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Global Asymptotic Stability of a Higher Order Difference Equation

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Abstract

The aim of this work is to investigate the global stability, periodic nature, oscillation and boundedness of the positive solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2r-1}}{B + Cx_{n-2l}x_{n-2k}} \qquad , n = 0, 1, 2, \ldots$$

where A, B, C are nonnegative real numbers and l, r, k are nonnegative integers, such that $l \leq k$ and $r \leq k$.

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1 Introduction

Difference equations have always played an important role in the construction and analysis of mathematical models of biology, ecology, physics, economic processes, etc. [6].

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, one can refer to the monographs [7, 4, 11, 5, 6] and the papers [2, 3, 12, 13, 14, 15, 10, 9, 8] and references therein.

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

In [1], we have discussed the asymptotic behavior of solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + C\prod_{i=l}^{k} x_{n-2i}} \qquad , n = 0, 1, 2, \ldots$$

where A, B, C are nonnegative real numbers and l, k are nonnegative integers, such that $l \leq k$

In this paper, we study the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{Ax_{n-2r-1}}{B + Cx_{n-2l}x_{n-2k}} \qquad , n = 0, 1, 2, \ldots \tag{1}$$

where A, B, C are nonnegative real numbers and l, r, k are nonnegative integers, such that $l \leq k$ and $r \leq k$.
The following particular cases can be obtained:

1. When $A = 0$, equation (1) reduces to $x_{n+1} = 0, n = 0, 1, 2, \ldots$ which has the trivial solution.

2. When $B = 0$, equation (1) reduces to

   $$x_{n+1} = \frac{Ax_{n-2r-1}}{Cx_{n-2l}x_{n-2k}} \qquad , n = 0, 1, 2, \ldots$$

   This equation can be reduced to the linear difference equation

   $$y_{n+1} - y_{n-2r-1} + y_{n-2l} + y_{n-2k} = \gamma,$$

   by taking

   $$x_n = e^{y_n}, \gamma = \ln \frac{A}{C}.$$

3. When $C = 0$, equation (1) reduces to $x_{n+1} = \frac{A}{B}x_{n-2r-1}, n = 0, 1, 2, \ldots$ Which is a linear difference equation.

For various values of l, r and k, we can get more equations.

2 Preliminaries

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}) \qquad , n = 0, 1, \dots \tag{2}$$

where $f : R^{k+1} \to R$.
Definition 2.1 *[11]*
An equilibrium point for equation (2) is a point $\bar{x} \in R$ such that $\bar{x} = f(\bar{x}, \bar{x}, \dots, \bar{x})$.
Definition 2.2 *[11]*

1. An equilibrium point $\bar{x}$ for equation (2) is called locally stable if for every $\epsilon > 0, \exists \delta > 0$ such that every solution$\{x_n\}$with initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in]\bar{x} - \delta, \bar{x} + \delta[$ is such that $x_n \in]\bar{x} - \epsilon, \bar{x} + \epsilon[, \forall n \in N$. Otherwise $\bar{x}$ is said to be unstable.

2. The equilibrium point $\bar{x}$ of equation (2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in]\bar{x} - \gamma, \bar{x} + \gamma[$, the corresponding solution $\{x_n\}$ tends to $\bar{x}$.

3. An equilibrium point $\bar{x}$ for equation (2) is called global attractor if every solution $\{x_n\}$ converges to $\bar{x}$ as $n \to \infty$.

4. The equilibrium point $\bar{x}$ for equation (2) is called globally asymptotically stable if it is locally asymptotically stable and global attractor.

The linearized equation associated with equation (2) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \dots, \bar{x}) y_{n-i} \qquad , n = 0, 1, 2, \dots \tag{3}$$

the characteristic equation associated with equation (3) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \dots, \bar{x}) \lambda^{k-i} = 0 \tag{4}$$

Theorem 2.1 *[11] Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of equation (2). Then the following statements are true:*

1. *If all roots of equation (4) lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.*

2. *If at least one root of equation (4) has absolute value greater than one, then $\bar{x}$ is unstable.*

3 Linearized stability analysis

Consider the difference equation

$$x_{n+1} = \frac{Ax_{n-2r-1}}{B + Cx_{n-2l}x_{n-2k}} \qquad , n = 0, 1, 2, \ldots$$

where A, B, C are nonnegative real numbers and l, r, k are nonnegative integers, such that $l \leq k$ and $r \leq k$.

The change of variables $x_n = \sqrt{\frac{A}{C}} y_n$ reduces equation (1) to the difference equation

$$y_{n+1} = \frac{y_{n-2r-1}}{\gamma + y_{n-2l}y_{n-2k}} \qquad , n = 0, 1, 2, \ldots \tag{5}$$

where $\gamma = \frac{B}{A}$.

Now we examine the equilibrium points of equation (5) and their local asymptotic behavior. Clearly equation (5) has two nonnegative equilibrium points $\bar{y} = 0$ and $\bar{y} = \sqrt{1-\gamma}$ when $\gamma < 1$ and $\bar{y} = 0$ only when $\gamma \geq 1$.

The linearized equation associated with equation (5) about $\bar{y}$ is

$$z_{n+1} - \frac{1}{\gamma + \bar{y}^2} z_{n-2r-1} + \frac{\bar{y}^2}{(\gamma + \bar{y}^2)^2}(z_{n-2l} + z_{n-2k}) = 0 \qquad , n = 0, 1, 2, \ldots \tag{6}$$

the characteristic equation associated with this equation is

$$\lambda^{2k+1} - \frac{1}{\gamma + \bar{y}^2}\lambda^{2k-2r-1} + \frac{\bar{y}^2}{(\gamma + \bar{y}^2)^2}(\lambda^{2k-2l} + 1) = 0. \tag{7}$$

We summarize the results of this section in the following theorem.

Theorem 3.1 *1. If $\gamma > 1$, then the zero equilibrium point is locally asymptotically stable.*

2. If $\gamma < 1$, then the equilibrium points $\bar{y} = 0$ and $\bar{y} = \sqrt{1-\gamma}$ are unstable (saddle points).

Proof

The linearized equation associated with equation (5) about $\bar{y} = 0$ is

$$z_{n+1} - \frac{1}{\gamma} z_{n-2r-1} = 0 \qquad , n = 0, 1, 2, \ldots$$

The characteristic equation associated with this equation is

$$\lambda^{2k+1} - \frac{1}{\gamma}\lambda^{2k-2r-1} = 0$$

so $\lambda = 0, \lambda = \pm \sqrt[2r+2]{\frac{1}{\gamma}}$.

1. *If $\gamma > 1$, then $|\lambda| < 1$ for all roots and $\bar{y} = 0$ is locally asymptotically stable.*

2. *If $\gamma < 1$, it follows that $\bar{y} = 0$ is unstable (saddle point). The linearized equation (6) about $\bar{y} = \sqrt{1-\gamma}$ becomes*

$$z_{n+1} - z_{n-2r-1} + (1-\gamma)(z_{n-2l} + z_{n-2k}) = 0 \qquad , n = 0, 1, 2, \ldots$$

   The associated characteristic equation is

$$\lambda^{2k+1} - \lambda^{2k-2r-1} + (1-\gamma)(\lambda^{2k-2l} + 1) = 0.$$

   Let $f(\lambda) = \lambda^{2k+1} - \lambda^{2k-2r-1} + (1-\gamma)(\lambda^{2k-2l} + 1)$. We can see that $f(\lambda)$ has a real root in $(-\infty, -1)$. Then the point $\bar{y} = \sqrt{1-\gamma}$ is unstable (saddle point).

4 Global behavior of equation (5)

Theorem 4.1 *If $\gamma > 1$, then the zero equilibrium point is globally asymptotically stable.*
Proof
Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of equation (5). Hence

$$y_{n+1} = \frac{y_{n-2r-1}}{\gamma + y_{n-2l}y_{n-2k}} < \frac{y_{n-2r-1}}{\gamma} \qquad , n = 0, 1, 2, \ldots$$

then

$$y_{2n(r+1)+i} < \frac{1}{\gamma^{n+1}} y_{i-2r-2}, i = 1, 2, \ldots 2r + 2.$$

Hence each of the subsequences $\{y_{2n(r+1)+i}\}_{n=0}^{\infty}$, $i = 1, 2 \ldots, 2r+2$, converges to zero. Therefore

$$\lim_{n\to\infty} y_n = 0.$$

In view of theorem (3.1), $\bar{y} = 0$ is globally asymptotically stable.

5 Semicycle analysis

Theorem 5.1 *Let $\{y_n\}_{n=-2k}^{\infty}$ be a nontrivial solution of equation (5) and let $\bar{y}$ denote the unique positive equilibrium of equation (5) such that either,*
(C_1) $y_{-2k}, y_{-2k+2}, \ldots, y_0 > \bar{y}$ and $y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} < \bar{y}$
Or
(C_2) $y_{-2k}, y_{-2k+2}, \ldots, y_0 < \bar{y}$ and $y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} > \bar{y}$
is satisfied, then $\{y_n\}_{n=-2k}^{\infty}$ oscillates about $\bar{y}$ with semicycles of length one.

Proof
Assume that condition (C_1) *is satisfied. Then we have*
$y_1 = \frac{y_{-2r-1}}{\gamma + y_{-2l}y_{-2k}} < \frac{\bar{y}}{\gamma+\bar{y}^2} = \bar{y}$,
$y_2 = \frac{y_{-2r}}{\gamma + y_{-2l+1}y_{-2k+1}} > \frac{\bar{y}}{\gamma+\bar{y}^2} = \bar{y}$, *by induction we obtain the result.*
Assume that condition (C_2) *is satisfied. Then we have*
$y_1 = \frac{y_{-2r-1}}{\gamma + y_{-2l}y_{-2k}} > \frac{\bar{y}}{\gamma+\bar{y}^2} = \bar{y}$,
$y_2 = \frac{y_{-2r}}{\gamma + y_{-2l+1}y_{-2k+1}} < \frac{\bar{y}}{\gamma+\bar{y}^2} = \bar{y}$, *by induction we obtain the result.*

6 case $r = k$

When $r = k$, equation (5) becomes

$$y_{n+1} = \frac{y_{n-2k-1}}{\gamma + y_{n-2l}y_{n-2k}} \qquad ,n = 0, 1, 2, \ldots \tag{8}$$

The following theorem summarizes the linearized stability analysis of equation (8).

Theorem 6.1 *1. If* $\gamma > 1$*, then the zero equilibrium point is locally asymptotically stable.*

2. If $\gamma < 1$*, then the equilibrium points* $\bar{y} = 0$ *is unstable (repeller) and* $\bar{y} = \sqrt{1-\gamma}$ *are unstable (saddle points).*

Proof
It is sufficient to consider the linearized equation

$$z_{n+1} + \frac{\bar{y}^2}{\gamma + \bar{y}^2}(z_{n-2l} + z_{n-2k}) - \frac{1}{\gamma + \bar{y}^2} z_{n-2k-1} = 0 \qquad ,n = 0, 1, 2, \ldots$$

and its associated characteristic equation

$$\lambda^{2k+2} + \frac{\bar{y}^2}{\gamma + \bar{y}^2}(\lambda^{2k-2l+1} + \lambda) - \frac{1}{\gamma + \bar{y}^2} = 0.$$

Therefore, the results follows.

Theorem 6.2 *The following statements are true:*

1. *Assume that* $\gamma > 1$*. Then the zero equilibrium point is globally asymptotically stable.*
2. *Assume that* $\gamma = 1$*. Then every solution of equation (8) converges to a periodic solution of equation (8) with period 2(k+1) and there exist periodic solutions of equation (8) with prime period 2(k+1).*

3. *Assume that $\gamma < 1$. Then there exist solutions of equation (8) which are neither bounded nor persist.*

Proof

1. *The proof is similar to that of theorem (4.1).*

2. *Assume that $\gamma = 1$. Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of equation (8). For $n \geq 0$ we have*

$$y_{n+1} = \frac{y_{n-2k-1}}{1 + y_{n-2l}y_{n-2k}} \leq y_{n-2k-1} \qquad , n = 0, 1, 2, \ldots$$

Hence the subsequences $\{y_{2n(k+1)+i}\}_{n=-1}^{\infty}$ are decreasing for each $1 \leq i \leq 2k+2$. Let

$$\lim_{n\to\infty} y_{(2k+2)n+i} = \rho_i \qquad i = 1, 2, \ldots, 2k+2.$$

It is clear that $\{\ldots, \rho_1, \rho_2, \ldots \rho_{2k+2}, \rho_1, \rho_2, \ldots \rho_{2k+2}, \ldots\}$ is a $2(k+1)$-periodic solution of equation (8).
Now let $\varphi_0, \varphi_1, \ldots, \varphi_k$ be distinct positive real numbers. It follows that the sequence

$$\ldots, \varphi_0, 0, \varphi_1, 0, \ldots, \varphi_k, 0, \varphi_0, 0, \varphi_1, 0, \ldots, \varphi_k, \ldots$$

is a periodic solution of equation (8) with prime period $2(k+1)$.

3. *Assume that $\gamma < 1$. Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a nontrivial solution of equation (8) and let $\bar{y}$ denote the unique positive equilibrium of equation (8) such that*
$0 < \bar{y} < y_{-2k}, y_{-2k+2}, \ldots, y_0$ *and* $0 < y_{-2k-1}, y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} < \bar{y}$ *is satisfied.*
It follows that for all $m \geq 0$ and $0 \leq j \leq k$, we have

$$y_{(2k+2)(m+1)+2j} > y_{(2k+2)m+2j}$$

and

$$y_{(2k+2)(m+1)+2j+1} < y_{(2k+2)m+2j+1}$$

Hence for each $0 \leq j \leq k$
$\lim_{m\to\infty} y_{(2k+2)m+2j} = L_{2j} \in (\sqrt{1-\gamma}, \infty)$ *and* $\lim_{m\to\infty} y_{(2k+2)m+2j+1} = L_{2j+1} \in [0, \sqrt{1-\gamma})$.
We show that for each $0 \leq j \leq k$, $L_{2j+1} = 0$.
For the sake of contradiction, suppose that there exists $j \in \{0, 1, \ldots, k\}$

with $L_{2j+1} \in (0, \sqrt{1-\gamma})$.
Then

$$L_{2j+1} = \lim_{m\to\infty} y_{(2k+2)(m+1)+2j+1}$$

$$= \lim_{m\to\infty} \frac{y_{(2k+2)m+2j+1}}{\gamma + y_{(2k+2)(m+1)+2j-2l}y_{(2k+2)m+2j+2}}$$

$$= \frac{L_{2j+1}}{\gamma + L_{2j-2l}L_{2j+2}}.$$

So as

$$\lim_{m\to\infty} y_{(2k+2)m+2j+1} = L_{2j+1} \in (0, \sqrt{1-\gamma})$$

we have

$$1 = \gamma + L_{2j-2l}L_{2j+2} > 1$$

which is a contradiction.
Thus it is true that for each $0 \le j \le k$, $L_{2j+1} = 0$, *and so*

$$\lim_{n\to\infty} y_{2n+1} = 0.$$

Now we show that for each $0 \le j \le k$, $L_{2j} = \infty$.
For the sake of contradiction, suppose that there exists $j \in \{0, 1, \ldots, k\}$ *with* $L_{2j} \in (\sqrt{1-\gamma}, \infty)$.
Then

$$L_{2j} = \lim_{m\to\infty} y_{(2k+2)(m+1)+2j}$$

$$= \lim_{m\to\infty} \frac{y_{(2k+2)m+2j}}{\gamma + y_{(2k+2)(m+1)+2j-2l-1}y_{(2k+2)m+2j+1}}$$

$$= \frac{L_{2j}}{\gamma}$$

So $\gamma = 1$, *which is a contradiction. Hence* $\lim_{n\to\infty} y_{2n} = \infty$, *and the proof is complete.*

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