Super-simple balanced incomplete block designs with block size 4 and index 5

Haitao Cao\textsuperscript{a}, Kejun Chen\textsuperscript{b}, Ruizhong Wei\textsuperscript{c,*}

\textsuperscript{a} School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing 210097, China
\textsuperscript{b} Department of Mathematics, Yancheng Teachers University, Jiangsu, 224002, China
\textsuperscript{c} Department of Computer Science, Lakehead University, Thunder Bay, ON, P7B 5E1 Canada

\begin{abstract}

In statistical planning of experiments, super-simple designs are the ones providing samples with maximum intersection as small as possible. Super-simple designs are also useful in other constructions, such as superimposed codes and perfect hash families etc. The existence of super-simple \((v, 4, \lambda)\)-BIBDs have been determined for \(\lambda = 2, 3, 4\) and 6. When \(\lambda = 5\), the necessary conditions of such a design are that \(v \equiv 1, 4 \mod 12\) and \(v \geq 13\). In this paper, we show that there exists a super-simple \((v, 4, 5)\)-BIBD for each \(v \equiv 1, 4 \mod 12\) and \(v \geq 13\).

\end{abstract}

\section{Introduction}

A group divisible design (or GDD), is a triple \((X, \mathcal{G}, \mathcal{B})\) which satisfies the following properties:
1. \(\mathcal{G}\) is a partition of a set \(X\) (of points) into subsets called \emph{groups};
2. \(\mathcal{B}\) is a set of subsets of \(X\) (called \emph{blocks}) such that a group and a block contain at most one common point;
3. Every pair of points from distinct groups occurs in exactly \(\lambda\) blocks.

The \emph{group type} (or \emph{type}) of a GDD is the multiset \(\{ |G| : G \in \mathcal{G} \}\). We shall use an “exponential” notation to describe types: so type \(g_1^{i_1} \cdots g_k^{i_k}\) denotes \(i_1\) occurrences of \(g_1, \ldots, g_k\) in the multiset. A GDD with block sizes from a set of positive integers \(K\) is called a \((K, \lambda)\)-GDD. When \(K = \{k\}\), we simply write \(k\) for \(K\). When \(\lambda = 1\), we simply write it as \(K\)-GDD. A \((K, \lambda)\)-GDD with group type \(1^t\) is called a \emph{pairwise balanced design}, denoted by \((v, K, \lambda)\)-PBD. A \((K, \lambda)\)-GDD with group type \(1^t\) is called a \emph{balanced incomplete block design}, and denoted by \((v, k, \lambda)\)-BIBD.

A \emph{transversal design}, TD\((k, \lambda; n)\), is a \((k, \lambda)\)-GDD of group type \(n^k\) and block size \(k\). When \(\lambda = 1\), we simply write TD\((k, n)\). It is well known that a TD\((k, n)\) is equivalent to \(k - 2\) mutually orthogonal Latin squares (MOLS) of order \(n\). For a list of lower bounds on the number of MOLS for orders up to 10,000, we refer the reader to [1]. We shall denote by \(N(n)\) the maximum number of MOLS of order \(n\).

In this paper, we shall employ the following known results.

\begin{lemma} ([1]) \end{lemma}

1. A TD\((q + 1, q)\) exists, consequently, a TD\((k, q)\) exists for any positive integer \(k (k \leq q)\), where \(q\) is a prime power.
2. A TD\((5, n)\) exists for all \(n \geq 4\) and \(n \neq 6, 10\).
3. A \((v, \{4, 5, 6\}, 1)\)-PBD exists for all \(v \geq 13\) and \(v \neq 14, 15, 18, 19, 23\).
4. A 4-GDD of type \(m^n\) exists if and only if \(u \geq 3, (u - 1)m \equiv 0 \mod 4\) and \(u(u - 1)m^2 \equiv 0 \mod 12\) except \((m, u) \in \{(2, 4), (6, 4)\}\).

* Corresponding author.
E-mail addresses: caohaitao@njnu.edu.cn (H. Cao), kejunchen1@yahoo.com.cn (K. Chen), wei@peace.lakeheadu.ca (R. Wei).

0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.07.003
A design is called simple if it contains no repeated blocks. A design is said to be super-simple if the intersection of any two blocks has at most two elements. When \( k = 3 \), a super-simple design is just a simple design. When \( \lambda = 1 \), the designs are necessarily super-simple. In this paper, when we talk about super-simple BIBDs, we usually mean the case \( k \geq 4 \) and \( \lambda > 1 \).

The term super-simple designs was introduced by Gronau and Mullin in [12]. The existence of super-simple designs is an interesting extremal problem by itself, but there are also some useful applications. For example, such super-simple designs are used in perfect hash families [18] and coverings [4], in the construction of new designs [3] and in the construction of superimposed codes [17]. In statistical planning of experiments, super-simple designs are the ones providing samples with a maximum intersection as small as possible.

It is well known that the following are the necessary conditions for the existence of a super-simple \((v, k, \lambda)\)-BIBD:

1. \( v \geq (k-2)\lambda + 2 \);
2. \( \lambda(v-1) \equiv 0 \pmod{k-1} \);
3. \( \lambda \nu(v-1) \equiv 0 \pmod{k(k-1)} \).

For arbitrary \( k \) and \( \lambda \), the above necessary conditions are asymptotically sufficient (see [13–15]). For the existence of super-simple \((v, 4, \lambda)\)-BIBDs, the necessary conditions are known to be sufficient for \( \lambda = 2, 3, 4, 6 \). Gronau and Mullin [12] solved the case for \( \lambda = 2 \), and the corrected proof appeared in [16]. The \( \lambda = 3 \) case was solved independently by Khodkar [16] and Chen [6]. The \( \lambda = 4 \) case was solved independently by Adams et al. [2] and Chen [7]. The \( \lambda = 6 \) case was solved by Chen, Cao and Wei [8]. A recent survey on super-simple \((v, 4, \lambda)\)-BIBDs with \( v \leq 32 \) and all admissible \( \lambda \) can be found in [5]. We summarize these known results in the following theorem.

**Theorem 1.2** ([12,16,6,27,8]). A super-simple \((v, 4, \lambda)\)-BIBD exists for \( \lambda = 2, 3, 4, 6 \) if and only if the following conditions are satisfied:

1. \( \lambda = 2 \), \( v \equiv 1 \pmod{3} \) and \( v \geq 7 \);
2. \( \lambda = 3 \), \( v \equiv 0 \pmod{4} \) and \( v \geq 8 \);
3. \( \lambda = 4 \), \( v \equiv 1 \pmod{3} \) and \( v \geq 10 \);
4. \( \lambda = 6 \), \( v \geq 14 \).

In this paper we investigate the existence of super-simple \((v, 4, 5)\)-BIBDs. Clearly, when \( k = 4 \) and \( \lambda = 5 \) the necessary condition becomes \( v \equiv 1 \pmod{12} \) and \( v \geq 13 \). We shall use direct and recursive constructions to show that the necessary condition is also sufficient.

Ref. [11] in Handbook of Combinatorial Designs was written when we are preparing this paper. So partial results of this paper are included in [11] without proofs and reference. Now we shall give a complete proof for the case \( \lambda = 5 \).

### 2. Recursive constructions

We shall use the following basic constructions, for which the proofs can be found in [7].

**Construction 2.1** (Weighting). Let \((X, \mathcal{G}, \mathcal{B})\) be a super-simple GDD with index \( \lambda_1 \), and let \( w : X \to Z^+ \cup \{0\} \) be a weight function on \( X \), where \( Z^+ \) is the set of positive integers. Suppose that for each block \( B \in \mathcal{B} \), there exists a super-simple \((k, \lambda_2)\)-GDD of type \( \{w(x) : x \in B\} \). Then there exists a super-simple \((k, \lambda_1 \lambda_2)\)-GDD of type \( \{\sum_{x \in G_i} w(x) : G_i \in \mathcal{G}\} \).

**Construction 2.2** (Breaking up Groups). If there exists a super-simple \((k, \lambda)\)-GDD of type \( h_1^{\nu_1} \cdots h_t^{\nu_t} \) and a super-simple \((h_i + \eta, k, \lambda)\)-BIBD for each \( i \) \((1 \leq i \leq t)\), then there exists a super-simple \((\sum_{i=1}^t h_i \eta + \eta, k, \lambda)\)-BIBD, where \( \eta = 0 \) or 1.

To present the next construction, we need the notation of \((v, w, k, \lambda)\)-IBIBD. An incomplete balanced incomplete block design \((v, w, k, \lambda)\)-IBIBD is a triple \((V, H, \mathcal{B})\) which satisfies the following properties:

1. \( V \) is a \( v \)-set of points, \( H \) is a \( w \)-subset of \( V \) (called a hole) and \( \mathcal{B} \) is a collection of \( k \)-subsets of \( V \) (called blocks);
2. \( |H \cap B| \leq 1 \) for all \( B \in \mathcal{B} \);
3. any two points of \( V \) appear either in \( H \) or in \( \lambda \) blocks of \( \mathcal{B} \) exactly.

Now we give a recursive construction for super-simple BIBDs by using incomplete super-simple BIBDs. It’s obvious that a \((v, w, k, \lambda)\)-IBIBD is a \((v, k, \lambda)\)-BIBD indeed when \( w \in \{0, 1\} \). So, the following construction can be considered as a generalization of Construction 2.2.

**Construction 2.3** (Filling in Holes). Suppose that there exists a super-simple \((k, \lambda)\)-GDD of type \( h_1 h_2 \cdots h_t \), a super-simple \((h_i + s, k, \lambda)\)-IBIBD for each \( i \) \((1 \leq i \leq t - 1)\), and a super-simple \((h_t + s, k, \lambda)\)-BIBD, then there exists a super-simple \((\sum_{i=1}^t h_i + s, k, \lambda)\)-BIBD.
Lemma 3.4. There exists a super-simple $\binom{v}{4, 5}$-BIBD for $v \equiv 1 \pmod{12}$. For convenience, we denote by $[a, b]$ the set of integers $c$ such that $a \leq c \leq b$, and $[a, b]_2$ the set of integers $c$ such that $a \leq c \leq b$ and $c \equiv 1 \pmod{12}$. In our proofs, we need the following result on super-simple TD$(4, \lambda; v)$ which can be found in Hartman [14].

Lemma 3.1 ([14]). A super-simple TD$(4, \lambda; v)$ exists if and only if $\lambda \leq v$ and $(\lambda, v)$ is neither $(1, 2)$ nor $(1, 6)$.

For the first two small values, Bluskov and Heinrich in [5] proved the following.

Lemma 3.2 ([5]). There exists a super-simple $(v, 4, 5)$-BIBD for $v = 13, 25$.

We shall first use direct constructions to obtain super-simple $(v, 4, 5)$-BIBDs for some small values $v$ and some super-simple $(4, 5)$-GDDs, which will be used as master designs or input designs in our recursive constructions. All of these designs have been found after computer-assisted searches. In fact, all of them have cyclic groups of automorphism of order $v$. So, they are cyclic designs.

The checking for super-simplicity can be done by a computer after developing the designs. But there are more economical ways to check the super-simplicity of cyclic designs. For details, we refer the reader to [5].

In computer searching, a method we used in computer program is applying multipliers of blocks. Since our constructions are over $\mathbb{Z}_n$, we can use both the addition and the multiplication of $\mathbb{Z}_n$. We say that $w \in \mathbb{Z}_n^*$ is a multiplier of the design, if for each base block $B = \{x_1, x_2, x_3, x_4\}$, there exists some $g \in \mathbb{Z}_n$ such that $C = w \cdot B + g = \{w \cdot x_1 + g, w \cdot x_2 + g, w \cdot x_3 + g, w \cdot x_4 + g\}$ is also a base block. We say that $w \in \mathbb{Z}_n^*$ is a partial multiplier of the design, if for each base block $B \in \mathcal{M}$, where $\mathcal{M}$ is a subset of all the base blocks, there exists some $g \in \mathbb{Z}_n$ such that $C = w \cdot B + g$ is also a base block.

In the computer program, we first choose a (partial) multiplier $w$. Our experiences tell us that choosing a $w$ which has long orbits in the multiplication group of $\mathbb{Z}_n$ usually gives better results. Then we start to find base blocks in the following way. When a base block $B$ is found, the algorithm requires that $wB, w^2B, \ldots, w^tB$ can also be different base blocks, where $s$ is a positive number. If we can find all the base blocks in this way, then $w^i, 1 \leq i \leq s$ are multipliers of the design. Otherwise, these are partial multipliers, and the algorithm tries to find the remaining base blocks. To decide the value of $s$ is also important for the success of the algorithm. In practice, we usually let $s$ be as large as possible at the beginning. Then the value of $s$ is reduced if the search time is too long.

Lemma 3.3. There exists a super-simple $(37, 4, 5)$-BIBD.

Proof. The point set is $\mathbb{Z}_{37}$. Below are the required base blocks.

\[
\begin{align*}
&\{0, 1, 2, 4\}, \{0, 1, 5, 6\}, \{0, 1, 7, 9\}, \{0, 2, 5, 14\}, \{0, 2, 16, 21\}.
&\{0, 3, 7, 20\}, \{0, 3, 11, 17\}.
\end{align*}
\]

The following super-simple GDDs will be used as master designs or input designs in our recursive constructions.

Lemma 3.4. There exists a super-simple $(4, 5)$-GDD of group type $3^9$.

Proof. Let the point set be $\mathbb{Z}_{4t}$ and let the groups be $\{i, 9 + i, 18 + i\} : 0 \leq i \leq 8$. The required base blocks are divided into two parts: $P$ and $R$, where $P$ consists of some base blocks with a partial multiplier $2$ of order $6$, (i.e., each base block of $P$ has to be multiplied by $2^i$ for $0 \leq i \leq 5$), and $R$ is the set of the remaining base blocks. We list $P$ and $R$ below. The desired super-simple design is generated by developing the base blocks modulo 27.

\[
\begin{align*}
P : &\{0, 1, 2, 5\},
R : &\{0, 1, 3, 15\}, \{0, 1, 7, 13\}, \{0, 3, 7, 20\}, \{0, 3, 11, 17\}.
\end{align*}
\]

Lemma 3.5. There exists a super-simple $(4, 5)$-GDD of group type $4^t$ for $t \in \{7, 10\}$.

Proof. For every $t \in \{7, 10\}$, let the point set be $\mathbb{Z}_4$, and let the group set be $\{i, t + i, 2t + i, 3t + i\} : 0 \leq i \leq t - 1$. Similar to the proof of Lemma 3.4, we only list $P, m, s$ and $R$ below. The desired super-simple design is generated by developing all the base blocks modulo $4t$.

\[
\begin{align*}
t = 7 &\{0, 1, 3, 11\}, m = 3, s = 6;
R : &\{0, 3, 8, 23\}, \{0, 2, 8, 12\}, \{0, 1, 12, 17\}\{0, 4, 13, 17\}.
\end{align*}
\]

\[
\begin{align*}
t = 10 &\{0, 1, 5, 7\}, \{0, 1, 2, 14\}, \{0, 1, 18, 33\}, m = 3, s = 4;
R : &\{0, 4, 11, 23\}, \{0, 5, 16, 24\}, \{0, 7, 24, 32\}.
\end{align*}
\]
Lemma 3.6. There exists a super-simple \((4, 5)\)-GDD of group type \((12)^t\) for any \(t \in [4, 11]\).

**Proof.** For \(t = 4\), a super-simple \((4, 5)\)-GDD of group type \((12)^4\) is given by Lemma 3.1.

For \(t = 9\), starting from a super-simple \((4, 5)\)-GDD of group type \((12)^9\), coming from Lemma 3.4 and applying Construction 2.1 with a TD\((4, 3)\) coming from Lemma 1.1, we obtain a super-simple \((4, 5)\)-GDD of group type \((12)^9\).

For \(t = 7, 10\), starting from a super-simple \((4, 5)\)-GDD of group type \((12)^t\) coming from Lemma 3.5 and applying Construction 2.1 with a TD\((4, 3)\) coming from Lemma 1.1, we obtain a super-simple \((4, 5)\)-GDD of group type \((12)^t\).

For every \(t \in [5, 6, 8, 11]\), let the point set be \(\mathbb{Z}_{2t}\), and let the group set be \(\{(i, t + i, 2t + i, \ldots, 11t + i) : 0 \leq i \leq t - 1\}\). Below are the required base blocks, which are divided into two parts, \(P\) and \(R\). Each of the base blocks of \(P\) has to be multiplied by \(m^t\) with \(0 \leq i \leq s - 1\). The required design is generated by developing the following base blocks modulo \(12t\).

- **Case 1:** \(t = 5\)
  \[
P = \{0, 1, 23, 39\}, \{0, 1, 13, 17\}, \{0, 2, 21, 53\}, \{0, 2, 18, 49\}, \{0, 2, 39, 48\}, \{0, 6, 53, 57\};
  \]
  \[
  R = \emptyset.
  \]

- **Case 2:** \(t = 6\)
  \[
P = \{0, 1, 2, 4\}, \{0, 1, 8, 39\}, \{0, 1, 59, 68\}, \{0, 8, 21, 35\}, m = 5, s = 4;
  \]
  \[
  R = \{0, 9, 26, 49\}, \{0, 14, 43, 69\}, \{0, 1, 16, 44\}, \{0, 4, 26, 69\}, \{0, 8, 46, 57\}, \{0, 14, 27, 34\},
  \]
  \[
  \{0, 16, 56, 59\}, \{0, 20, 40, 57\}, \{0, 10, 21, 49\}.
  \]

- **Case 3:** \(t = 8\)
  \[
P = \{0, 1, 2, 85\}, \{0, 1, 7, 14\}, \{0, 1, 11, 28\}, \{0, 3, 9, 36\}, \{0, 3, 25, 46\}, m = 5, s = 5;
  \]
  \[
  R = \{0, 4, 23, 77\}, \{0, 18, 22, 76\}, \{0, 9, 61, 79\}, \{0, 19, 33, 77\}, \{0, 4, 37, 57\}, \{0, 26, 28, 79\},
  \]
  \[
  \{0, 20, 26, 54\}, \{0, 20, 43, 50\}, \{0, 4, 22, 66\}, \{0, 4, 14, 34\}.
  \]

- **Case 4:** \(t = 11\)
  \[
P = \{0, 1, 2, 58\}, \{0, 1, 5, 6\}, \{0, 1, 7, 9\}, \{0, 2, 5, 8\}, \{0, 2, 6, 9\}, \{0, 3, 9, 94\}, \{0, 4, 24, 60\},
  \]
  \[
  \{0, 9, 26, 38\}, \{0, 9, 29, 76\}, \{0, 10, 23, 46\}, m = 25, s = 5;
  \]
  \[
  R = \emptyset. \quad \Box
  \]

Lemma 3.7. There exists a super-simple \((v, 4, 5)\)-BIBD for any \(v = 12t + 1\), where \(t \geq 4\) and \(t \neq 12\), \(14, 15, 18, 19, 23\).

**Proof.** By Lemma 1.1, a \((t, \{4, 5, 6\}, 1)\)-PBD exists for all \(t \geq 13\) and \(t \neq 14, 15, 18, 19, 23\). Applying Construction 2.1 with a super-simple \((4, 5)\)-GDD of group type \((12)^t\), \(h = 4, 5, 6\), coming from Lemma 3.6, we get a super-simple \((4, 5)\)-GDD of group type \((12)^t\) for all \(t \geq 13\) and \(t \neq 14, 15, 18, 19, 23\). Combining with Lemma 3.6, we have obtained a super-simple \((4, 5)\)-GDD of group type \((12)^t\) for each \(t \geq 4\) and \(t \neq 12, 14, 15, 18, 19, 23\). Since there exists a super-simple \((12 + 1, 4, 5)\)-BIBD from Lemma 3.2, by Construction 2.2 we obtain a super-simple \((12t + 1, 4, 5)\)-BIBD. \(\Box\)

Lemma 3.8. There exists a super-simple \((v, 4, 5)\)-BIBD for any \(v = 12t + 1 + t = 12, 14, 15\).

**Proof.** For \(v = 12 \times 12 + 1 = 145\), starting from a super-simple \((4, 5)\)-GDD of group type \((12)^4\) coming from Lemma 3.6 and applying Construction 2.1 with a TD\((4, 3)\) coming from Lemma 1.1, we get a super-simple \((4, 5)\)-GDD of group type \((36)^4\). Since there exists a super-simple \((36 + 1, 4, 5)\)-BIBD from Lemma 3.3, a super-simple \((145, 4, 5)\)-BIBD is obtained by Construction 2.2.

For \(v = 12 \times 14 + 1 = 169\), starting from a \((4, 1)\)-GDD of group type \((12)^1\) coming from Lemma 1.1 and applying Construction 2.1 with a super-simple \((4, 5)\)-GDD of group type \((6)^6\) coming from Lemma 3.1, we obtain a super-simple \((4, 5)\)-GDD of group type \((24)^7\). Since there exists a super-simple \((24 + 1, 4, 5)\)-BIBD from Lemma 3.2, by Construction 2.2 we obtain a super-simple \((169, 4, 5)\)-BIBD.

For \(v = 12 \times 15 + 1 = 181\), starting from a super-simple \((4, 5)\)-GDD of group type \((12)^5\) coming from Lemma 3.7 and applying Construction 2.1 with a TD\((4, 3)\) coming from Lemma 1.1, we obtain a super-simple \((4, 5)\)-GDD of group type \((36)^5\). Since there exists a super-simple \((36 + 1, 4, 5)\)-BIBD from Lemma 3.3, by Construction 2.2 we obtain a super-simple \((181, 4, 5)\)-BIBD. \(\Box\)

Lemma 3.9. There exists a super-simple \((v, 4, 5)\)-BIBD for any \(v = 12t + 1 + t = 18, 19, 23\).

**Proof.** For \(t = 18, 19\), take a TD\((5, 4)\) and remove one or two points from the last group to obtain two \((4, 5)\)-GDDs of group type \((4^3)^1\) or \((4^2)^1\). Applying Construction 2.1 with super-simple \((4, 5)\)-GDDs of group type \((12)^4\) and \((12)^5\) coming from Lemma 3.7, we obtain two super-simple \((4, 5)\)-GDDs of group type \((48)^4(24)^1\) and \((48)^3(36)^1\). Since there exists a super-simple \((24 + 1, 4, 5)\)-BIBD and a super-simple \((36 + 1, 4, 5)\)-BIBD from Lemmas 3.2 and 3.3, by Construction 2.2 we obtain a super-simple \((12 \times 18 + 1, 4, 5)\)-BIBD and a super-simple \((12 \times 19 + 1, 4, 5)\)-BIBD.
For \( t = 23 \), take a TD(5, 5) and remove two points from the last group to obtain a \([4, 5]\)-GDDs of group type \( 5^4 3 \). Applying Construction 2.1 with super-simple \([4, 5]\)-GDDs of group type \((12)^4 \) and \((12)^5 \) coming from Lemma 3.7, we obtain a super-simple \([4, 5]\)-GDD of group type \((60 + 1, 4, 5)\)-BIBD and a super-simple \((36 + 1, 4, 5)\)-BIBD from Lemmas 3.7 and 3.3, by Construction 2.2 we obtain a super-simple \((12 \times 23 + 1, 4, 5)\)-BIBD.

\[ \Box \]

Combining Lemmas 3.2, 3.3 and 3.7–3.9, we have the following theorem.

**Theorem 3.10.** A super-simple \((v, 4, 5)\)-BIBD exists for any \( v \equiv 1 \text{ (mod } 12) \) and \( v \geq 13 \).

4. \( v \equiv 4 \text{ (mod } 12) \)

In this section, we shall prove that there exists a super-simple \((v, 4, 5)\)-BIBD for every \( v \equiv 4 \text{ (mod } 12) \) and \( v \geq 16 \). We shall distinguish four cases, \( v \equiv 4, 16, 28, 40 \text{ (mod } 48) \).

**Lemma 4.1.** If there exists a super-simple \((v, 4, 5)\)-BIBD, then there exists a super-simple \((4v, 4, 5)\)-BIBD.

**Proof.** A super-simple TD(4, 5; \( v \)) exists from Lemma 3.1. Since there exists a super-simple \((v, 4, 5)\)-BIBD, by Construction 2.2 we get a super-simple \((4v, 4, 5)\)-BIBD.

**Lemma 4.2.** There exists a super-simple \((48t + 4, 4, 5)\)-BIBD for all \( t \geq 1 \).

**Proof.** For each \( t \geq 1 \), we have \( 48t + 4 = 4(12t + 1) \). By Theorem 3.10 there exists a super-simple \((12t + 1, 4, 5)\)-BIBD, the conclusion follows from Lemma 4.1.

For some small values \( v \), we construct the super-simple designs by direct constructions. We have the following.

**Lemma 4.3.** There exists a super-simple \((v, 4, 5)\)-BIBD for any \( v \in \{16, 28, 40, 88, 124\} \).

**Proof.** For each \( v \in \{16, 28, 40\} \), a super-simple \((v, 4, 5)\)-BIBD was shown in [5]. For each \( v \in \{40, 88, 124\} \), let the point set be \( \mathbb{Z}_v \). Below are the required base blocks, which are divided into two parts, \( P \) and \( R \). Each of the base blocks of \( P \) has to be multiplied by \( m^t \) with \( 0 \leq i \leq s - 1 \). The required designs are obtained by developing the following base blocks modulo \( v \). Here, the last base block \([0, v/4, 2v/4, 3v/4]\) has a short orbit of order \( v/4 \).

\[
\begin{align*}
v &= 40, \\
P &\{0, 1, 2, 4\}, \{0, 1, 5, 7\}, \{0, 1, 6, 17\}, m = 3, s = 3; \\
R &\{0, 14, 16, 26\}, \{0, 12, 13, 23\}, \{0, 13, 20, 36\}, \{0, 10, 21, 35\}, \{0, 12, 19, 32\}, \{0, 8, 21, 32\}, \\
&\{0, 2, 10, 25\}, \{0, 10, 20, 30\}. \\
v &= 88, \\
P &\{0, 1, 3, 32\}, \{0, 2, 5, 41\}, \{0, 5, 24, 39\}, \{0, 4, 52, 82\}, m = 7, s = 7; \\
R &\{0, 50, 61, 72\}, \{0, 11, 15, 43\}, \{0, 55, 65, 66\}, \{0, 22, 51, 68\}, \{0, 25, 38, 44\}, \{0, 25, 33, 55\}, \\
&\{0, 11, 18, 44\}, \{0, 13, 37, 68\}, \{0, 22, 44, 66\}. \\
v &= 124, \\
P &\{0, 1, 2, 12\}, \{0, 1, 5, 21\}, \{0, 2, 20, 59\}, \{0, 22, 46, 75\}, m = 3, s = 10; \\
R &\{0, 41, 48, 90\}, \{0, 4, 49, 63\}, \{0, 31, 81, 83\}, \{0, 13, 27, 112\}, \{0, 12, 86, 117\}, \{0, 70, 96, 109\}, \\
&\{0, 21, 84, 120\}, \{0, 6, 31, 93\}, \{0, 4, 32, 55\}, \{0, 29, 84, 107\}, \{0, 16, 42, 78\}, \{0, 31, 62, 93\}. \Box
\end{align*}
\]

To obtain the other three classes of super-simple \((48t + s, 4, 5)\)-BIBDs, \( s \in \{16, 28, 40\} \), we need the following lemma on \( 4\)-GDDs which can be found in [9].

**Lemma 4.4** ([9, 10]). (i) There exists a \( 4\)-GDD of group type \( 2^u m^1 \) for each \( u \geq 6, u \equiv 0 \text{ (mod } 3) \) and \( m \equiv 2 \text{ (mod } 3) \) with \( 2 \leq m \leq u - 1 \) except for \((u, m) = (6, 5)\) and possibly excepting \((u, m) \in \{(21, 17), (33, 23), (33, 29), (39, 35), (57, 44)\}\). (ii) There exists a \( 4\)-GDD of group type \( 4^u m^1 \) for each \( u \geq 6, u \equiv 0 \text{ (mod } 3) \) and \( m \equiv 1 \text{ (mod } 3) \) with \( 1 \leq m \leq 2(u - 1) \). (iii) There exists a \( 4\)-GDD of group type \( 6^u 3^1 \) for each \( u \geq 4 \).

**Lemma 4.5.** There exists a super-simple \((48t + 16, 4, 5)\)-BIBD for each \( t \geq 0 \).

**Proof.** By Lemma 4.3, there exists a super-simple \((16, 4, 5)\)-BIBD. Consequently, there exists a super-simple \((64, 4, 5)\)-BIBD by Lemma 4.1.

For \( t \geq 2 \), by Lemma 4.4(i) there exists a \( 4\)-GDD of group type \( 2^{3t + 1} \). Starting from this GDD and applying Construction 2.1 with a super-simple \((4, 5)\)-GDD of group type \( 8^4 \) coming from Lemma 3.1, we obtain a super-simple \((4, 5)\)-GDD of group type \((16)^{3t + 1}\). Since there exists a super-simple \((16, 4, 5)\)-BIBD, by Construction 2.2 we obtain a super-simple \((48t + 16, 4, 5)\)-BIBD. \( \Box \)
Lemma 4.6. There exists a super-simple \((48t + 40, 4, 5)\)-BIBD for each \(t \geq 0\).

**Proof.** For \(t = 0, 1\), a super-simple \((40, 4, 5)\)-BIBD and a \((88, 4, 5)\)-BIBD were provided in Lemma 4.3.

For \(t = 2\), starting from a super-simple \((4, 5)\)-GDD of group type \(3^9\) coming from Lemma 3.4 and applying Construction 2.1 with a TD(4, 5) coming from Lemma 1.1, we get a super-simple \((4, 5)\)-GDD of group type \((15)^9\). Since there exists a super-simple \((15 + 1, 4, 5)\)-BIBD, by Construction 2.2 we obtain a super-simple \((136, 4, 5)\)-BIBD.

For \(t \geq 3\), by Lemma 4.4(i) there exists a 4-GDD of group type \(2^{3L}5^1\). Starting from this GDD and applying Construction 2.1 with a super-simple \((4, 5)\)-GDD of group type \(8^4\) coming from Lemma 3.1, we obtain a super-simple \((4, 5)\)-GDD of group type \((16)^{3L}(40)^1\). Since there exist a super-simple \((16, 4, 5)\)-BIBD and a super-simple \((40, 4, 5)\)-BIBD, by Construction 2.2 we obtain a super-simple \((48t + 40, 4, 5)\)-BIBD. □

Now we consider the last case \(v \equiv 28 \text{ (mod 48)}\). To solve this class, we need the following super-simple \((52, 4, 4, 5)\)-IBIBD.

**Lemma 4.7.** There exists a super-simple \((52, 4, 4, 5)\)-IBIBD.

**Proof.** Let the point set \(V = Z_{48} \cup H, H = \{\infty_1, \infty_2, \infty_3, \infty_4\}\). The required block set \(B\) will contain three parts of blocks.

The first part contains \(12\) blocks \(B_i(0 \leq i \leq 11)\) which are generated from the base block \(B_0 = \{0, 12, 24, 36\}\), where \(B_i = B_0 + i\). The second part contains \(48 \times 16\) blocks which can be obtained from the following \(16\) base blocks by \(+ 1 \text{ mod } 48\):

\[
\begin{array}{cccccccc}
0 & 10 & 20 & 33 & 0 & 3 & 18 & 29 \\
0 & 6 & 12 & 19 & 0 & 6 & 14 & 21 \\
0 & 9 & 22 & 36 & 0 & 1 & 3 & 20 \\
0 & 2 & 13 & 29 & 0 & 4 & 20 & 27 \\
\end{array}
\]

The last part contains \(20 \times 16\) blocks which can be obtained as follows. For each block in the following, we can obtain 16 blocks by \(+ 3 \text{ mod } 48\), these 16 blocks form a partition or a parallel class of \(Z_{48}\). In this way, we obtain 10 parallel classes \(P_t(1 \leq i \leq 10)\) of \(Z_{48}\). Further, let \(Q_t = \{B + 1 : B \in P_t\}\). Thus we can obtain another 10 parallel classes \(Q_t(1 \leq i \leq 10)\) of \(Z_{48}\). Now add \(\infty_1\) to each block in \(P_i(1 \leq i \leq 5)\), \(\infty_2\) to each block in \(P_i(6 \leq i \leq 10)\), \(\infty_3\) to each block in \(Q_i(1 \leq i \leq 5)\), and \(\infty_4\) to each block in \(Q_i(6 \leq i \leq 10)\). These blocks form the last part of our construction.

So we have obtained \(12 + 48 \times 16 + 20 \times 16 = 1100\) blocks. It is checked by the computer that these blocks form a super-simple \((52, 4, 4, 5)\)-IBIBD. □

**Lemma 4.8.** There exists a super-simple \((4, 5)\)-GDD of group type \(9^5\).

**Proof.** Let the point set be \(Z_{45}\) and the group set be \(\{(i, 5 + i, 10 + i, \cdots, 40 + i) : 0 \leq i \leq 4\}\). Below are the required base blocks, which are divided into two parts, \(P\) and \(R\). Each of the base blocks of \(P\) has to be multiplied by \(2^i\) with \(0 \leq i \leq 5\). The desired super-simple design is obtained by developing the base blocks modulo 45.

\[
P : \{0, 1, 2, 8\}, \{0, 2, 9, 26\}; \\
R : \{0, 3, 9, 21\}, \{0, 3, 22, 36\}, \{0, 6, 17, 24\}. \]

**Lemma 4.9.** There exists a super-simple \((v, 4, 5)\)-BIBD for each \(v = 76, 172\).

**Proof.** For \(v = 76\), starting from a \((4, 1)\)-GDD of group type \(3^5\) coming from Lemma 1.1 and applying Construction 2.1 with a super-simple \((4, 5)\)-GDD of group type \(5^4\) coming from Lemma 3.1, we obtain a super-simple \((4, 5)\)-GDD of group type \((15)^5\). Since there exists a super-simple \((15 + 1, 4, 5)\)-BIBD from Lemma 3.2, by Construction 2.2 we obtain a super-simple \((76, 4, 5)\)-BIBD.

For \(v = 172\), remove one point from the last group of a TD(5, 4) to obtain a \((4, 5)\)-GDD of group type \(4^3\). Applying Construction 2.1 with super-simple \((4, 5)\)-GDDs of group type \(4^3\) and \(5^9\) coming from Lemmas 3.1 and 4.8, we obtain a super-simple \((4, 5)\)-GDDS of group type \((36)^{4}(27)^1\). Since there exists a super-simple \((36 + 1, 4, 5)\)-BIBD and a super-simple \((27 + 1, 4, 5)\)-BIBD from Theorem 3.10 and Lemma 4.3, by Construction 2.2 we obtain a super-simple \((172, 4, 5)\)-BIBD. □

**Lemma 4.10.** There exists a super-simple \((48t + 28, 4, 5)\)-BIBD for each \(t \geq 0\).

**Proof.** For \(t = 0, 2\), a super-simple \((48t + 28, 4, 5)\)-BIBD exists by Lemma 4.3. For \(t = 1, 3\), a super-simple \((48t + 28, 4, 5)\)-BIBD exists by Lemma 4.9.

For \(t \geq 4\), by Lemma 4.4(iii) there exists a 4-GDD of group type \(6^3\). Starting from this GDD and applying Construction 2.1 with a super-simple \((4, 5)\)-GDD of group type \(8^4\) coming from Lemma 3.1, we obtain a super-simple \((4, 5)\)-GDD of group type \((48)^3(24)^1\). Since there exist a super-simple \((28, 4, 5)\)-BIBD and a super-simple \((52, 4, 5)\)-IBIBD by Lemma 4.7, by Construction 2.3 we obtain a super-simple \((48t + 40, 4, 5)\)-BIBD. □
Combining Lemmas 4.2, 4.5, 4.6 and 4.10, we have the following theorem.

**Theorem 4.11.** A super-simple \((v, 4, 5)\)-BIBD exists for any \(v \equiv 4 \pmod{12}\) and \(v \geq 16\).

Combining Theorems 3.10 and 4.11, we have proved our main result as follows.

**Theorem 4.12.** A super-simple \((v, 4, 5)\)-BIBD exists if and only if \(v \equiv 1, 4 \pmod{12}\) and \(v \geq 13\).

**Acknowledgements**

The first author’s research was supported by the National Natural Science Foundation of China under Grant No. 10501023 and 60673070. The second author’s research was supported by National Natural Science Foundation of China under Grant 10771193. The third author’s research was supported by NSERC Grant 239135-06.

**References**


