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Haar Wavelet Solutions of Nonlinear Oscillator Equations

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Abstract

In this paper, we present a numerical scheme using uniform Haar wavelet approximation and quasilinearization process for solving some nonlinear oscillator equations. In our proposed work, quasilinearization technique is first applied through Haar wavelets to convert a nonlinear differential equation into a set of linear algebraic equations. Finally, to demonstrate the validity of the proposed method, it has been applied on three type of nonlinear oscillators namely Duffing, Van der Pol and Duffing-van der Pol. The obtained responses are presented graphically and compared with available numerical and analytical solutions found in the literature. The main advantage of uniform Haar wavelet series with quasilinearization process is that it captures the behavior of the nonlinear oscillators without any iteration. The numerical problems are considered with force and without force to check the efficiency and simple applicability of method on nonlinear oscillator problems.

Keywords: Haar wavelets, Nonlinear oscillators, Multiresolution analysis, Operational matrix, Quasilinearization Process

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1. Introduction

Nonlinear problems are of interest to many scientists and engineers, because most of physical systems in the real world are inherently nonlinear in nature. Many nonlinear differential equations arise in physical, chemical and biological contexts. Finding innovative methods to solve and analyze these equations has been an interesting subject in the field of differential equations and dynamical systems [1].

Considerable attention has been directed toward the chaos, chaotic systems and solutions of nonlinear oscillator differential equations since they play crucial role in natural and physical simulations. These problems are important for wavelet analysis, applied mathematics, physics and engineering sciences [2,3]. The chaotic behavior can be observed in natural and man-made systems. Some of chaotic systems are represented by Duffing, Van der Pol and Duffing-van der Pol equations which are important mathematical models for dynamical systems having a single unstable fixed point, along with a single stable limit cycle. These systems are highly sensitive to initial conditions because small differences in initial conditions yield widely diverging outcomes [4,5]. Up to now various aspects of nonlinear oscillators have been studied in the literature such as, the vibration amplitude control, synchronization dynamics and additive resonances etc. [6-10]. Recently, Duffing equation has been enhanced by Ahmad et al. [11] in the field of the prediction of diseases. A cautious measurement and analysis of a strongly chaotic voice has the potential to serve as an early warning system for more serious chaos and possible commencement of a disease. In fact, the success at analyzing and predicting the commencement of chaos in a signal and its simulation by equations lie in the problems as the Duffing, Van der Pol and Duffing-van der Pol play very important role. In this work, we have studied these equations by Haar wavelet method.

In this work, we consider a more general nonlinear oscillator system of the form

\[
\epsilon u(t) + (\delta + \beta u^p(t))u(t) - \mu u(t) + \alpha u^q(t) = g(F, \omega, t), \quad p, q \in \mathbb{N}
\]  

with the initial conditions

\[
u(0) = \gamma_0, \quad u'(0) = \gamma_1
\]

Depending on the parameters chosen, the Eq. (1) can take a number of special forms, where differentiation is with respect to independent time variable \(t\) and all parameters \(\epsilon, \delta, \beta, \mu\) and \(\alpha\) are
real constants. Here $\omega$ is an angular frequency and $g(F,\omega,t)$ represents the periodic driving function of time with period $T = 2p/\omega$.

Among the periodically forced self-excited oscillators, one of the most extensively studied examples is the Duffing-van der Pol oscillator whose mathematical expression is assumed in the form of the second order differential equation. The Eq. (1) is referred to the Duffing-van der Pol oscillator [12] when $p = 2, q = 3$ and other parameters which are involved in Eq.(1) are non-zero with periodic deriving function $g(F,\omega,t) = F\cos(\omega t)$ as

$$\varepsilon u''(t) + (\delta + \beta u^2(t))u'(t) - \mu u(t) + \alpha u^3(t) = F\cos(\omega t)$$ \hspace{1cm} (3)

The choice of $q = 3, \beta, \delta = 0$ and $g(F,\omega,t) = F\sin(\omega t)$ leads Eq. (1) to the Duffing oscillator [13,14] represented by

$$\varepsilon u''(t) - \mu u(t) + \alpha u^3(t) = F\sin(\omega t), \hspace{0.5cm} \varepsilon, \mu \text{ and } \alpha \neq 0$$ \hspace{1cm} (4)

It is the true equation of a forced, damped pendulum if $F \neq 0$ and unforced damped pendulum if $F = 0$. The Duffing equation is interesting because it allows us to examine what happens when we force on oscillator near its resonant frequency and to investigate a balance between the linear behavior and weak dissipative or nonlinear effects. In most physical systems, damping or nonlinear effects become important when the amplitude grows sufficiently large.

The choice $\alpha = 0$ and $p = 2$ leads Eq. (1) to the Van der Pol oscillator [15] governed by the second-order differential equation

$$\varepsilon u'(t) + (\delta + \beta u^2(t))u'(t) - \mu u(t) = F\sin(\omega t), \hspace{0.5cm} \delta, \beta \neq 0$$ \hspace{1cm} (5)

This model was proposed by engineer Balthasar van der Pol (1889-1959) for an electrical circuit with a triode valve in 1920 when he was an engineer working for Philips Company (in the Netherlands) and was later extensively studied as a host of a rich class of dynamical behavior.

There are various techniques to solve nonlinear oscillators. For details, one may refer to survey articles [11] which includes finite difference method, finite element method, boundary value approach, various forms of spline and wavelet methods to obtain approximate solutions of nonlinear equations of various types. Most scientific problems in solid mechanics problems are inherently nonlinear. Except a limited number of these problems, most of them do not have analytical solutions. Some of them are solved using a numerical techniques and the analytical perturbation method. In the perturbation method, the small parameter is inserted in the equation. Therefore, finding the small parameter and exerting it into the equation are deficiencies of this
method [7]. In addition it requires a large amount of calculations resulting in a major computational difficulty for getting the accurate results for small damping parameter and tackling the nonlinearity. Duffing van der Pol equation is investigated recently by Ji and Zhang [8] while Duffing equation by Liu, Wu [13, 14] and Van der Pol equation by Lepik [15]. But their solutions are restricted to very small parameters.

In recent years the wavelet approach is becoming increasingly popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used in numerical solution of differential equations. Out of these, the Haar wavelets have gained popularity among researchers due to their useful properties such as simple applicability, orthogonality and compact support. Concept of Haar wavelets for solving differential equations is used by Chen and Hsiao in [16]. G. Hariharan [17] has derived solutions of partial differential equations by Haar wavelets. S.A. Yousefi et al. have solved fractional optimal control problems by using Legendre multiwavelet collocation method[18]. Integro differential equations are also solved by M. Lakestani et al. using trigonometric wavelets [19]. Second order boundary value problems and non-linear Lane Emden equations have been successfully solved by using the Haar wavelet quasilinearization method in [20, 21]. An attempt is made in this paper to use Haar wavelet quasilinearization technique to solve nonlinear oscillator differential equations. Only problem we face is that when we increase number of points, the corresponding coefficient matrix becomes ill-conditioned.

The paper is organized as follows. In next section we have discussed the brief introduction on preliminaries of Haar wavelets. In section 3, quasilinearization process is discussed to deal the nonlinearity in the equations and Haar wavelet method is also described here. Convergence of method is discussed in section 4. The proposed method is implemented on different nonlinear oscillator differential equations in section 5. These nonlinear oscillators as Duffing, Van der Pol and Duffing-van der Pol which are encountered in structural dynamics. A comparison of Haar wavelet solutions with available ones is shown graphically and numerical results are depicted in tables for different parameters. Section 6 concludes the present work.

2. Haar Wavelet Preliminaries

Among the different wavelet families which are defined by an analytical expression, mathematically most simple are the Haar wavelets. Due to the simplicity the Haar wavelets are
very effective for solving ordinary differential and partial differential equations. In 1910, Alfred Haar[22] introduced the notion of wavelets in the form of a rectangular pulse pair function. His initial theory has been expanded recently into a wide variety of applications, but primarily it allows for the representation of various functions by a combination of step functions and wavelets over specified interval widths. The Haar wavelet is the only real valued function which is symmetric, orthogonal and have a compact support[23]. Here first Haar wavelet function is defined as

\[
h_1(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq \frac{1}{2} \\
-1 & \text{if } 0 \leq t \leq \frac{1}{2} \\
0 & \text{elsewhere}
\end{cases}
\]  

(6)

The following definitions illustrate the translation-dilation of Haar wavelet function \( h_t(t) \).

**Translation and dilation operators:** Let \( h \in L^2(R) \). For \( k \in Z \), let \( T_k : L^2(R) \to L^2(R) \) be given by \((T_k h)(t) = h(t-k)\) and \( D_j : L^2(R) \to L^2(R) \) be given by \((D^j h)(t) = 2^j h(2^j t)\) operators \( T_k \) and \( D^j \) are called translation and dilation operator.

**Orthonormal Haar Wavelet:** A function \( h \in L^2(R) \) is called an orthonormal wavelet for \( L^2(R) \) if \( \{D^j T_k h : j,k \in Z \} = \left\{ 2^j h(2^j t - k) : j,k \in Z \right\} \) is an orthonormal basis for \( L^2(R) \). Index \( j \) refers to dilation and \( k \) refers to translation.

Thus Haar wavelet family \( h_t(t) \) is orthogonal square waves family which is obtained by translation and dilation operators as[16]

\[
h_t(t) = \begin{cases} 
1 & t \in \left[ \xi_1, \xi_2 \right] \\
-1 & t \in \left[ \xi_2, \xi_3 \right] \\
0 & \text{elsewhere}
\end{cases}
\]

(7)

Here \( \xi_1 = \frac{k}{m} \), \( \xi_2 = \frac{k+0.5}{m} \) and \( \xi_3 = \frac{k+1}{m} \)

(8)

The collocation points \( t_i = \frac{l-\frac{1}{2}}{2m} \), \( l=1,2, \ldots, 2m \). For \( i \geq 2 \), \( i = 2^j + k + 1 \), \( j \geq 0 \), \( 0 \leq k \leq 2^j - 1 \)
Here $m$ is the level of the wavelet, we assume the maximum level of resolution as index $J$, then $m = 2^j, (j = 0, 1, 2, \ldots, J)$; in case of minimal values $m = 1, k = 0$ then $i = 2$. For any fixed level $m$, there are $m$ series of $i$ to fill the interval $[0,1)$ corresponding to that level and for a provided $J$, the index number $i$ can reach the maximum value $M = 2^j + 1$, when including all levels of wavelets. Also for the ease of implementation, we have used the same notations for Haar wavelets and their integrals as [24] and matrix form of Haar wavelet family $h_i(t)$ for $j = 1, 2m = 4$ is given as

$$H = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 
\end{bmatrix}$$  \tag{9}$$

We can find the required derivatives in terms of operational matrix. The operational matrix $p_{i,n}(t)$ of order $2m \times 2m$ can be obtained by integration of Haar wavelet. Integrals can be evaluated from Eq. (7) and the first two integrals of them are given below.

$$p_{i,n}(t) = \begin{cases}
\frac{t - k}{m} & \text{if } t \in \left[k \frac{k + 0.5}{m}, \frac{k + 1}{m}\right] \\
\frac{k + 1}{m} - t & \text{if } t \in \left[k \frac{k + 0.5}{m}, \frac{k + 1}{m}\right] \\
0 & \text{elsewhere}
\end{cases}$$  \tag{10}$$

$$p_{i,n}(t) = \begin{cases}
\frac{1}{4m^2} \left(t - \frac{k}{m}\right)^2 & \text{if } t \in \left[k \frac{k + 0.5}{m}, \frac{k + 1}{m}\right] \\
\frac{1}{4m^2} \left(- \frac{1}{2} \left(\frac{k + 1}{m} - t\right)^2 & \text{if } t \in \left[k \frac{k + 0.5}{m}, \frac{k + 1}{m}\right] \\
0 & \text{elsewhere}
\end{cases}$$  \tag{11}$$

3. Description of Haar Wavelet Technique

The sequence $\{h_i\}_{i=0}^{\infty}$ is a complete orthonormal system in $L^2[0,1]$ and by using the concept of multiresolution analysis (MRA) as an example the space $V_j$ can be defined like

$$V_j = sp\{\psi_{j,k}\}_{j=0,1,2,\ldots} = W_{j-1} \oplus V_{j-1} = W_{j-2} \oplus W_{j-1} \oplus V_{j-2} \oplus \ldots \oplus W_{j-1} W \oplus V_0.$$  \tag{12}$$
The linearly independent functions \( h_{j,k}(t) \) spanning \( W_j \) are called wavelets. The Haar basis has the very important property of multiresolution analysis that \( V_{j+1} = V_j \oplus W_j \). Original signal can be expressed as a linear combination of the box basis functions in \( V_j \). The orthogonality property puts a strong limitation on the construction of wavelets and allows us to transform any square integral function on the interval time \([0,1]\) into Haar wavelets series as

\[
f(t) = a_0 h_0(t) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^{j-1}} a_{2^j+k} h_{2^j+k}(t), \quad t \in [0,1]
\]  

(13)

Similarly the highest derivative can be written as wavelet series \( \sum_{i=-\infty}^{\infty} a_i h_i(t) \). In applications, Haar series are always truncated to \( 2m \) terms, that is \( \sum_{i=0}^{2m-1} a_i h_i(t) \) [25,26].

The presented technique is based on integral operational matrices of Haar wavelet approximated series and quasilinearization process[27]. The quasilinearization process[28] is application of the Newton Raphson Kantrovich approximation method in function space. The idea and advantage of the quasilinearization are based on the fact that linear equations can often be solved analytically or numerically while there are no useful techniques for obtaining the general solution of a nonlinear equation in terms of a finite set of particular solutions.

Consider an \( n^{th} \) order nonlinear ordinary differential equation

\[
L^{(n)} u(t) = f(u(t), u^{(1)}(t), u^{(2)}(t), \ldots, u^{(n)}(t), t)
\]

with the initial conditions

\[
u(0) = \lambda_0, u^{(1)}(0) = \lambda_1, u^{(2)}(0) = \lambda_2, \ldots, u^{(n-1)}(0) = \lambda_{n-1}
\]

(15)

Here \( L^{(n)} \) is the linear \( n^{th} \) order ordinary differential operator, \( f \) is nonlinear function of \( u(t) \) and its \( (n-1) \) derivatives are \( u^{(s)}(t), s = 0, 1, 2, \ldots, n-1 \).

The quasilinearization prescription determines the \((r+1)^{th}\) iterative approximation to the solution of Eq.(14) with Eq.(15) and its linearized form is given by Eq.(16)

\[
L^{(n)} u_{r+1}(t) = f(u_r(t), u_r^{(1)}(t), u_r^{(2)}(t), u_r^{(3)}(t), \ldots, u_r^{(n-1)}(t), t) + \sum_{s=0}^{n-1} (u_r^{(s)}(t) - u_r^{(s)}(t))
\]

\[
f_u^{(s)}(u_r(t), u_r^{(1)}(t), u_r^{(2)}(t), u_r^{(3)}(t), \ldots, u_r^{(n-1)}(t), t)
\]

(16)
where $u^{(0)}(t) = u(t)$. The functions $f_{u^{(k)}} = \frac{\partial f}{\partial u^k}$ are functional derivatives of the functions. The zero\(^{th}\) approximation $u_0(t)$ is chosen from mathematical or physical considerations.

By using quasi-linearization process, we get the following two equations for linearization of Eq.(1)

$$u^2(t)u'(t) = u^2_r(t)u^{(1)}_r(t) + (u_{r+1}(t) - u_r(t))2u_r(t)u^{(1)}_r(t) + (u^{(1)}_{r+1}(t) - u^{(1)}_r(t))u^2_r(t)$$

$$u^3(t) = -2u^2_r(t) + 3u^2_r(t)u^{(1)}_r(t)$$

Thus, Eq. (1) becomes after quasilinearization

$$\varepsilon u_{r+1}^{(2)}(t) + \delta u_{r+1}^{(1)}(t) + \beta (u^{(0)}_r(t)u^{(1)}_r(t) + u_{r+1}(t)pu_r(t)u^{(1)}_r(t) - u_r(t)pu_r(t)u^{(1)}_r(t) + (u^{(1)}_{r+1}(t) - u^{(1)}_r(t))u^0_r(t))$$

$$- \mu u_{r+1}(t) + \alpha u^2_r(t) + \alpha qu^{(1)}_r(t)u_{r+1}(t) - \alpha qu^{(1)}_r(t)u_r(t) = g(F, \omega, t)$$

$$u^{(0)}_{r+1}(t) = \sum_{i=0}^{2m} a_i h_i(t)$$

$$u^{(n-1)}_{r+1}(t) = \sum_{i=0}^{2m} a_i p_{i,n}(t) + t^n u^{(n-1)}(0) + \cdots + u^{(n-1)}_{r+1}(0)$$

Finally on applying Haar wavelet technique and using the Eq.(21), Eq. (19) becomes

$$\sum_{i=0}^{2m} a_i \left( \varepsilon \sum_{i=0}^{2m} h_i(t_i) + (\delta + 1) \sum_{i=0}^{2m} p_{i,n}(t_i) + (qu^{(1)}_r(t_i) + pu^{(1)}_r(t_i)u^{(1)}_r(t_i)) \sum_{i=0}^{2m} p_{i,2}(t_i) \right) + \delta u^{(1)}_r(0)$$

$$+ \beta (u^{(0)}_r(t_i)u^{(1)}_r(t_i) + (t, u^{(1)}_r(t_i) + u^{(1)}_r(t_i)pu^{(1)}_r(t_i) - u_r(t_i)pu^{(1)}_r(t_i) + (u^{(1)}_{r+1}(t_i) - u^{(1)}_r(t_i))u^0_r(t_i))$$

$$u^2_r(t_i) - u^2_r(t_i) - \mu u_r(t_i) + \alpha u^2_r(t_i) + (t, u^{(1)}_r(t_i) + u_r(t_i))qu^{(1)}_r(t_i) - u_r(t_i)qu^{(1)}_r(t_i) = g(F, \omega, t_i)$$

Now we will simplify Eq. (22) as the linear matrix system for getting the coefficients $a_i$, and finally get the Haar wavelet solution (HWS) of Eq. (1).

4. Convergence of Haar Wavelet Approximation

A function $u \in L^2(R)$, MRA of $L^2(R)$ generates a sequence of subspaces $V_j, V_{j+1}, V_{j+2}$, such that the projections of $u$ onto these spaces give finer and finer approximations of the function $u$ as $J \to \infty$, then the corresponding error at $J^{th}$ level may be defined as

$$e_j(t) = \left| u(t) - u_j(t) \right| = \left| u(t) - \sum_{i=0}^{2^{j+1}} a_i h_i(t) \right| = \left| \sum_{i=2^{j+1}}^{\infty} a_i h_i(t) \right|$$

(23)
We can analyze the error for nonlinear oscillator Eq. (1). Convergence of the method may be discussed on the same lines as given in Saeedi et al. [29]. We can also discuss the convergence of the method for nonlinear oscillator problem if we know the exact solution.

**Theorem.** Suppose that \( f(x) \) satisfies a Lipschitz condition on \([0,1]\), that is, \( \exists M > 0, \forall x, y \in [0,1] : |f(x) - f(y)| \leq M|x - y| \). \( M \) is the Lipschitz constant.

The error bound for \( e_j(x) \) is also obtained as

\[
\|e_j(x)\|_2 \leq \left( \frac{M}{2^{j+i}\sqrt{3}} \right)
\]

Then the Haar wavelet method will be convergent in the sense that \( e_j(x) \) goes to zero as \( j \) goes to infinity and Lipschitz constant \( M \) may be small not too large. Moreover, the convergence is of order one, that is,

\[
\|e_j(x)\|_2 = O\left( \frac{1}{2^{j+i}} \right)
\]

**Proof.** See Saeedi et al.[29]

5. Applications and Numerical Problems

In this section, the wide applicability and efficiency of the Haar wavelet method are manifested further through a set of experiments on numerical problems. We consider the following oscillators in particular cases and comparison will be made with existing available solutions in literature. All computations are carried out by programming in C++ and MATLAB R2007b at maximum level of resolution \( J = 4 \) and \( 2m = 32 \).

**Problem 5.1. Duffing oscillator**

Consider the Duffing oscillator from Eq. (4) as following

\[
\varepsilon u'(t) - \mu u(t) + \alpha u^3(t) = F \sin(\omega t)
\]

(24)

where \( \varepsilon, \mu, \alpha, \) and \( F \) are given parameters, \( \omega \) is also a given constant which represents the enforcing frequency. The analytic solution is given by means of a trigonometric series \([13, 14]\).

\[
u(t) \equiv a_1 \sin(\omega t) + a_2 \sin(3\omega t) + a_3 \sin(5\omega t) + \ldots \]

(25)

Here we consider two cases of Duffing equation with different parameters as follows.
5.1.1. Unforced Duffing oscillator

Unforced Duffing oscillator represents the free vibration of pendulum. The frequency of the oscillations depends on the initial displacement of the pendulum. By taking the parameters $\varepsilon = 1, \mu = -1, \alpha = -\frac{1}{6}, \omega = 0.7$ and $F = 0$ as considered in [13] and initial conditions $u(0) = 0$ and $u'(0) = 1.6376$ in Eq. (24), we get

$$u'(t) + u(t) - \frac{1}{6}u^3(t) = 0$$

(26)

The trigonometric series solution up to three terms of Eq.(26) is

$$u(t) = 2.058\sin(0.7t) + 0.0816\sin(2.1t) + 0.00337\sin(3.5t).$$

Using the Eqs. (18, 21) in Eq.(26), we get

$$\sum_{i=0}^{2m} a_i h_i(t_i) + \left(1 - 0.5(1.6376)^2 \sin^2(t_i)\right) \sum_{i=0}^{2m} a_i p_{i,2}(t_i) = 1.6376 \left(\frac{1}{3}(1.6376)^2 \sin^2(t_i) + 0.5(1.6376)^2 - \sin^2(t_i) \right)$$

(27)

and applying the procedure mentioned in section 3, finally Haar wavelet solution of Eq. (26) can be obtained by getting the values of coefficients which are computed after solving the Eq.(27). Computed values of $u, u'$ and $u''$ are compared with trigonometric series solution and depicted graphically in Fig. 5.1.1a and the absolute errors in the solutions for $\alpha = -\frac{1}{6}$ are given in Table 1.

For $\alpha = -10$, wavelet solution is shown in Fig. 5.1.1b and in Table 2.

**Fig. 5.1.1a.** Comparison of Haar wavelet solution with trigonometric series solution for $\alpha = -\frac{1}{6}$. 
5.1.2. Forced Duffing oscillator

By taking the parameters $\varepsilon=1, \mu=-1, \alpha=\frac{-1}{6}, \omega=1$ and $F=2$ in Eq. (24), we get the forced Duffing system with initial conditions given as in [13],

$$u'(t) - \frac{1}{6}u^3(t) + u(t) = 2\sin(t), \quad \text{with } u(0) = 0 \text{ and } u'(0) = -2.7676$$  \hfill (28)

Trigonometric series solution up to three terms is given below

$$u(t) \cong 2.5425\sin(t) - 0.07139\sin(3t) - 0.00219\sin(5t).$$

Fig. 5.1.2a. Comparison of Haar wavelet solution with trigonometric series solution for $\alpha=\frac{-1}{6}$. 

Fig. 5.1.1b. Haar wavelet solution for $\alpha=-10$. 

![Graph](image-url)
Fig. 5.1.2b. Haar wavelet solution for $\alpha = 10$.

Comparison of computed $u, u'$ and $u''$ are depicted graphically in Fig. 5.1.2a and the absolute errors in the solutions are given in Table 1 for $\alpha = -\frac{1}{6}$. Haar wavelet solution for $\alpha = 10$ is also shown in Fig. 5.1.2b and in Table 2.

Problem 5.2. Duffing-van der Pol Oscillator

5.2.1. Unforced Duffing-van der Pol Oscillator [30,31]

Consider Eq. (3) with parameters $\epsilon = 1, \delta = 0.1, \beta = 0.1, \mu = -1, \alpha = 0.01$ and $F = 0$ as below

$$u''(t) - 0.1(1 - u^2(t))u'(t) + u(t) + 0.01u^3(t) = 0, \quad \text{with } u(0) = 2 \text{ and } u'(0) = 0$$

(29)

By quasilinearization, we get the form

$$u_r^{(2)}(t) - 0.1u_{r+1}^{(1)}(t) + u_{r+1}(t) = -0.1u_r^2(t)u'_{r+1}(t) + 0.2u_r^3(t)u'_r(t) - 0.2u'_r(t)u_r(t)u_{r+1}(t) + 0.02u'_{r+1}(t) - 0.03u_r^2(t)u_{r+1}(t)$$

(30)

By using the Eq.(19) in Eq. (27), we obtain

$$\sum_{i=0}^{2m} a_i \left[ \sum_{i=0}^{2m} h_i(t_i) + \sum_{i=0}^{2m} p_{i,1}(t_i)(-0.1 + 0.4 \cos^2(t_i)) + \sum_{i=0}^{2m} p_{i,2}(t_i)(1 + 0.4 \cos^2(t_i)(4 \cos(t_i) + 0.3)) \right]$$

$$= -2 - 8 \sin(2t_i) \cos(t_i) - \cos^2(t_i)(15.4 \cos(t_i) + 0.24)$$

(31)

After solving system of Eq. (31), we have computed solutions $u$ and $u'$. Comparison of obtained
solutions with those solutions which have been obtained by Adomian decomposition method[29] is shown in Fig. 5.2.1. The absolute errors in the solutions are also reported in Table 1.

![Unforced Duffing-van der Pol](image)

**Fig. 5.2.1.** Comparison of Haar wavelet solution with Adomain decomposition method solution.

### 5.2.2. Forced Duffing-van der Pol Oscillator [31,32]

Consider Eq. (3) with initial conditions $u(0) = 1, \dot{u}(0) = 0$ and parameters are given as below $\epsilon = 1, \delta = -0.1, \beta = 0.1, \mu = -0.5, \alpha = 0.5$ and $F = 0.5, \omega = 0.79$.

Haar wavelet solutions are shown graphically in Figs. 5.2.2a, 5.2.2b and reported in Table 3 for different values of parameters $\mu$ and $\alpha$. Comparison of solutions with those available in [32] are given in Table 4, Table 5 and Table 6 for different values of parameters.

![Forced Duffing-van der Pol](image)

**Fig 5.2.2a.** The effect of different values of $\alpha$ keeping $\mu = -0.5$. 
Problem 5.3. Van der Pol Oscillator[15]

5.3.1. Consider the Van der Pol oscillator from Eq. (5) with following initial conditions and parameters $\varepsilon = 1, \delta = -0.05, \beta = 0.05, \mu = -1, \alpha = 0$ and $F = 0$.

$$u'(t) - 0.05(1-u(t)^2)u(t) + u(t) = 0, \quad \text{with} \quad u(0) = 0 \text{ and } u'(0) = 0.5$$  (32)

Computed Haar wavelet solution is depicted in Fig. 5.3.1. Comparison of results with available solutions are also reported in Table 7.

\[\text{Fig. 5.3.1. Haar wavelet solution.}\]

5.3.2. From Eq. (5), consider the Van der Pol oscillator with the following initial conditions and parameters $\delta = 1, \beta = -1, \mu = -1, \alpha = 0$ and $F = 0$.

$$\varepsilon u'(t) + (1-u(t)^2)u(t) + u(t) = 0, \quad \text{with} \quad u(0) = 0.5 \text{ and } u'(0) = 1$$  (33)
Haar wavelet solutions of Eq.(33) are shown in Figures 5.3.2a, 5.3.2b, 5.3.2c, 5.3.2d, 5.3.2e for different values of $\varepsilon$. Also see Table 8 for solutions.

Fig. 5.3.2a. Haar wavelet solution.

Fig. 5.3.2b. Haar wavelet solution.

Fig. 5.3.2c. Haar wavelet solution.
5. Conclusion
The aim of this paper is to represent a Haar wavelet method to solve well known nonlinear oscillator differential equations such as Duffing, Van der Pol and Duffing-van der Pol with different parameters. To overcome the nonlinearities, quasilinearization is used. It is observed that the quasilinearization makes easier procedure for the Haar wavelet method to handle nonlinearity in a shorter time of computations. There is no need of iterations for achieving sufficient accuracy in numerical results. Therefore, it is suggested that quasilinearization can effectively be used to solve the nonlinear oscillator differential equations. In our method, when we increase number of points $m = 2^j$, then coefficient matrix becomes ill-conditioned and it becomes difficult to find direct solutions. The Haar wavelet collocation method computes the solutions only at odd points. However, results can be obtained at any point of the domain. The
obtained numerical solutions are in very good coincidence with those solutions which are available in literature computed by other methods and indicate that the proposed method is feasible and convergent. The effects of constant parameters on responses of system for Haar wavelet method are also shown in figures. Therefore, it is recommended to use Haar wavelets to compute solutions of nonlinear vibration problems.

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References

Table 1
Computed absolute errors at different points for problems 5.1.1 and 5.1.2.

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Table 2
Computed Haar wavelet solution for different values of $\alpha$ for Problems 5.1.1 and 5.1.2.

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<th>Prob. 5.1.1 for $\alpha = -10$</th>
<th>Prob. 5.1.1 for $\alpha = -10$</th>
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<th>Prob. 5.1.2 for $\alpha = 10$</th>
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<td>$u(t)$</td>
<td>$u(t)$</td>
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Table 3
Computed Haar wavelet Solution $u(t)$ for problem 5.2.2 at different values of $\mu$ and $\alpha$.

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Table 4
Comparison of Haar wavelet solution with those solutions available in [15] by different methods for problem 5.2.2 with parameters $\delta = 0.1, \beta = 0.1, \mu = 0.5, \alpha = 0.5$ and $F = 0.5, \omega = 0.79$.

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Table 5
Comparison of Haar wavelet solution with those solutions available in [15] by different methods for problem 5.2.2 with parameters for following parameters: $\delta = 0.1, \beta = 0.1, \mu = -0.5, \alpha = 0.5$ and $F = 0.5, \omega = 0.79$.

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Table 6
Comparison of Haar wavelet solution with those solutions available in [15] by different methods for problem 5.2.2 with parameters $\delta = 0.1, \beta = 0.1, \mu = 0.5, \alpha = -0.5$ and $F = 0.5, \omega = 0.79$. 
### Table 7
Comparison of Haar wavelet solution $u(t)$ with those obtained by different numerical methods [33] for Prob. 5.3.1.

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### Table 8
Computed Haar wavelet solution $u(t)$ for different values of $\varepsilon$ for prob. 5.3.2.

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