{x}, \ddot{x} \in B$, with $l_{h_1}, l_{h_2}, l_{h_3}, l_{h_4}$ and $l_{\Phi}$ being positive constants. Now, set

$$\alpha_1 = (v - c_d m_1 \dot{\theta}_1), \quad \alpha_2 = \frac{e}{\epsilon} - c_d (l_{h_1} + l_{h_2} \omega) (l_{f_1} + \beta_2)$$

$$\alpha_3 = \mu N$$

$$\beta_1 = c_d (l_{m_1} + m_1 \dot{\theta}_1) + c_d (l_{h_1} + l_{h_2} \omega) (l_{f_1} + b_2 l_{s_2} + m_1 \dot{\theta}_1)$$

$$\beta_2 = c_d (m_0 m_1 + m_1 \dot{\theta}_1)$$

$$\beta_3 = c_d [(l_{h_1} + l_{h_2} \omega) + b_2 (l_{f_1} + b_2 l_{s_2}) \dot{\theta}_1] + (1/\epsilon) c_d m_0 m_1 \dot{\theta}_1 + c_d \dot{\epsilon}_1$$

$$\beta_4 = c_d [(l_{h_1} + l_{h_2} \omega) + l_{h_3}] (l_{f_1} + b_2 l_{s_2}) \dot{\theta}_1$$

$$\gamma = \dot{\epsilon}_2 + (\epsilon/\epsilon)$$

(73)

The following result gives sufficient conditions to assure the uniform boundedness of the augmented closed-loop non-linear singularly perturbed system (64).

**Theorem 2:** Consider a non-linear singularly perturbed system (1) and (3) for which a composite control (46)-(48) is designed such that (30) and (45) are satisfied. Suppose that an observer (56) with exponential rate of convergence is designed. Thus, if there exist some numbers $0 < \theta_i < 1, i = 1, 2, 3, 4$, such that

$$\mu_{cw} = \min \left\{ \frac{a' \cdot b' \cdot c'}{e_2 \cdot e_2 \cdot \epsilon N} \right\} > 0$$

(74)

where

$$a' = \alpha_1 - \frac{\beta_1 \theta_1}{2} - \frac{\beta_2 \theta_2}{2}$$

$$b' = \alpha_2 - \frac{\beta_1}{2 \theta_1} - \frac{\beta_3 \theta_3}{2} - \frac{\beta_4 \theta_4}{2}$$

$$c' = \alpha_3 - \frac{\beta_2}{2 \theta_2} - \frac{\beta_3}{2 \theta_3}$$

(75)

for sufficiently small $\epsilon$, then the augmented closed-loop non-linear singularly perturbed system (64) is locally ultimately bounded.

**Proof:** Let us consider the Lyapunov function candidate

$$L(x, \eta, \epsilon) = V_f(x) + W_f(\eta) + V_o(\epsilon)$$

where $V_o(\epsilon) = \epsilon^T S_1 \epsilon$.

Differentiating $V_f(x)$ along the first dynamics of (64), one obtains

$$\dot{V}_f \leq -v_1 \|x\|^2 + v_2 + \left\| \frac{\partial V_f}{\partial x} \right\| f_1(x, \eta)$$

(76)

On the other hand, in §2 it was shown, using the function $W_f$, that the system (36) is locally uniformly bounded. Moreover, in Khalil (1996), it is shown that when $\tilde{x}$ is replaced by slow state variable $x$, the local stability property of the state $\eta_{\text{apx}} = z - h(\tilde{x})$ is kept. Then, differentiating $W_f(\eta)$ along the trajectories of the second dynamics in (64), one gets

$$\frac{\partial W_f}{\partial \eta} \left[ g_c(x, \eta, x_{sd}) + g_c(x, x_{sd}) \tilde{x}_{sd} \right] \leq -v_1 \|\eta\|^2 + v_2$$

Thus,

$$W_f \leq -v_1 \|x\|^2 + v_2 + \left\| \frac{\partial W_f}{\partial \eta} \right\| f_1(x, \eta, x_{sd}) + g_c(x, x_{sd}) \tilde{x}_{sd}$$

(77)

In the same way, when $V_c(\epsilon)$ is differentiated along the trajectories of the third dynamics of (64), and using the results of §3 (Theorem 2.7), one obtains

$$V_c(\epsilon) \leq -\mu N \|\epsilon\|^2$$

(78)
When using (27), (42), (67)-(72) in the inequalities (76), (77) and (78) together with the property $ah \leq (k/2)a^2 + (1/2k)h^2$, with $k \in (0,1)$, one obtains

$$L \leq -a'\|x\|^2 - b'\|\eta\|^2 - c'\|e\|^2 + d' \leq -\mu_{co}L + \lambda$$

(79)

where $d' = \gamma + \beta_4/2\theta_4$. Let $\mu_{co}$ be defined by (74), it then follows that

$$L \leq -\mu_{co}L + d'$$

which implies that

$$L(x(t), \eta(t), e(t)) \leq \left[ L(x_0, \eta_0, e_0) - \frac{d'}{\mu_{co}} \right] e^{-\mu_{co}(t-t_0)} + \frac{d'}{\mu_{co}}$$

Then, the states $x$, $\eta$ and $e$ are locally ultimately bounded and will converge to the ball given by

$$(x, \eta, e) = \left\{(x, \eta, e) : L(x, \eta, e) \leq \frac{d'}{\mu_{co}}\right\}$$

This ends the proof. □

4. Application to the model of the manipulator

4.1. Model of the manipulator

A robot manipulator basically consists of $n$ links interconnected at $n$ joints into an open kinematic chain. Each link is driven by an actuator, which may be electric, hydraulic or pneumatic. The actuator may be located directly at the joint that it actuates or it may drive the link through a remote transmission of some sort. In a rigid robot model it is assumed that the coupling between the actuators and links are perfectly rigid. By contrast, in a flexible joint robot model it is assumed that the links are rigid but that the actuators are elastically coupled to the links.

In this work, we consider a robot manipulator with elastic joints which is represented by the equations (see, for example, Nicosia et al. 1988, Spong 1990)

$$H_1(q_1, \dot{q}_1) + B_1 \dot{q}_1 + C_1(q_1, \dot{q}_1) + G(q_1) + K(q_1 - q_2) = 0$$

$$H_2 \ddot{q}_2 - K(q_1 - q_2) = u$$

(80)

where $q_1$ and $q_2$ are the link positions and the actuator rotor positions, respectively, while $u$ is the input force from the actuator (motor torque). $H_1(q_1)$ is an $n \times n$ matrix which represents the inertial properties of the rigid links, $H_2$ is an $n \times n$ matrix which denotes a constant diagonal matrix depending on the rotor inertias of the motors and on the gear ratios, $B_1$ is the motor viscous friction, $C_1(q_1, \dot{q}_1)$ is the Coriolis and centrifugal terms, $K = \text{diag}(k_1, \ldots, k_n)$ is the joint stiffness matrix, $k_i > 0$ being the elastic constant of joint $i$ and $G(q_1)$ is the gravity vector.

4.2. Two time scale representation

Consider the following assignment of variables: for the slow variables $x_1 = q_1, x_2 = \dot{q}_1$, for the fast variables $z_1 = K(q_1 - q_2), z_2 = \dot{z}_1$. If it is assumed that the diagonal matrix $K$ has all large and similar elements, then we can extract a large common scalar factor $1/\varepsilon^2 \gg 1$ from $K$ such that $K (1/\varepsilon^2) = (1/\varepsilon^2) \text{diag}(r_1, \ldots, r_n)$. Then, it is possible to write the model (80) in the standard singularly perturbed form (1) and (2) with

$$f_1(x) = \left(\begin{array}{c} x_2 \\ \-H_1^{-1}(x_1) \{B_1 x_2 + C_1(x_1, x_2) x_2 + G_1(x_1)\} \end{array}\right)$$

$$F_1(x) = \left(\begin{array}{cc} 0 & 0 \\ -H_1^{-1}(x_1) & 0 \end{array}\right), \quad g_1(x) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right)$$

$$f_2(x) = \left(\begin{array}{c} \\-RH_1^{-1}(x_1) \{B_1 x_2 + C_1(x_1, x_2) x_2 + G_1(x_1)\} \\ 0 \end{array}\right)$$

$$F_2(x) = \left(\begin{array}{cc} 0 & I_{n \times n} \\ -R H_1^{-1}(x_1) + H_2^{-1} & 0_{n \times n} \end{array}\right), \quad g_2(x) = \left(\begin{array}{c} 0 \\ -R H_2^{-1} \end{array}\right)$$

4.3. Control law design

Taking $\varepsilon = 0$, one obtains a unique root which is given by

$$z_1 = (H_1^{-1}(x_1) + H_2^{-1})^{-1}$$

$$\times \left(\begin{array}{c} \-
H_1^{-1}(x_1) \{B_1 x_2 + C_1(x_1, x_2) x_2 + G_1(x_1)\} \\
\ \-H_2^{-1} u \end{array}\right)$$

$$z_2 = 0$$

and replacing the above equations, the slow reduced system then takes the form

$$\dot{x}_s = f(x_s) + g(x_s) u_s, \quad x_s(t_0) = x_0$$

with

$$f(x_s) = \left(\begin{array}{c} x_{z_2} \\ f_2(x_s) \end{array}\right)$$

$$0$$

$$g(x_s) = \left(\begin{array}{c} H_1^{-1}(x_1) (H_1^{-1}(x_1) + H_2^{-1})^{-1} H_2^{-1} \end{array}\right)$$

where
\[ f_2(x_s) = -H_1^{-1}(x_s) \{ B_1x_n + C_1(x_n, x_s) x_n + G_1(x_n) \} \]

\[ H_1^{-1}(x_s) (H_1^{-1}(x_s) + H_2^{-1})^{-1} \times (-H_1^{-1}(x_s) \{ B_1x_n + C_1(x_n, x_s) x_n + G_1(x_n) \}) \]

Since it is desired that the link position tracks a reference signal, the following slow switching function is chosen

\[ \sigma_s = S(x_s - x_{sd}) \quad (82) \]

where \( S \) is a constant matrix and \( x_{sd} = \text{col}(x_n, x_s) \) is the reference signal. It follows that the slow control is given by

\[ u_s = -[Sg(x_s)]^{-1} [SF(x_s) - SX_{sd} + L_s(x_s) S(x_s - x_{sd})] \quad (83) \]

On the other hand, in this application one just needs to stabilize the fast variables to the origin. Then, we have

\[ \frac{d\eta_{fpx}}{d\tau} = F_2(\tilde{x}) \eta_{fpx} + g_2(\tilde{x}) u_f \quad (84) \]

with

\[ F_2(\tilde{x}) = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ -R(H_1^{-1}(\tilde{x}_1) + H_2^{-1}) & 0_{n \times n} \end{pmatrix} \]

\[ g_2(\tilde{x}) = \begin{pmatrix} 0_{n \times n} \\ -R H_2^{-1} \end{pmatrix} \]

Thus, we choose the fast switching function as

\[ \sigma_f = S \eta_{fpx} \quad (85) \]

where \( S \) is constant matrix of real coefficients. Finally, the fast control is given by

\[ u_f = -[Sg_2(\tilde{x})]^{-1} [SF_2(\tilde{x}) \eta_{fpx} + L_f(\eta_{fpx}) S \eta_{fpx}] \quad (86) \]

4.1. Non-linear observer design

Now, in order to estimate the rotor position and the angular velocities of the link and rotor, a high gain observer, proposed in 42, is designed. By making the change of coordinates \( \zeta_1 = x_1, \zeta_2 = x_2, \zeta_3 = z_1 \) and \( \zeta_4 = z_2 \), one can write the singular perturbed system associated with the manipulator in the form (53) with \( y = x_1 \) the measurable output, \( \zeta = \text{col}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \) and

\[ F(y) = \begin{pmatrix} 0 & I_{n \times n} & 0 & 0 \\ 0 & 0 & -H_1^{-1}(\zeta_1) & 0 \\ 0 & 0 & 0 & I_{n \times n} / \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ G(u, y, \zeta) = \begin{pmatrix} g_2(u, y, \zeta) \\ 0 \\ 0 \end{pmatrix} \quad (87) \]

where

\[ g_2(u, y, \zeta) = -H_1^{-1}(\zeta_1) \{ B_1 \zeta_2 + C_1 (\zeta_1, \zeta_2) \zeta_2 + G_1(\zeta_1) \} \]

\[ g_4(\zeta_1, \zeta_2) = \frac{1}{\varepsilon} R H_2^{-1}(\zeta_1) \{ B_4 \zeta_2 + C_4 (\zeta_1, \zeta_2) \zeta_2 + C_4(\zeta_1, \zeta_2) \} \]

\[ - \frac{1}{\varepsilon} R H_2^{-1}(\zeta_1 + u) \]

Taking into account the properties of the inertia matrix one has that \( m_1 L_{x,n} \leq H_1(\zeta_1) \leq m_2 L_{x,n} \) and \( m_1 L_{n,n} \leq H_1^{-1}(\zeta_1) \leq m_4 L_{n,n} \). For \( m_1 > 0, i = 1, \ldots, 4 \) then it is possible construct an observer for the state variable \( \zeta \) of the form

\[ \dot{\zeta}(t) = F(y) \dot{\zeta} + G(u, y, \dot{\zeta}) - S_\theta^{-1}(y) C^T [\zeta_1, \zeta_1] \quad (88) \]

where \( S_\theta = \Omega(y) S_\theta \Omega(y) \), with \( S_\theta \) is the unique solution of (58).

4.5. Simulation results

In order to illustrate the performance of the proposed scheme, we now show some simulation results when the controller-observer scheme is applied to the model of a two degrees of freedom flexible joint robot represented by the expressions (see Nicosia et al. 1988, Spong 1990):

\[ \dot{x}_{11} = x_{21}, \quad \dot{x}_{12} = x_{22} \]

\[ \dot{x}_{21} = [k_1(x_31 - x_{11}) + c_2 x_{21} x_{22} \sin(2x_{12})] \]

\[ \div [c_1 + c_2 \cos^2(x_{12})] \]

\[ \dot{x}_{22} = (k_2/c_3)(x_{32} - x_{12}) - (d_1/c_1) \cos(x_{12}) \]

\[ + (c_2/2c_3) x_{21} \sin(x_{12}) \]

\[ \dot{x}_{31} = x_{41}, \quad \dot{x}_{32} = x_{42} \]

\[ \dot{x}_{41} = (k_1/j_1)(x_{11} - x_{31}) + (1/j_1)u_1 \]

\[ \dot{x}_{42} = (k_2/j_2)(x_{12} - x_{32}) + (1/j_2)u_2 \]
where \( x_{11} = q_{11}, x_{12} = q_{12} \) are the link angular positions, \( x_{21} = \dot{q}_{11}, x_{22} = \dot{q}_{12} \) are the link angular velocities, \( x_{31} = q_{21}, x_{32} = q_{22} \) are the rotor positions and \( x_{41} = \dot{q}_{21}, x_{42} = \dot{q}_{22} \) are the rotor velocities. The parameters of the model are \( c_1 = 0.084, c_2 = 22.058, c_3 = 10.53, j_1 = 23.296, j_2 = 70.656, d_1 = 291.106, k_1 = 29800, k = 14210. \) The matrices \( S, S', L_s(x_s) \) and \( L_f(\eta_{lpx}) \) are given by

\[
S = \begin{pmatrix}
s_1 & 0 & s_2 & 0 \\
0 & s_3 & 0 & s_4 \\
0 & s_3 & 0 & s_4
\end{pmatrix}, \\
S' = \begin{pmatrix}
s_1 & 0 & s_2 & 0 \\
0 & s_3 & 0 & s_4 \\
0 & s_3 & 0 & s_4
\end{pmatrix},
\]

\[
L_s(x_s) = \begin{pmatrix}
l_f & 0 \\
l_f & 0 \\
l_f & 0
\end{pmatrix}, \\
L_f(\eta_{lpx}) = \begin{pmatrix}
l_f & 0 \\
l_f & 0 \\
l_f & 0
\end{pmatrix}
\]

where \( s_1 = 1, s_2 = 2, l_f = 100, s_3 = 1, s_4 = 2, l_f = 100, \\
s_1 = 0.1, s_2 = 1.0, l_f = 1, s_3 = 0.1, s_4 = 1.0, l_f = 1.0, \\
\theta = 15. \) The reference signal to be tracked was set as \( x_{sd} = 2 + 2 \sin (t), x_{sd} = 2 + 2 \cos (t). \)

The time closed-loop plots showing the dynamic behaviour of the link angular positions \( (x_{11} = q_{11}, x_{12} = q_{12}) \), the reference signals \( (x_{sd}, x_{sd}) \), the link angular velocities \( (x_{21} = \dot{q}_{11}, x_{22} = \dot{q}_{12}) \) and the time derivative of the reference signals \( (\dot{x}_{sd}, \dot{x}_{sd}) \) are plotted in figures 1–4. Also the composite control \( u_1 = u_{1f} + u_{1f} \) and \( u_2 = u_{2f} + u_{2f} \) are plotted in figure 5. In order to illustrate the behaviour of tracking errors, we plot in figures 6–9. \( e_{track1} = x_{d1} - x_{sd1}, e_{track2} = x_{d2} - x_{sd2}, e_{track3} = x_{d3} - x_{sd3}, e_{track4} = x_{d4} - x_{sd4}. \) From these plots, one can note that a good tracking performance is obtained. The estimation errors \( e_{est1} = x_{11} - \hat{x}_{11}, e_{est2} = x_{12} - \hat{x}_{12}, e_{est3} = x_{21} - \hat{x}_{21} \) and \( e_{est4} = x_{22} - \hat{x}_{22} \) are given in figures 10–13.
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Figure 6. Position tracking error $e_{\text{track}} = x_{sd} - q_1$.

Figure 10. Estimating error $e_{\text{est}} = y_{11} - x_{11}$.

Figure 7. Position tracking error $e_{\text{track}} = x_{nl} - q_2$.

Figure 11. Estimating error $e_{\text{est}} = y_{12} - x_{12}$.

Figure 8. Velocity tracking error $e_{\text{track}} = x_{21} - \dot{x}_{sd}$.

Figure 12. Estimating error $e_{\text{est}} = y_{21} - x_{21}$.

Figure 9. Velocity tracking error $e_{\text{track}} = x_{22} - \dot{x}_{sd}$.

Figure 13. Estimating error $e_{\text{est}} = y_{22} - x_{22}$.
5. Conclusions
A non-linear control-observer structure based on a class of non-linear singularly perturbed systems and canonical representations has been presented. A composite control using singular perturbation methods and sliding-modes and high gain observer were applied to the model of a single-link flexible joint robot manipulator. A good trajectory tracking performance approach was applied for the control of such electromechanical devices. Moreover, an analysis of stability of the clod-loop augmented system was presented.

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