Quanlin Li
Constructive Computation in Stochastic Models with Applications
The *RG*-Factorizations

Constructive Computation in Stochastic Models with Applications The RG-Factorizations provides a unified, constructive and algorithmic framework for numerical computation of many practical stochastic systems. It summarizes recent important advances in computational study of stochastic models from several crucial directions, such as stationary computation, transient solution, asymptotic analysis, reward processes, decision processes, sensitivity analysis as well as game theory. Graduate students, researchers, and practicing engineers in the field of operations research, management sciences, applied probability, computer networks, manufacturing systems, transportation systems, insurance and finance, risk management and biological sciences will find this book valuable.

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With 23 figures

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Quan-Lin Li

随机模型构造性计算理论及其应用：*RG*–分解方法

Constructive Computation in Stochastic Models with Applications

The *RG*-Factorization

With 23 figures

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内 容 简 介

本书介绍了随机模型中计算技术的主要基础理论，总结了近十年来国内外所取得的新成果与进展。它构造性地建立了一般马尔可夫过程的 *RG*-分解，其中 *RG*-分解是马尔可夫过程与高斯消元法的完美结合，为求解无限维（或大型）线性方程组提供了有效途径。全书共分为三个部分。第一部分描述了如何把排队系统、可靠性工程、制造系统、计算机网络、交通系统、服务系统等应用随机模型转化为块型结构的马尔可夫过程，这为研究许多实际系统的性能评价、优化与决策提供了统一的数学理论框架。第二部分提供了研究随机模型的计算理论基础，包括 Censoring 不变性、*RG*-分解、*RG*-对偶性、谱分析、稳态计算、瞬态计算、渐近性分析、敏感性分析等。第三部分研究了随机模型中的一些热点问题，例如拟平稳分布、连续状态马尔可夫过程、马尔可夫报酬过程、马尔可夫决策过程、演化博弈论等。

本书的读者对象为代数、应用概率、运筹学、管理科学、制造系统、计算机网络、交通系统、服务系统、生物工程等领域中高年级大学生、研究生、科技人员与工程技术人员。

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To my friend Marcel F. Neuts for his pioneering
contributions to stochastic models

Preface

Stochastic systems are involved in many practical areas, such as applied probability, queueing theory, reliability, risk management, insurance and finance, computer networks, manufacturing systems, transportation systems, supply chain management, service operations management, genomic engineering and biological sciences. When analyzing a stochastic system, block-structured stochastic models are found to be a useful and effective mathematical tool. In the study of the block-structured stochastic models, this book provides a unified, constructive and algorithmic framework on two important directions: performance analysis and system decision. Different from those books given in the literature, the framework of this book is systematically organized by means of the UL- and LU-types of *RG*-factorizations, which are completely developed in this book and have been extensively discussed by the author. The *RG*-factorizations can be applied to provide effective solutions for the block-structured Markov chains, and are shown to be also useful for optimal design and dynamical decision making of many practical systems, such as computer networks, transportation systems and manufacturing systems. Besides, this book uses the *RG*-factorizations to deal with some recent interesting topics, for example, tailed analysis, continuous-state Markov chains, quasi-stationary distributions, Markov reward processes, Markov decision processes, sensitivity analysis, evolutionary game and stochastic game. Note that all these different problems can be dealt with by a unified computational method through the *RG*-factorizations. Specifically, this book pays attention to optimization, control, decision making and game of the block-structured stochastic models, although available results on these directions are still few up to now.

The block-structured stochastic models began with studying the matrix-geometric stationary probability of a Quasi-Birth-And-Death (QBD) process which was first proposed to analyze two-dimensional queues or computer systems, e.g., see Evans (1967) and Wallace (1969). The initial attention was directed toward performance computation. Neuts (1978) extended the results of the QBD processes to Markov chains of $GI/M/1$ type for the first time. Based on the phase type (PH) distribution given in Neuts (1975), Neuts (1981) opened an interesting and crucial door in numerical analysis of stochastic models, which has become increasingly important for dealing with large-scale and complex stochastic systems due to advent more powerful computational ability under fast development of computer technology and communication networks. For a complete understanding of stochastic models, it is necessary to review two key advances. Firstly, Neuts (1981) considered Markov chains of $GI/M/1$ type whose stationary probability vectors are the matrix-geometric form, called *matrix-geometric solution*. For the matrix-geometric solution, the matrix R, the minimal nonnegative solution to the nonlinear matrix equation $R = \sum_{k=0}^{\infty} R^k A_k$, plays an important role. He indicated that numerical computation of Markov chains of $GI/M/1$ type can be transformed to that of the matrix R, and then an infinite-dimensional computation for the stationary probability vector is transformed to another finite-dimensional computation for the censored Markov chain to level 0. Readers may refer to Neuts (1981), Latouche and Ramaswami (1999), Bini, Latouche and Meini (2005) and others therein. Secondly, as a companion research for Markov chains of $GI/M/1$ type, Neuts (1989) provided a detailed discussion for Markov chains of $M/G/1$ type whose stationary probability vector has a complicated form, called *matrix-iterative solution*. Although the two types of Markov chains have different block structure, the matrix-iterative solution has many properties similar to those in the matrix-geometric solution, for example, the matrix-iterative solution is determined by the minimal nonnegative solution G to another key nonlinear matrix equation $G = \sum_{k=0}^{\infty} A_k G^k$.

These results given in Neuts' two books (1981, 1989) are simple, perfect and computable. However, Markov chains of $GI/M/1$ type and Markov chains of $M/G/1$ type are two important examples in the study of block-structured stochastic models, while analysis of many practical stochastic systems needs to use more general block-structured Markov chains, e.g., see retrial queues given in Artalejo and Gómez-Corral (2008) and other stochastic models given in Chapters 1 and 7 of this book. Under the situation, these practical examples motivate us in this book to develop a more general algorithmic framework for studying the block-structured stochastic models, including generalization of the matrix-geometric solution and the matrix-iterative solution from the level independence to the level dependence. It is worthwhile to note that such a generalization is not easy and simple, it needs and requires application of new mathematical methods. During the two decades, the censoring technique is indicated to be a key method for dealing with more general block-structured Markov chains. Grassmann and Heyman (1990) first used the censoring technique to find some basic relationships between the matrix-geometric solution and the matrix-iterative solution from a more general model: Markov chains of $GI/G/1$ type. Furthermore, Heyman (1995) applied the censoring technique to provide an LU-decomposition for any ergodic stochastic matrix of infinite size, Li (1997) gave the LU-decomposition for Markov chains of $GI/M/1$

type and also for Markov chains of $M/G/1$ type, and Zhao (2000) obtained the UL-type RG-factorization for Markov chains of $GI/M/1$ type. From these works, it may be clear that finding such matrix decomposition for general Markov chains is a promissing direction for numerical solution of block-structred stochastic models. Along similar lines, we have systematically developed the UL- and LU-types of RG-factorizations for any irreducible Markov chains in the past ten years, e.g., see Li and Cao (2004), Li and Zhao (2002, 2004) and Li and Liu (2004). This book summarizes many important results and basic relations for block-structured Markov chains by means of the RG-factorizations. The RG-factorizations are derived for any irreducible Markov chains in terms of the Wiener-Hopf equations, while some useful iterative relations among the R-, U- and G-measures are organized in the Wiener-Hopf equations. Specifically, the iterative relations are sufficiently helpful for dealing with performance computation and system decision. On the other hand, this book also provides new probabilistic interpretations for those results obtained by Neuts' method. We may say that the RG-factorizations begin a new era in the study of block-structured stochastic models with an algebraic and probabilistic combination.

The main contribution of this book is to construct a unified computational framework to study stochastic models both from stationary solution and from transient solution. When a practical system is described as a block-structured Markov chain, performance computation and system decision can always be organized as a system of linear equations: $xA=0$ or $xA=b$ where $b\neq 0$. This book provides two different computational methods to deal with the system of linear equations. At the same time, it is seen from the computational process that the middle diagonal matrix of the RG-factorizations plays an important role based on the state classification of Markov chains.

Method Ⅰ In this method the matrix A can be shown to have a UL-type RG-factorization

$$A=(I-R_U)\cdot \mathrm{diag}\,(\Theta_0,\Theta_1,\Theta_2,...)\cdot(I-G_L),$$

where the size of the matrix Θ_0 is always small and finite in level 0. This book summarizes two important conclusions:

(1) If the block-structured Markov chain is positive recurrent, then the matrix Θ_0 is singular and all the other matrices Θ_k for $k\geqslant 1$ are invertible. In this case, the UL-type RG-factorization can be used to solve the system of linear equations: $xA=0$ given in Section 2.4, and then such a solution can be used to obtain stationary performance analysis.

(2) If the block-structured Markov chain is transient, then the matrix Θ_k is invertible for $k\geqslant 0$. In this case, the UL-type RG-factorization is used to solve the system of linear equations: $xA=b$ with

$$x=b(I-G_L)^{-1}\cdot \mathrm{diag}\,(\Theta_0^{-1},\Theta_1^{-1},\Theta_2^{-1},...)\cdot(I-R_U)^{-1},$$

which leads to transient performance analysis.

Method Ⅱ In this method the matrix A can be shown to have an LU-type RG-factorization

$$A = (I - \overline{R}_L) \cdot \text{diag}\,(\varLambda_0, \varLambda_1, \varLambda_2, \ldots) \cdot (I - \overline{G}_U),$$

where the matrix $\varLambda_k$ is invertible for $k \geqslant 0$. Therefore, the LU-type RG-factorization can be used to deal with the system of linear equations: $xA = b$ with

$$x = b(I - \overline{G}_U)^{-1} \cdot \text{diag}\,(\varLambda_0^{-1}, \varLambda_1^{-1}, \varLambda_2^{-1}, \ldots) \cdot (I - \overline{R}_L)^{-1},$$

which further leads to transient performance analysis of a stochastic model.

This book has grown out of my research and lecture notes on the matrix-analytic methods since 1997. Although I have made an effort to introduce explanations and definitions for mathematical tools, crucial concepts and basic conclusions in this book, it is still necessary for readers to have a better mathematical background, including probability, statistics, Markov chains, Markov renewal processes, Markov decision processes, queueing theory, game theory, matrix analysis and numerical computation. Readers are assumed to be familiar with the basic materials or parts of them.

The organization of this book is strictly logical and more complete from performance computation to system decision. This book contains eleven chapters whose structured relationship is shown in Fig. 0.1. Chapters 1 and 7 introduce motivating examples from different research areas, such as queueing theory, computer networks and manufacturing systems. The examples are first described as the block-structured Markov chains, then they will help readers to understand the basic structure of practical stochastic models. Chapters 2, 3, 5, 6 and 9 systematically develop the construction of the RG-factorizations for Markov chains, Markov renewal processes and β-discounted transition matrices. Chapters 4, 8, 10 and 11 apply the RG-factorizations to deal with some current interesting topics including tailed analysis, Markov chains on a continuous state space, transient solution, Markov reward processes, sensitivity analysis and game theory, respectively. Finally, we also provide two useful appendices which may be basically helpful for readers to understand the contents of this book. Every chapter consists of a brief summary, a main body and a discussion with "Notes in the Literature". At the same time, every chapter also contains a number of problems whose purpose is to help readers understand the corresponding concepts, results and conclusions.

It is hoped that this book will be useful for the first-year graduate students or advanced undergraduates, as well as researchers and engineers who are interested in, for example, applied probability, queueing theory, reliability, risk management, insurance and finance, communication networks, manufacturing systems, transportation systems, supply chain management, service operations management, performance evaluation, system decision, and game theory with applications. We suggest a full semester course with two or three hours per week. Shorter courses can be also based on part of the chapters, for instance, engineering students or researchers may only study Chapters 1, 2, 6, 8, 10 and 11.

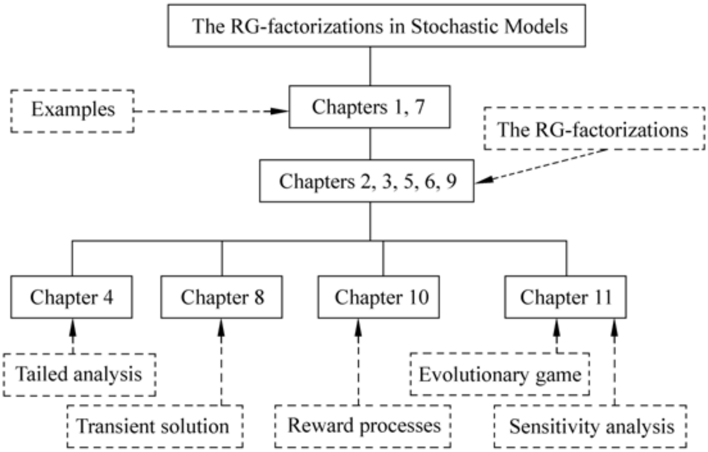

Figure 0.1 Organization of this book

It is a pleasure to acknowledge Marcel F. Neuts for his pioneering work which developed an important area: Numerical computation of stochastic models, his comments and suggestions are valuable and useful for improving the presentation of this book. I also thank Yiqiang Zhao for his cooperation on the *RG*-factorizations and block-structured stochastic models from 1999 to 2003. In fact, some sections of this book directly come from his or our collaboration works. Special thanks go to Jinhua Cao, Ming Tan, Naishuo Tian and Dequan Yue who encouraged me in the study of stochastic models during my master and Ph.D. programs. I am grateful to my friends J.R. Artalejo, N.G. Bean, Xiuli Chao, A. Dudin, A. Gómez-Corral, Fuzhou Gong, Xianping Guo, Qiming He, Zhengting Hou, Guanghui Hsu, Ninjian Huang, Haijun Li, Wei Li, Zhaotong Lian, Chuang Lin, Ke Liu, Zhiming Ma, Zhisheng Niu, T. Takine, Peter G. Taylor, Jeffery D. Tew, Jingting Wang, Dinghua Shi, Deju Xu, Susan H. Xu, David D. Yao and Hanqin Zhang for their great help and valuable suggestions on the matrix-analytic methods. I am indebted to Xiren Cao, Liming Liu and Shaohui Zheng for the financial support for my visits to Hong Kong University of Science and Technology in the recent years. Their valuable comments and suggestions helped me to develop new and interesting fields, such as perturbation analysis and Markov decision processes. I thank my master and Ph.D. students, such as Dejing Chen, Shi Chen, Yajuan Li, Jinqi Wang, Yang Wang, Yitong Wang, Xiaole Wu, Jia Yan and Qinqin Zhang. This book has benefited from the financial support provided by National Natural Science Foundation of China (Grant No. 10671107, Grant No. 10871114, Grant No. 60736028) and the National Grand Fundamental Research (973) Program of China (Grant No. 2006CB805901). I thank all of my colleagues in the Department of Industrial Engineering, Tsinghua

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May 30, 2009

Contents

1 Stochastic Models

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Abstract In this chapter, we provide an introduction to Markov chains, practical examples for block-structured Markov chains, QBD processes, UL- and LU-types of *RG*-factorizations for QBD processes, the phase type (PH) distribution, the Markovian arrival process (MAP) and the matrix-exponential distribution. We list some necessary definitions and useful results, which are basic and crucial preliminaries in the study of stochastic models.

Keywords stochastic model, block-structured Markov chain, QBD process, *RG*-factorization, phase type distribution, Markovian arrival process, matrix-exponential distribution.

In this chapter, we provide some basic and useful preliminaries in the study of stochastic models. This contains a simple introduction to Markov chains with discrete state space, motivating practical examples for how to construct block-structured Markov chains, applying the censoring technique to deal with a QBD process with either finitely-many levels or infinitely-many levels, the UL- and LU-types of *RG*-factorizations for the QBD process, the PH distribution, the Markovian arrival process and the matrix-exponential distribution. These contents are organized in seven sections. Here, we mainly list the main definitions and results without proofs. Readers may refer to Neuts [92,94] and others for the proofs, if necessary.

1.1 Stochastic Systems

In this section, we show that Markov chain is a useful mathematical tool in the study of stochastic systems. We provide some useful discussions for Markov chains with discrete state space that are described as different types of block-structured Markov chains. These are useful for understanding the sequence of this book.

Modern science and technology has created our beautiful life and suitable working space. We use various natural or man-made systems on a daily basis. Important examples of such systems include manufacturing systems, communication networks and transportation systems. From the ordinary observations, we can easily find that random factors usually exist in these different systems. Thus, it is not only necessary but also important for studying the real systems under stochastic conditions. To do this, we now provide a simple introduction to stochastic processes, and specifically, Markov chains.

Let $X(t)$, which is either scalar or vector, be the state of a stochastic system at time t. Then $\{X(t):t\geqslant 0\}$ in general is a stochastic process. The stochastic process is a family of random variables $X(t)$ for $t\in T$, where T is a non-null set. If $T=[0, +\infty]$ or [0,a] with $a>0$, then the stochastic process called continuous-time; if $T=\{0, 1, 2, ...\}$, then the stochastic process is discrete-time. In this case, we write the Markov chain $\{X(t)\}$ or $\{X_t\}$. On the other hand, the stochastic process is distinguished by its state which is denoted as a real number. The range of possible state values for the random variables $X(t)$ for $t\in T$ is called the state space Ω. If $\Omega=[0, +\infty]$ or $[0,b]$ with $b>0$, then the stochastic process is said to have a continuous state space; if $\Omega=\{0, 1, 2, ...\}$ or $\{0, 1, 2, ..., M\}$ with $M>0$, then the stochastic process is called to have a discrete state space. In general, a stochastic process always has a complicated behavior that is difficult to be analyzed in detail. From practical applications, we only need to consider a subset of stochastic processes analyzed above.

1.1.1 The Markov Property

A discrete-time stochastic process has the Markov property if

$$P\{X_{n+1} \mid X_n, X_{n-1}, ..., X_0\} = P\{X_{n+1} \mid X_n\}.$$

Similarly, a continuous-time stochastic process has the Markov property if

$$P\{X(t+u) \mid X(s), 0\leqslant s\leqslant u\} = P\{X(t+u) \mid X(u)\}.$$

Based on the Markov property, we consider an important type of stochastic processes: Markov chains on discrete state space, which are either discrete-time or continuous-time.

1.1.2 A Discrete-Time Markov Chain with Discrete State Space

Let $\{X_n\}$ be a discrete-time Markov chain with discrete state space, that is, $T=\{0, 1, 2, ...\}$ and $\Omega=\{0, 1, 2, ...\}$. The probability of X_{n+1} being in state j given that X_n is in state i is called the one-step transition probability and is denoted by

$$P_{i,j}^{n,n+1} = P\{X_{n+1} = j \mid X_n = i\}.$$

If all the one-step transition probabilities are independent of time $n \geqslant 0$, then the Markov chain is said to be homogeneous; otherwise it is said to be non-homogeneous.

In this book, we mainly analyze the homogeneous Markov chains for simplicity of description. In this case, let $P_{i,j} = P_{i,j}^{n,n+1}$ for all $i,j \in \Omega$. We write

$$P^{①} = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & \cdots \\ P_{2,0} & P_{2,1} & P_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Let

$$P_{i,j}^{(n)} = P\{X_n = j \mid X_0 = i\}$$

and $P^{(n)} = \left(P_{i,j}^{(n)}\right)_{i,j \geqslant 0}$. Then it is easy to check that $P^{(n)} = P^n$ for each $n \geqslant 2$.

State j is said to be accessible from state i if $P_{i,j}^{(n)} > 0$ for some integer $n \geqslant 1$, denoted as $i \rightarrow j$. Two states i and j are called communication if each one is accessible to the other, writen as $i \longleftrightarrow j$. It is clear that the communication is an equivalence relation:

(D1) reflexivity $i \longleftrightarrow i$;

(D2) symmetry if $i \longleftrightarrow j$, then $j \longleftrightarrow i$;

(D3) transitivity if $i \longleftrightarrow j$ and $j \longleftrightarrow k$, then $i \longleftrightarrow k$.

We can now partition the state space Ω into some equivalent classes C_k for $k = 1, 2, \ldots, K$. Then $\Omega = \bigcup_{k=1}^{K} C_k$, where $C_i \bigcap C_j = \varnothing$ for $i \neq j$, and $\varnothing$ is an empty set. In each equivalent class, all the states can communicate with each other.

If the state space Ω has only one equivalent class, then the Markov chain is said to be irreducible; otherwise it is called to be reducible.

Now, we define the period of state i, written as $d(i)$, to be the greatest common divison (g.c.d) of all integers $n \geqslant 1$ for which $P_{i,i}^{(n)} \geqslant 0$. Specifically, if $P_{i,i}^{(n)} = 0$ for all $n \geqslant 1$, then $d(i) = 0$. In an equivalent class $C \subset \Omega$, it is easy to prove that $d(i) = d(j)$ for all $i, j \in C$. Furthermore, for an irreducible Markov chain, we always have $d(i) = d(j)$ for all $i, j \in \Omega$.

If $d(i) = 1$, then state i is said to be aperiodic; otherwise it is periodic. For an irreducible Markov chain, it is said to be aperiodic if there exists a state which is aperiodic; while it has period d if there exists a state which is of period d.

---

① From this section, notation for vector or matrix will not use the bold form.

In what follows we provide state classification of Markov chains. We write

$$f_{i,i}^{(n)} = P\{X_n = i, X_v \neq i \text{ for all } v = 1,2,\ldots,n-1 \mid X_0 = i\}.$$

Clearly, $f_{i,i}^{(1)} = P_{i,i}$ and for $n \geqslant 1$,

$$P_{i,i}^{(n)} = \sum_{k=0}^{n} f_{i,i}^{(k)} P_{i,i}^{(n-k)}.$$

Let

$$f_{i,i} = \sum_{n=0}^{\infty} f_{i,i}^{(n)}.$$

Then we say that a state i is recurrent if $f_{i,i} = 1$; otherwise it is transient.

The following proposition provides a necessary and sufficient condition under which a state i is recurrent. The proof is standard and is omitted here.

Proposition 1.1 *A state i is recurrent if and only if* $\sum_{n=1}^{\infty} P_{i,i}^{(n)} = +\infty$; *while a state i is transient if and only if* $\sum_{n=1}^{\infty} P_{i,i}^{(n)} < +\infty$.

It is easy to see that in an equivalent class $C \subset \Omega$, if a state $i \in C$ is recurrent, then each state $j \in C$ is also recurrent. Furthermore, for an irreducible Markov chain, it is clear that if a state $i \in \Omega$ is recurrent, then each state $j \in \Omega$ is also recurrent. Such a discussion is also valid for the transient case.

We can further classify the recurrent class of states into two subsets: positive recurrence and null recurrence. To do that, we write

$$m_i = \sum_{n=0}^{\infty} n f_{i,i}^{(n)}.$$

We say that a state i is positive recurrent if $m_i < +\infty$; otherwise it is null recurrent.

In an equivalent class $C \subset \Omega$, it is easy to prove that if a state $i \in C$ is positive recurrent, then each state $j \in C$ is also positive recurrent. Furthermore, for an irreducible Markov chain, it is easy to see that if a state $i \in \Omega$ is positive recurrent, then each state $j \in \Omega$ is also positive recurrent. Such a discussion is also valid for the null recurrent case.

For an irreducible Markov chain, we now consider the limiting

$$\pi_i = \lim_{n \to \infty} P_{i,i}^{(n)}$$

with normalization condition $\sum_{i \in C} \pi_i = 1$ or $\sum_{i \in \Omega} \pi_i = 1$.

We state the basic limit theorem for Markov chains as follows.

Theorem 1.1 *For an irreducible, aperiodic and positive recurrent Markov chain, there must exist the unique limiting distribution vector* $\pi=(\pi_0, \pi_1, \pi_2, ...)$ *with* $\pi_i=1/m_i$ *for all* $i\in\Omega$*. At the same time,*

$$\lim_{n\to\infty} P_{i,j}^{(n)} = \pi_j$$

is independent of $i\in\Omega$*.*

If the Markov chain is either null recurrent or transient, then

$$\pi=(\pi_0, \pi_1, \pi_2, ...)=0.$$

If the Markov chain with period d is irreducible and positive recurrent, then

$$\lim_{n\to\infty} P_{i,j}^{(nd)} = \pi_j(d),$$

which is independent of the initial state $i\in\Omega$.

For an irreducible, aperiodic and positive recurrent Markov chain, it is clear that

$$\lim_{n\to\infty} P^n = \lim_{n\to\infty} P^{(n)} = e\pi,$$

where e is a column vector of ones with switable size. Thus

$$e\pi P = \lim_{n\to\infty} P^n P = \lim_{n\to\infty} P^{n+1} = e\pi$$

which obviously leads to $\pi P=\pi$ and $\pi e=1$. That is, the limiting distribution vector π of the Markov chain P is the unique positive solution to the system of equations $\pi P=\pi$ and $\pi e=1$.

For an irreducible finite-state Markov chain P, it is either positive recurrent or transient. When P is positive recurrent, the matrix $I-P+e\pi$ is invertible and is called the fundamental matrix of the Markov chain; when P is transient, the matrix $I-P$ is invertible.

For an irreducible finite-state Markov chain P, the limit

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{n=0}^{N} P^n$$

must exist and is called *the Cesaro limit*. We denote the Cesaro limit by P^*. Specifically, $P^*=e\pi$ if P is aperiodic and positive recurrent. For the Cesaro limit, it is easy to check that

$$PP^*=P^*P=P^*P^*=P^*$$

and

$$P^n-P^*=[P-P^*]^n, \quad n\geqslant 1.$$

1.1.3 A Continuous-Time Markov Chain with Discrete Space

We denote by $\{X(t)\}$ a continuous-time Markov chain with $T=[0, +\infty)$ and $\Omega=\{0, 1, 2, ...\}$. We write

$$P_{i,j}(t) = P\{X(t+u) = j \mid X(u) = i\}$$

which is independent of $u \geqslant 0$. The Markov property asserts that $P_{i,j}(t)$ satisfies

(C1) $P_{i,j}(t) \geqslant 0$ for all $i, j \in \Omega$,

(C2) $\sum_{j \in \Omega} P_{i,j}(t) = 1$ for all $i \in \Omega$,

(C3) $P_{i,j}(s+t) = \sum_{k \in \Omega} P_{i,k}(s) P_{k,j}(t)$ for all $s, t \geqslant 0$ (Chapman-Kolmogorov relation),

and

(C4) $\lim_{t \to 0^+} P_{i,j}(t) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

Let

$$P(t) = \begin{pmatrix} P_{0,0}(t) & P_{0,1}(t) & P_{0,2}(t) & \cdots \\ P_{1,0}(t) & P_{1,1}(t) & P_{1,2}(t) & \cdots \\ P_{2,0}(t) & P_{2,1}(t) & P_{2,2}(t) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

From (C1) to (C4), it is easy to check that

$$\lim_{h \to 0} P(t+h) = P(t).$$

Therefore, $P(t)$ is also a matrix of continuous functions for $t \geqslant 0$. Actually, $P(t)$ is a matrix of differentiable functions for $t \geqslant 0$ in which the limits

$$\lim_{h \to 0^+} \frac{1 - P_{i,i}(h)}{h} = q_i, \quad i \in \Omega$$

and

$$\lim_{h \to 0^+} \frac{P_{i,i}(h)}{h} = q_{i,j}, \quad i \neq j \text{ and } i, j \in \Omega$$

exist, where $0 \leqslant q_i < +\infty$ for $i \in \Omega$, and $0 \leqslant q_{i,j} < +\infty$ for $i \neq j$ and $i, j \in \Omega$. Note that

$$1 - P_{i,i}(h) = \sum_{j \in \Omega - \{i\}} P_{i,j}(h),$$

which leads to that for $i \in \Omega$,

$$q_i = \sum_{j\in\Omega-\{i\}} q_{i,j}.$$

At the same time, the rates q_i and $q_{i,j}$ furnish an infinitesimal description as follows:
For $h>0$ sufficiently small, $i \neq j$ and $i, j \in \Omega$,

$$P\{X(t+h) = j \mid X(t) = i\} = q_{i,j}h + o(h),$$

and for $i \in \Omega$,

$$P\{X(t+h) = i \mid X(t) = i\} = 1 - q_i h + o(h).$$

In contrast to the infinitesimal description, the sojourn description is given as follows. If the Markov chain is in state i, then it sojourns there for a duration that is exponentially distributed with parameter q_i, and can jump to the next state j with probability $q_{i,j}/q_i$ for $j \neq i$.

We write

$$Q = \begin{pmatrix} -q_0 & q_{0,1} & q_{0,2} & q_{0,3} & \cdots \\ q_{1,0} & -q_1 & q_{1,2} & q_{1,3} & \cdots \\ q_{2,0} & q_{2,1} & -q_2 & q_{2,3} & \cdots \\ q_{3,0} & q_{3,1} & q_{3,2} & -q_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

which is called the infinitesimal generator of the continuous-time Markov chain. Clearly

$$\lim_{h\to 0^+} \frac{P(h)-I}{h} = Q.$$

Note that

$$\frac{P(t+h)-P(t)}{h} = \frac{P(h)-I}{h}P(t) = P(t)\frac{P(h)-I}{h},$$

we obtain

$$P'(t) = QP(t) = P(t)Q,$$

where

$$P'(t) = \frac{\mathrm{d}}{\mathrm{d}t}P(t) = \lim_{h\to 0^+} \frac{P(t+h)-P(t)}{h}.$$

Therefore

$$P(t) = \exp\{Qt\}$$

by means of the initial condition $P(0)=I$. The continuous-time Markov chain may also be called the Markov chain Q.

During a similar analysis of the discrete-time Markov chains, we can introduce some basic concepts for the continuous-time case, such as, the irreducible and reducible, and the aperiodic and periodic. At the same time, we can also give the state classification: transience, null recurrence and positive recurrence.

If the continuous-time Markov chain is irreducible, aperiodic and positive recurrent, then the limits $\lim_{t\to+\infty} P_{i,j}(t)$ exist and is independent of the initial state $i \in \Omega$, denoted as π_j. Let $\pi=(\pi_0, \pi_1, \pi_2, \ldots)$. Then

$$\lim_{t\to+\infty} P(t) = \lim_{t\to+\infty} \exp\{Qt\} = e\pi$$

and clearly, the limiting distribution vector is the unique positive solution to the system of equations $\pi Q=0$ and $\pi e=1$.

1.1.4 A Continuous-Time Birth Death Process

As a useful example, we now discuss a special continuous-time Markov chain: A continuous-time birth death process whose infinitesimal generator is given by

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \\ & & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $\lambda_i > 0$ for $i \geqslant 0$ and $\mu_j > 0$ for $j \geqslant 1$.

If the birth death process Q is irreducible, aperiodic and positive recurrent, then the limiting distribution is given by

$$\pi_j = \frac{\xi_j}{\sum_{k=0}^{\infty} \xi_k}, \quad j \geqslant 0,$$

where

$$\xi_0 = 1, \quad \xi_k = \xi_{k-1}\frac{\lambda_{k-1}}{\mu_k} = \frac{\lambda_0\lambda_1 \ldots \lambda_{k-1}}{\mu_1\mu_2 \ldots \mu_k}, \quad k \geqslant 1.$$

If $\lambda_0 = 0, \lambda_j > 0$ and $\mu_j > 0$ for $j \geqslant 1$, then it is clear that the birth death process is transient with absorbing state 0. We denote by g_m the probability of absorption into state 0 from the initial state m for $m \geqslant 1$. Then we have

$$g_m = \frac{\sum_{i=m}^{\infty} \eta_i}{\sum_{i=0}^{\infty} \eta_i}, \quad m \geqslant 1,$$

where

$$\eta_0 = 1, \quad \eta_k = \eta_{k-1} \frac{\mu_k}{\lambda_k} = \frac{\mu_1 \mu_2 \ldots \mu_k}{\lambda_1 \lambda_2 \ldots \lambda_k}, \quad k \geqslant 1.$$

1.1.5 Block-Structured Markov Chains

A practical stochastic system always indicates special block structure of its corresponding Markov chain. This motivates us to develop constructively numerical computation of stochastic models on the line of Neuts [92, 94]. In what follows we summarize the useful block structure of Markov chains applied to real stochastic systems recently.

In the study of stochastic models, Neuts opened a key door for developing numerical theory of stochastic models. It is necessary for us to review the two books by Neuts in 1981 and 1989, respectively. In the first book, Neuts [92] studied a level-independent QBD process whose transition matrix is given by

$$P = \begin{pmatrix} B_1 & B_0 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.1}$$

By using the QBD process, Chapter 3 of Neuts [92] analyzed the *M*/*PH*/1, *PH*/*M*/*c* and *PH*/*PH*/1 queues; Chapter 5 studied buffer models including tandem queues and a multiprogramming model; and Chapter 6 discussed queues in random environment such as a queue with repairable server, a finitesource priority queue, and a queue with paired customers. As an important generalization, Chapter 1 of Neuts [92] systemically analyzed a level-independent Markov chain of *GI*/*M*/1 type whose transition matrix is given by

$$P = \begin{pmatrix} B_1 & B_0 & & & & \\ B_2 & A_1 & A_0 & & & \\ B_3 & A_2 & A_1 & A_0 & & \\ B_4 & A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.2}$$

Chapter 4 of Neuts [92] discussed the *GI*/*PH*/1, *GI*/*PH*/*c*, *D*/*PH*/1 and *SM*/*PH*/1 queues.

In the second book, Neuts [94] analyzed a level-independent Markov chain of *M*/*G*/1 type whose transition matrix is given by

$$P = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 & \cdots \\ B_0 & A_1 & A_2 & A_3 & \cdots \\ & A_0 & A_1 & A_2 & \cdots \\ & & A_0 & A_1 & \cdots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{1.3}$$

Applying the Markov chains of *M*/*G*/1 type, Neuts [94] studied the *M*/*SM*/1 queue and its variants in Chapter 4, the *BMAP*/*G*/1 queue in Chapter 5, and several practical systems such as a data communication model, a poor man's satellite, and a series queue with two servers in Chapter 6.

Up to now, there have been many applied problems that are described as more general block-structured Markov chains. In what follows we list some basic examples from the block-structured Markov chains.

(1) A level-dependent QBD process whose transition matrix is given by

$$P = \begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & & \\ & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & \\ & & A_2^{(3)} & A_1^{(3)} & A_0^{(3)} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.4}$$

(2) A Markov chain of *GI*/*G*/1 type whose transition matrix is given by

$$P = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ D_{-1} & A_0 & A_1 & A_2 & \cdots \\ D_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ D_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.5}$$

(3) A level-dependent Markov chain of *GI*/*M*/1 type whose transition matrix is given by

$$P = \begin{pmatrix} A_{0,0} & A_{0,1} & & & & \\ A_{1,0} & A_{1,1} & A_{1,2} & & & \\ A_{2,0} & A_{2,1} & A_{2,2} & A_{2,3} & & \\ A_{3,0} & A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.6}$$

(4) A level-dependent Markov chain of $M/G/1$ type whose transition matrix is given by

$$P = \begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & \cdots \\ A_{1,0} & A_{1,1} & A_{1,2} & A_{1,3} & \cdots \\ & A_{2,1} & A_{2,2} & A_{2,3} & \cdots \\ & & A_{3,2} & A_{3,3} & \cdots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{1.7}$$

(5) A general block-structured Markov chain with infinitely-many levels whose transition matrix is given by

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} & \cdots \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} & \cdots \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{1.8}$$

(6) A general block-structured Markov chain with finitely-many levels whose transition matrix is given by

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots & P_{0,M} \\ P_{1,0} & P_{1,1} & P_{1,2} & \cdots & P_{1,M} \\ P_{2,0} & P_{2,1} & P_{2,2} & \cdots & P_{2,M} \\ \vdots & \vdots & \vdots & & \vdots \\ P_{M,0} & P_{M,1} & P_{M,2} & \cdots & P_{M,M} \end{pmatrix}. \tag{1.9}$$

Specifically, a Markov chain of $M/G/1$ type with finitely-many levels whose transition matrix is given by

$$P = \begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,M-1} & A_{0,M} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,M-1} & A_{1,M} \\ & A_{2,1} & \cdots & A_{2,M-1} & A_{2,M} \\ & & \ddots & \vdots & \vdots \\ & & & A_{M,M-1} & A_{M,M} \end{pmatrix}; \tag{1.10}$$

a Markov chain of $GI/M/1$ type with finitely-many levels whose transition matrix is given by

$$P=\begin{pmatrix} A_{0,0} & A_{0,1} & & & & \\ A_{1,0} & A_{1,1} & A_{1,2} & & & \\ A_{2,0} & A_{2,1} & A_{2,2} & A_{2,3} & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ A_{M,1} & A_{M,2} & A_{M,3} & A_{M,4} & \cdots & A_{M,M} \end{pmatrix}; \tag{1.11}$$

a QBD process with finitely-many levels whose transition matrix is given by

$$P=\begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & & \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & & & \\ & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_2^{(M-1)} & A_1^{(M-1)} & A_0^{(M-1)} \\ & & & & A_2^{(M)} & A_1^{(M)} \end{pmatrix}; \tag{1.12}$$

a special QBD process with finitely-many levels whose transition matrix is given by

$$P=\begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & & & & \\ A_2^{(1)} & B & C & & & & & \\ & A & B & C & & & & \\ & & A & B & C & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & A & B & C & \\ & & & & & A & B & A_0^{(M-1)} \\ & & & & & & A_2^{(M)} & A_1^{(M)} \end{pmatrix}; \tag{1.13}$$

This book will provide a unified algorithmic framework for dealing with the Markov chains given in Eq. (1.1) to Eq. (1.13).

1.2 Motivating Practical Examples

In this section, we choose some simple practical systems under exponential-type assumptions to indicate how to organize the block-structured Markov chains given in Eq. (1.1) to Eq. (1.13). At the same time, more general examples will be arranged in Chapter 7 and in Problems of each chapter. Note that similar examples are considerable in the literature for studying queueing systems, communication networks, manufacturing systems and transportation systems etc.

1.2.1 A Queue with Server Vacations

Queues with server vacations are of an important type of stochastic models, which have been extensively applied to communication networks, manufacturing systems and transportation systems. The first example analyzes a single-sever vacation model. We consider a single-server queue with server vacations, where the arrival process is a Poisson process with arrival rate $\lambda > 0$, the service and vacation times are i.i.d. with exponential distributions $F(x) = 1 - e^{-\mu x}$ and $V(y) = 1 - e^{-\gamma y}$, respectively. We assume that the server can take at most N consecutive vacations. After the N consecutive vacations, the server has to enter an idle period, even though there is no customer in the waiting room. All the random variables defined above are assumed to be mutually independent.

For this system, we denote the number of customers in the system and the state of server at time t, by $Q(t)$ and $Z(t)$ respectively. For example, $Q(t) = 0, 1, 2, \ldots$, and

$$Z(t) = \begin{cases} I, 0, & \text{if the server is idle,} \\ W, m, & \text{if the server is busy with } m \text{ customers in the system,} \\ V, m, n, & \text{if the server has been at the } n\text{th consecutive vacation} \\ & \text{and there are } m \text{ customers in the system.} \end{cases}$$

It is clear from Fig. 1.1 that the Markov chain $\{(Q(t), Z(t)), t \geqslant 0\}$ is a QBD process Q given in Eq. (1.1) whose block-entries are given by

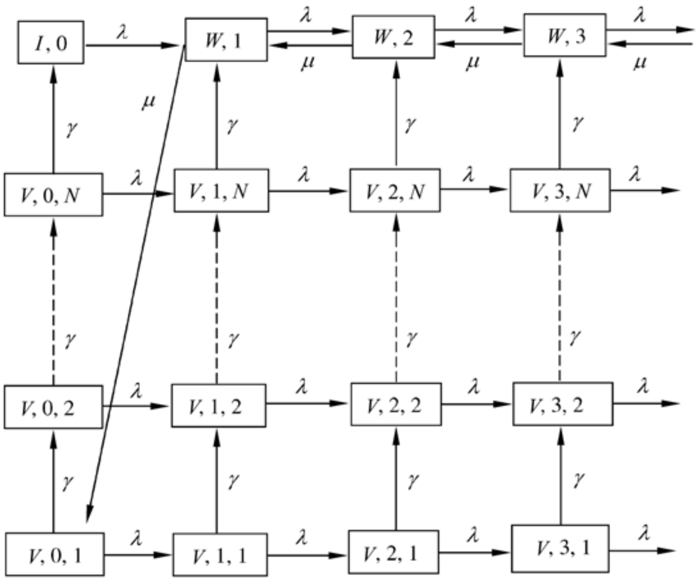

Figure 1.1 Relation of state transitions

$$B_1 = \begin{pmatrix} -\lambda & & & & & \\ \gamma & -(\gamma+\lambda) & & & & \\ & \gamma & -(\gamma+\lambda) & & & \\ & & \ddots & \ddots & & \\ & & & \gamma & -(\gamma+\lambda) & \\ & & & & \gamma & -(\gamma+\lambda) \end{pmatrix}, \quad B_2 = \begin{pmatrix} \\ \\ \\ \\ \\ \mu \end{pmatrix},$$

$$B_0 = A_0 = \mathrm{diag}(\lambda, \lambda, \lambda, \ldots, \lambda, \lambda),\ A_2 = \mathrm{diag}(\mu, 0, 0, \ldots, 0, 0),$$

$$A_1 = \begin{pmatrix} -(\lambda+\mu) & & & & & \\ \gamma & -(\gamma+\lambda) & & & & \\ & \gamma & -(\gamma+\lambda) & & & \\ & & \ddots & \ddots & & \\ & & & \gamma & -(\gamma+\lambda) & \\ & & & & \gamma & -(\gamma+\lambda) \end{pmatrix}.$$

1.2.2 A Queue with Repairable Servers

The second example considers a simple repairable queueing system. We consider a queueing system with N identical servers and a repairman. The arrival process is a Poisson process with arrival rate $\lambda > 0$, the service times are i.i.d. and are exponentially distributed with service rate μ, and the life and repair times of each server are exponentially distributed with parameters α and β, respectively. We assume that the service discipline is FCFS and the repair discipline is as good as new after repaired. All the random variables defined above are assumed to be mutually independent.

Let $Q(t)$ and $Z(t)$ be the number of customers in the system and the number of the available servers at time t, respectively. Then the Markov chain $\{(Q(t), Z(t)), t \geqslant 0\}$ is a QBD process Q given in Eq. (1.12) whose block-entries are given by

$$B_1 = \begin{pmatrix} \mathcal{A}_0 & \mathcal{C}_0 & & & & \\ B_1 & \mathcal{A}_1 & \mathcal{C}_1 & & & \\ & \mathcal{B}_2 & \mathcal{A}_2 & \mathcal{C}_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathcal{B}_{N-1} & \mathcal{A}_{N-1} & \mathcal{C}_{N-1} \\ & & & & \mathcal{B}_N & \mathcal{A}_N \end{pmatrix},$$

$$B_2 = (0, 0, 0, \ldots, 0, \mathcal{A}_2), \quad B_0 = (0, 0, 0, \ldots, 0, \mathcal{C}_N)^{\mathrm{T}},$$

where

$$
\begin{gathered}
\mathcal{C}_i = \lambda I, \quad 0 \leqslant i \leqslant N, \\
\mathcal{B}_i = \operatorname{diag}(\min\{i, j\}\ \mu), \quad 0 \leqslant i, j \leqslant N, \\
\mathcal{A}_0 = \Theta - \lambda I, \quad \mathcal{A}_i = \Theta - \mathcal{B}_i - \mathcal{C}_i, \quad 1 \leqslant i \leqslant N, \\
A_0 = \lambda I, \quad A_2 = \operatorname{diag}(0, \mu, 2\mu, ..., N\mu), \\
A_1 = \Theta - A_0 - A_2,
\end{gathered}
$$

$$
\Theta = \begin{pmatrix}
-\beta & \beta & & & & \\
\alpha & -(\alpha+\beta) & \beta & & & \\
 & 2\alpha & -(2\alpha+\beta) & \beta & & \\
 & & \ddots & \ddots & & \\
 & & & (N-1)\alpha & -[(N-1)\alpha+\beta] & \beta \\
 & & & & N\alpha & -N\alpha
\end{pmatrix}.
$$

1.2.3 A Call Center

The third example discusses a call center which is modeled by means of retrial queues. We consider a call center which is described as a retrial queue with impatient customers. This system is structured in Fig. 1.2, which indicates that this system contains two areas: a service and waiting area, and an orbit. In this system, there

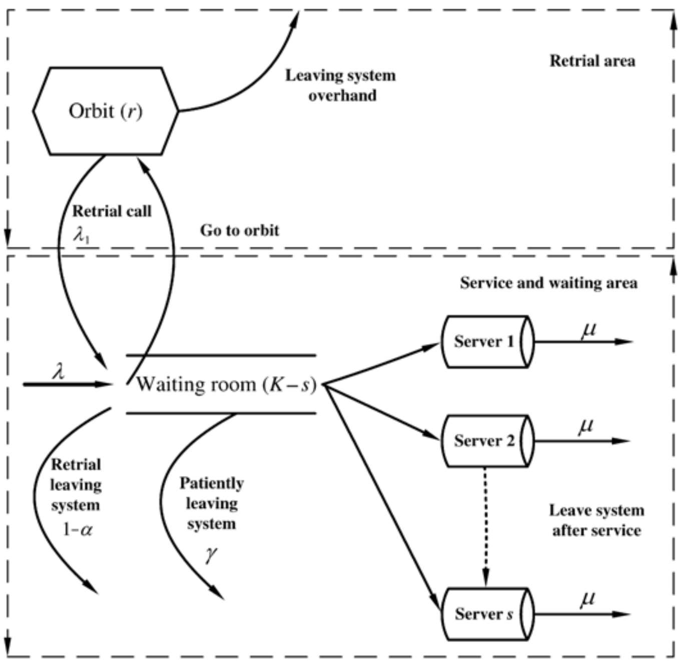

Figure 1.2 Queueing model for a call center

are s parallel identical servers, the sizes of the waiting room and the orbit are $K-s$ and r, respectively. The arrival process is a Poisson process with arrival rate λ, the service times are i.i.d. and are exponentially distributed with service rate μ, the retrial time and the patient waiting time are exponential with parameters λ_1 and γ, respectively. The service discipline is FCFS. Once the waiting time of a customer exceeds the patient waiting time, it immediately leaves this system. If an arrival customer can not enter the service and waiting area, then he has to go to the orbit for a retrial purpose with probability α, or he will immediately leave this system with probability $1-\alpha$. Once an arrival customer cannot enter the orbit, he immediately leave this system. All the random variables defined above are assumed to be mutually independent.

Let $Q(t)$ and $Z(t)$ be the number of customers in the service and waiting area, and the number of customers in the orbit at time t, respectively. Then the Markov chain $\{(Q(t), Z(t)), t \geqslant 0\}$ is a QBD process with finitely-many levels whose infinitesimal generator is given by

$$Q = \begin{pmatrix} A_0 & C & & & & \\ B_1 & A_1 & C & & & \\ & B_2 & A_2 & C & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_{K-1} & A_{K-1} & C \\ & & & & B_K & A_K \end{pmatrix},$$

where for $0 \leqslant i \leqslant K-1$,

$$A_i = \mathrm{diag}(a_{i,0},\ a_{i,1},\ a_{i,2}, \ldots, a_{i,r})$$

with for $0 \leqslant j \leqslant r$,

$$a_{i,j} = \begin{cases} -(\lambda + j\lambda_1 + i\mu), & 0 \leqslant i \leqslant s, \\ -[\lambda + j\lambda_1 + s\mu + (i-s)\gamma], & s+1 \leqslant i \leqslant K-1; \end{cases}$$

$$A_k = \begin{pmatrix} f_{K,0} & \alpha\lambda & & & & \\ d_1 & f_{K,1} & \alpha\lambda & & & \\ & d_2 & f_{K,2} & \alpha\lambda & & \\ & & \ddots & \ddots & \ddots & \\ & & & d_{r-1} & f_{K,r-1} & \alpha\lambda \\ & & & & d_r & f_{K,r} \end{pmatrix},$$

with for $0 \leqslant j \leqslant r$,

$$f_{K,j} = \begin{cases} -[\alpha\lambda + (1-\alpha)j\lambda_1 + s\mu + (K-s)\gamma], & 0 \leqslant j \leqslant r-1, \\ -[(1-\alpha)r\lambda_1 + s\mu + (i-s)\gamma], & j = r, \end{cases}$$

and for $1 \leqslant j \leqslant r$,

$$d_j = (1-\alpha) j \lambda_1;$$

$$B_i = b_i I,$$

with for $0 \leqslant i \leqslant K$,

$$b_j = \begin{cases} i\mu, & 0 \leqslant i \leqslant s, \\ s\mu + (i-s)\gamma, & s+1 \leqslant i \leqslant K; \end{cases}$$

and

$$C = \begin{pmatrix} \lambda & & & & \\ \lambda_1 & \lambda & & & \\ & 2\lambda_1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & r\lambda_1 & \lambda \end{pmatrix}.$$

1.2.4 A Two-Loop Closed Production System

The fourth example considers a simple closed loop production line, and constructs a block-structured infinitesimal generator. In Fig. 1.3, the rectangles and circles represent the machines and buffers in a production line, respectively. There are always plenty of raw materials which can be sent to machine M_1. Note that each operation on the three machines must be performed with the support of some carts. Operations on machines M_1 and M_3 need carts A and B, respectively; while operations on machine M_2 need carts A and B simultaneously. The manufacturing processes on the three machines are given as follows. For the first machine, the raw material is first loaded into an empty cart A which comes from buffer B_a, then the cart A with the raw material is sent to machine M_1. After processing on M_1, the cart A carries the products to buffer B_1 for the further operation on machine M_2. If the arriving cart A finds that machine M_2 is available and there is also an empty cart B in buffer B_b, then the carts A and B enter machine M_2 together for product processing. Once the operation on machine M_2 is finished, all the products are loaded to cart B from cart A. At the same time, the cart A returns to buffer B_a while cart B brings the products to buffer B_2. Once machine M_3 is available, the cart B enters machine M_3 for the final processing. After all these processes, these products must leave this system from machine M_3, and the empty cart B returns to buffer B_b. The three machines are all reliable and always produce no bad parts. Machine M_1 is starved if buffer B_a is empty, Machine M_2 is starved if either buffer B_1 is empty or buffer B_b is empty, Machine M_3 is starved if B_2 is empty. We assume that the service times of machine M_i are i.i.d. and are exponentially distributed with parameter μ_i for $i=1, 2, 3$; the total numbers of carts A and carts

B are m and n, respectively. All the random variables defined above are assumed to be mutually independent.

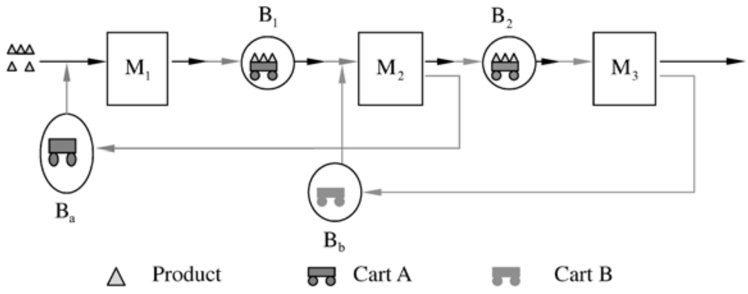

Figure 1.3 A two-loop closed production system

Let $s(t)$ be the state of machine M_2 at time t: If the machine is working, $s(t)=1$, otherwise $s(t)=0$. Let $N_1(t)$ and $N_3(t)$ denote the total number of carts A in both buffer B_a and machine M_1 at time t, and the total number of carts B in both buffer B_2 and machine M_3 at time t, respectively. It is seen that this system is described as a continuous-time Markov chain $\{X(t), t\geqslant 0\}$, where $X(t)=(s(t), N_1(t), N_3(t))$. Note that the total number of carts A is m, the number of carts A in buffer B_1 is $m-N_1(t)-s(t)$. Similarly, the total number of carts B is n, the numbers of carts B in buffer B_b is $n-N_3(t)-s(t)$. When $s(t)=0$, machine M_2 is idle due to the condition that either buffer B_1 or buffer B_b is empty. The state space of the Markov chain $\{X(t), t\geqslant 0\}$ is expressed by

$$\begin{aligned}\Omega = &\{(0,m,n)\} \cup \{(0,i,n): 0 \leqslant i \leqslant m-1\} \cup \{(0,m,j): 0 \leqslant j \leqslant n-1\} \\ &\cup \{(1,i,j): 0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant n-1\}.\end{aligned}$$

It is easy to check that the infinitesimal generator of the Markov chain $\{X(t), t\geqslant 0\}$ is given by

$$Q=\begin{pmatrix} A & B & & & & & & E \\ & C & D_1 & D_2 & D_3 & \cdots & D_{m-1} & D_m \\ & F_1 & M & N & & & & \\ & F_2 & T & P & N & & & \\ & \vdots & & \ddots & \ddots & \ddots & & \\ & F_{m-2} & & & T & P & N & \\ & F_{m-1} & & & & T & P & N \\ K & & & & & & T & P \end{pmatrix}, \qquad (1.14)$$

where

$$A=\begin{pmatrix} -\mu_1 & & & & \\ \mu_3 & -(\mu_1+\mu_3) & & & \\ & \mu_3 & -(\mu_1+\mu_3) & & \\ & & \ddots & \ddots & \\ & & & \mu_3 & -(\mu_1+\mu_3) \end{pmatrix}_{(n+1)\times(n+1)},$$

$$B=\begin{pmatrix} & \\ & \\ & \\ & \\ & \mu_1 \end{pmatrix}_{(n+1)\times m}, \quad E=\begin{pmatrix} \mu_1 & & & \\ & \mu_1 & & \\ & & \ddots & \\ & & & \mu_1 \\ 0 & 0 & \dots & 0 \end{pmatrix}_{(n+1)\times n},$$

$$C=\begin{pmatrix} -\mu_3 & & & & \\ \mu_1 & -(\mu_1+\mu_3) & & & \\ & \mu_1 & -(\mu_1+\mu_3) & & \\ & & \ddots & \ddots & \\ & & & \mu_1 & -(\mu_1+\mu_3) \end{pmatrix}_{m\times m},$$

D_j is a matrix of size $m\times n$ with the (j, n)th element being μ_3 and all the other elements being 0 for $1\leqslant j\leqslant m$, F_i is a matrix of size $n\times m$ with the $(n, i+1)$th element being μ_2 and all the other elements being 0 for $1\leqslant i\leqslant m-1$, $T=\mu_1 I_{n\times n}$,

$$K=\begin{pmatrix} 0 & \mu_2 & & & \\ 0 & & \mu_2 & & \\ & & & \ddots & \\ 0 & & & & \mu_2 \end{pmatrix}_{n\times(n+1)}, \quad N=\begin{pmatrix} 0 & \mu_2 & & \\ & 0 & \ddots & \\ & & \ddots & \mu_2 \\ & & & 0 \end{pmatrix}_{n\times n},$$

$$M=\begin{pmatrix} -\mu_2 & & & & \\ \mu_3 & -(\mu_2+\mu_3) & & & \\ & \mu_3 & -(\mu_2+\mu_3) & & \\ & & \ddots & \ddots & \\ & & & \mu_3 & -(\mu_2+\mu_3) \end{pmatrix}_{n\times n}$$

and

$$P=\begin{pmatrix} -(\mu_1+\mu_2) & & & & \\ \mu_3 & -(\mu_1+\mu_2+\mu_3) & & & \\ & \mu_3 & -(\mu_1+\mu_2+\mu_3) & & \\ & & \ddots & \ddots & \\ & & & \mu_3 & -(\mu_1+\mu_2+\mu_3) \end{pmatrix}_{n\times n}.$$

For simplification of description, the infinitesimal generator given in Eq. (1.14) can be further rewritten as

$$Q=\begin{pmatrix}
A_0^{(0)} & A_1^{(0)} & A_2^{(0)} & A_3^{(0)} & A_4^{(0)} & \cdots & A_{m-1}^{(0)} & A_m^{(0)} \\
A_{-1}^{(1)} & A_0^{(1)} & A_1^{(1)} & & & & & \\
A_{-2}^{(2)} & A_{-1}^{(2)} & A_0^{(2)} & A_1^{(2)} & & & & \\
A_{-3}^{(3)} & & A_{-1}^{(3)} & A_0^{(3)} & A_1^{(3)} & & & \\
A_{-4}^{(4)} & & & A_{-1}^{(4)} & A_0^{(4)} & A_1^{(4)} & & \\
\vdots & & & & \ddots & \ddots & \ddots & \\
A_{-(m-1)}^{(m-1)} & & & & & A_{-1}^{(m-1)} & A_0^{(m-1)} & A_1^{(m-1)} \\
A_{-m}^{(m)} & & & & & & A_{-1}^{(m)} & A_0^{(m)}
\end{pmatrix},$$

where

$$A_0^{(0)}=\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},\quad A_j^{(0)}=\begin{pmatrix} 0 \\ D_j \end{pmatrix} \text{ for } 1\leqslant j\leqslant m-1,\quad A_m^{(0)}=\begin{pmatrix} E \\ D_m \end{pmatrix},$$

$$A_{-i}^{(i)}=(0, F_i) \text{ for } 1\leqslant i\leqslant m-1,\quad A_{-m}^{(m)}=(K,0),$$

$$A_0^{(1)}=M,\quad A_0^{(i)}=P \text{ for } 2\leqslant i\leqslant m,$$

$$A_1^{(i)}=N \text{ for } 1\leqslant i\leqslant m-1,\quad A_{-1}^{(j)}=T \text{ for } 2\leqslant i\leqslant m.$$

1.2.5 An E-mail Queueing System Under Attacks

The fifth example provides a queueing model to describe an email queueing system with three types of attacks, as shown in Fig. 1.4, by means of an irreducible continuous-time QBD process with finitely-many levels.

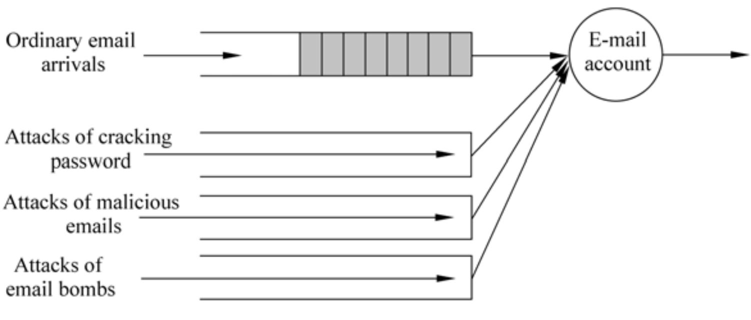

Figure 1.4 Three types of attacks on an email account

The ordinary emails in each email account form a basic queue that is expressed by the email information units. The email arrival is a Poisson process with rate λ, and each email information unit is dealt by the user through a time length that has an exponential distribution with mean $1/\mu$. These email information units are processed by the user according to FCFS. Assume that each email account has a capacity of at most N email information units.

The attack of cracking password is abstracted as an input to the second queue. By cracking the password of the email account, the attacker can get some useful and valuable email information for either his personal interests, business or other purposes. The events that occur for the succesful acquisition of the password are treated as attacking processes or a customer arrival process that is a Poisson process with rate α_c. Such a customer remains in the email account with a time period x_c that is exponential with mean $1/\gamma_c$. Since the attackers are only interested in some useful and valuable email information units, we assume that this kind of email attacks can grab the email information with a probability $P_c(x_c)$ when they remain in the email account.

The attack of sending malicious emails attached with Trojan virus is also modeled as an input process in the third queue. The Trojan Horse is planted if the email user clicks the attachment. Based on this, such attackers can enter the email system for reading the email account's private and valuable information. The malicious emails are regarded as a customer arrival process that is a Poisson process with rate α_m. Some attached Trojan virus is defended by the fairwall or it is ignored by the email user. Let P_{in} denote the probability that the malicious attachment is clicked by the email user, which means that the attacker gets an access into the email account. This type of attacks remains in the email system for a time length x_m which is exponential with mean $1/\gamma_m$. Let $P_m(x_m)$ be the probability that the attacker is able to obtain the email information.

The fourth input process is the attack of email bombs. The successfully deployed email bombs are treated as the customers whose arrivals are a Poisson process with rate α_b. Once an email bomb arrives, the email account is crashed for a time period whose length has an exponential distribution with mean $1/\beta_b$.

It is worth noting that the attacks of cracking password and sending malicious emails will not change the ordinary email queue. However, the attacks of email bombs can change the behavior of the ordinary email queue.

Let $n(t)$, $s(t)$ and $r(t)$ denote the number of email information units, the state of the email account and the types of attacks in the email system at time t, respectively, where $0 \leqslant n(t) \leqslant N$, N is the maximum number of email information units, $s(t) \in \{I, W, F\}$ and $r(t) \in \{C, M, CM\}$. We provide a simple interpretation for the elements I, W, F, C, M and CM. Firstly, I, W and F stand for Idle, Working and Fail of the email account, respectively. Secondly, C, M and CM describe the attacks of cracking password, the attacks of malicious email, and the co-existing of the attacks of cracking password and malicious emails, respectively. Obviously, $\{(n(t), s(t), r(t)) : t \geqslant 0\}$ is a QBD process whose state space Ω is given by

$$\Omega = \bigcup_{k=0}^{N} L_k,$$

where

$$L_0 = \{(0, I), (0^{(C)}, I), (0^{(M)}, I), (0^{(CM)}, I), (0, F)\},$$

and for $1 \leqslant i \leqslant N$,

$$L_i = \{(i, W), (i^{(C)}, W), (i^{(M)}, W), (i^{(CM)}, W), (i, F)\}.$$

Based on the state space $\Omega = \bigcup_{k=0}^{N} L_k$, it is easy to see that the QBD process has the following infinitesimal generator

$$Q = \begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & & \\ & \ddots & \ddots & \ddots & \\ & & A_2^{(N-1)} & A_1^{(N-1)} & A_0^{(N-1)} \\ & & & A_2^{(N)} & A_1^{(N)} \end{pmatrix}, \tag{1.15}$$

where

$$A_1^{(i)} = \begin{pmatrix} -(\alpha_c + \alpha_m + \xi) & \alpha_c & \alpha_m & 0 & \alpha_b \\ \gamma_c & -(\alpha_m + \xi + \gamma_c) & 0 & \alpha_m & \alpha_b \\ \gamma_m & 0 & -(\alpha_c + \xi + \gamma_m) & \alpha_c & \alpha_b \\ 0 & \gamma_m & \gamma_c & -(\xi + \gamma_m + \gamma_c) & \alpha_b \\ \beta_b & 0 & 0 & 0 & -\beta_b \end{pmatrix}$$

with

$$\xi = \alpha_b + \lambda + \mu,$$

for $1 \leqslant i \leqslant N-1$, and

$$A_1^{(0)} = \begin{pmatrix} -(\alpha_c + \alpha_m + \eta) & \alpha_c & \alpha_m & 0 & \alpha_b \\ \gamma_c & -(\alpha_m + \eta + \gamma_c) & 0 & \alpha_m & \alpha_b \\ \gamma_m & 0 & -(\alpha_c + \eta + \gamma_m) & \alpha_c & \alpha_b \\ 0 & \gamma_m & \gamma_c & -(\eta + \gamma_m + \gamma_c) & \alpha_b \\ \beta_b & 0 & 0 & 0 & -\beta_b \end{pmatrix}$$

with

$$\eta = \alpha_b + \lambda;$$

$$A_1^{(N)} = \begin{pmatrix} -(\alpha_c + \alpha_m + \zeta) & \alpha_c & \alpha_m & 0 & \alpha_b \\ \gamma_c & -(\alpha_m + \zeta + \gamma_c) & 0 & \alpha_m & \alpha_b \\ \gamma_m & 0 & -(\alpha_c + \zeta + \gamma_m) & \alpha_c & \alpha_b \\ 0 & \gamma_m & \gamma_c & -(\zeta + \gamma_m + \gamma_c) & \alpha_b \\ \beta_b & 0 & 0 & 0 & -\beta_b \end{pmatrix}$$

with

$$\zeta = \alpha_b + \mu,$$
$$A_2^{(j)} = \text{diag}\,(\mu, \mu, \mu, \mu, 0), \quad 1 \leqslant j \leqslant N,$$
$$A_0^{(k)} = \text{diag}\,(\lambda, \lambda, \lambda, \lambda, 0), \quad 0 \leqslant k \leqslant N-1.$$

1.3 The QBD Processes

Based on the above motivating examples, this section analyzes a continuous-time QBD process with either finitely-many levels or infinitely-many levels. We construct the UL-and LU-types of *RG*-factorizations for the QBD process; while the *RG*-factorizations for any irreducible Markov chains will be systemically developed in Chapters 2, 3, 5, 6 and 9 of this book. In addition, we iteratively define the *R*- and *G*-measures, both of which are a direct generalization of the matrices *R* and *G* given in Neuts [92, 94], respectively.

We considers an irreducible continuous-time QBD process with infinitely-many levels whose infinitesimal generator is given by

$$Q = \begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & & \\ & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}; \tag{1.16}$$

and an irreducible continuous-time QBD process with finitely-many levels whose infinitesimal generator is given by

$$Q = \begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & & \\ & \ddots & \ddots & \ddots & \\ & & A_2^{(N)} & A_1^{(N)} & A_0^{(N)} \\ & & & A_2^{(N+1)} & A_1^{(N+1)} \end{pmatrix}. \tag{1.17}$$

1.3.1 Heuristic Expressions

We first consider an irreducible continuous-time level-independent QBD process with infinitely-many levels given in Eq. (1.16) with $A_0^{(i)} = A_0$ and $A_1^{(i)} = A_1$ for $i \geqslant 1$, and $A_2^{(j)} = A_2$ for $j \geqslant 2$. Note that the repeated row has the non-zero blocks A_2, A_1 and A_0, it is easy to conjecture that the stationary probability vector, $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$, of the QBD process Q has the following matrix-geometric form:

$$\pi_k = \pi_1 R^{k-1}, \quad k \geqslant 1,$$

where the matrix R is the minimal nonnegative solution to the matrix equation $R = \sum_{k=0}^{\infty} R^k A_k$, and π_0 and π_1 need to satisfy the following three conditions:

$$\pi_0 A_1^{(0)} + \pi_1 A_2^{(1)} = 0,$$
$$\pi_0 A_0^{(0)} + \pi_1 (A_1 + RA_2) = 0$$

and

$$\pi_0 e + \pi_1 (I - R)^{-1} e = 1.$$

This was described as Theorem 3.1 in Chapter 3 of Neuts [92]. It is easy to see that $\sum_{k=0}^{\infty} \pi_k e = 1$ if and only if the spectral radious: $sp(R) < 1$. Thus, the spectral analysis of the rate matrix R is a key in the study of the level-independent QBD processes.

Then, we analyze an irreducible continuous-time level-independent QBD process with finitely-many levels given in Eq. (1.17) with $A_0^{(i)} = A_0$ for $1 \leqslant i \leqslant N-1$, $A_1^{(j)} = A_1$ for $1 \leqslant j \leqslant N$, and $A_2^{(k)} = A_2$ for $2 \leqslant k \leqslant N$. Note that this QBD process has the top and bottom boundaries. We take a common matrix as follows:

$$\Omega = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & A_2 & A_1 & A_0 & & & \\ & & A_2 & A_1 & A_0 & & \\ & & & A_2 & A_1 & A_0 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

When seeing the matrix Ω from top to bottom, the rate matrix R is the minimal nonnegative solution to the matrix equation

$$A_0 + RA_1 + R^2 A_2 = 0;$$

and from bottom to top, the rate matrix $\overline{R}$ is the minimal nonnegative solution to the matrix equation

$$\overline{R}^2 A_0 + \overline{R} A_1 + A_2 = 0.$$

Note that the repeated row has only non-zero blocks A_2, A_1, A_0. It is easy to conjecture that the stationary probability vector, $\pi = (\pi_0, \pi_1, ..., \pi_N, \pi_{N+1})$, of the QBD process Q has the following matrix-geometric form:

$$\pi_k = \pi_1 R^{k-1} + \pi_N \overline{R}^{N-k}, \quad 1 \leqslant k \leqslant N.$$

In this case, we may check that π_0, π_1, π_N and π_{N+1} need to satisfy the following five conditions:

$$\pi_0 A_1^{(0)} + \pi_1 A_2^{(1)} + \pi_N \overline{R}^{N-1} A_2^{(1)} = 0,$$
$$\pi_0 A_0^{(0)} + \pi_1 (A_1 + RA_2) + \pi_N \overline{R}^{N-2} (\overline{R}A_1 + A_2) = 0,$$
$$\pi_1 R^{N-1} A_0^{(N)} + \pi_N A_0^{(N)} + \pi_{N+1} A_1^{(N+1)} = 0,$$
$$\pi_1 R^{N-2} (A_0 + RA_1) + \pi_N (A_1 + \overline{R}A_0) + \pi_{N+1} A_2^{(N+1)} = 0$$

and

$$\pi_0 e + \pi_{N+1} e + \pi_1 \sum_{k=0}^{N-1} R^k e + \pi_N \sum_{k=0}^{N-1} \overline{R}^k e = 1.$$

Note that the three matrices $A_0^{(0)}$, $A_1^{(0)}$ and $A_2^{(1)}$ and the three matrices $A_0^{(N)}$, $A_1^{(N+1)}$ and $A_2^{(N+1)}$ are the top boundary blocks and the bottom boundary blocks, respectively.

1.3.2 The LU-Type *RG*-Factorization

We first construct the LU-type *RG*-factorization[①] for the QBD process with finitely-many levels. For the irreducible QBD process with $N+2$ levels given in Eq. (1.17), we write the *U*-measure as

$$\overline{U}_0 = A_1^{(0)}$$

and

$$\overline{U}_k = A_1^{(k)} + A_2^{(k)} (-\overline{U}_{k-1}^{-1}) A_0^{(k-1)}, \quad 1 \leqslant k \leqslant N+1.$$

It is easy to check that $\overline{U}_k$ is the infinitesimal generator of a Markov chain with m_k states for $0 \leqslant k \leqslant N+1$. Also, the Markov chain $\overline{U}_k$ is transient, and thus the matrix $\overline{U}_k$ is invertible for $0 \leqslant k \leqslant N$; while the Markov chain $\overline{U}_{N+1}$ is positive recurrent (resp. transient) if and only if the Markov chain Q is positive recurrent (resp. transient).

Based on the *U*-measure $\{\overline{U}_k\}$, we can define the LU-type *R*- and *G*-measures respectively as

$$\overline{R}_k = A_2^{(k)} (-\overline{U}_{k-1}^{-1}), \quad 1 \leqslant k \leqslant N+1,$$

and

$$\overline{G}_k = (-\overline{U}_k^{-1}) A_0^{(k)}, \quad 0 \leqslant k \leqslant N.$$

Note that the matrix sequence $\{\overline{R}_k, 1 \leqslant k \leqslant N\}$ is the unique nonnegative

① From this section, notation for the LU-type measures will use $\overline{U}$, $\overline{R}$ and $\overline{G}$.

solution to the system of matrix equations

$$\overline{R}_{k+1}\overline{R}_k A_0^{(k-1)} + \overline{R}_{k+1}A_1^{(k)} + A_2^{(k+1)} = 0, \quad 1 \leqslant k \leqslant N,$$

with the boundary condition

$$\overline{R}_1 = A_2^{(1)}(-\overline{U}_0^{-1}).$$

Hence

$$\overline{R}_{k+1} = -A_2^{(k+1)}[\overline{R}_k A_0^{(k-1)} + A_1^{(k)}]^{-1}, \quad 1 \leqslant k \leqslant N.$$

The matrix sequence $\{\overline{G}_k, 1 \leqslant k \leqslant N\}$ is the unique nonnegative solution to the system of matrix equations

$$A_0^{(k)} + A_1^{(k)}\overline{G}_k + A_2^{(k)}\overline{G}_{k-1}\overline{G}_k = 0, \quad 1 \leqslant k \leqslant N,$$

with the boundary condition

$$\overline{G}_0 = (-\overline{U}_0^{-1})A_0^{(0)}.$$

Thus

$$\overline{G}_k = -[A_1^{(k)} + A_2^{(k)}\overline{G}_{k-1}]^{-1} A_0^{(k)}, \quad 1 \leqslant k \leqslant N.$$

For the QBD process with finitely-many levels given in Eq. (1.17), we can easily obtain the LU-type RG-factorization as follows:

$$Q = (I - \overline{R}_L)\overline{U}_D(I - \overline{G}_U), \tag{1.18}$$

where

$$\overline{R}_L = \begin{pmatrix} 0 & & & & \\ \overline{R}_1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \overline{R}_N & 0 & \\ & & & \overline{R}_{N+1} & 0 \end{pmatrix},$$

$$\overline{U}_D = \text{diag}\,(\overline{U}_0, \overline{U}_1, ..., \overline{U}_N, \overline{U}_{N+1}),$$

$$\overline{G}_U = \begin{pmatrix} 0 & \overline{G}_0 & & & \\ & 0 & \overline{G}_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & \overline{G}_N \\ & & & & 0 \end{pmatrix}.$$

We now extend the LU-type RG-factorization Eq. (1.18) to that for the irreducible QBD process with infinitely-many levels given in Eq. (1.16). Clearly, this extension is easy to achieve by assuming only letting $N \to \infty$. Note that the matrix $\overline{U}_k$ is invertible for all $k \geqslant 0$.

For the QBD process with infinitely-many levels given in Eq. (1.16), the LU-type RG-factorization is given by

$$Q = (I - \overline{R}_L)\overline{U}_D(I - \overline{G}_U), \tag{1.19}$$

where

$$\overline{R}_L = \begin{pmatrix} 0 & & & & \\ \overline{R}_1 & 0 & & & \\ & \overline{R}_2 & 0 & & \\ & & \overline{R}_3 & 0 & \\ & & & \ddots & \ddots \end{pmatrix},$$

$$\overline{U}_D = \text{diag}\,(\overline{U}_0, \overline{U}_1, \overline{U}_2, \overline{U}_3, \ldots),$$

$$\overline{G}_U = \begin{pmatrix} 0 & \overline{G}_0 & & & \\ & 0 & \overline{G}_1 & & \\ & & 0 & \overline{G}_2 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

1.3.3 The UL-Type *RG*-Factorization

For the irreducible QBD process with $N+2$ levels given in Eq. (1.17), we write the U-measure as

$$U_{N+1} = A_1^{(N+1)}$$

and

$$U_k = A_1^{(k)} + A_0^{(k)} U_{k+1} A_2^{(k+1)}, \quad 0 \leqslant k \leqslant N.$$

It is easy to check that U_k is the infinitesimal generator of a Markov chain with m_k states for $0 \leqslant k \leqslant N+1$. Also, the Markov chain U_k is transient, and thus the matrix U_k is invertible for $1 \leqslant k \leqslant N+1$; while the Markov chain U_0 is positive recurrent (resp. transient) if and only if the Markov chain Q is positive recurrent (resp. transient).

Based on the U-measure $\{U_k\}$, we can respectively define the UL-type R- and G-measures as

$$R_k = A_0^{(k)}(-U_{k+1}^{-1}), \quad 0 \leqslant k \leqslant N,$$

and

$$G_k = (-U_k^{-1})A_2^{(k)}, \quad 1 \leqslant k \leqslant N+1.$$

Note that the matrix sequence $\{R_k, 0 \leqslant k \leqslant N-1\}$ is the unique nonnegative solution to the system of matrix equations

$$A_0^{(k)} + R_k A_1^{(k+1)} + R_k R_{k+1} A_2^{(k+2)} = 0, \quad 0 \leqslant k \leqslant N-1,$$

with the boundary condition

$$R_N = A_0^{(N)}(-U_{N+1}^{-1}).$$

Hence

$$R_k = -A_0^{(k)}[A_1^{(k+1)} + R_{k+1}A_2^{(k+2)}]^{-1}, \quad 0 \leqslant k \leqslant N-1.$$

Similarly, the matrix sequence $\{G_k, 1 \leqslant k \leqslant N\}$ is the unique nonnegative solution to the system of matrix equations

$$A_0^{(k)} G_{k+1} G_k + A_1^{(k)} G_k + A_2^{(k)} = 0, \quad 1 \leqslant k \leqslant N,$$

with the boundary condition

$$G_{N+1} = (-U_{N+1}^{-1})A_2^{(N+1)}.$$

Thus

$$G_k = -[A_0^{(k)} G_{k+1} + A_1^{(k)}]^{-1} A_2^{(k)}, \quad 1 \leqslant k \leqslant N.$$

For the QBD process with finitely-many levels given in Eq. (1.17), the UL-type RG-factorization is given by

$$Q = (I - R_U)U_D(I - G_L), \tag{1.20}$$

where

$$R_U = \begin{pmatrix} 0 & R_0 & & & & \\ & 0 & R_1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & R_N & \\ & & & & 0 & \end{pmatrix},$$

$$U_D = \text{diag}\,(U_0, U_1, ..., U_N, U_{N+1}),$$

$$G_L = \begin{pmatrix} 0 & & & & \\ G_1 & 0 & & & \\ & \ddots & \ddots & & \\ & & G_N & 0 & \\ & & & G_{N+1} & 0 \end{pmatrix}.$$

Finally, we simply describe the UL-type RG-factorization for the QBD process with infinitely-many levels given in Eq. (1.16).

Let the matrix sequences $\{R_k\}$ and $\{G_k\}$ be the minimal nonnegative solutions to the systems of matrix equations

$$A_0^{(l)} + R_l A_1^{(l+1)} + R_l R_{l+1} A_2^{(l+2)} = 0, \quad l \geqslant 0,$$

and

$$A_0^{(k)} G_{k+1} G_k + A_1^{(k)} G_k + A_2^{(k)} = 0, \quad k \geqslant 1,$$

respectively, Thus, we obtain

$$U_k = A_1^{(k)} + R_k A_2^{(k)} = A_1^{(k)} + A_0^{(k)} G_{k+1}, \quad k \geqslant 0.$$

It is easy to check that U_k is the infinitesimal generator of a Markov chain with m_k states for $0 \leqslant k \leqslant N+1$. Also, the Markov chain U_k is transient, and thus the matrix U_k is invertible for $k \geqslant 1$; while the Markov chain U_0 is positive recurrent (resp. transient) if and only if the Markov chain Q is recurrent (resp. transient).

For the QBD process with infinitely-many levels given in Eq. (1.16), the UL-type RG-factorization is given by

$$Q = (I - R_U) U_D (I - G_L), \tag{1.21}$$

where

$$R_U = \begin{pmatrix} 0 & R_0 & & & \\ & 0 & R_1 & & \\ & & 0 & R_2 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix},$$

$$U_D = \mathrm{diag}\,(U_0, U_1, U_2, \ldots),$$

$$G_L = \begin{pmatrix} 0 & & & & \\ G_1 & 0 & & & \\ & G_2 & 0 & & \\ & & G_3 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

1.3.4 Linear QBD-Equations

Based on the UL- and LU-types of RG-factorizations, we provide a constructive approach to solve a linear QBD-equation

$$XQ = b \quad \text{or} \quad QX = b, \tag{1.22}$$

where Q is the infinitesimal generator of an irreducible continuous-time level-dependent QBD process with either finitely-many levels or infinitely-many levels. When $b=0$, the Eq. (1.22) is homogeneous, otherwise it is nonhomogeneous. Note that we will provide a more general discussion for such systems of infinite-dimensional linear equation in Chapter 9.

1.3.4.1 The Homogeneous Linear QBD-Equations

For the homogeneous linear QBD-equations, a well-known example is the equation $\pi Q=0$, where the QBD process Q is positive recurrent and π is its stationary probability vector.

For convenience of description, we denote by $\pi_k^{(LU)}$ and $\pi_k^{(UL)}$ the stationary probability vector obtained by means of the LU- and UL-type of RG-factorizations, respectively.

Case Ⅰ Infinitely-many levels

In this case, we apply the UL-type of RG-factorization to derive the stationary probability vector of a QBD process with infinitely-many levels. Let the matrix sequence $\{R_l, l\geqslant 0\}$ be the minimal nonnegative solution to the system of matrix equations

$$A_0^{(l)} + R_l A_1^{(l+1)} + R_l R_{l+1} A_2^{(l+2)} = 0, \quad l \geqslant 0.$$

Using the RG-factorization Eq. (1.21), we have

$$\pi_0 = \kappa v, \tag{1.23}$$

$$\pi_k = \kappa v R_0 R_1 \ldots R_{k-1}, \quad k \geqslant 1, \tag{1.24}$$

where κ is a normalization constant, and v is the stationary probability vector of the censored chain $U_0 = A_1^{(0)} + R_0 A_2^{(1)}$ to level 0.

Case Ⅱ Finitely-many levels

In this case, we use the LU- and UL-types of RG-factorizations to derive the stationary probability vector of a QBD process with finitely-many levels, which leads to the following three expressions.

Expression 1.1 Using the LU-type RG-factorization Eq. (1.18), we get

$$\pi_k^{(LU)} = \kappa v_{N+1} \overline{R}_{N+1} \overline{R}_N \ldots \overline{R}_{k+1}, \quad 0 \leqslant k \leqslant N, \tag{1.25}$$

and

$$\pi_{N+1}^{(LU)} = \kappa v_{N+1}, \tag{1.26}$$

where κ is a normalization constant and v_{N+1} is the stationary probability vector of the censored chain $\overline{U}_{N+1} = A_1^{(N+1)} + \overline{R}_{N+1} A_0^{(N)}$ to level $N+1$.

Expression 1.2 According to the UL-type of *RG*-factorization Eq. (1.20), we have

$$\pi_0^{(UL)} = \varphi v_0 \tag{1.27}$$

and

$$\pi_k^{(UL)} = \varphi v_0 R_0 R_1 \dots R_{k-1}, \quad 1 \leqslant k \leqslant N+1, \tag{1.28}$$

where φ is a normalization constant and v_0 is the stationary probability vector of the censored chain $U_0 = A_1^{(0)} + R_0 A_2^{(1)}$ to level 0.

Expression 1.3 By means of a property of linear combination for the solutions of a system of linear equations, it follows from Eq. (1.25) to Eq. (1.28) that the stationary probability vector $\pi = (\pi_0, \pi_1, ..., \pi_N, \pi_{N+1})$ is given by

$$\pi_0 = u_{N+1} \overline{R}_{N+1} \overline{R}_N \dots \overline{R}_1 + u_0, \tag{1.29}$$

$$\pi_k = u_{N+1} \overline{R}_{N+1} \overline{R}_N \dots \overline{R}_{k+1} + u_0 R_0 R_1 \dots R_{k-1}, \quad 1 \leqslant k \leqslant N, \tag{1.30}$$

$$\pi_{N+1} = u_{N+1} + u_0 R_0 R_1 \dots R_N, \tag{1.31}$$

where the row vectors u_{N+1} and u_0 are uniquely determined by the equations $(u_{N+1}, u_0) Q^E = 0$ and

$$u_{N+1} \left(I + \sum_{k=0}^{N} \prod_{j=N+1}^{k+1} \overline{R}_j \right) e + u_0 \left(I + \sum_{k=1}^{N+1} \prod_{j=0}^{k-1} R_j \right) e = 1,$$

where

$$Q^E = \begin{pmatrix} \overline{R}_{N+1} \overline{R}_N \dots \overline{R}_2 [A_2^{(1)} + \overline{R}_1 A_1^{(0)}] & A_1^{(N+1)} + \overline{R}_{N+1} A_0^{(N)} \\ A_1^{(0)} + R_0 A_2^{(1)} & R_0 R_1 \dots R_{N-1} [A_0^{(N)} + R_N A_1^{(N+1)}] \end{pmatrix}.$$

Note that Q^E is the infinitesimal generator of the censored Markov chain of Q to the set $E = \{\text{Level } 0, \text{Level } N+1\}$.

1.3.4.2 The Nonhomogeneous Linear QBD-Equations

As an illustrating example, we now use LU-type of *RG*-factorization to solve a nonhomogeneous linear QBD-equation: $XQ = b$ or $QX = b$. The maximal non-positive inverse $Q_{\max}^{-1}$ of the matrix Q has probabilistic meaning, hence the performance measures of a stochastic system can be expressed in terms of the blocks of $Q_{\max}^{-1}$. Therefore, it is necessary to express each block of $Q_{\max}^{-1}$ in order to give the solution $X = bQ_{\max}^{-1}$ or $X = Q_{\max}^{-1} b$. We set $Q_{\max}^{-1} = (q_{m,n})_{m,n \geqslant 0}$ partitioned according to the levels.

Based on the LU-type R-measure $\{\overline{R}_k, k \geqslant 1\}$ and G-measure $\{\overline{G}_k, k \geqslant 0\}$, we write

$$X_k^{(l)} = \overline{R}_l \overline{R}_{l-1} \overline{R}_{l-2} \ldots \overline{R}_{l-k+1}, \quad l \geqslant k \geqslant 1, \tag{1.32}$$

and

$$Y_k^{(l)} = \overline{G}_l \, \overline{G}_{l+1} \overline{G}_{l+2} \ldots \overline{G}_{l+k-1}, \quad k \geqslant 1, l \geqslant 0. \tag{1.33}$$

The following Theorem provides expressions for each block in the maximal non-positive inverse $Q_{\max}^{-1}$ by means of the LU-type RG-factorization $Q = (I - \overline{R}_L)\, \overline{U}_D (I - \overline{G}_U)$. In this case, we have

$$Q_{\max}^{-1} = (I - \overline{G}_U)^{-1} \, \overline{U}_D^{-1} (I - \overline{R}_L)^{-1}$$

Theorem 1.2 (1) *For the irreducible QBD process with infinitely-many levels given in Eq.* (*1.16*),

$$q_{m,n} = \begin{cases} \overline{U}_m^{-1} X_{m-n}^{(m)} + \sum\limits_{i=1}^{\infty} Y_i^{(m)} \overline{U}_{i+m}^{-1} X_{i+m-n}^{(i+m)}, & \text{if } 0 \leqslant n \leqslant m-1, \\ \overline{U}_m^{-1} + \sum\limits_{i=1}^{\infty} Y_i^{(m)} \overline{U}_{i+m}^{-1} X_i^{(i+m)}, & \text{if } n = m, \\ Y_{n-m}^{(m)} \overline{U}_n^{-1} + \sum\limits_{i=n-m+1}^{\infty} Y_i^{(m)} \overline{U}_{i+m}^{-1} X_{i-(n-m)}^{(i+m)}, & \text{if } n \geqslant m+1. \end{cases}$$

(2) *If the irreducible QBD process with finitely-many levels given in Eq.* (*1.17*) *is transient, then*

$$q_{m,n} = \begin{cases} \overline{U}_m^{-1} X_{m-n}^{(m)} + \sum\limits_{i=1}^{N+1-m} Y_i^{(m)} \overline{U}_{i+m}^{-1} X_{i+m-n}^{(i+m)}, & \text{if } 0 \leqslant m \leqslant N, \\ & 0 \leqslant n \leqslant m-1, \\ \overline{U}_m^{-1} + \sum\limits_{i=1}^{N+1-m} Y_i^{(m)} \overline{U}_{i+m}^{-1} X_i^{(i+m)}, & \text{if } 0 \leqslant m \leqslant N, \ n = m, \\ Y_{n-m}^{(m)} \overline{U}_n^{-1} + \sum\limits_{i=n-m+1}^{N+1-m} Y_i^{(m)} \overline{U}_{i+m}^{-1} X_{i-(n-m)}^{(i+m)}, & \text{if } 0 \leqslant m \leqslant N, \\ & m+1 \leqslant n \leqslant N+1, \\ \overline{U}_{N+1}^{-1} X_{N+1-n}^{(N+1)}, & \text{if } m = N+1, \\ & 0 \leqslant n \leqslant N, \\ \overline{U}_{N+1}^{-1}, & \text{if } m = n = N+1. \end{cases}$$

Theorem 1.2 indicates that if an irreducible QBD process has infinitely-many levels, then the existence of the maximal non-positive inverse is irrespective of whether the QBD process is recurrent or transient.

When the QBD process is transient, we can use the UL-type RG-factorization $Q=(I-R_U)U_D(I-G_L)$ to provide expression for the maximal non-positive inverse as follows:

$$Q_{\max}^{-1}=(I-G_L)^{-1}U_D^{-1}(I-R_U)^{-1}$$

whose detail is similar to those in Theorem 1.2. Note that Appendix A.3 provides expressions for the inverse of a general block-structured matrix.

1.4 Phase-Type Distributions

In this section, we provide an intuitive understanding for constructing a univariate PH distribution, and also list some basic properties of the PH distribution. Furthermore, we study a multivariate PH distribution, and give some important properties.

1.4.1 The Exponential Distribution

The exponential distribution is extensively used to deal with various stochastic models. The key to these applications is that the exponential distribution has the memoryless property.

Let X be a random variable with the exponential distribution function $F(x)=1-\exp\{-\lambda x\}$ for $0<\lambda<+\infty$. The probability density function of $F(x)$ is given by $f(x)=\lambda\exp\{-\lambda x\}$. It is easy to check that for all $t, s\geqslant 0$,

$$P\{X>t+s\mid X>s\}=P\{X>t\},$$

which is called the memoryless property of the exponential random variable X. The memoryless property can effectively simplify various conditional probabilities involved in a stochastic model so that mathematical analysis for the stochastic model can be simplified sufficiently.

It is crucial to find a more general distribution which retains the memoryless property. As a minor generalization of the exponential distribution, we let X be a random variable with distribution function $G(x)=1-\alpha\exp\{-\lambda x\}$ for $0<\alpha\leqslant 1$ and $0<\lambda<+\infty$. The probability density function of $G(x)$ is given by $g(x)=\lambda\alpha\exp\{-\lambda x\}$. It is easy to check that the random variable Y also has the memoryless property. However, there does not exist any other distribution with the memoryless property except for the two distributions. Therefore, it is necessary to find a class of new distributions which can retain some similar properties to the memoryless property of the exponential distribution.

1.4.2 The Erlang Distribution

Based on the exponential distribution, A.K. Erlang use the stage method to construct the so-called Erlang distribution of order n, denoted as $E[n,\lambda]$. Let X_i be a random variable with distribution function $F(x)=1-\exp\{-\lambda x\}$ for $1\leqslant i\leqslant n$. We assume that $X_1, X_2, ..., X_n$ are i.i.d.. Thus the random variable $Z_n=\sum_{k=1}^{n} X_k$ is Erlang of order n with distribution function

$$F_n(x)=P\{Z_n\leqslant x\}=1-\exp\{-\lambda x\}\sum_{k=0}^{n-1}\frac{(\lambda x)^k}{k!}$$

and the probability density function

$$f_n(x)=\exp\{-\lambda x\}\frac{\lambda(\lambda x)^{n-1}}{(n-1)!}.$$

For the Erlang distribution of order n, Fig. 1.5(a) expresses the random variable $Z_n=\sum_{k=1}^{n} X_k$. Based on this, it is clear that the Erlang distribution of order n can be understood as $F_n(x)=1-\alpha e^{Tx}e$, where $\alpha=(1,0,0,...,0)$, T is the infinitesimal generator of an n-state Markov chain, given by

$$T=\begin{pmatrix}-\lambda & \lambda & & & \\ & -\lambda & \lambda & & \\ & & \ddots & \ddots & \\ & & & -\lambda & \lambda \\ & & & & -\lambda\end{pmatrix}, \quad T^0=\begin{pmatrix}0\\0\\ \vdots \\0\\ \lambda\end{pmatrix}. \tag{1.34}$$

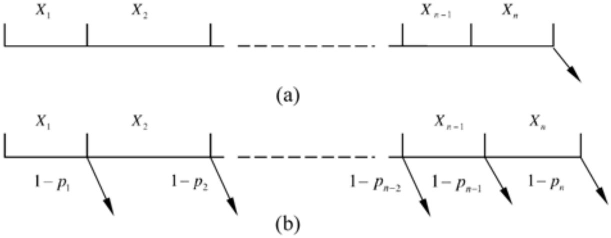

Figure 1.5 The stage method

As a minor generalization of the Erlang distribution, we assume that these exponential random variables X_1, X_2, ..., X_n are independent but differently distributional with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. In this situation, a generalized Erlang distribution is expressed by $F_n(x) = 1 - \alpha e^{Tx} e$, where $\alpha = (1, 0, 0, \ldots, 0)$,

$$T = \begin{pmatrix} -\lambda_1 & \lambda_1 & & & \\ & -\lambda_2 & \lambda_2 & & \\ & & \ddots & \ddots & \\ & & & -\lambda_{n-1} & \lambda_{n-1} \\ & & & & -\lambda_n \end{pmatrix}, \quad T^0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda_n \end{pmatrix}. \tag{1.35}$$

A further generalization of the Erlang distribution, which is seen from Fig. 1.5(b), can be given by $\alpha = (1, 0, 0, \ldots, 0)$ and

$$T = \begin{pmatrix} -\lambda_1 & \lambda_1 p_1 & & & \\ & -\lambda_2 & \lambda_2 p_2 & & \\ & & \ddots & \ddots & \\ & & & -\lambda_{n-1} & \lambda_{n-1} p_{n-1} \\ & & & & -\lambda_n \end{pmatrix}, \quad T^0 = \begin{pmatrix} \lambda_1 (1 - p_1) \\ \lambda_2 (1 - p_2) \\ \vdots \\ \lambda_{n-1} (1 - p_{n-1}) \\ \lambda_n \end{pmatrix}. \tag{1.36}$$

1.4.3 The PH Distribution

The above three examples given in Eq. (1.34) to Eq. (1.36) indicate that a distribution can be expressed by means of a Markov chain T with an initial probability vector α. Hence, this motivates us to propose a PH distribution under a unified framework (α, T). The PH distribution is characterized by an absorbing Markov chain with finite states, which is measured by the time that the underlying Markov chain spends in all the transient states until the first absorption. The phase number of the PH distribution is equal to the number of transient states in the Markov chain T, and a row vector α is the initial probability vector of the Markov chain T.

We consider a continuous-time Markov chain with state space $\Omega = \{1, 2, \ldots, m, m+1\}$ whose infinitesimal generator is given by

$$Q = \begin{array}{c} \\ (1, \ldots, m) \\ m+1 \end{array} \begin{array}{c} (1, \ldots, m) \quad m+1 \\ \begin{pmatrix} T & T^0 \\ 0 & 0 \end{pmatrix} \end{array} \tag{1.37}$$

where $T^0 \gneq 0$ and $Te + T^0 = 0$. It is clear that state $m+1$ is an absorbing state and all the others are transient. Let (α, α_{m+1}) be the initial probability vector of the Markov chain, where $\alpha e + \alpha_{m+1} = 1$. For the Markov chain given in Eq. (1.37), the distribution $F(x)$ of the time X until absorption into the absorbing state $m+1$ is called a PH distribution with representation (α, T). If $T + T^0\alpha$ is the infinitesimal generator of an irreducible Markov chain, then this representation (α, T) is called an irreducible representation. In this case, it follows from Lemma 2.2.2 in Chapter 2 of Neuts [92] that

$$F(x) = 1 - \alpha \exp\{Tx\}e. \tag{1.38}$$

Obviously, $F(0) = 1 - \alpha e = \alpha_{m+1}$. If $\alpha = 0$, then $F(x) = 1$ for all $x \geqslant 0$; if $0 < \alpha e < 1$, then $F(0) = \alpha_{m+1}$; if $\alpha_{m+1} = 0$, then $F(0) = 0$.

In general, if a stochastic model can be analyzed when the relevant distributions are exponential, then the stochastic model with corresponding PH distributions may also admit an algorithmic solution. Table 1.1 provides some useful relations between the exponential distribution and the PH distribution. Note that LST denotes the Laplace-Stieltjes transform of $F(x)$, $\tilde{f}^*(s) = \int_0^{+\infty} \exp\{-sx\} \mathrm{d}F(x)$.

Table 1.1 A comparison for the exponential and PH distributions

Class of distributions	Exponential Distribution	PH Distribution
Distribution function	$F(x) = 1 - \alpha \exp\{-\lambda x\}$	$F(x) = 1 - \alpha \exp\{Tx\}e$
Distribution parameters	α, λ	α, T, m
Density function	$f(x) = \alpha\lambda \exp\{-\lambda x\}$	$f(x) = \alpha \exp\{Tx\}T^0$
Moments	$\mu_i = (-1)^{i+1} i! \alpha \lambda^{-i}$	$\mu_i = (-1)^i i! \alpha T^{-i} e$
LST	$\tilde{f}^*(s) = 1 - \alpha + \alpha\lambda(s+\lambda)^{-1}$	$\tilde{f}^*(s) = \alpha_{m+1} + \alpha(sI - T)^{-1}T^0$

In what follows, we provide four useful properties for the PH distribution, while the proofs may refer to Chapter 2 of Neuts [92].

Property 1.1 For a PH distribution with representation (α,T), the states 1, 2, ..., m are transient if and only if the matrix T is invertible, and $T^{-1} < 0$.

Property 1.2 For a PH distribution with irreducible representation (α,T), α exp $\{Tx\} > 0$ and exp $\{Tx\}T^0 > 0$ for each $x \geqslant 0$. At the same time, $\alpha(sI - T)^{-1} > 0$ and $(sI - T)^{-1}T^0 > 0$ for $s \geqslant 0$.

Property 1.3 For two PH distributions with irreducible representations (α, T) and (β, S), the matrix $T \oplus S$ is invertible.

Property 1.4 For a PH distribution with reducible representation (α,T), we may delete rows and columns of T corresponding to a subset of state space $\{1, 2, ..., m\}$ to obtain a smaller, irreducible representation (α_1, T_1).

As a companion distribution, we now consider a discrete-time PH distribution. Consider a discrete-time Markov chain with state space $\Omega = \{1, 2, ..., m, m+1\}$ whose transition probability matrix is given by

$$P = \begin{array}{c} \\ (1,\dots,m) \\ m+1 \end{array} \begin{array}{c} (1,\dots,m)\ m+1 \\ \begin{pmatrix} T & T^0 \\ 0 & 1 \end{pmatrix} \end{array} \tag{1.39}$$

where $T^0 \gneq 0$ and $Te + T^0 = e$. State $m+1$ is an absorbing state and all the others are transient. Let (α, α_{m+1}) be the initial probability vector of the Markov chain, where $\alpha e + \alpha_{m+1} = 1$. For the Markov chain given in (1.39), the distribution $\{p_k, k \geqslant 0\}$ of the number N of state transitions until absorption into the absorbing state $m+1$ is called a discrete-time PH distribution with representation (α, T). If $T + T^0\alpha$ is the transition probability matrix of an irreducible Markov chain, then this representation (α, T) is called an irreducible representation. In this case, it follows from Chapter 2 of Neuts [92] that

$$p_k = \begin{cases} \alpha_{m+1}, & k = 0, \\ \alpha T^{k-1} T^0, & k \geqslant 1. \end{cases} \tag{1.40}$$

If a stochastic model can be analyzed when the relevant distributions are geometric, then the stochastic model with corresponding discrete-time PH distributions may admit an algorithmic solution. Table 1.2 provides some useful relations between the geometric distribution and the discrete-time PH distribution. We write $P^*(z) = \sum_{k=0}^{\infty} z^k p_k$.

Table 1.2 A comparison for the geometric and PH distributions

Class of Distributions	Geometric Distribution	PH Distribution
Distribution function	$p_k = \begin{cases} 1-\alpha, & k = 0, \\ \alpha t^{k-1}(1-t), & k \geqslant 1. \end{cases}$	$p_k = \begin{cases} \alpha_{m+1}, & k = 0, \\ \alpha T^{k-1} T^0, & k \geqslant 1. \end{cases}$
Distribution parameters	α, t	α, T, m
Moments	$\mu_i = i! \alpha t^{i-1} (1-t)^{-i}$	$\mu_i = i! \alpha T^{i-1} (I-T)^{-i} e$
PGF	$P^*(z) = 1 - \alpha + z\alpha(1-zt)^{-1}(1-t)$	$P^*(z) = a_{m+1} + z\alpha(I-zt)^{-1} T^0$

Now, we provide some closure properties of the PH distributions, the corresponding proofs may refer to Chapter 2 of Neuts [92]. For convenience of description, we introduce a notation: If the distribution of the random variable X is of phase type of order m with irreducible representation (α, T), we write as $X \sim \mathrm{PH}(\alpha, T; m)$; or $F(x) \sim \mathrm{PH}(\alpha, T; m)$ for the continuous-time case, while $\{p_k\} \sim \mathrm{PH}(\alpha, T; m)$ for the discrete-time case.

Property 1.5 Let X_R and X_A be the stationary residual life time and stationary age of the random variable X, respectively. If $X \sim \mathrm{PH}(\alpha, T; m)$, then X_R, $X_A \sim$

$\mathrm{PH}(\theta, T; m)$, where θ is the stationary probability vector of the Markov chain $T + T^0\alpha$ for $\alpha e = 1$.

Property 1.6 If X~$\mathrm{PH}(\alpha, T; m)$ and Y~$\mathrm{PH}(\beta, S; n)$, then X+Y~$\mathrm{PH}(\gamma, L; m+n)$, where $\gamma = (\alpha, a_{m+1}\beta)$ and

$$L = \begin{pmatrix} T & T^0\beta \\ 0 & S \end{pmatrix}.$$

Property 1.7 If X~$\mathrm{PH}(\alpha, T; m)$ and Y~$\mathrm{PH}(\beta, S; n)$, then min $\{X, Y\}$~PH $(\alpha \otimes \beta, T \oplus S; m+n)$, and max $\{X, Y\}$~$\mathrm{PH}(\gamma, L; mn+m+n)$, where

$$\gamma = (\alpha \otimes \beta, \beta_{n+1}\alpha, \alpha_{m+1}\beta),$$
$$L = \begin{pmatrix} T \oplus S & I \oplus S^0 & T^0 \oplus I \\ 0 & T & 0 \\ 0 & 0 & S \end{pmatrix}.$$

Property 1.8 Let $p = (p_1, p_2, ..., p_k)$ be a probability vector for $pe = 1$, and $F_j(x)$~$\mathrm{PH}(\alpha_j, T_j; m_j)$ for $j = 1, 2, ..., k$. Then $G(x) = \sum_{j=1}^{k} p_j F_j(x)$~$\mathrm{PH}(\gamma, L; \sum_{j=1}^{k} m_j)$, where $\gamma = (p_1\alpha_1, p_2\alpha_2, ..., p_k\alpha_k)$ and $L = \mathrm{diag}\,(T_1, T_2, ..., T_n)$.

It is worthwhile to note that infinite mixtures of PH distributions are generally not of phase type. An important and useful exception is given in Properties 1.9 and 1.10. Let $F^{n*}(x)$ be the nth-fold convolution of the function $F(x)$, that is, $F^{0*}(x) = 1$, $F^{1*}(x) = F(x)$ and $F^{n*}(x) = \int_0^x F^{(n-1)*}(x-u)\mathrm{d}F(u)$ for $n \geqslant 2$.

Property 1.9 If $\{s_k\}$~$\mathrm{PH}(\beta, S; n)$ and $F(x)$~$\mathrm{PH}(\alpha, T; m)$, then the infinite mixture $G(x) = \sum_{k=0}^{\infty} s_k F^{k*}(x)$~$\mathrm{PH}(\gamma, L; mn)$, where

$$\gamma = \alpha \otimes \beta(I - \alpha_{m+1}S)^{-1},$$
$$L = T \otimes I + (1-\alpha_{m+1})T^0\alpha \otimes (I - \alpha_{m+1}S)^{-1}S.$$

The height of the jump γ_{mn+1} at time 0 and the vector L^0 are given by

$$\gamma_{mn+1} = \beta_{n+1} + \alpha_{m+1}\beta(I - \alpha_{m+1}S)^{-1}S^0,$$
$$L^0 = T^0 \otimes (I - \alpha_{m+1}S)^{-1}S^0.$$

Property 1.10 If $\{s_k\}$~$\mathrm{PH}(\beta, S; n)$ and $\{p_v\}$~$\mathrm{PH}(\alpha, T; m)$, then for the infinite mixture $P_k = \sum_{k=0}^{\infty} s_k \{p_v\}^{k*}$, $\{P_k\}$ ~$\mathrm{PH}(\gamma, L; mn)$, where

$$\gamma = \alpha \otimes \beta (I - \alpha_{m+1} S)^{-1},$$
$$L = T \otimes I + (1 - \alpha_{m+1}) T^0 \alpha \otimes (I - \alpha_{m+1} S)^{-1} S^0.$$

In many applications, we need to consider the following integral

$$a_k = \int_0^{+\infty} \exp\{-\lambda x\} \frac{(\lambda x)^k}{k!} \mathrm{d}F(x), \quad k \geqslant 0.$$

The following Property 1.11 describes a closure property of the PH distribution for the sequence $\{a_k\}$.

Property 1.11 If $F(x) \sim \mathrm{PH}(\alpha, T; m)$, then the sequence $\{a_k\} \sim \mathrm{PH}(\gamma, L; m)$, where

$$\gamma = \lambda \alpha (\lambda I - T)^{-1},$$
$$L = \lambda (\lambda I - T)^{-1}.$$

The height of the jump γ_{mn+1} at time 0 and the vector L^0 are given by

$$\gamma_{m+1} = \alpha_{m+1} + \alpha (\lambda I - T)^{-1} T^0,$$
$$L^0 = (\lambda I - T)^{-1} T^0.$$

For the two random variables N and X, we write $a_n = P\{N = n\}$ for $n \geqslant 0$ and

$$P\{X = j \mid N = n\} = \binom{n}{j} p^j (1-p)^{n-j}.$$

Then it is clear that

$$P\{X = j\} = \sum_{n=0}^{\infty} P\{N = n\} P\{X = j \mid N = n\}$$
$$= \sum_{n=0}^{\infty} a_n \binom{n}{j} p^j (1-p)^{n-j}.$$

The following Property 1.12 indicates that the random variable X is of phase type if the random variable N is of phase type.

Property 1.12 If $N \sim \mathrm{PH}(\alpha, T; m)$, then $X \sim \mathrm{PH}(\gamma, L; m)$, where

$$\gamma = p\alpha [I - (1-p)T]^{-1},$$
$$L = pT[I - (1-p)T]^{-1}.$$

The height of the jump γ_{mn+1} at time 0 and the vector L^0 are given by

$$\gamma_{m+1} = \alpha_{m+1} + \alpha (1-p) p [I - (1-p)T]^{-1} T^0,$$
$$L^0 = [I - (1-p)T]^{-1} T^0.$$

In the rest of this section, we discuss a multivariate PH distribution which is always useful in modeling real situations that involve n nonnegative dependent random variables. Our analysis focuses on the following two cases: The bivariate PH distribution and the multivariate PH distribution.

Definition 1.1 *For a Markov chain with state space Ω, a set $\Gamma \subset \Omega$ is said to be stochastically closed if once the Markov chain enters the set Γ, it never leaves again.*

For the stochastically closed sets of the Markov chain, we have following useful properties:

(1) If Γ_1 and Γ_2 are two nonempty stochastically closed sets of the state space Ω, then $\Gamma_1 \bigcup \Gamma_2$ and $\Gamma_1 \bigcap \Gamma_2$ are all stochastically closed.

(2) If Γ is a nonempty stochastically closed set of the state space Ω, then the infinitesimal generator can be simplified as

$$T = \begin{array}{c} \\ \Gamma^c \\ \Gamma \end{array} \begin{array}{c} \begin{array}{cc} \Gamma^c & \Gamma \end{array} \\ \begin{pmatrix} T_{1,1} & T_{1,2} \\ 0 & T_{2,2} \end{pmatrix} \end{array}.$$

1.4.4 The Bivariate PH Distribution

Consider a continuous-time and right-continuous Markov chain $\{X(t)\}$ with state space $\Omega = \{1, 2, ..., m, m+1\}$ whose infinitesimal generator is given in (1.37). Let Γ_1 and Γ_2 be two nonempty stochastically closed sets of the state space Ω such that $\Gamma_1 \bigcap \Gamma_2 = \{m+1\}$. We define

$$Y_k = \inf\{t : X(t) \in \Gamma_k\}$$

with a convention: $\inf \varnothing = +\infty$ for the empty set $\varnothing$. Let

$$\overline{F}(x_1, x_2) = P\{Y_1 > x_1, Y_2 > x_2\}.$$

Then (Y_1, Y_2) or $F(x_1, x_2)$ is said to be of bivariate phase type.

For the bivariate PH distribution, we have the following closed-form expression:

$$F(x_1, x_2) = \begin{cases} \alpha \mathrm{e}^{Tx_2} g_2 \mathrm{e}^{T(x_1 - x_2)} g_1 e, & 0 \leqslant x_2 \leqslant x_1, \\ \alpha \mathrm{e}^{Tx_1} g_1 \mathrm{e}^{T(x_2 - x_1)} g_2 e, & 0 \leqslant x_1 \leqslant x_2, \end{cases}$$

where $g_k = \mathrm{diag}(g_k(1), g_k(2), ..., g_k(m))$

$$g_k(i) = \begin{cases} 1, & i \in \Gamma_k^c, \\ 0, & i \in \Gamma_k. \end{cases}$$

It is obvious that $\overline{F}(x, x) = \alpha \mathrm{e}^{Tx} g_2 g_1 e$ and $\overline{F}(0, 0) = \alpha g_2 g_1 e$.

Now, we list some useful properties for the bivariate PH distribution as follows:

Property 1.13 The probability density function of the bivariate PH distribution is given by

$$f(x_1, x_2) = \begin{cases} \alpha \mathrm{e}^{Tx_2}[T, g_2]\mathrm{e}^{T(x_1 - x_2)} g_1 e, & 0 \leqslant x_2 \leqslant x_1, \\ \alpha \mathrm{e}^{Tx_1}[T, g_1]\mathrm{e}^{T(x_2 - x_1)} g_2 e, & 0 \leqslant x_1 \leqslant x_2, \end{cases}$$

where the operator $[A, B] = AB - BA$.

Property 1.14 Let $\tilde{f}(s_1, s_2) = E[\exp\{-(s_1 Y_1 + s_2 Y_2)\}]$. Then

$$\tilde{f}(s_1, s_2) = \alpha[(s_1 + s_2)I - T]^{-1}\{[T, g_2](s_1 I - T)^{-1} T g_1 + [T, g_1](s_2 I - T)^{-1} T g_2 \\ - T g_1 g_2 + [T, g_1] + [T, g_2]\}e.$$

Property 1.15

$$E[Y_1 Y_2] = \alpha(T^{-1} g_1 T^{-1} g_2 + T^{-1} g_2 T^{-1} g_1)e.$$

1.4.5 The Multivariate PH Distribution

Consider a continuous-time and right-continuous Markov chain $\{X(t)\}$ with state space $\Omega = \{1, 2, ..., m, m+1\}$ whose infinitesimal generator is given in Eq. (1.37). Let $\Gamma_1, \Gamma_2, ..., \Gamma_n$ be n nonempty stochastically closed sets of the state space Ω such that $\bigcap_{i=1}^{n} \Gamma_i = \{m+1\}$. We define

$$Y_k = \inf\{t : X(t) \in \Gamma_k\}.$$

Let

$$\overline{F}(x_1, x_2, ..., x_n) = P\{Y_1 > x_1, Y_2 > x_2, ..., Y_n > x_n\}.$$

Then $Y = (Y_1, Y_2, ..., Y_n)$ or $F(x_1, x_2, ..., x_n)$ is said to be of n-variate phase type. For the multivariate PH distribution, we call $(\alpha, T; \Gamma_1, \Gamma_2, ..., \Gamma_n)$ its irreducible representation, or Y~PH$(\alpha, T; \Gamma_1, \Gamma_2, ..., \Gamma_n)$.

For the nonnegative vector $(x_1, x_2, ..., x_n)$, we write $x_{i_1} \geqslant x_{i_2} \geqslant \cdots \geqslant x_{i_n} \geqslant 0$. Thus, for the multivariate PH distribution we can write the closed-form expression as follows:

$$\overline{F}(x_1, x_2, ..., x_n) = \alpha \exp\{T x_{i_n}\} g_{i_n} \prod_{k=n}^{2} \exp\{T(x_{i_{k-1}} - x_{i_k})\} g_{i_{k-1}} e,$$

the probability density function is given by

$$f(x_1, x_2, ..., x_n) = (-1)^n \alpha \exp\{Tx_{i_n}\}[T, g_{i_n}] \prod_{k=n}^{3} \exp\{T(x_{i_{k-1}} - x_{i_k})\}[T, g_{i_{k-1}}]$$
$$\cdot \exp\{T(x_{i_1} - x_{i_2})\} g_{i_1} e$$

and the joint mean is given by

$$E[Y_1 Y_2 \cdots Y_n] = (-1)^n \alpha \sum_{\omega \in \Theta} \prod_{i=1}^{n} T^{-1} g_{\omega(i)} e,$$

where Θ is the set of all permutations of $\{1, 2, ..., n\}$, and $\omega = (\omega(1), \omega(2), ..., \omega(n))$.

For the multivariate PH distribution, we now list its useful properties as follows:

Property 1.16 If Y~PH$(\alpha, T; \Gamma_1, \Gamma_2, ..., \Gamma_n)$ and Z~PH$(\beta, S; \Im_1, \Im_2, ..., \Im_m)$, then

$$(Y, Z) \sim \text{PH}(\alpha \otimes \beta, T \oplus S; \Gamma_1, \Gamma_2, ..., \Gamma_n, \Im_1, \Im_2, ..., \Im_m).$$

Property 1.17 If $(Y_1, Y_2, ..., Y_n)$~PH$(\alpha, T; \Gamma_1, \Gamma_2, ..., \Gamma_n)$, then

$$(Y_{i_1}, Y_{i_2}, ..., Y_{i_k}) \sim \text{PH}(\alpha, T; \Gamma_{i_1}, \Gamma_{i_2}, ..., \Gamma_{i_k})$$

for each subset $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$.

1.4.6 The Discrete-Time Multivariate PH Distribution

Consider a discrete-time Markov chain $\{X_k\}$ with state space $\Omega = \{1, 2, ..., m, m+1\}$ whose transition probability matrix is given in Eq. (1.39). Let $\Gamma_1, \Gamma_2, ..., \Gamma_n$ be n nonempty stochastically closed sets of the state space Ω such that $\bigcap_{i=1}^{n} \Gamma_i = \{m+1\}$. We define

$$N_k = \inf\{k : X_k \in \Gamma_k\}.$$

Let

$$p(k_1, k_2, ..., k_n) = P\{N_1 = k_1, N_2 = k_2, ..., N_n = k_n\}.$$

Then $N = (N_1, N_2, ..., N_n)$ or $\{p(k_1, k_2, ..., k_n)\}$ is said to be of discrete-time multivariate phase type. For the discrete-time multivariate PH distribution, we call $(\alpha, T; \Gamma_1, \Gamma_2, ..., \Gamma_n)$ its irreducible representation.

When $k_{i_1} \geqslant k_{i_2} \geqslant ... \geqslant k_{i_n} \geqslant 0$, for the discrete-time multivariate PH distribution we can write the closed-form expression as follows:

$$p(k_1, k_2, ..., k_n) = \alpha T^{k_{i_n}} [T, g_{i_n}] \prod_{j=n}^{3} T^{k_{i_{j-1}} - k_{i_j}} [T, g_{k_{i_{j-1}}}] T^{k_{i_1} - k_{i_2}} g_{k_{i_1}} e$$

and

$$E[N_1 N_2 \cdots N_n] = \alpha \sum_{\omega \in \Theta} \prod_{i=1}^{n} (I-T)^{-1} g_{\omega(i)} e.$$

1.5 The Markovian Arrival Processes

In this section, we provide several important types of point processes and list their useful properties. Specifically, we indicate how to construct a MAP in terms of a Markov chain with finite states. Finally, we study multivariate Markovian arrival processes.

The analysis of this section contains twofold: Renewal processes and non-renewal processes. First, we provide a unified framework of Markov chains for analyzing two renewal counting processes: The Poisson process and the PH renewal process.

1.5.1 The Poisson Process

There is an arrival process with the kth arrival epoch τ_k for $k \geqslant 0$, where $\tau_0 = 0$. Let $X_n = \tau_n - \tau_{n-1}$ for $n \geqslant 1$. Then X_n is the nth interarrival time. Obviously, $\tau_k = \sum_{n=1}^{k} X_n$ for $k \geqslant 0$. We assume that the random variables X_n for $n \geqslant 1$ are i.i.d.. Figure 1.6 shows a useful relation between the arrival epoch and the interarrival time.

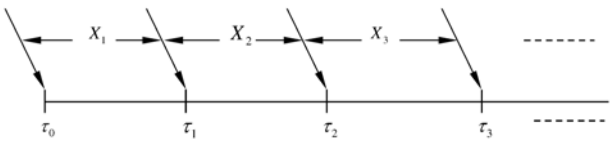

Figure 1.6 The arrival epoch and the interarrival time

Let $N(t)$ be the arrival number of the arrival process in the time interval $(0;t]$. In general, we denote by $\{N(t), t \geqslant 0\}$ the arrival process, which is also called a counting process.

We provide a definition for the Poisson process as follows.

Definition 1.2 *A Poisson process with rate λ is a counting process $\{N(t), t \geqslant 0\}$ for which*

(1) $N(0) = 0$;

(2) *the process increments* $N(\tau_1) - N(\tau_0), N(\tau_2) - N(\tau_1), ..., N(\tau_n) - N(\tau_{n-1})$ *are mutually independent; and*

(3) *for any* $s \geqslant 0$, $t > 0$,

$$P\{N(t+s) - N(s) = k\} = \exp\{-\lambda t\}\frac{(\lambda t)^k}{k!}, \quad k \geqslant 0,$$

which is independent of the initial epoch $s \geqslant 0$.

The following proposition provides an intuitive understanding of the Poisson process in terms of the exponential distribution. The proof is clear and is omitted here.

Proposition 1.2 *The random variables* X_n *for* $n \geqslant 1$ *are i.i.d. with exponential distribution function* $F(x) = 1 - \exp\{-\lambda t\}$ *if and only if the counting process* $\{N(t), t \geqslant 0\}$ *is a Poisson process with rate* λ.

Note that the exponential interarrival times X_n for $n \geqslant 1$ are i.i.d., the Poisson process with rate λ may be regarded as a pure birth process whose infinitesimal generator is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & & & \\ & -\lambda & \lambda & & \\ & & -\lambda & \lambda & \\ & & & \ddots & \ddots \end{pmatrix}.$$

We write

$$P(k;t) = P\{N(t) = k\}.$$

Let

$$\overline{P}(t) = \begin{pmatrix} P(0;t) & P(1;t) & P(2;t) & P(3;t) & \cdots \\ & P(0;t) & P(1;t) & P(2;t) & \cdots \\ & & P(0;t) & P(1;t) & \cdots \\ & & & P(0;t) & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Then it is easy to check that

$$\overline{P}'(t) = \overline{P}(t)Q \quad \text{or} \quad \overline{P}'(t) = Q\overline{P}(t),$$

with the initial condition $\overline{P}(0) = I$. A simple computation can yield $\overline{P}(t) = \exp\{Qt\}$, which yields

$$P(k;t) = \exp\{-\lambda t\}\frac{(\lambda t)^k}{k!}, \quad k \geqslant 0.$$

1.5.2 The PH Renewal Process

An only different assumption from that for the Poisson process analyzed above is that the distribution function $F(x)$ is taken as a PH distribution of order m with irreducible representation (α, T). In this case, we need to introduce a matrix sequence $\{P(k; t)\}$ with respect to the m phases. Let $P(k;t) = (P_{j,j'}(k;t))_{1 \leqslant j,j' \leqslant m}$, where $P_{j,j'}(k;t)$ is a conditional probability that the Markov chain $T + T^0\alpha$ is in the phase j' at time t and that k renewals occur in $[0, t)$, given that the Markov chain starts in the phase j at time 0.

The matrix sequence $\{P(k; t)\}$ satisfies the forward Chapman-Kolmogorov differential equations

$$P'(0;t) = P(0;t)T, \tag{1.41}$$

$$P'(k;t) = P(k;t)T + P(k-1;t)T^0\alpha, \quad k \geqslant 1, \tag{1.42}$$

with the initial condition $P(k; 0) = \delta_{0,k}I$ for $k \geqslant 0$; or the backward Chapman-Kolmogorov differential equations

$$P'(0;t) = TP(0;t), \tag{1.43}$$

$$P'(k;t) = TP(k;t) + T^0\alpha P(k-1;t), \quad k \geqslant 1. \tag{1.44}$$

Let $P^*(z;t) = \sum_{k=0}^{\infty} z^k P(k;t)$. Then $P^*(z;0) = I$, and it follows from Eq. (1.41) and Eq. (1.42), or Eq. (1.43) and Eq. (1.44) that

$$P^*(z;t) = \exp\{(T + zT^0\alpha)t\}. \tag{1.45}$$

Next, we consider the factorial moment matrices

$$V_n(t) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} P(k;t), \quad n \geqslant 0.$$

It follows from Eq. (1.41) and Eq. (1.42) that

$$V_0'(t) = V_0(t)(T + T^0\alpha), \tag{1.46}$$

$$V_n'(t) = V_n(t)(T + T^0\alpha) + nV_{n-1}(t)T^0\alpha, \quad n \geqslant 1; \tag{1.47}$$

or from Eq. (1.43) and Eq. (1.44) that

$$V_0'(t) = (T + T^0\alpha)V_0(t), \tag{1.48}$$

$$V_n'(t) = (T + T^0\alpha)V_n(t) + T^0\alpha n V_{n-1}(t), \quad n \geqslant 1; \tag{1.49}$$

with the initial conditions $V_0(0) = I$ and $V_n(0) = 0$ for $n \geqslant 1$.

Now, we use a Markov chain to understand Eq. (1.41) to Eq. (1.44), and Eq. (1.46) to Eq. (1.49).

Let

$$Q=\begin{pmatrix} T & T^0\alpha & & & \\ & T & T^0\alpha & & \\ & & T & T^0\alpha & \\ & & & \ddots & \ddots \end{pmatrix} \tag{1.50}$$

and

$$\overline{P}(t)=\begin{pmatrix} P(0;t) & P(1;t) & P(2;t) & P(3;t) & \cdots \\ & P(0;t) & P(1;t) & P(2;t) & \cdots \\ & & P(0;t) & P(1;t) & \cdots \\ & & & P(0;t) & \cdots \\ & & & & \ddots \end{pmatrix}.$$

It is clear that $\overline{P}'(t)=\overline{P}(t)Q$ or $\overline{P}'(t)=Q\overline{P}(t)$ with the initial condition $\overline{P}(0)=I$, which can lead to Eq. (1.41) to Eq. (1.44).

Let

$$\tilde{Q}=\begin{pmatrix} T+T^0\alpha & nT^0\alpha & & & \\ & T+T^0\alpha & nT^0\alpha & & \\ & & T+T^0\alpha & nT^0\alpha & \\ & & & \ddots & \ddots \end{pmatrix}$$

and

$$\overline{V}(t)=\begin{pmatrix} V_0(t) & V_1(t) & V_2(t) & V_3(t) & \cdots \\ & V_0(t) & V_1(t) & V_2(t) & \cdots \\ & & V_0(t) & V_1(t) & \cdots \\ & & & V_0(t) & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Obviously, $\overline{V}'(t)=\overline{V}(t)Q$ or $\overline{V}'(t)=Q\overline{V}(t)$ with the initial condition $\overline{V}(0)=I$, which can yield Eq. (1.46) to Eq. (1.49).

It is necessary to provide some computations for the matrix sequence $\{V_n(t)\}$. Referring to Sec. 5.1 of Neuts [94], we have

(1)

$$V_0(t)=\exp\{(T+T^0\alpha)t\}.$$

(2) Let θ be the stationary probability vector of the Markov chain $T+T^0\alpha$, $\lambda=\theta T^0$. Then $\theta V_1(t)e=\lambda t$.

(3)

$$\alpha V_1(t)e = \lambda t - 1 - \lambda\alpha \exp\{(T + T^0\alpha)t\}T^{-1}e.$$

For the PH renewal process, we take the renewal density function as

$$u(t) = \sum_{k=0}^{\infty} f^{k*}(t),$$

where $f^{k*}(t)$ is the kth-fold convolution of the probability density function $f(t)$ of the PH distribution $F(t)$. It is worthwhile to note that $E[N(t)] = 1 + \int_0^t u(x)\mathrm{d}x$.

The following proposition expresses the renewal density function, while the proof is easy and is omitted here.

Proposition 1.3 *For a PH renewal process with PH irreducible representation* (α, T), *then the renewal density function is given by*

$$u(t) = \alpha \exp\{(T + T^0\alpha)t\}T^0.$$

In the PH renewal process, we define the excess life $\xi(t)$ and the age $\eta(t)$ at time t, respectively, both of which are expressed in Fig. 1.7. The following proposition provides expressions for the distribution function of $\xi(t)$, while the associated discussion for $\eta(t)$ is similar.

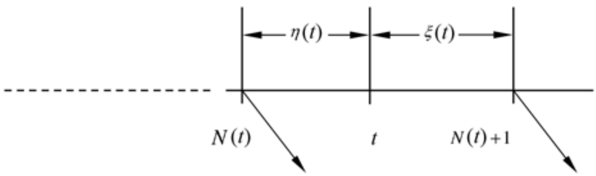

Figure 1.7 The excess life and the age at time t

Proposition 1.4 *For a PH renewal process with PH irreducible representation* (α, T), *the excess life* $\xi(t)$ *at time* t *is of phase type with irreducible representation* $(\gamma(t), T)$, *where* $\gamma(t) = \alpha \exp\{(T + T^0\alpha)t\}$.

Let

$$A_k = \int_0^{+\infty} P(k;t)\mathrm{d}H(t), \quad k \geqslant 0,$$

and

$$\tilde{A}_k = \int_0^{+\infty} P(k;t)[1 - H(t)]\mathrm{d}t, \quad k \geqslant 0.$$

The following proposition provides a PH description for the two matrix sequences $\{A_k\}$ and $\{\tilde{A}_k\}$, respectively.

Proposition 1.5 *Suppose* $H(t)$ *is a PH distribution with irreducible representation* (β, S).

(1) $\{\alpha A_k e, k \geqslant 0\}$ *is a discrete-time PH distribution with irreducible representation* (γ, L) *of order mn, where*

$$\gamma = -(\alpha \otimes \beta)(T \oplus S)^{-1}(T^0\alpha \otimes I),$$
$$L = -(T \oplus S)^{-1}(T^0\alpha \otimes I),$$
$$\gamma_{mn+1} = -(\alpha \otimes \beta)(T \oplus S)^{-1}(e \otimes S^0),$$
$$L^0 = -(T \oplus S)^{-1}(e \otimes S^0).$$

(2) $\{\alpha \tilde{A}_k e, k \geqslant 0\}$ *is a discrete-time PH distribution with irreducible representation* (γ, L) *of order mn,*

$$\gamma = -(\alpha \otimes \theta)(T \oplus S)^{-1}(T^0\alpha \otimes I),$$
$$L = -(T \oplus S)^{-1}(T^0\alpha \otimes I),$$
$$\gamma_{mn+1} = -(\alpha \otimes \theta)(T \oplus S)^{-1}(e \otimes S^0),$$
$$L^0 = -(T \oplus S)^{-1}(e \otimes S^0),$$

where θ *is the stationary probability vector of the Markov chain* $T + T^0\alpha$.

1.5.3 The Markovian Modulated Poisson Process

We consider an arrival process $\{(N(t), J(t)): t \geqslant 0\}$ which extends the Poisson process to a multiple Poisson process depending on m environment states. This arrival process is written as MMPP and is depicted in Fig. 1.8, which gives an obviously physical interpretation.

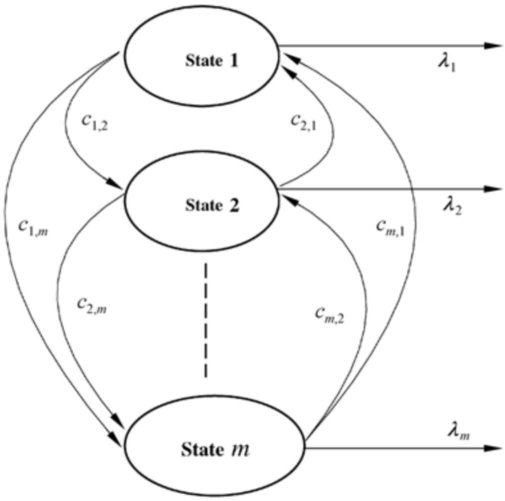

Figure 1.8 The state transitions in an MMPP

From Fig. 1.8, it is seen that the random environment $\{J(t):t\geqslant 0\}$ is a Markov chain with m environment states whose infinitesimal generator is given by $\mathcal{C}=(c_{i,j})_{1\leqslant i,j\leqslant m}$. When $J(t)=i$, the counting process $\{N(t):t\geqslant 0\}$ is a Poisson process with rate λ_i for $1\leqslant i\leqslant m$. To express such an arrival process in terms of a Markov chain, we write

$$C=\mathcal{C}-\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m)$$

and

$$D=\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m).$$

Using the matrix pair (C, D), the MMPP can be described as a block-structured pure birth process with environment block C and a birth rate block D. This corresponding infinitesimal generator is similar to that of the PH renewal process with environment block $C=T$ and birth rate block $D=T^0\alpha$, see Eq. (1.50).

1.5.4 The Markovian Modulated PH Process

We extend the MMPP to a more general arrival process: Markovian modulated PH process (MMPHP). An only difference from the MMPP is that when $J(t)=i$, the counting process $\{N(t):t\geqslant 0\}$ is a discrete-time PH renewal process with PH irreducible representation (α_i, T_i) of order n_i for $1\leqslant i\leqslant m$. In this case, the matrix pair (C, D) is respectively given by

$$C=\begin{pmatrix} c_{1,1}I+T_1 & c_{1,2}e\alpha_2 & c_{1,3}e\alpha_3 & \cdots & c_{1,m}e\alpha_m \\ c_{2,1}e\alpha_1 & c_{2,2}I+T_2 & c_{2,3}e\alpha_3 & \cdots & c_{2,m}e\alpha_m \\ c_{3,1}e\alpha_1 & c_{3,2}e\alpha_2 & c_{3,3}I+T_3 & \cdots & c_{3,m}e\alpha_m \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m,1}e\alpha_1 & c_{m,2}e\alpha_2 & c_{m,3}e\alpha_3 & \cdots & c_{m,m}I+T_m \end{pmatrix}$$

and

$$D=\operatorname{diag}(T_1^0\alpha_1, T_2^0\alpha_2, T_3^0\alpha_3, \ldots, T_m^0\alpha_m).$$

1.5.5 The Markovian Arrival Processes

Observing the PH renewal process, the MMPP and the MMPHP, it is easy to see that each of them corresponds to a matrix pair (C, D), which then is used to express a block-structured pure birth process whose infinitesimal generator is given by

$$Q = \begin{pmatrix} C & D & & & \\ & C & D & & \\ & & C & D & \\ & & & \ddots & \ddots \end{pmatrix}. \tag{1.51}$$

We now consider the MAP, which may have an intuitive interpretation such that the elements of the matrices C and D are more general. The matrix C has negative diagonal elements and nonnegative off-diagonal elements; the matrix D is non-zero and nonnegative; and the matrix $C+D$ is the infinitesimal generator of an irreducible Markov chain with m states.

We provide a physical interpretation that the MAP can be obtained from a generalization of the MMPP. When the Markov environment process $J(t)=i$, not only does the corresponding Poisson process depend on the present state i, but it is also related to jumping to the next state j. Thus, the arrival rate of the MAP should be given by $\lambda_{i,j}$ for $1 \leqslant i, j \leqslant m$, e.g.. See Fig. 1.9 for a clear illustration.

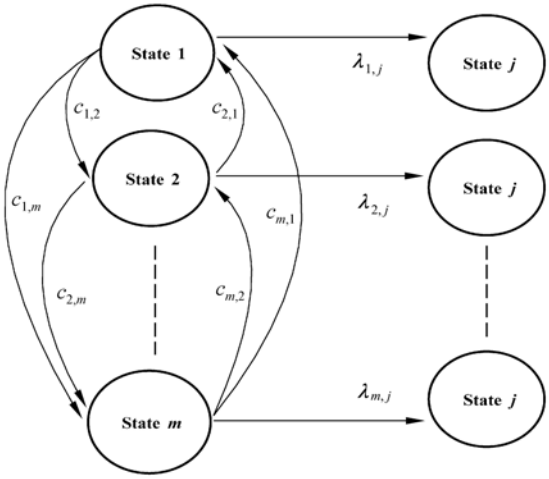

Figure 1.9 The state transitions in a MAP

Under the situation, we obviously have

$$C = \mathcal{C} - \text{diag}\left(\sum_{j=1}^{m} \lambda_{1,j}, \sum_{j=1}^{m} \lambda_{2,j}, \ldots, \sum_{j=1}^{m} \lambda_{m,j} \right)$$

and

$$D = \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,m} \\ \lambda_{2,1} & \lambda_{2,2} & \cdots & \lambda_{2,m} \\ \vdots & \vdots & & \vdots \\ \lambda_{m,1} & \lambda_{m,2} & \cdots & \lambda_{m,m} \end{pmatrix}.$$

Let θ be the stationary probability vector of the Markov chain $C+D$. The stationary arrival rate of the MAP is given by $\lambda = \theta De$. We write $P(k; t) = (P_{j,j'}(k;t))_{1 \leqslant j,j' \leqslant m}$, where $P_{j,j'}(k;t)$ is a conditional probability that the Markov chain $C+D$ is in the phase j' at time t and that k renewals occur in $[0, t)$, given that the Markov chain starts in the phase j at time 0. The matrix sequence $\{P(k; t)\}$ satisfies the forward Chapman-Kolmogorov differential equations

$$P'(0;t) = P(0;t)C,$$
$$P'(k;t) = P(k;t)C + P(k-1;t)D, \quad k \geqslant 1,$$

with the initial condition $P(k;0) = \delta_{0,k} I$ for $k \geqslant 0$; or the backward Chapman-Kolmogorov differential equations

$$P'(0;t) = CP(0;t),$$
$$P'(k;t) = CP(k;t) + DP(k-1;t), \quad k \geqslant 1.$$

Let $P^*(z;t) = \sum_{k=0}^{\infty} z^k P(k;t)$. Then it is easy to see that

$$P^*(z;t) = \exp\{(C + zD)t\}.$$

Note that the MAP is not a renewal process, it is necessary to discuss for the inter-dependent structure of the MAP. To do end, we assume that an arrival occurs at time 0. Let τ_k be the kth arrival epoch of the MAP for $k \geqslant 0$, where $\tau_0 = 0$. Let $X_n = \tau_n - \tau_{n-1}$. Then X_n is the nth interarrival time of the MAP for $n \geqslant 1$. In general, these random variables X_n for $n \geqslant 1$ are not independent but they are identically distributed with marginal densities given by

$$f(t) = \theta \exp\{Ct\} De.$$

Define the matrix $W(t)$ with the (i, j)th element $W_{i,j}(t)$ which is a conditional probability density for an interarrival time $[0,t)$, terminating at phase j and beginning from phase i. It is easy to check that $W(t) = \exp\{Ct\}D$. The transition probability matrix of the MAP evolves the phase of the environment stochastic process $\{J(t); t \geqslant 0\}$ from one arrival epoch to the next one is given by

$$W = \int_0^{+\infty} W(t)\mathrm{d}t = \int_0^{+\infty} \exp\{Ct\} D\mathrm{d}t = -C^{-1}D.$$

It is clear that $\theta W = \theta$. The joint probability density function of the two random variables X_l and X_{l+k} is given by

$$f_{k,l}(x, y) = \theta \exp\{Ct\} DW^{k-1} \exp\{Ct\} De.$$

Therefore, we obtain

$$E[X_l X_{l+k}] = \int_0^{+\infty}\int_0^{+\infty} xyf_{k,l}(x,y)\mathrm{d}x\mathrm{d}y = \theta C^{-1}W^k C^{-1}e.$$

Note that

$$E[X_l]E[X_{l+k}] = (\theta C^{-1}e)^2 = \theta C^{-1}e\theta W^k C^{-1}e,$$

the correlation between the two random variables X_l and X_{l+k} is given by

$$\rho_k = \frac{\theta C^{-1}(I - e\theta)W^k C^{-1}e}{\theta C^{-1}(2I - e\theta)C^{-1}e}.$$

The joint probability density of the k random variables $X_1, X_2, ..., X_k$ is given by

$$f(x_1, x_2, \ldots, x_k) = \alpha \exp\{Cx_1\}D\exp\{Cx_2\}D\ldots\exp\{Cx_k\}De.$$

The MAPs have an important closure property of superposition, which is similar to that of the Poisson processes. We describe this closure property of the MAPs in the following proposition, while the proof is omitted here.

Proposition 1.6 *If* $\{(N_i(t), J_i(t)), t \geqslant 0\}$ *is a MAP with matrix descriptor* (C_i, D_i) *of size* m_i *for* $1 \leqslant i \leqslant K$, *then* $\left\{\left(\sum_{i=1}^{K} N_i(t), \sum_{i=1}^{K} J_i(t)\right), t \geqslant 0\right\}$ *is also a MAP with matrix descriptor* $(C_1 \oplus C_2 \oplus \ldots \oplus C_K, D_1 \oplus D_2 \oplus \ldots \oplus D_K)$ *of size* $\sum_{i=1}^{K} m_i$.

1.5.6 The Batch Markovian Arrival Process

We extend the MAP to a batch Markovian arrival process (BMAP) with matrix descriptor $(C, D_1, D_2, D_3, ...)$, where each element of the matrix D_k denotes the arrival rate of batch size k. Thus, the corresponding block-structured batch pure birth process has the following infinitesimal generator:

$$Q = \begin{pmatrix} C & D_1 & D_2 & D_3 & \cdots \\ & C & D_1 & D_2 & \cdots \\ & & C & D_1 & \cdots \\ & & & \ddots & \ddots \end{pmatrix}.$$

For the BMAP, let $P(k; \mathrm{t}) = (P_{j,j'}(k;t))_{1 \leqslant j,j' \leqslant m}$, where $P_{j,j'}(k;t)$ is a conditional probability that the Markov chain $C + \sum_{k=1}^{\infty} D_k$ is in the phase j' at time t and that k renewals occur in $[0, t)$, given that the Markov chain started in the phase j at time 0.

The matrix sequence $\{P(k;\,t)\}$ satisfies the forward Chapman-Kolmogorov differential equations

$$P'(0;t) = P(0;t)C,$$
$$P'(k;t) = P(k;t)C + \sum_{i=1}^{k-1} P(k-i;t)D_i, \quad k \geqslant 1,$$

with the initial condition $P(k;\,0) = \delta_{0,k} I$ for $k \geqslant 0$; or the backward Chapman-Kolmogorov differential equations

$$P'(0;t) = CP(0;t),$$
$$P'(k;t) = CP(k;t) + \sum_{i=1}^{k-1} D_i P(k-i;t), \quad k \geqslant 1.$$

Let $P^*(z;t) = \sum_{k=0}^{\infty} z^k P(k;t)$. Then it is easy to see that

$$P^*(z;t) = \exp\left\{\left(C + \sum_{k=1}^{\infty} z^k D_k\right)t\right\}.$$

The BMAPs also have the closure property for superposition, which is described in the following proposition, while the proof is omitted here.

Proposition 1.7 *If* $\{(N_i(t),\, J_i(t)),\, t \geqslant 0\}$ *is a BMAP with matrix descriptor* $(C^{(i)}, D_1^{(i)}, D_2^{(i)}, \ldots)$ *of size* m_i *for* $i = 1, 2$, *then* $\{(N_1(t) + N_2(t),\, J_1(t) + J_2(t)),\, t \geqslant 0\}$ *is also a BMAP with matrix descriptor* $(C^{(1)} \oplus C^{(2)}, D_1^{(1)} \oplus D_1^{(2)}, D_2^{(1)} \oplus D_2^{(2)}, \ldots)$ *of size* $m_1 + m_2$.

1.5.7 The Multivariate Markovian Arrival Process

Now, we consider a K-dimensional Markovian arrival processes (MMAP[K]) and a K-dimensional batch Markovian arrival process (MBMAP[K]). An MMAP[K] $\{\mathcal{N}(t), J(t) : t \geqslant 0\}$ is constructed by K different classes of arrivals (for example, customers, products and orders), where

$$\mathcal{N}(t) = (N_1(t), N_2(t), \ldots, N_K(t)),$$

$N_i(t)$ and $J(t)$ denote the number of the ith class of arrivals in the time interval $[0,\, t)$ for $1 \leqslant i \leqslant K$, and the phase of the Markov environment at time t, respectively. In general, an MMAP[K] can be described as an irreducible matrix descriptor $(C, D^{(1)}, D^{(2)}, \ldots, D^{(K)})$ of size m, where C has negative diagonal elements and nonnegative off-diagonal elements; the matrix $D^{(i)} \geqslant 0$ is the rate matrix of the ith class of arrivals for $1 \leqslant i \leqslant K$; and the matrix $C + \sum_{i=1}^{K} D^{(i)}$ is the infinitesimal generator of an irreducible Markov chain with m states.

Let θ be the stationary probability vector of the Markov chain $C+\sum_{i=1}^{K} D^{(i)}$. The stationary arrival rate of the ith class of arrivals is given by $\lambda_i = \theta D^{(i)} e$ and the total stationary arrival rate of the MMAP[K] is given by $\lambda = \sum_{i=1}^{K} \lambda_i$. Under the stable conditions, the probability that an arbitrary arrival is of the ith class is given by λ_i / λ for $1 \leqslant i \leqslant K$.

We write

$$\begin{aligned} P_{j,j'}(n_1, n_2, ..., n_K; t) = P\{&N_1(t) = n_1, N_2(t) = n_2, ..., N_K(t) = n_K; \\ &J(t) = j' \mid N_1(0) = 0, N_2(0) = 0, ..., N_K(0) = 0, J(0) = j\} \end{aligned}$$

and

$$P(n_1, n_2, ..., n_K; t) = (P_{j,j'}(n_1, n_2, ..., n_K; t))_{1 \leqslant j,j' \leqslant m}.$$

The matrix sequence $\{P(n_1, n_2, ..., n_K; t)\}$ satisfies the forward Chapman-Kolmogorov differential equations

$$P'(0, 0, ..., 0; t) = P(0, 0, ..., 0; t)C$$

and for $n_i \geqslant 1$ and $1 \leqslant i \leqslant K$,

$$\begin{aligned} P'(n_1, n_2, ..., n_K; t) = &P(n_1, n_2, ..., n_K; t)C \\ &+ \sum_{i=1}^{K} P(n_1, ..., n_{i-1}, n_i - 1, n_{i+1}, ..., n_K; t) D^{(i)}, \end{aligned}$$

with the initial condition

$$P(n_1, n_2, \ldots, n_K; 0) = \begin{cases} I, & n_1 = n_2 = \ldots = n_K = 0 \\ 0, & \text{otherwise;} \end{cases}$$

or the backward Chapman-Kolmogorov differential equations

$$P'(0, 0, ..., 0; t) = CP(0, 0, ..., 0; t)$$

and for $n_i \geqslant 1$ and $1 \leqslant i \leqslant K$,

$$\begin{aligned} P'(n_1, n_2, ..., n_K; t) = &CP(n_1, n_2, ..., n_K; t) \\ &+ \sum_{i=1}^{K} D^{(i)} P(n_1, ..., n_{i-1}, n_i - 1, n_{i+1}, ..., n_K; t). \end{aligned}$$

Let

$$P^*(z_1, z_2, \ldots, z_K; t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_K=0}^{\infty} z_1^{n_1} z_2^{n_2} \ldots z_K^{n_K} P(n_1, n_2, \ldots, n_K; t).$$

Then

$$P^*(z_1, z_2, \ldots, z_K; t) = \exp\left\{\left(C + \sum_{i=1}^{K} z_i D^{(i)}\right)t\right\}.$$

In what follows we provide some useful properties for the MMAP[K]:

(1) The arrival process of the ith class is an ordinary MAP with irreducible matrix descriptor $\left(C + \sum_{j \neq i} D^{(j)}, D^{(i)}\right)$.

(2) The arrivals process of classes i_1 to i_L is an MMAP[L] with irreducible matrix descriptor $\left(C + \sum_{j \neq i_1 \text{ to } i_L} D^{(j)}, D^{(i_1)}, D^{(i_2)}, \ldots, D^{(i_L)}\right)$.

(3) The superposition of the two MMAP[K] and MMAP[L] is still an MMAP[M], where the number M depends on the practical system.

1.5.8 The Multivariate Batch Markovian Arrival Process

An MBMAP[K] $\{(N(t), J(t)) : t \geqslant 0\}$ is constructed by K different classes of arrivals with stochastic batch size, where $N(t) = (N_1(t), N_2(t), ..., N_K(t))$, $N_i(t)$ and $J(t)$ denote the number of the ith class of arrivals in the time interval $[0, t)$ for $1 \leqslant i \leqslant K$, and the phase of the Markovian enviornment at time t, respectively.

Let $\mathcal{N}_+^K$ be a set of non-zero K-tuples of nonnegative integers. A generic element of $\mathcal{N}_+^K$ is denoted as $n = (n_1, n_2, ..., n_K)$. An MBMAP[K] can be described as an irreducible matrix descriptor $(C, D_n, n \in \mathcal{N}_+^K)$ of size m, where C has negative diagonal elements and nonnegative off-diagonal elements; the matrix $D_n \gneqq 0$ is the rate matrix of arrivals of batch size n for $n \in \mathcal{N}_+^K$; and the matrix $C + \sum_{n \in \mathcal{N}_+^K} D_n$ is the infinitesimal generator of an irreducible Markov chain with m states.

We write

$$C^{(k)} = C + \sum_{n_k = 0,\, n \in \mathcal{N}_+^K} D_n$$

and

$$D_r^{(k)} = \sum_{n_k = r,\, n \in \mathcal{N}_+^K} D_n.$$

Then the arrival process of the kth class is a BMAP with irreducible matrix descriptor $(C^{(k)}, D_1^{(k)}, D_2^{(k)}, \ldots)$. Let θ be the stationary probability vector of the Markov chain $C + \sum_{n \in \mathcal{N}_+^K} D_n$. The stationary arrival rate of the kth class of arrivals is given by $\lambda_k = \theta \sum_{r=1}^{\infty} r D_r^{(k)} e$. Let

$$D_r = \sum_{\substack{n_1+n_2+\ldots+n_K=r, \\ n \in \mathcal{N}_+^K}} D_n.$$

Then the counting process $\{(N(t)e, J(t))\}$ is a BMAP with irreducible matrix descriptor $(C, D_1, D_2, \ldots)$ and the stationary arrival rate of the BMAP is given by $\lambda = \sum_{i=1}^{K} \lambda_i$. Under the stable conditions, the probability that an arbitrary arrival is of the kth class is given by λ_k / λ for $1 \leqslant k \leqslant K$.

We write

$$P_{j,j'}(\boldsymbol{n}; t) = P\{N(t) = \boldsymbol{n}; J(t) = j' \mid N(0) = \mathbf{0}, J(0) = j\}$$

and

$$P(\boldsymbol{n}; t) = (P_{j,j'}(\boldsymbol{n}; t))_{1 \leqslant j, j' \leqslant m},$$

where $\boldsymbol{n} = (n_1, n_2, \ldots, n_K)$
The matrix sequence $\{P(\boldsymbol{n}; t)\}$ satisfies the forward Chapman-Kolmogorov differential equations

$$P'(\mathbf{0}; t) = P(\mathbf{0}; t)C$$

and for $\boldsymbol{n} \in \mathcal{N}_+^K$,

$$P'(\boldsymbol{n}; t) = P(\boldsymbol{n}; t)C + \sum_{\boldsymbol{h} \leqslant \boldsymbol{n}, \boldsymbol{h} \in \mathcal{N}_+^K} P(\boldsymbol{n} - \boldsymbol{h}; t) D_h;$$

or the backward Chapman-Kolmogorov differential equations

$$P'(\mathbf{0}; t) = CP(\mathbf{0}; t)$$

and for $\boldsymbol{n} \in \mathcal{N}_+^K$,

$$P'(\boldsymbol{n}; t) = CP(\boldsymbol{n}; t) + \sum_{\boldsymbol{h} \leqslant \boldsymbol{n}, \boldsymbol{h} \in \mathcal{N}_+^K} D_h P(\boldsymbol{n} - \boldsymbol{h}; t).$$

Let $z = (z_1, z_2, \ldots, z_K)$, $z^{\boldsymbol{n}} = z_1^{n_1} z_2^{n_2} \ldots z_K^{n_K}$ and $P^*(z; t) = \sum_{\boldsymbol{n} \in \mathcal{N}_+^K} z^{\boldsymbol{n}} p(\boldsymbol{n}; t)$.

Then

$$P^*(z;t) = \exp\left\{\left(C + \sum_{\boldsymbol{n}\in\mathcal{N}_+^K} z^{\boldsymbol{n}} D_{\boldsymbol{n}}\right)t\right\}.$$

For the stationary version of the MBMAP[K], we list the useful results as follows:

(1) Backward looking: The probability that the last arrival before an arbitrary time t has the batch size $\boldsymbol{n}$ is given by $\theta D_{\boldsymbol{n}}(-C)^{-1}e$ for $\boldsymbol{n}\in\mathcal{N}_+^K$.

(2) Forward looking: The probability that the last arrival after an arbitrary time t has the batch size $\boldsymbol{n}$ is given by $\theta(-C)^{-1}D_{\boldsymbol{n}}e$ for $\boldsymbol{n}\in\mathcal{N}_+^K$.

(3) At the arrival: The probability that an arbitrary arrival has the batch size $\boldsymbol{n}$ is given by $\theta D_{\boldsymbol{n}}e/\lambda$ for $\boldsymbol{n}\in\mathcal{N}_+^K$.

We now extend the above probabilities to that of r consecutive arrivals with batch sizes $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_r$, respectively. We list the useful results as follows:

(1) Backward looking: The probability that the last arrival before an arbitrary time t has the batch sizes $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_r$ is given by

$$\theta D_{\boldsymbol{n}_1}(-C)^{-1}D_{\boldsymbol{n}_2}(-C)^{-1}\ldots D_{\boldsymbol{n}_r}(-C)^{-1}e.$$

(2) Forward looking: The probability that the last arrival after an arbitrary time t has the batch sizes $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_r$ is given by

$$\theta(-C)^{-1}D_{\boldsymbol{n}_1}(-C)^{-1}D_{\boldsymbol{n}_2}\ldots(-C)^{-1}D_{\boldsymbol{n}_r}e.$$

(3) At the arrival: The probability that an arbitrary arrival has the batch sizes $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_r$ is given by $\theta D_{\boldsymbol{n}_1}D_{\boldsymbol{n}_2}\ldots D_{\boldsymbol{n}_r}e/\lambda$.

1.6 Matrix-Exponential Distribution

In this section, we discuss a useful distribution: Matrix-exponential distribution, which is a formal generalization of the PH distribution. We analyze some useful properties of this distribution and its renewal process.

Now, we extend the PH distribution with probability density function

$$f(x) = \hat{\alpha}\exp\{\hat{T}x\}\hat{T}^0$$

to a more general distribution: Matrix-exponential distribution with probability density function

$$b(x) = \alpha\exp\{Tx\}s.$$

Thus, a matrix-exponential distribution can be determined by a triple (α, T, s) of size m, denoted as MED (α, T, s). The matrix-exponential distribution satisfies two basic conditions: $\alpha s \geqslant 0$ and $\alpha T^{-1}s = 1$. Specifically, if α is a probability vector, T is an infinitesimal generator of the Markov chain with an absorbing state and $s=-Te,$ then the MED (α, T, s) is a PH distribution.

It is easy to check that the MED (α, T, s) has the probability distribution function

$$B(x)=1+\alpha \exp\{Tx\}T^{-1}s,$$

the Laplace transform

$$\tilde{b}(s)=\alpha(sI-T)^{-1}s,$$

which is rational, and the nth moment

$$\mu_n=(-1)^{n+1}n!\alpha T^{-(n+1)}s.$$

Proposition 1.8 *The following statements are equivalent:*

(1) $b(x)=\alpha \exp\{Tx\}s,$

(2) $\tilde{b}(s)$ *is rational, and*

(3) $b(x)=\sum_{j=0}^{n} c_j x^j \exp\{\eta_j x\}.$

The following theorem provides a useful relation between the matrix-exponential distribution and the probability distribution with rational Laplace transform.

Theorem 1.3 *The Laplace transform of a matrix-exponential distribution can be written as*

$$\tilde{b}(s)=\frac{b_1+b_2 s+\ldots+b_n s^{n-1}}{s^n+a_1 s^{n-1}+\ldots+a_{n-1}s+a_n}$$

for some $n \geqslant 1$ *and some constants* $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$. *At the same time, the matrix-exponential distribution has the matrix descriptor* (α, T, s) *of size n, where*

$$\alpha=(b_1, b_2, \ldots, b_n),$$

$$T=\begin{pmatrix} & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{pmatrix}$$

and

$$s=(0,0,0,0,0,1)^{\mathrm{T}}.$$

Let the random variable X be matrix-exponential with matrix descriptor $(\alpha, T, s),$ and

$$F_y(x)=P\{X \leqslant x+y \mid X>y\}.$$

Then $F_y(x)$ is also matrix-exponential with matrix descriptor (α_y, T, s), where

$$\alpha_y = \frac{\alpha \exp\{Ty\}}{\alpha \exp\{Ty\} T^{-1} s}.$$

If A is an invertible matrix of size m, then

$$\begin{aligned}\text{MED}\,(\alpha, T, s) &\sim \text{MED}\,(\alpha A, A^{-1}TA, A^{-1}s) \\ &\sim \text{MED}\,(\alpha A^{-1}, ATA^{-1}, As).\end{aligned} \tag{1.52}$$

For an arbitrary MED (β, R, r) of size m, it is necessary to give a closed PH representation (α, T, T^0). To do this, it follows from Eq. (1.52) that MED $(\beta, R, r) \sim (\beta A, A^{-1}RA, A^{-1}r)$. Let $\alpha = \beta A, T = A^{-1}RA$ and $T^0 = -Te = A^{-1}r$. Therefore, $-A^{-1}RAe = A^{-1}r$, which leads to $Ae = -R^{-1}r$. To determine the matrix $A = (a_{i,j})$, we write $(r_1, r_2, ..., r_m) = -R^{-1}r$. Using $Ae = -R^{-1}r$, we may take a special matrix A as follows:

(1) If $r_i \neq 0$, then we take

$$a_{i,j} = \begin{cases} r_i, & i = j, \\ 0, & i \neq j. \end{cases}$$

(2) If $r_i = 0$, then we take

$$a_{i,j} = \begin{cases} 1, & i = j, \\ -1, & i = j-1, \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to check that the matrix A is invertible and $Ae = -R^{-1}r$.

Now, we study the minimal representation of the matrix-exponential distribution. Let

$$R_p = \text{span}\,\{s, Ts, T^2 s, ..., T^p s\},$$
$$R_\infty = \text{span}\,\{T^n s : n = 0, 1, 2, ...\},$$
$$R_e = \text{span}\,\{\exp\{Tx\} s : x \geqslant 0\};$$

and

$$L_p = \text{span}\,\{\alpha, \alpha T, \alpha T^2, ..., \alpha T^p\},$$
$$L_\infty = \text{span}\,\{\alpha T^n : n = 0, 1, 2, ...\},$$
$$L_e = \text{span}\,\{\alpha \exp\{Tx\} : x \geqslant 0\}.$$

It is easy to check that $R_p = R_\infty = R_e$ and $L_p = L_\infty = L_e$.

Let (α, T, s) and (β, T, s) be two representations of a matrix-exponential distribution. If dim $(R_p) = p$, then $\alpha = \beta$. On the other hand, let (α, T, s) and (α, T, t) be two representations of a matrix-exponential distribution. If dim $(L_p) = p$, then $s = t$.

The following theorem provides a necessary and sufficient condition under which the representation (α, T, s) of a matrix-exponential distribution is minimal.

Theorem 1.4 *A representation* (α, T, s) *of a matrix-exponential distribution of size m is minimal if and only if dim* $(R_m) = \dim (L_m) = m$.

For a representation (α, T, s) of a matrix-exponential distribution, we can often construct its minimal representation. The steps for finding the minimal representation are given in the following theorem.

Theorem 1.5 *Suppose* (α, T, s) *of size m is a matrix representation of a matrix-exponential distribution.*

(1) *Let* $p = \dim (R_m) < m$. *We denote by* $x_1, x_2, \ldots, x_p$ *a basis for the linear space* R_m, *and define an* $m \times p$ *matrix* $A = (x_1, x_2, \ldots, x_p)$ *and the change of basis transformation B of size* $p \times m$ *such that* $Bx_i = e_i$ *for* $1 \leqslant i \leqslant p$, *where* $e_1, e_2, \ldots, e_p$ *denote the Euclidian basis vectors. Then* $(\alpha A, BTA, Bs)$ *is a p-dimensional representation of the matrix-exponential distribution.*

(2) *Let* $q = \dim (L_m) < m$. *We denote by* $y_1, y_2, \ldots, y_p$ *a basis for the linear space* L_m, *and define a* $p \times m$ *matrix* $C = (y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, \ldots, y_p^{\mathrm{T}})^{\mathrm{T}}$ *and the change of basis transformation D of size* $p \times m$ *such that* $x_i D = e_i$ *for* $1 \leqslant i \leqslant p$, *where* $e_1, e_2, \ldots, e_p$ *denote the Euclidian basis vectors. Then* $(\alpha D, CTD, Cs)$ *is a p-dimensional representation of the matrix-exponential distribution.*

Now, we consider a renewal process where the interarrival distribution $B(x)$ is matrix-exponential with matrix representation (α, T, s).

Let the renewal density function be $u(x) = \sum_{n=0}^{\infty} b^{*n}(x)$. Then

$$u(x) = \alpha \exp\{(T + s\alpha)x\}s.$$

Furthermore, we consider a delayed renewal process with $b_0(x) = \beta \exp\{Tx\}s$ and $b(x) = \alpha \exp\{Tx\}s$. Then the renewal density function is given by

$$u_0(x) = \beta \exp\{(T + s\alpha)x\}s.$$

Let $\xi(t)$ be the excess life time at time t. Then $\xi(t)$ is matrix-exponential with matrix representation (β_t, T, s), where $\beta_t = \alpha \exp\{(T + s\alpha)t\}$; or another matrix representation (α, T, r_t), where $r_t = \exp\{(T + s\alpha)t\}s$. Furthermore, for a delayed renewal process with $b_0(x) = \beta \exp\{Tx\}s$ and $b(x) = \alpha \exp\{Tx\}s$, $\xi(t)$ is also matrix-exponential with matrix representation (γ_t, T, s), where $\gamma_t = \beta \exp\{(T + s\alpha)t\}$.

1.7 Notes in the Literature

For Markov chains, we may refer to Kemeny and Snell [64], Kemeny, Snell and Knapp [65], Anderson [7], Meyn and Tweedie [84], Karlin and Taylor [61, 62], Kijima [66], Taylor and Karlin [121]. The QBD process is an important example in Markov chains, and provides a useful mathematical tool for studying stochastic models such as queueing systems, manufacturing systems, communication networks and transportation systems. Readers may refer to Chapter 3 of Neuts [92], Hajek [40], Gaver, Jacobs and Latouche [39], Latouche and Ramaswami [69], Neuts [96], Naoumov [86, 87], Bright and Taylor [29, 30], Ramaswami [110], Ramaswami and Taylor [111, 112], Meini [83], Kijima and Makimoto [68], Latouche and Ramaswami [70], He, Meini and Rhee [43], Li and Cao [73] and references therein. Specifically, some stochastic models have motivated the need for studying a level-dependent QBD process. Important examples include the retrial queues, see, for example, Artalejo [9], Anisimov and Artalejo [8], Breuer, Dudin and Klimenok [28] and Artalejo and Gómez-Corral [10].

The PH distribution plays an important role in numerical computation of stochastic models. Since the introduction of the PH distribution by Neuts [89], research on the PH distribution has been greatly motivated by many practical applications. Chapter 2 in Neuts [92] provided a detailed analysis for the PH distribution with many useful properties. Subsequent papers have been published on this theme from some different points of view. The properties of the PH distribution are studied by, such as, Assafe and Levikson [17], Neuts [93], Assaf and Langberg [15], Aldous and Shepp [2], Sengupta [118, 119], Bean and Nielsen [18], Hipp [51]. For the structured representation, readers may refer to Commault and Chemla [33, 34], Mocana and Commault [85], Commault and Mocanu [35, 36], He and Zhang [45 – 47]. For the phase structure and the minimal irreducible representation of the PH distribution, readers may refer to O'Cinnerde [102 – 106], Maier [80, 81], Maier and O'Cinnerde [82]. Specifically, O'Cinnerde [107] provided an overview on this direction. For the PH approximation, readers may refer to Altiok [5, 4], Bobbio and Telek [24], Bobbio, Horváth and Telek [22, 23], Bobbio, Horváth, Scarpa and Telek [21], Horváth and Telek [52], He, Wu and Li [50], Li, Lin and Li [78]. The statistical analysis of PH distributions has been given by some researchers, such as, Johnson [55 – 58], Bobbio and Cumani [20], Olsson [108], Rydén [117], Asmussen [12], Faddy [38]. The PH distribution has been extended to several useful classes as follows. Shanthikumar [120], Ahn and Ramaswami [1] analyzed a bilateral PH distribution, and Li, Wang and Zhou [73] studied a symmetric PH distributions. For the multivariate PH distribution, readers may refer to Assaf, Langberg, Savits and Shaked [16], Kulkarni [67], Li [72], Cai and Li [31], Asimit and Jones [11].

When the interarrival time is of phase type, Neuts [90] studied a PH-renewal process. Neuts and Latouche [98] discussed the superposition of two PH-renewal processes. Kao and Smith [59, 60] analyzed excess-, current-, and total-life distributions of phase-type renewal processes. Neuts [91] extended the PH renewal process to a versatile Markovian point process, which is called batch Markovian arrival process, e.g., see Neuts [94]. Lucantoni [79] provided a simple representation for the BMAP. Narayana and Neuts [88] considered the first two moment matrices of the counts for the MAP. Neuts [95] analyzed the burstiness of the MAP. For the

MAP and BMAP, readers may refer to some crucial works such as Liu and Neuts [77], Neuts, Liu and Narayana [99], Neuts [97], Johnson and Narayana [54], Nishimura and Sato [101], Johnson, Liu and Narayana [53], Breuer [27], Andersen, Neuts and Nielsen [6], Kawanishi [63], Nielsen, Nilsson, Thygesen and Beyer [100], Rydén [113 – 115]. The MAP describes bursty traffic in modern communication networks. Readers may refer to Chapter 5 in Neuts [94], Ramaswami [109], Lucantoni [79], Neuts [96], Alfa [3], Latouche and Ramaswami [70], Lee and Jeon [71], Chakravarthy [32], Li and Zhao [76, 75]. To describe multivariate and inter-dependent arrival processes, He and Neuts [44] provided a Markovian arrival process with marked transitions (MMAP[K]). Readers may refer to applications of the MMAP[K] such as He [42, 43].

Asmussen and Bladt [13] discussed matrix-exponential distributions. Asmussen and Perry [14] provided an operational calculus for matrix-exponential distributions. Readers may refer to Bladt [19], Fackrell [37], van de Liefvoort and Heindl [122], He and Zhang [47 – 49].

In this chapter, we refer to more references such as Kemeny and Snell [64], Kemeny, Snell and Knapp [65], Anderson [7], [92, 94], Li and Cao [73], Lucantoni [79], Assaf, Langberg, Savits and Shaked [16], He and Neuts [44] and Asmussen and Bladt [13].

Problems

1.1 Consider an $M/M/c$ queue with server multiple vacations. In the following two different cases, please write the infinitesimal generators of the corresponding systems.

(1) The vacation process of the c servers is Synchronous Start and Synchronous End.

(2) The vacation process of the c servers is Independent Start and Independent End.

1.2 Consider a two-node closed queueing network depicted in Fig. 1.10 There are N customers in the system, the service times of servers 1 and 2 are i.d.d. and

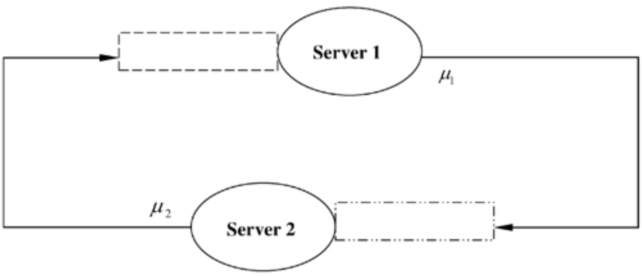

Figure 1.10 A closed queueing network with server vacations

are exponentially distributed with service rates μ_1 and μ_2, respectively. If there is no customer in server 1 or 2, then the server enters a vacation state with multiple vacation discipline. The vacation times of servers 1 and 2 are exponentially distributed with vacation rates γ_1 and γ_2, respectively. Please write the infinitesimal generators of the queueing system.

1.3 Consider an $M/M/c$ queue with c repairable servers and N repairmen for $2 \leqslant N \leqslant c$. The parameters and relative assumptions are introduced in Section 1.2 for "A queue with repairable servers". Please write the infinitesimal generators of the queueing system.

1.4 In Problem 1.3, if the c repairable servers may be different with parameters μ_i, α_i and β_i for $1 \leqslant i \leqslant c$, please write the infinitesimal generators of the corresponding systems.

1.5 Consider a bound call center with two classes A and B of customers and three different groups of agents, as depicted in Fig. 1.11. The A- and B-classes of customers arrive at the system according to Poisson processes with arrival rates λ_1 and λ_2, respectively. The sizes of the A- and B-queues are M and N, respectively. If the A- or B-queue is fully loaded, the arriving customer has to be lost immediately. Once the waiting time of a customer exceeds the patient waiting time, then the arriving customer has to be lost immediately. The patient waiting times of the A- and B-customers are exponentially distributed with rates α_1 and α_2, respectively. The service times of A-specialist, B-specialist and Generalist are all i.i.d. and are exponentially distributed with service rates μ_1, μ_2 and μ_0, respectively. Please write the infinitesimal generators of the queueing system.

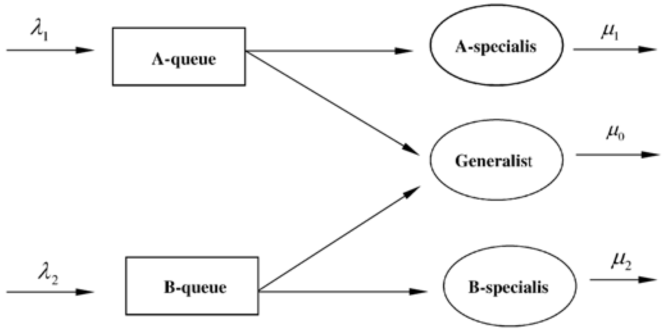

Figure 1.11 A call center with two classes of customers

1.6 In Fig. 1.12, the arrival of repaired locomotives is a Poisson process with arrival rate λ. Once the locomotive enters the repair shop, it is immediately disconnected into m different types of parts or it has to wait for a disconnected room on a queueing line. The sub-warehouse size of the ith type of parts, including the failed or repaired ones, is allocated to be N_i, thus $N = \sum_{i=1}^{m} N_i$ is the size of the repair shop. The repair time of each ith type of part is exponentially distributed with repair rate β_i. We assume that the parts are as good as new after repair. When the disconnected process of the arriving locomotive is completed, each corresponding part is taken from its sub-warehouse, if it exists, and is installed in the suitable position of the locomotive. After being installed, the locomotive leaves the repair shop immediately. We assume that all the other time, except for the repair times, are so short that they can be ignored as zero. Please write the infinitesimal generators of the corresponding system.

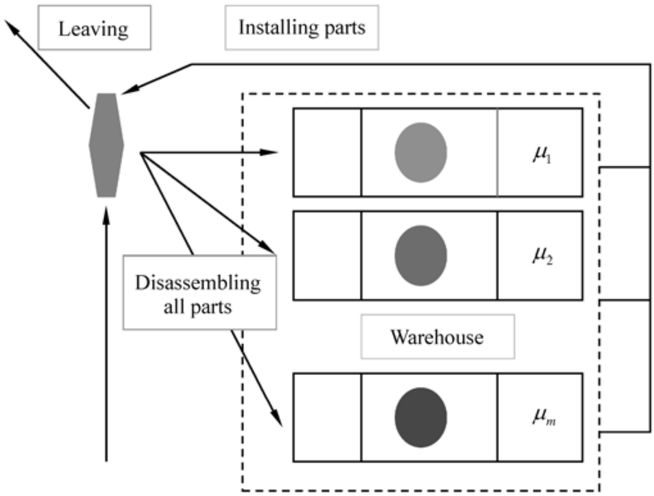

Figure 1.12 A locomotive repair system

1.7 If a continuous-time QBD process Q with either infinitelymany levels or finitely mang levels is transient, please use the UL-type RG-factorization to compute the inverse of the infinitesimal generator Q.

1.8 Prove that (1) $g_1 g_2 = g_2 g_1$, and (2) $T^{-1}(g_1 + g_2)T = g_1 + g_2$.

1.9 Let Γ_1 and Γ_2 be two nonempty stochastically closed sets of the state space Ω such that $\Gamma_1 \cap \Gamma_2 = \{m+1\}$. We write $E_0 = \Gamma_1^c \cap \Gamma_2^c, E_1 = \Gamma_1^c \cap \Gamma_2$ and $E_2 = \Gamma_1 \cap \Gamma_2^c$.

(1) Prove that

$$T = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{ccc} E_0 & E_1 & E_2 \end{array} \\ \begin{pmatrix} T_{0,0} & T_{0,1} & T_{0,2} \\ & T_{1,1} & \\ & & T_{2,2} \end{pmatrix} \end{array}.$$

(2) Using the above block structure, please compute $\tilde{F}(x_1, x_2), f(x_1, x_2)$, $\tilde{f}(s_1, s_2)$ and $E[Y_1 \quad Y_2]$.

1.10 For an $M/PH/1/c$ queue, prove that its departure point process is a MAP and write the associated matrix descriptor (C, D). Provide some numerical examples to indicate effects of the service density, $\rho = \lambda / \mu$, on the behavior of the MAP.
1.11 For a $PH/PH/1$ queue, use a finite-state MAP to provide an approximate algorithm for computing the mean and variance of its departure point process. Provide some numerical examples to indicate the effects of the service density, $\rho = \lambda / \mu$, on the behavior of the departure point process.
1.12 Provide concrete examples for applying the multivariate MAP to describe some multivariate dependent arrival processes in practical areas.

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2 Block-Structured Markov Chains

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Abstract In this chapter, the censoring technique is applied to be able to deal with any irreducible block-structured Markov chain, which is either discrete-time or continuous-time. The *R*-, *U*- and *G*-measures are iteratively defined from two different censored directions: UL-type and LU-type. An important censoring invariance for the *R*- and *G*-measures is obtained. Using the censoring invariance, the Wiener-Hopf equations are derived, and then the UL- and UL-types of *RG*-factorizations are given. The stationary probability vector is given an *R*-measure expression; while the transient probability can be computed by means of the *R*-, *U*- and *G*-measures. Finally, the *A*- and *B*-measures are proposed in order to discuss the state classification of the block-structured Markov chain.

Keywords stochastic model, block-structured Markov chain, the censoring technique, *R*-measure, *U*-measure, *G*-measure, *A*-measure, *B*-measure, censoring invariance, Wiener-Hopf equation, *RG*-factorization, state classification, stationary probability vector, transient probability, the first passage time.

In this chapter, the censoring technique is applied to be able to deal with any irreducible block-structured Markov chain with either finitely-many levels or infinitely-many levels, which is either discrete-time or continuous-time. Three probabilistic measures: *R*-, *U*-and *G*-measures, are defined from two different censored directions: UL-type and LU-type, and an important censoring invariance for the *R*-and *G*-measures is obtained. Based on the censoring invariance, the Wiener-Hopf equations are derived, and the UL-and UL-types of *RG*-factorizations for the transition matrix are given. Furthermore, the *A*- and *B*-measures are proposed in order to discuss the state classification of the block-structured Markov chain. This chapter systemically develops the decomposition theory: the *RG*-factorizations, for any irreducible Markov chains. Based on the *RG*-factorizations, effective algorithms can be designed under a unified, constructive computational framework in order to deal with performance computation and system decision for practical stochastic models in many applied areas.

This chapter is organized as follows. Section 2.1 applies the censoring technique to deal with any irreducible discrete-time block-structured Markov chain. Some important properties of the censored Markov chain are given. For the discrete-time block-structured Markov chain, Sections 2.2 and 2.3 derive the UL-and LU-types of *RG*-factorizations, respectively. Section 2.4 provides an *R*-measure expression for the stationary probability vector of any positive recurrent block-structured Markov chain. Note that the *R*-measure expression is more effective than that in the literature. Specifically, those crucial formulae given in Neuts [26, 27] are simply re-derived by means of the *R*-measure expression. Section 2.5 defines *A*- and *B*-measures for any irreducible block-structured Markov chain, and constructs expressions for the *A*- and *B*-measures by means of the *R*-, *U*- and *G*-measures, respectively. Based on the *A*- and *B*-measures, necessary and sufficient conditions for the state classification of the block-structured Markov chain are obtained. Section 2.6 discusses the block-structured Markov chains with finitely-many levels. Section 2.7 gives the UL-and LU-types of *RG*-factorizations for any irreducible continuous-time block-structured Markov chain, and some useful results are summarized simply. Finally, Section 2.8 summarizes the references related to the results of this chapter.

2.1 The Censoring Chains

In this section, the censoring technique is applied to be able to deal with any irreducible discrete-time block-structured Markov chain. Also, some important properties for the censored Markov chains are given.

We consider an irreducible discrete-time block-structured Markov chain $\{X_n, n\geqslant 0\}$ on the state space $\Omega=\{(k,j): k\geqslant 0, 1\leqslant j\leqslant m_k\}$ whose transition probability matrix is given by

$$P=\begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & \cdots \\ P_{2,0} & P_{2,1} & P_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \tag{2.1}$$

where $P_{i,j}$ is a matrix of size $m_i\times m_j$ whose (r,r')th entry is given by

$$(P_{i,j})_{r,r'}=P\{X_{n+1}=(j,r')\,|\,X_n=(i,r)\}.$$

In this chapter, we always assume that the Markov chain P is irreducible and stochastic (or substochastic). Note that the stochastics and substochastics are denoted as $Pe=e$ and $Pe\lneqq e$, respectively, where e is a column vector of ones with suitable size. Under the block structure, the state space Ω is partitioned as $\Omega = \bigcup_{i=0}^{\infty} L_i$ with $L_i = \{(i,1),(i,2),\ldots,(i,m_i)\}$. For state (i, k), i is called the *level variable* and k the *phase variable*. We also write $L_{\leqslant k} = \bigcup_{i=0}^{k} L_i$ for the set of all the states in the levels up to k, and $L_{\geqslant k}$ for the complement of $L_{\leqslant(k-1)}$.

Now, we define a censored chain for an irreducible Markov chain whose transition probability matrix consists of scalar entries. We then treat a block-structured Markov chain as a special case.

Definition 2.1 *Suppose that $\{X_n, n\geqslant 0\}$ is an irreducible Markov chain on the state space $\Omega = \{0,1,2,\ldots\}$. Let E be a non-empty subset of Ω. If the successive visits of X_n to the subset E take place at the n_k th step of state transition. We write $X_k^E = X_{n_k}$ for $k\geqslant 1$. Then the sequence $\{X_k^E, k\geqslant 1\}$ is called the censored chain with censoring set E.*

For convenience of description, we write $P^{[\leqslant n]}$ for the censored transition probability matrix P^E if the censored set $E = L_{\leqslant n}$, in particular, $P^{[<+\infty]} = P$ and $P^{[0]} = P^{[\leqslant 0]}$. Similarly, $P^{[\geqslant n]}$ is the censored transition probability matrix with the censored set $E = L_{\geqslant n}$, specifically, $P^{[\geqslant 0]} = P$.

Let $E^c = \Omega - E$. According to the subsets E and E^c, the transition probability matrix P is partitioned as

$$P = \begin{array}{cc} & \begin{array}{cc} E & E^c \end{array} \\ \begin{array}{c} E \\ E^c \end{array} & \begin{pmatrix} T & U \\ V & W \end{pmatrix} \end{array}. \tag{2.2}$$

Lemma 2.1 *If P is irreducible and $V \gneq 0$, then each element of $\widehat{W} = \sum_{n=0}^{\infty} W^n$ is finite.*

Proof If P is irreducible, then W is strictly substochastic due to $V \gneq 0$. The Markov chain W may be regarded as having absorbing state set E so that the expected number of visits to each state in E^c is finite. Hence

$$\widehat{W} = \sum_{n=0}^{\infty} W^n < +\infty,$$

where $\widehat{W}$ is the minimal nonnegative inverse of $I - W$, denoted as $(I-W)_{\min}^{-1}$. This completes the proof.

The matrix $\widehat{W}$ is referred to as the fundamental matrix of W. Similarly, if $U \neq 0$, then $\widehat{T} = \sum_{n=0}^{\infty} T^n$, and specifically, $\widehat{T} = (I-T)^{-1}$ when the order of the matrix T is finite.

In the following, we show that the censored chain $\{X_k^E, k\geqslant 1\}$ is also a Markov chain, and derive its transition probability matrix.

Theorem 2.1 *The censored chain* $\{X_k^E, k \geqslant 1\}$ *is a Markov chain whose transition probability matrix is given by*

$$P^E = T + U\widehat{W}V. \tag{2.3}$$

Proof It follows from the Markov property of P that the (i, j)th entry of the transition probability matrix of the censored chain $\{X_k^E, k \geqslant 1\}$ is given by

$$P_{i,j}^E = P\{X_{n+1}^E = j \mid X_n^E = i\} = P\{X_1^E = j \mid X_0^E = i\}.$$

Thus, the censored chain $\{X_k^E, k \geqslant 1\}$ is obviously a Markov chain.

In what follows we explicitly express P^E in terms of the original transition probability matrix. To do this, we consider the following two possible cases:

Case Ⅰ $n_1 = 1$. In this case, $i, j \in E, X_1^E = X_1$ and

$$P\{X_1^E = j \mid X_0^E = i\} = T_{i,j}. \tag{2.4}$$

Case Ⅱ $n_1 = k$ for $k \geqslant 2$. In this case, $i, j \in E, X_1^E = X_k$ and

$$\begin{aligned} &P\{X_1^E = j \mid X_0^E = i\} = P\{X_k = j, X_l \notin E \\ &\text{for } l = 1, 2, ..., k-1 \mid X_0 = i\} = (UW^{k-2}V)_{i,j}. \end{aligned} \tag{2.5}$$

It follows from Eq. (2.4) and Eq. (2.5) that

$$\begin{aligned} P\left\{X_1^E = j \mid X_0^E = i\right\} &= T_{i,j} + \left(\sum_{k=2}^{\infty} UW^{k-2}V\right)_{i,j} \\ &= T_{i,j} + (U\widehat{W}V)_{i,j}. \end{aligned}$$

This completes the proof.

Remark 2.1 *The censored chain* $\{X_k^{E^c}, k \geqslant 1\}$ *is a Markov chain whose transition probability matrix is given by*

$$P^{E^c} = W + V\widehat{T}U.$$

Note that the two censored Markov chains $\{X_k^E, k \geqslant 1\}$ *and* $\{X_k^{E^c}, k \geqslant 1\}$ *have different utilities, which can lead to two different types of RG-factorizations later.*

Based on the above discussion on the censored chains, a probabilistic interpretation for each component of the matrix P^E is listed as follows.

(1) $\widehat{W}_{i,j}$ is the expected number of visits to state $j \in E^c$ before entering E, given that the Markov chain starts in state $i \in E^c$.

(2) $(U\widehat{W})_{i,j}$ is the expected number of visits to state $j \in E^c$ before returning to E, given that the Markov chain starts in state $i \in E$.

(3) $(\widehat{W}V)_{i,j}$ is the probability that upon entering E the first state visited is $j \in E$, given that the Markov chain starts in state $i \in E^c$.

(4) $(U\widehat{W}V)_{i,j}$ is the probability that upon returning to E the first state visited is $j \in E$, given that the Markov chain starts in state $i \in E$.

Define the z-transformation for the censored Markov chain as

$$(P^E)^*(z) = \left((P_{i,j}^E)^*(z)\right), \ i, j \in E,$$

where

$$(P_{i,j}^E)^*(z) = \sum_{k=0}^{\infty} z^k P\{X_k^E = j \mid X_0^E = i\}.$$

The following corollary provides a useful result for studying the censored Markov chain, the proof of which is obvious from Eq. (2.4) and Eq. (2.5).

Corollary 2.1

$$(P^E)^*(z) = zT + z^2 U \cdot \widehat{zW} \cdot V,$$

where

$$\widehat{zW} = \sum_{n=0}^{\infty} z^n W^n.$$

Based on Definition 2.1, we can summarize the following five useful properties for the censored Markov chains. These properties can be easily proved by means of the sample path analysis for the censored Markov chains.

Property 2.1 For $E_1 \subset E_2$, $P^{E_1} = (P^{E_2})^{E_1}$.

Property 2.2 P is irreducible if and only if P^E is irreducible for every subset E of Ω.

Property 2.3 P is recurrent if and only if P^E is recurrent for every subset $E \subset \Omega$.

Property 2.4 P is transient if and only if P^E is transient for every subset $E \subset \Omega$.

Property 2.5 Suppose P is irreducible.

(1) P is recurrent if and only if P^E is recurrent for some subsets $E \subset \Omega$. Specifically, if P is recurrent, then $P^{[\leqslant n]}$ is positive recurrent for each $n \geqslant 0$.

(2) P is transient if and only if P^E is transient for some subsets $E \subset \Omega$. Specifically, if P is transient, then $P^{[\leqslant n]}$ contains at least an absorbing state for each $n \geqslant 0$.

2.2 The UL-type *RG*-Factorization

In this section, three UL-type probabilistic measures: the R-, U- and G-measures, are defined by means of the censored chain, an important censoring invariance for the R- and G-measures is given, and a UL-type RG-factorization for the transition probability matrix is derived.

We first define the R- and G-measures for the Markov chain P given in Eq. (2.1). Note that the R- and G-measures are crucial for studying block-structured Markov chains.

For $0 \leqslant i < j$, $R_{i,j}(k)$ is an $m_i \times m_j$ matrix whose (r,r')th entry $(R_{i,j}(k))_{r,r'}$ is the probability that starting in state (i, r), the Markov chain makes its kth transition for a visit into state (j,r') without visiting any states in $L_{\leqslant(j-1)}$ during intermediate steps; or

$$(R_{i,j}(k))_{r,r'} = P\{X_k = (j,r'), X_l \notin L_{\leqslant(j-1)} \\ \text{for } l = 1,2,\ldots,k-1 \mid X_0 = (i,r)\}.$$

Write $R_{i,j} = \sum_{k=1}^{\infty} R_{i,j}(k)$. Then the (r,r')th entry of $R_{i,j}$ is the expected number of visits to state (j,r') before hitting any states in $L_{\leqslant(j-1)}$, given that the Markov chain starts in state (i, r).

For $0 \leqslant j < i$, $G_{i,j}(k)$ is an $m_i \times m_j$ matrix whose (r,r')th entry $(G_{i,j}(k))_{r,r'}$ is the probability that starting in state (i, r), the Markov chain makes its kth transition for a visit into state (j,r') without visiting any states in $L_{\leqslant(i-1)}$ during intermediate steps; or

$$(G_{i,j}(k))_{r,r'} = P\{X_k = (j,r'), X_l \notin L_{\leqslant(i-1)} \\ \text{for } l = 1,2,\ldots,k-1 \mid X_0 = (i,r)\},$$

let us consider $G_{i,j} = \sum_{k=1}^{\infty} G_{i,j}(k)$. Then the (r,r')th entry of $G_{i,j}$ is the probability of hitting state (j,r') when the Markov chain enters $L_{\leqslant(i-1)}$ for the first time, given that the Markov chain starts in state (i, r).

The two matrix sequences $\{R_{i,j}\}$ and $\{G_{i,j}\}$ are called the R- and G-measures of the Markov chain P, respectively. We show that the R- and G-measures can be expressed in terms of the transition probability matrix. To see this, partition the transition probability matrix P according to the three subsets $L_{\leqslant(n-1)}$, L_n and $L_{\geqslant(n+1)}$ as

$$P = \begin{pmatrix} T & U_0 & U_1 \\ V_0 & T_0 & U_2 \\ V_1 & V_2 & T_1 \end{pmatrix}. \tag{2.6}$$

Let

$$W = \begin{pmatrix} T_0 & U_2 \\ V_2 & T_1 \end{pmatrix} \tag{2.7}$$

and

$$W^n = \begin{pmatrix} D_{11}(n) & D_{12}(n) \\ D_{21}(n) & D_{22}(n) \end{pmatrix}, \quad n \geqslant 0.$$

Partition $\widehat{W} = \sum_{n=0}^{\infty} W^n$ accordingly as

$$\widehat{W} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}. \tag{2.8}$$

We write

$$R_{<n} = (R_{0,n}^T, R_{1,n}^T, R_{2,n}^T, ..., R_{n-1,n}^T)^T, \tag{2.9}$$

where the superscript T stands for the transpose of a matrix, and

$$G_{<n} = (G_{n,0}, G_{n,1}, G_{n,2}, ..., G_{n,n-1}). \tag{2.10}$$

For convenience, for a matrix $B = (B_0, B_1, B_2, ...)$ or $B = (B_0^T, B_1^T, B_2^T, ...)^T$, let $(B)^{\langle i \rangle}$ denote the ith block-entry B_i of the matrix B and $(B)_{r,r'}^{\langle i \rangle}$ the (r,r')th entry in the ith block-entry of B.

Lemma 2.2 *For* $n \geqslant 1$,

$$R_{<n} = U_0 H_{11} + U_1 H_{21} \tag{2.11}$$

and

$$G_{<n} = H_{11} V_0 + H_{12} V_1. \tag{2.12}$$

Proof We only prove Eq. (2.11), while Eq. (2.12) can be proved similarly.

For $0 \leqslant i \leqslant n-1$, we consider two possible cases for $(R_{i,n}(k))_{r,r'}$ as follows:

Case Ⅰ $k=1$. In this case,

$$(R_{i,n}(k))_{r,r'} = P\{X_1 = (n,r') \mid X_0 = (i,r)\} = (U_0)_{r,r'}^{\langle i \rangle}. \tag{2.13}$$

Case Ⅱ $k \geqslant 2$. In this case,

$$\begin{aligned} (R_{i,n}(k))_{r,r'} &= P\{X_k = (n,r'), X_l \notin L_{\leqslant (j-1)} \\ &\qquad \text{for } l = 1, 2, ..., k-1 \mid X_0 = (i, r)\} \\ &= (U_0 D_{11}(k-1) + U_1 D_{21}(k-1))_{r,r'}^{\langle i \rangle}. \end{aligned} \tag{2.14}$$

Noting that $D_{11}(0) = I$ and $D_{21}(0) = 0$, it follows from Eq. (2.13) and Eq. (2.14) that

$$
\begin{aligned}
(R_{i,n})_{r,r'} &= \sum_{k=1}^{\infty}(R_{i,n}(k))_{r,r'} \\
&= (U_0)_{r,r'}^{\langle i\rangle} + \sum_{k=2}^{\infty}(U_0 D_{11}(k-1) + U_1 D_{21}(k-1))_{r,r'}^{\langle i\rangle} \\
&= \left(\sum_{k=0}^{\infty}(U_0 D_{11}(k) + U_1 D_{21}(k))\right)_{r,r'}^{\langle i\rangle} \\
&= (U_0 H_{11} + U_1 H_{21})_{r,r'}^{\langle i\rangle}.
\end{aligned}
$$

This completes the proof.

For the Markov chain P, let W_n be the southeast corner of P from level n, i.e., $W_n = (P_{i,j})_{i,j\geqslant n}$. Let $\widehat{W}_n = \sum_{k=0}^{\infty} W_n^k$, and $\widehat{W}_n^{(k,\cdot)}$ and $\widehat{W}_n^{(\cdot,l)}$ be the kth block-row and the lth block-column of $\widehat{W}_n$, respectively.

The following corollary easily follows from Lemma 2.2. Such expressions are crucial in our study later.

Corollary 2.2 *For* $0\leqslant i<j$,

$$
R_{i,j} = (P_{i,j}, P_{i,j+1}, P_{i,j+2}, \ldots)\, \widehat{W}_j^{(\cdot,1)}, \tag{2.15}
$$

and for $0\leqslant j<i$,

$$
G_{i,j} = \widehat{W}_i^{(1,\cdot)} (P_{i,j}^{\mathrm{T}}, P_{i+1,j}^{\mathrm{T}}, P_{i+2,j}^{\mathrm{T}}, \ldots)^{\mathrm{T}}. \tag{2.16}
$$

From either Lemma 2.2 or Corollary 2.2, it is clear that the R- and G-measures depend on the entries of the fundamental matrix $\widehat{W}$ given in Eq. (2.8). In this case, it is necessary to provide expression for the fundamental matrix $\widehat{W}$. The result in this lemma can be viewed as a generalized version of the expression for the minimal nonnegative inverse of a block-structured matrix of infinite size, while the proof is easy and is omitted here. For the matrix W given in Eq. (2.7), the following lemma expresses the four block-entries of the fundamental matrix $\widehat{W}$.

Lemma 2.3 *For the block structure given in Eq.* (2.8)*, we have*

$$
\begin{aligned}
H_{1,1} &= (I - T_0 - U_2 \widehat{T}_1 V_2)^{-1}, \\
H_{1,2} &= (I - T_0 - U_2 \widehat{T}_1 V_2)^{-1} U_2 \widehat{T}_1, \\
H_{2,1} &= \widehat{T}_1 V_2 (I - T_0 - U_2 \widehat{T}_1 V_2)^{-1}, \\
H_{2,2} &= \widehat{T}_1 + \widehat{T}_1 V_2 (I - T_0 - U_2 \widehat{T}_1 V_2)^{-1} U_2 \widehat{T}_1.
\end{aligned}
$$

Symmetrically,

$$H_{1,1} = \widehat{T}_0 + \widehat{T}_0 U_2 (I - T_1 - V_2 \widehat{T}_0 U_2)^{-1} V_2 \widehat{T}_0,$$
$$H_{1,2} = \widehat{T}_0 U_2 (I - T_1 - V_2 \widehat{T}_0 U_2)^{-1},$$
$$H_{2,1} = (I - T_1 - V_2 \widehat{T}_0 U_2)^{-1} V_2 \widehat{T}_0,$$
$$H_{2,2} = (I - T_1 - V_2 \widehat{T}_0 U_2)^{-1}.$$

The following theorem provides expressions for $R_{<n}$ and $G_{<n}$.

Theorem 2.2 *For* $n \geqslant 1$,

$$R_{<n} = (U_0 + U_1 \widehat{T}_1 V_2) \sum_{l=0}^{\infty} \sum_{k=0}^{l} \binom{l}{k} T_0^k (U_2 \widehat{T}_1 V_2)^{l-k}$$

and

$$G_{<n} = \sum_{l=0}^{\infty} \sum_{k=0}^{l} \binom{l}{k} T_0^k (U_0 + U_1 \widehat{T}_1 V_2)^{l-k} (V_0 + U_2 \widehat{T}_1 V_1).$$

Proof It follows from Lemmas 2.2 and 2.3 that

$$R_{<n} = (U_0 + U_1 \widehat{T}_1 V_2)(I - T_0 - U_2 \widehat{T}_1 V_2)^{-1}$$

and

$$G_{<n} = (I - T_0 - U_2 \widehat{T}_1 V_2)^{-1} (V_0 + U_2 \widehat{T}_1 V_1).$$

The inverse transform for the above two equations immediately leads to the desired result.

The following theorem provides an important property: *Censoring invariance* for the R- and G-measures. We denote by $R_{i,j}^{[\leqslant n]}$ and $G_{i,j}^{[\leqslant n]}$ the R- and G-measures of the censored Markov chain $P^{[\leqslant n]}$, respectively.

Theorem 2.3 (1) *For* $0 \leqslant i < j \leqslant n$, $R_{i,j}^{[\leqslant n]} = R_{i,j}$.

(2) *For* $0 \leqslant j < i \leqslant n$, $G_{i,j}^{[\leqslant n]} = G_{i,j}$.

Proof We only prove (1), while (2) can be proved similarly.

First, we assume that $j = n$ and P is partitioned according to the three subsets $L_{<n}$, L_n and $L_{>n}$ as in Eq. (2.6). It follows from Theorem 2.1 that

$$\begin{aligned} P^{[\leqslant n]} &= \begin{pmatrix} T & U_0 \\ V_0 & T_0 \end{pmatrix} + \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \widehat{T}_1 (V_1, V_2) \\ &= \begin{pmatrix} T + U_1 \widehat{T}_1 V_1 & U_0 + U_1 \widehat{T}_1 V_2 \\ V_0 + U_2 \widehat{T}_1 V_1 & T_0 + U_2 \widehat{T}_1 V_2 \end{pmatrix}. \end{aligned} \tag{2.17}$$

Using the definition of the R-measure, simple calculations from $P^{[\leqslant n]}$ lead to

$$\begin{aligned}R_{<n}^{[\leqslant n]} &= (U_0 + U_1\widehat{T}_1V_2)(I - T_0 - U_2\widehat{T}_1V_2)^{-1} \\ &= (U_0 + U_1\widehat{T}_1V_2)\sum_{l=0}^{\infty}[T_0 + U_2\widehat{T}_1V_2]^l \\ &= (U_0 + U_1\widehat{T}_1V_2)\sum_{l=0}^{\infty}\sum_{k=0}^{l}\binom{l}{k}T_0^k(U_2\widehat{T}_1V_2)^{l-k}. \end{aligned} \tag{2.18}$$

Therefore, $R_{<n}^{[\leqslant n]} = R_{<n}$ according to Theorem 2.2.

If $j<n$, we need to construct a useful relation among matrices P, $P^{[\leqslant n]}$ and $P^{[\leqslant j]}$. We first censor the matrix P in the set $L_{\leqslant j}$, $R_{i,j}^{[\leqslant j]} = R_{i,j}$ based on the fact just proved. Then, we censor the matrix $P^{[\leqslant n]}$ in the set $L_{\leqslant n}$. According to Property 1, the censored matrix $P^{[\leqslant j]}$ can be obtained by the censored matrix $P^{[\leqslant n]}$, thus $R_{i,j}^{[\leqslant n]} = R_{i,j}^{[\leqslant j]}$ based on the fact just proved. Therefore, $R_{i,j}^{[\leqslant n]} = R_{i,j}$ for $j<n$. This completes the proof.

It is worthwhile to note that the censoring invariance is a key to organizing the Wiener-Hopf equations for any irreducible Markov chain, and also leads to construct the RG-factorizations.

Let

$$P^{[\leqslant n]} = \begin{pmatrix} \phi_{0,0}^{(n)} & \phi_{0,1}^{(n)} & \cdots & \phi_{0,n}^{(n)} \\ \phi_{1,0}^{(n)} & \phi_{1,1}^{(n)} & \cdots & \phi_{1,n}^{(n)} \\ \vdots & \vdots & & \vdots \\ \phi_{n,0}^{(n)} & \phi_{n,1}^{(n)} & \cdots & \phi_{n,n}^{(n)} \end{pmatrix}, \quad n \geqslant 0, \tag{2.19}$$

be block-partitioned according to levels.

The following lemma provides useful equations among the block entries of the censored Markov chains, which can essentially lead to the Wiener-Hopf equations for the Markov chain.

Lemma 2.4 *For $n\geqslant 0$, $0\leqslant i, j\leqslant n$,*

$$\phi_{i,j}^{(n)} = P_{i,j} + \sum_{k=n+1}^{\infty}\phi_{i,k}^{(k)}\sum_{l=0}^{\infty}[\phi_{k,k}^{(k)}]^l\phi_{k,j}^{(k)}.$$

Proof Consider the censored matrix $P^{[\leqslant n]}$ based on $P^{[\leqslant (n+1)]}$. It follows from Theorem 2.1 that

$$P^{[\leqslant n]} = \begin{pmatrix} \phi_{0,0}^{(n+1)} & \phi_{0,1}^{(n+1)} & \cdots & \phi_{0,n}^{(n+1)} \\ \phi_{1,0}^{(n+1)} & \phi_{1,1}^{(n+1)} & \cdots & \phi_{1,n}^{(n+1)} \\ \vdots & \vdots & & \vdots \\ \phi_{n,0}^{(n+1)} & \phi_{n,1}^{(n+1)} & \cdots & \phi_{n,n}^{(n+1)} \end{pmatrix} + \begin{pmatrix} \phi_{0,n+1}^{(n+1)} \\ \phi_{1,n+1}^{(n+1)} \\ \vdots \\ \phi_{n,n+1}^{(n+1)} \end{pmatrix} \sum_{l=0}^{\infty}\left[\phi_{n+1,n+1}^{(n+1)}\right]^l\left(\phi_{n+1,0}^{(n+1)}, \phi_{n+1,1}^{(n+1)}, \ldots, \phi_{n+1,n}^{(n+1)}\right).$$

Therefore, using Theorem 2.1 we obtain

$$\begin{aligned}
\phi_{i,j}^{(n)} &= \phi_{i,j}^{(n+1)} + \phi_{i,(n+1)}^{(n+1)} \sum_{l=0}^{\infty} \left[\phi_{n+1,n+1}^{(n+1)} \right]^l \phi_{n+1,j}^{(n+1)} \\
&= \phi_{i,j}^{(n+2)} + \phi_{i,(n+2)}^{(n+2)} \sum_{l=0}^{\infty} \left[\phi_{n+2,n+2}^{(n+2)} \right]^l \phi_{n+2,j}^{(n+2)} \\
&\quad + \phi_{i,(n+1)}^{(n+1)} \sum_{l=0}^{\infty} \left[\phi_{n+1,n+1}^{(n+1)} \right]^l \phi_{n+1,j}^{(n+1)} \\
&= \cdots = P_{i,j} + \sum_{k=n+1}^{\infty} \phi_{i,k}^{(k)} \sum_{l=0}^{\infty} \left[\phi_{k,k}^{(k)} \right]^l \phi_{k,j}^{(k)},
\end{aligned}$$

where $P_{i,j} = \phi_{i,j}^{(\infty)}$. This completes the proof.

The following lemma provides expressions for the *R*- and *G*-measures.

Lemma 2.5 (1) *For* $0 \leqslant i < j$,

$$R_{i,j} = \phi_{i,j}^{(j)} \sum_{l=0}^{\infty} \left[\phi_{j,j}^{(j)} \right]^l.$$

(2) *For* $0 \leqslant j < i$,

$$G_{i,j} = \sum_{l=0}^{\infty} \left[\phi_{i,i}^{(i)} \right]^l \phi_{i,j}^{(i)}.$$

Proof Applying Corollary 2.2 to the censored chain $P^{[\leqslant j]}$ gives that

$$R_{i,j}^{[\leqslant j]} = \phi_{i,j}^{(j)} \sum_{l=0}^{\infty} \left[\phi_{j,j}^{(j)} \right]^l, \quad 0 \leqslant i < j,$$

and

$$G_{i,j}^{[\leqslant j]} = \sum_{l=0}^{\infty} \left[\phi_{i,i}^{(i)} \right]^l \phi_{i,j}^{(i)}, \quad 0 \leqslant j < i.$$

The rest of the proof follows from the censoring invariance for the *R*- and *G*-measures proved in Theorem 2.3.

We define the *U*-measure as

$$\Psi_n = \phi_{n,n}^{(n)}, \quad n \geqslant 0.$$

The following theorem provides an equivalent form to the equations in Lemma 2.4, and is called the Wiener-Hopf equations for any irreducible Markov chain in terms of the *R*- and *G*-measures.

Theorem 2.4 (1) *For* $0 \leqslant i < j$,

$$R_{i,j}(I - \Psi_j) = P_{i,j} + \sum_{k=j+1}^{\infty} R_{i,k}(I - \Psi_k)\, G_{k,j}.$$

(2) *For* $0 \leqslant j < i$,

$$(I - \Psi_i)\, G_{i,j} = P_{i,j} + \sum_{k=i+1}^{\infty} R_{i,k} (I - \Psi_k)\, G_{k,j}.$$

(3) *For* $n \geqslant 0$,

$$\Psi_n = P_{n,n} + \sum_{k=n+1}^{\infty} R_{n,k} (I - \Psi_k)\, G_{k,n}.$$

Proof We only prove (1), while (2) and (3) can be proved similarly.

It follows from Lemma 2.5 that

$$R_{i,j} (I - \Psi_j) = \phi_{i,j}^{(j)}$$

and

$$(I - \Psi_i)\, G_{i,j} = \phi_{i,j}^{(i)}.$$

Using Lemma 2.4 and the censoring invariance in Theorem 2.3 leads to

$$\begin{aligned}
\phi_{i,j}^{(j)} &= P_{i,j} + \sum_{k=j+1}^{\infty} \phi_{i,k}^{(k)} \sum_{l=0}^{\infty} \left[\phi_{k,k}^{(k)} \right]^l \phi_{k,j}^{(k)} \\
&= P_{i,j} + \sum_{k=j+1}^{\infty} R_{i,k}^{[\leqslant k]} (I - \Psi_k)\, G_{k,j}^{[\leqslant k]} \\
&= P_{i,j} + \sum_{k=j+1}^{\infty} R_{i,k} (I - \Psi_k)\, G_{k,j}.
\end{aligned}$$

This completes the proof.

Based on the Wiener-Hopf equations given in Theorem 2.4, a UL-type *RG*-factorization for the transition probability matrix *P* is obtained in the following theorem. The UL-type *RG*-factorization always holds for any irreducible Markov chain, and it is very useful for computing the stationary probability vector which can help to analyze stationary performance measures of a practical stochastic system.

Theorem 2.5 *For the Markov chain P given in Eq.* (2.1),

$$I - P = (I - R_U)(I - \Psi_D)(I - G_L), \tag{2.20}$$

where

$$R_U = \begin{pmatrix} 0 & R_{0,1} & R_{0,2} & R_{0,3} & \cdots \\ & 0 & R_{1,2} & R_{1,3} & \cdots \\ & & 0 & R_{2,3} & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$\Psi_D = \mathrm{diag}\,(\Psi_0, \Psi_1, \Psi_2, \Psi_3, \ldots)$$

and

$$G_L = \begin{pmatrix} 0 & & & & \\ G_{1,0} & 0 & & & \\ G_{2,0} & G_{2,1} & 0 & & \\ G_{3,0} & G_{3,1} & G_{3,2} & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof We only prove Eq. (2.20) for the entries in the first block-row and first block-column. The rest can be proved similarly.

The entry (0, 0) on the right-hand side is

$$I - \Psi_0 + \sum_{k=1}^{\infty} R_{0,k} (I - \Psi_k)\, G_{k,0},$$

which is equal to $I - P_{0,0}$ according to (3) of Theorem 2.4.

The entry (0, l) with $l \geqslant 1$ on the right-hand side is

$$-R_{0,l}(I - \Psi_l) + \sum_{k=l+1}^{\infty} R_{0,k} (I - \Psi_k)\, G_{k,l},$$

which is equal to $-P_{0,l}$ according to (1) of Theorem 2.4.

Finally, to see that the entry (l, 0) with $l \geqslant 1$ on the right-hand side is equal to the corresponding entry on the left-hand side, it follows from (2) of Theorem 2.4 that

$$-(I - \Psi_l)\, G_{l,0} + \sum_{k=l+1}^{\infty} R_{l,k} (I - \Psi_k)\, G_{k,0} = -P_{l,0}.$$

This completes the proof.

Now, we consider some important examples for simplifying the UL-type RG-factorization, which is necessary in the study of practical systems, and also provides some new understanding for the Neuts' results, e.g., see [26, 27].

2.2.1 Level-Dependent Markov Chains of *M*/*G*/1 Type

For the Markov chain P of $M/G/1$ type given in Eq. (1.7), we write

$$W_k = \begin{pmatrix} A_{k,k} & A_{k,k+1} & A_{k,k+2} & \cdots \\ A_{k+1,k} & A_{k+1,k+1} & A_{k+1,k+2} & \cdots \\ & A_{k+2,k+1} & A_{k+2,k+2} & \cdots \\ & & \ddots & \ddots \end{pmatrix}, \quad k \geqslant 1.$$

We denote by $\left(\left(\widehat{W}_{1,1}^{(k)}\right)^{\mathrm{T}},\left(\widehat{W}_{2,1}^{(k)}\right)^{\mathrm{T}},\ldots\right)^{\mathrm{T}}$ the first block-column of the matrix $\widehat{W}_k=\sum_{l=0}^{\infty}[W_k]^l$. It follows from Eq. (2.17) and Eq. (2.19) that for $0\leqslant i\leqslant n$ and $0\leqslant j\leqslant n-1$,

$$\phi_{i,j}^{(n)}=\begin{cases}A_{i,j}, & i-j\leqslant 1,\\ 0, & i-j\geqslant 2,\\ A_{i,n}+\sum_{l=1}^{\infty}A_{i,n+l}\widehat{W}_{l,1}^{(n+1)}A_{n+1,n}, & 0\leqslant i\leqslant n.\end{cases}$$

Therefore,

$$R_{i,j}=\sum_{l=0}^{\infty}A_{i,j+l}\widehat{W}_{l+1,1}^{(j)},\quad 0\leqslant i\leqslant j, \tag{2.21}$$

$$G_k\overset{\text{def}}{=}G_{k,k-1}=\widehat{W}_{1,1}^{(k)}A_{k,k-1},\quad k\geqslant 1, \tag{2.22}$$

$G_{i,j}=0$ for $0\leqslant j\leqslant i-2$, and

$$\varPsi_k=A_{k,k}+\sum_{i=1}^{\infty}A_{k,k+i}G_{k+i}G_{k+i-1}\ldots G_{k+1},\quad k\geqslant 0. \tag{2.23}$$

The following lemma provides expressions for the first block-column in the fundamental matrix $\widehat{W}_k$ in terms of $\widehat{W}_{1,1}^{(k)}$ and the G-measure $\{G_k\}$.

Lemma 2.6 *For $k\geqslant 1$ and $j\geqslant 2$,*

$$\widehat{W}_{j,1}^{(k)}=G_{k+j-1}G_{k+j-2}\ldots G_{k+1}\widehat{W}_{1,1}^{(k)} \tag{2.24}$$

and

$$(I-\varPsi_k)\widehat{W}_{1,1}^{(k)}=\widehat{W}_{1,1}^{(k)}(I-\varPsi_k)=I. \tag{2.25}$$

Proof Eq. (2.25) is clear from Lemma 2.3 and the expression $\varPsi_k=T_0+U_2\widehat{T}_1V_2$. Thus we only need to prove Eq. (2.24).

Let

$$W_k=\begin{pmatrix}T_0 & U_2\\ V_2 & T_1\end{pmatrix},$$

where

$$T_0 = A_{k,k}, \quad T_1 = W_{k+1},$$
$$U_2 = (A_{k,k+1}, A_{k,k+2}, A_{k,k+3}, \ldots),$$
$$V_2 = (A_{k,k-1}^T, 0, 0, 0, \ldots)^{\mathrm{T}}.$$

It follows from Lemma 2.3 that

$$H_{2,1}^{(k)} = \widehat{T}_1 V_2 (I - T_0 - U_2 \widehat{T}_1 V_2)^{-1} = \widehat{T}_1 V_2 H_{1,1}^{(k)} = \widehat{W}_{k+1} V_2 \widehat{W}_{1,1}^{(k)}.$$

Note that

$$\widehat{W}_{k+1} V_2 = \begin{pmatrix} \widehat{W}_{1,1}^{(k+1)} A_{k+1,k} \\ \widehat{W}_{2,1}^{(k+1)} A_{k+1,k} \\ \widehat{W}_{3,1}^{(k+1)} A_{k+1,k} \\ \vdots \end{pmatrix} = \begin{pmatrix} G_{k+1} \\ H_{2,1}^{(k+1)} A_{k+1,k} \end{pmatrix},$$

and

$$H_{2,1}^{(k)} = \begin{pmatrix} G_{k+1} \\ H_{2,1}^{(k+1)} A_{k+1,k} \end{pmatrix} \widehat{W}_{1,1}^{(k)},$$

we obtain

$$\widehat{W}_{2,1}^{(k)} = G_{k+1} \widehat{W}_{1,1}^{(k)},$$
$$\widehat{W}_{3,1}^{(k)} = \widehat{W}_{2,1}^{(k+1)} A_{k+1,k} \widehat{W}_{1,1}^{(k)} = G_{k+2} \widehat{W}_{1,1}^{(k+1)} A_{k+1,k} \widehat{W}_{1,1}^{(k)} = G_{k+2} G_{k+1} \widehat{W}_{1,1}^{(k)},$$

by induction, we have

$$\widehat{W}_{j,1}^{(k)} = \widehat{W}_{j-1,1}^{(k+1)} A_{k+1,k} \widehat{W}_{1,1}^{(k)} = G_{k+j-1} G_{k+j-2} \ldots G_{k+1} \widehat{W}_{1,1}^{(k)}.$$

This completes the proof.

The following theorem provides a useful expression or interpretation for the R- and G-measures.

Theorem 2.6 (1) *For* $0 \leqslant i < j$,

$$R_{i,j} = \left(A_{i,j} + \sum_{l=1}^{\infty} A_{i,j+l} G_{l+j} G_{l+j-1} \ldots G_{j+1} \right) (I - \Psi_j)^{-1}. \qquad (2.26)$$

(2) *The matrix sequence* $\{G_i\}$ *is the minimal nonnegative solution to the system of matrix equations*

$$G_i = A_{i,i-1} + \sum_{l=0}^{\infty} A_{i,i+l} G_{i+l} G_{i+l-1} \ldots G_i, \quad i \geqslant 1. \qquad (2.27)$$

Proof (1) Using Lemma 2.6, we obtain

$$
\begin{aligned}
R_{i,j} &= \sum_{l=1}^{\infty} A_{i,j+l} \widehat{W}_{l,1}^{(j)} \\
&= A_{i,i+l} + \sum_{l=1}^{\infty} A_{i,j+l} G_{j+l-1} G_{j+l-2} \cdots G_{k+1} \widehat{W}_{1,1}^{(k)} \\
&= \left(A_{i,j} + \sum_{l=1}^{\infty} A_{i,j+l} G_{l+j} G_{l+j-1} \cdots G_{j+1} \right) (I - \varPsi_j)^{-1}.
\end{aligned}
$$

(2) Since $G_k = \widehat{W}_{1,1}^{(k)} A_{k,k-1}$, we obtain that $(I - \varPsi_k)\, G_k = A_{k,k-1}$ or $G_k = A_{k,k-1} + \varPsi_k G_k$. Note that

$$
\varPsi_k = A_{k,k} + \sum_{i=1}^{\infty} A_{k,k+i} G_{k+i} G_{k+i-1} \cdots G_{k+1},
$$

it is easy to obtain the desired result given in Eq. (2.26). This completes the proof.

2.2.2 Level-Independent Markov Chains of *M*/*G*/1 Type

For the Markov chain *P* of *M*/*G*/1 type given in Eq. (1.3), let the matrix *G* be the minimal nonnegative solution to the matrix equation $G = \sum_{k=0}^{\infty} A_k G^k$. Then the *G*-measure is given by

$$
G_k = \begin{cases} G_1, & k = 1, \\ G, & k \geqslant 2, \end{cases}
$$

where

$$
G_1 = (I - \varPsi)^{-1} B_0;
$$

the *U*-measure

$$
\varPsi_0 = B_1 + \sum_{k=2}^{\infty} B_k G^{k-2} G_1
$$

and for $k \geqslant 1$

$$
\varPsi = \varPsi_k = \sum_{k=1}^{\infty} A_k G^{k-1};
$$

and the *R*-measure

$$
R_{0,j} = \left(\sum_{k=j+1}^{\infty} B_k G^{k-1} \right) (I - \varPsi)^{-1}, \quad j \geqslant 1,
$$

and for $i \geqslant 1$

$$R_{i,j} = \left(\sum_{k=j+1}^{\infty} A_k G^{k-1} \right) (I - \Psi)^{-1}, \quad j \geqslant 1.$$

2.2.3 Level-Dependent Markov Chains of *GI*/*M*/1 Type

Using an analysis similar to the Markov chain of *M*/*G*/1 type, we can deal with a Markov chain of *GI*/*M*/1 type given in Eq. (1.6). We provide a simple introduction to the *R*-, *U*- and *G*-measures. To show this, we write

$$W_k = \begin{pmatrix} A_{k,k} & A_{k,k+1} & & \\ A_{k+1,k} & A_{k+1,k+1} & A_{k+1,k+2} & \\ A_{k+2,k} & A_{k+2,k+1} & A_{k+2,k+2} & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad k \geqslant 1.$$

We denote by $\left(\left(\widehat{W}_{1,1}^{(k)} \right), \left(\widehat{W}_{1,2}^{(k)} \right), \ldots \right)$ the first block-row of the matrix $\widehat{W}_k = \sum_{l=0}^{\infty} [W_k]^l$.

Hence we have

$$R_k \overset{\text{def}}{=} R_{k,k+1} = A_{k,k+1} \widehat{W}_{1,1}^{(k+1)}, \quad k \geqslant 0,$$

$R_{i,j} = 0$ for $j \geqslant i + 2$,

$$G_{i,j} = \sum_{l=1}^{\infty} \widehat{W}_{1,l}^{(i)} A_{i+l,j}, \quad 0 \leqslant j < i,$$

and

$$\Psi_k = A_{k,k} + \sum_{i=1}^{\infty} R_k R_{k+1} \ldots R_{k+i-1} A_{k+i,k}, \quad k \geqslant 0.$$

It is clear that for $k \geqslant 1$ and $j \geqslant 2$,

$$\widehat{W}_{1,j}^{(k)} = \widehat{W}_{1,1}^{(k)} R_k R_{k+1} \ldots R_{k+j-2}.$$

Therefore, the matrix sequence $\{R_k\}$ is the minimal nonnegative solution to the system of matrix equations

$$R_k = A_{k,k+1} + \sum_{i=1}^{\infty} R_k R_{k+1} \ldots R_{k+i-1} A_{k+i,k+1}, \quad k \geqslant 0,$$

$$G_{i,j} = (I - \Psi_i)^{-1} \left(A_{i,j} + \sum_{l=1}^{\infty} R_i R_{i+1} \ldots R_{i+l-1} A_{i+l,j} \right).$$

2.2.4 Level-Independent Markov Chains of *GI*/*M*/1 Type

For the Markov chain P of $M/G/1$ type given in Eq. (1.2), let the matrix R be the minimal nonnegative solution to the matrix equation $R=\sum_{k=0}^{\infty} R^k A_k$. Then the R-measure is given by

$$R_k = \begin{cases} R_1, & k=1, \\ R, & k \geqslant 2, \end{cases}$$

where

$$R_1 = B_0 (I-\Psi)^{-1};$$

the U-measure

$$\Psi_0 = B_1 + \sum_{k=2}^{\infty} R_1 R^{k-2} B_k$$

and for $k \geqslant 1$

$$\Psi = \Psi_k = \sum_{k=1}^{\infty} R^{k-1} A_k;$$

and the G-measure

$$G_{0,j} = (I-\Psi)^{-1} \left(\sum_{k=j+1}^{\infty} R^{k-1} B_k \right), \quad j \geqslant 1,$$

and for $i \geqslant 1$

$$G_{i,j} = (I-\Psi)^{-1} \left(\sum_{k=j+1}^{\infty} R^{k-1} A_k \right), \quad j \geqslant 1.$$

2.2.5 The QBD Processes

For the QBD process, the matrix sequences $\{R_k\}$ and $\{G_k\}$ are the minimal nonnegative solutions to the systems of matrix equations

$$R_k = A_{k+1}^{(k)} + R_k A_{k+1}^{(k+1)} + R_k R_{k+1} A_{k+1}^{(k+2)}, \quad k \geqslant 0,$$

and

$$G_k = A_{k-1}^{(k)} + A_k^{(k)} G_k + A_{k+1}^{(k)} G_{k+1} G_k, \quad k \geqslant 1,$$

respectively. The RG-factorization Eq. (2.20) can be simplified as

$$R_U = \begin{pmatrix} 0 & R_0 & & & \\ & 0 & R_1 & & \\ & & 0 & R_2 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix} \tag{2.28}$$

and

$$G_L = \begin{pmatrix} 0 & & & & \\ G_1 & 0 & & & \\ & G_2 & 0 & & \\ & & G_3 & 0 & \\ & & & \ddots & \ddots \end{pmatrix} \tag{2.29}$$

2.3 The LU-Type *RG*-Factorization

In this section, three LU-type probabilistic measures: the R-, U- and G-measures, are defined in another censored direction, and an LU-type RG-factorization for the transition probability matrix is derived. Note that the method used in this section is similar to that in Section 2.2.

For the block-structured Markov chain P given in Eq. (2.1), we use the same partition given in Eq. (2.2). Let

$$P^{[\geqslant n]} = W + V(I-T)^{-1}U.$$

The block-entry expression of the matrix $P^{[\geqslant n]}$ is written as

$$P^{[\geqslant n]} = \begin{pmatrix} \eta_{n,n}^{(n)} & \eta_{n,n+1}^{(n)} & \eta_{n,n+2}^{(n)} & \cdots \\ \eta_{n+1,n}^{(n)} & \eta_{n+1,n+1}^{(n)} & \eta_{n+1,n+2}^{(n)} & \cdots \\ \eta_{n+2,n}^{(n)} & \eta_{n+2,n+1}^{(n)} & \eta_{n+2,n+2}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Lemma 2.7 *For* $i, j \geqslant n+1$, *we have*

$$\eta_{i,j}^{(n+1)} = P_{i,j} + \sum_{k=0}^{n} \eta_{i,k}^{(k)} \left[I - \eta_{k,k}^{(k)} \right]^{-1} \eta_{k,j}^{(k)}.$$

Proof Since

$$P^{[\geqslant(n+1)]} = \begin{pmatrix} \eta_{n+1,n+1}^{(n)} & \eta_{n+1,n+2}^{(n)} & \eta_{n+1,n+3}^{(n)} & \cdots \\ \eta_{n+2,n+1}^{(n)} & \eta_{n+2,n+2}^{(n)} & \eta_{n+2,n+3}^{(n)} & \cdots \\ \eta_{n+3,n+1}^{(n)} & \eta_{n+3,n+2}^{(n)} & \eta_{n+3,n+3}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$
$$+ \begin{pmatrix} \eta_{n+1,n}^{(n)} \\ \eta_{n+2,n}^{(n)} \\ \eta_{n+3,n}^{(n)} \\ \vdots \end{pmatrix} \left[I - \eta_{n,n}^{(n)} \right]^{-1} \left(\eta_{n,n+1}^{(n)}, \eta_{n,n+2}^{(n)}, \eta_{n,n+3}^{(n)}, \dots \right),$$

we obtain

$$\begin{aligned} \eta_{i,j}^{(n+1)} &= \eta_{i,j}^{(n)} + \eta_{i,n}^{(n)} \left[I - \eta_{n,n}^{(n)} \right]^{-1} \eta_{n,j}^{(n)} \\ &= \eta_{i,j}^{(n-1)} + \eta_{i,n-1}^{(n-1)} \left[I - \eta_{n-1,n-1}^{(n-1)} \right]^{-1} \eta_{n-1,j}^{(n-1)} + \eta_{i,n}^{(n)} \left[I - \eta_{n,n}^{(n)} \right]^{-1} \eta_{n,j}^{(n)} \\ &\cdots \\ &= \eta_{i,j}^{(0)} + \sum_{k=0}^{n} \eta_{i,k}^{(k)} \left[I - \eta_{k,k}^{(k)} \right]^{-1} \eta_{k,j}^{(k)}. \end{aligned}$$

Note that $\eta_{i,j}^{(0)} = P_{i,j}$ for all $i, j \geqslant 0$. This completes the proof.

We define the U-measure as

$$\Phi_n = \eta_{n,n}^{(n)}, \quad n \geqslant 0, \tag{2.30}$$

the R-measure as

$$\overline{R}_{i,j} = \eta_{i,j}^{(j)} \left[I - \eta_{j,j}^{(j)} \right]^{-1}, \quad 0 \leqslant j < i,$$

and the G-measure as

$$\overline{G}_{i,j} = \left[I - \eta_{i,i}^{(i)} \right]^{-1} \eta_{i,j}^{(i)}, \quad 0 \leqslant i < j.$$

It is obvious that

$$\overline{R}_{i,j} = \eta_{i,j}^{(j)} (I - \Phi_j)^{-1}, \quad 0 \leqslant j < i, \tag{2.31}$$

and

$$\overline{G}_{i,j} = (I - \Phi_i)^{-1} \eta_{i,j}^{(i)}, \quad 0 \leqslant i < j. \tag{2.32}$$

The following theorem provides the important Wiener-Hopf equations for the R-, U- and G-measures.

Theorem 2.7 *The LU-type R-, U- and G-measures defined above satisfy the following Wiener-Hopf equations,*

$$\overline{R}_{i,j}(I-\Phi_j)=P_{i,j}+\sum_{k=0}^{j-1}\overline{R}_{i,k}(I-\Phi_k)\,\overline{G}_{k,j},\quad 0\leqslant j<i, \tag{2.33}$$

$$(I-\Phi_i)\,\overline{G}_{i,j}=P_{i,j}+\sum_{k=0}^{i-1}\overline{R}_{i,k}(I-\Phi_k)\,\overline{G}_{k,j},\quad 0\leqslant i<j, \tag{2.34}$$

and

$$\Phi_n=P_{n,n}+\sum_{k=0}^{n-1}\overline{R}_{n,k}(I-\Phi_k)\,\overline{G}_{k,n},\quad n\geqslant 0. \tag{2.35}$$

Proof We only prove Eq. (2.33), while Eq. (2.34) and Eq. (2.35) can be proved similarly.

It follows from Eq. (2.31) that

$$\overline{R}_{i,j}(I-\Phi_j)=\eta_{i,j}^{(j)}. \tag{2.36}$$

By Lemma 2.7, we have

$$\eta_{i,j}^{(j)}=P_{i,j}+\sum_{k=0}^{j-1}\eta_{i,k}^{(k)}\left[I-\eta_{k,k}^{(k)}\right]^{-1}\eta_{k,j}^{(k)}. \tag{2.37}$$

From Eq. (2.31), Eq. (2.32) and Eq. (2.37) we obtain

$$\eta_{i,j}^{(j)}=P_{i,j}+\sum_{k=0}^{j-1}\overline{R}_{i,k}(I-\Phi_k)\,\overline{G}_{k,j},$$

which, together with Eq. (2.36), leads to the stated result.

By the Wiener-Hopf equations Eq. (2.33), Eq. (2.34) and Eq. (2.35), the following theorem constructs an LU-type *RG*-factorization, which is different from the UL-type *RG*-factorization given in Theorem 2.5.

Theorem 2.8 *For the Markov chain P defined in Eq.* (2.1),

$$I-P=(I-\overline{R}_L)\,(I-\Phi_D)\,(I-\overline{G}_U), \tag{2.38}$$

where

$$\overline{R}_L=\begin{pmatrix} 0 & & & & \\ \overline{R}_{1,0} & 0 & & & \\ \overline{R}_{2,0} & \overline{R}_{2,1} & 0 & & \\ \overline{R}_{3,0} & \overline{R}_{3,1} & \overline{R}_{3,2} & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Phi_D=\mathrm{diag}\,(\Phi_0,\Phi_1,\Phi_2,\Phi_3,...)$$

and

$$\overline{G}_U = \begin{pmatrix} 0 & \overline{G}_{0,1} & \overline{G}_{0,2} & \overline{G}_{0,3} & \cdots \\ & 0 & \overline{G}_{1,2} & \overline{G}_{1,3} & \cdots \\ & & 0 & \overline{G}_{2,3} & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Proof We prove Eq. (2.38) for the block-entries of the first two block-rows. The rest can be proved similarly.

For the first block-row, the entry (0, 0) is

$$I - \Phi_0 = I - \eta_{0,0}^{(0)} = I - P_{0,0},$$

and the entry $(0, l)$ for $l \geqslant 1$ is

$$\begin{aligned} -(I - \Phi_0)\, \overline{G}_{0,l} &= -\left[I - \eta_{0,0}^{(0)}\right]\left[I - \eta_{0,0}^{(0)}\right]^{-1} \eta_{0,l}^{(0)} \\ &= -\eta_{0,l}^{(0)} = -P_{0,l} \end{aligned}$$

by means of Eq. (2.32). For the second block-row, the entry (1, 0) is

$$\begin{aligned} -\overline{R}_{1,0}(I - \Phi_0) &= -\eta_{1,0}^{(0)}\left[I - \eta_{0,0}^{(0)}\right]^{-1}\left[I - \eta_{0,0}^{(0)}\right] \\ &= -\eta_{1,0}^{(0)} = -P_{1,0} \end{aligned}$$

by Eq. (2.31). By Lemma 2.7, the entry Eq. (1, 1) is

$$\begin{aligned} \overline{R}_{1,0}(I - \Phi_0)\, \overline{G}_{0,1} + I - \Phi_1 &= \eta_{1,0}^{(0)}\left[I - \eta_{0,0}^{(0)}\right]^{-1}\left[I - \eta_{0,0}^{(0)}\right]\left[I - \eta_{0,0}^{(0)}\right]^{-1} \eta_{0,1}^{(0)} \\ &\quad + I - \eta_{1,1}^{(1)} \\ &= \eta_{1,0}^{(0)}\left[I - \eta_{0,0}^{(0)}\right]^{-1} \eta_{0,1}^{(0)} + I - \eta_{1,1}^{(1)} \\ &= I - P_{1,1}, \end{aligned}$$

and the entry $(1, k)$ for $k \geqslant 2$ is

$$\begin{aligned} \overline{R}_{1,0}(I - \Phi_0)\, \overline{G}_{0,k} - (I - \Phi_1)\, \overline{G}_{1,k} &= \eta_{1,0}^{(0)}\left[I - \eta_{0,0}^{(0)}\right]^{-1}\left[I - \eta_{0,0}^{(0)}\right]\left[I - \eta_{0,0}^{(0)}\right]^{-1} \eta_{0,k}^{(0)} \\ &\quad - \left[I - \eta_{1,1}^{(1)}\right]\left[I - \eta_{1,1}^{(1)}\right]^{-1} \eta_{1,k}^{(1)} \\ &= \eta_{1,0}^{(0)}\left[I - \eta_{0,0}^{(0)}\right]^{-1} \eta_{0,k}^{(0)} - \eta_{1,k}^{(1)} \\ &= P_{1,k}. \end{aligned}$$

This completes the proof.

The LU-type RG-factorization, given in Theorem 2.5, can be applied to solve the inhomogeneous systems of infinite-dimensional linear equations: $x(I-P)=b$ or $(I-P)x=b$ where $b \neq 0$. The LU-type RG-factorization, given in Theorem 2.8, is useful in the study of transient solutions, such as, the first passage times and the sojourn times.

In what follows we analyze some important examples for illustrating applications of the LU-type RG-factorization.

2.3.1 Level-Dependent Markov Chains of *M*/*G*/1 Type

For the Markov chain P of $M/G/1$ type given in Eq. (1.7), the following corollary provides an expression for the matrix $\overline{R}_L$.

Corollary 2.3 *For the Markov chain P of M/G/*1 *type given in Eq.* (1.7),

$$\overline{R}_L = \begin{pmatrix} 0 & & & & \\ \overline{R}_{1,0} & 0 & & & \\ & \overline{R}_{2,1} & 0 & & \\ & & \overline{R}_{3,2} & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Proof Note that

$$\begin{aligned} P^{[\geqslant n]} &= W + V(I-T)^{-1}U \\ &= \begin{pmatrix} A_{n,n} & A_{n,n+1} & A_{n,n+2} & \cdots \\ A_{n+1,n} & A_{n+1,n+1} & A_{n+1,n+2} & \cdots \\ & A_{n+2,n+1} & A_{n+2,n+2} & \cdots \\ & & \ddots & \ddots \end{pmatrix} + \begin{pmatrix} A_{n,n-1} \\ \\ \\ \end{pmatrix}(I-T)^{-1}U \\ &= \begin{pmatrix} A_{n,n} & A_{n,n+1} & A_{n,n+2} & \cdots \\ A_{n+1,n} & A_{n+1,n+1} & A_{n+1,n+2} & \cdots \\ & A_{n+2,n+1} & A_{n+2,n+2} & \cdots \\ & & \ddots & \ddots \end{pmatrix} + \begin{pmatrix} * * * \cdots \\ \\ \end{pmatrix}, \end{aligned}$$

where $*$ denotes a non-zero block-entry, we obtain that $\eta_{i,j}^{(n)} = 0$ for all $n \leqslant j \leqslant i-2$, which lead to

$$\overline{R}_{i,j} = \eta_{i,j}^{(j)} \left[I - \eta_{j,j}^{(j)} \right]^{-1} = 0, \quad 0 \leqslant j \leqslant i-2,$$

and

$$\overline{R}_L = \begin{pmatrix} 0 & & & & \\ \overline{R}_{1,0} & 0 & & & \\ & \overline{R}_{2,1} & 0 & & \\ & & \overline{R}_{3,2} & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

This completes the proof.

2.3.2 Level-Dependent Markov Chains of *GI*/*M*/1 Type

For the Markov chain P of $GI/M/1$ type given in Eq. (1.6), the G-measure can be simplified as

$$\overline{G}_{i,j} = 0, \quad 0 \leqslant i \leqslant j-2.$$

Thus, we have

$$\overline{G}_U = \begin{pmatrix} 0 & \overline{G}_{0,1} & & & \\ & 0 & \overline{G}_{1,2} & & \\ & & 0 & \overline{G}_{2,3} & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

2.3.3 The QBD Processes

We consider a discrete-time QBD process whose transition probability matrix is given in Eq. (1.4). The LU-type RG-factorization is given by

$$I - P = (I - \overline{R}_L)\,(I - \varPhi_D)\,(I - \overline{G}_U), \tag{2.39}$$

where

$$\varPhi_D = \text{diag}\,(\varPhi_0, \varPhi_1, \varPhi_2, \varPhi_3, \ldots),$$

$$\overline{R}_L = \begin{pmatrix} 0 & & & & \\ \overline{R}_1 & 0 & & & \\ & \overline{R}_2 & 0 & & \\ & & \overline{R}_3 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

and

$$\overline{G}_U = \begin{pmatrix} 0 & \overline{G}_0 & & & \\ & 0 & \overline{G}_1 & & \\ & & 0 & \overline{G}_2 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

2.4 The Stationary Probability Vector

In this section, we apply the *RG*-factorizations to provide an *R*-measure expression for the stationary probability vector of any positive-recurrent discrete-time Markov chain with block structure.

We always assume that the block-structured Markov chain P given in Eq. (2.1) is irreducible and positive recurrent, thus there must exist the stationary probability vector $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$. Obviously, $\pi(I - P) = 0$ and $\pi e = 1$. Based on Theorem 2.5, we have

$$\pi(I - P) = \pi\,(I - R_U)\,(I - \Psi_D)\,(I - G_L).$$

We write $x = (x_0, x_1, x_2, \ldots)$ and

$$x = \pi(I - R_U).$$

Then

$$x(I - \Psi_D)\,(I - G_L) = 0,$$

which leads to

$$\begin{cases} x_0(I - \Psi_0) - \sum_{k=1}^{\infty} x_k (I - \Psi_k)\, G_{k,0} = 0, \\ x_i(I - \Psi_i) - \sum_{k=i+1}^{\infty} x_k (I - \Psi_k)\, G_{k,i} = 0, \quad \text{for } i \geqslant 1. \end{cases} \tag{2.40}$$

Note that Ψ_0 is the transition probability matrix of the censored chain $P^{[0]}$. Since P is positive recurrent, $P^{[0]}$ is also positive recurrent. Hence there exists the stationary probability vector x_0 such that $x_0(I - \Psi_0) = 0$ and $x_0 e = 1$. It is easy to check that $(\tau x_0, 0, 0, \ldots)$ is a non-zero nonnegative solution to the systems of Eq. (2.40), where τ is a constant.

Solving the simplified system of linear equations $\pi(I - R_U) = (\tau x_0, 0, 0, \ldots)$, we obtain $\pi = (\tau x_0, 0, 0, \ldots)\,(I - R_U)^{-1}$. Therefore, we can obtain the following theorem.

Theorem 2.9 *For the discrete-time block-structured Markov chain given in*

Eq. (2.1)*, the stationary probability vector is given by*

$$\begin{cases} \pi_0 = \tau x_0, \\ \pi_k = \sum_{i=0}^{k-1} \pi_i R_{i,k}, \quad k \geqslant 1, \end{cases}$$

where x_0 is the stationary probability vector of the censored Markov chain Ψ_0 to level 0 *and the scalar τ is determined by $\sum_{k=0}^{\infty} \pi_k e = 1$ uniquely.*

Remark 2.2 *Although the matrix $I - G_L$ must be invertible, the equation $y(I - G_L) = 0$* or *$(I - G_L)y = 0$ may have a non-zero nonnegative solution. This is different from the system of linear equations with finite dimensions. This is the reason why we need to take a special non-zero nonnegative solution to Eq.* (2.40)*. As an illustration, we consider*

$$G_L = \begin{pmatrix} 0 & & & & \\ G_1 & 0 & & & \\ & G & 0 & & \\ & & G & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

When G is irreducible and stochastic, it is easy to see that $(gG_1, g, g, \ldots)$ is a non-zero nonnegative solution to the equation $y(I - G_L) = 0$, where $g = gG$ and $ge = 1$.

When a level-dependent Markov chain of *GI*/*M*/1 type is irreducible and positive recurrent, the stationary probability vector is given by

$$\begin{cases} \pi_0 = \tau x_0, \\ \pi_k = \pi_{k-1} R_{k-1,k}, \quad k \geqslant 1, \end{cases}$$

or

$$\begin{cases} \pi_0 = \tau x_0, \\ \pi_k = \pi_0 \prod_{i=1}^{k} R_{i-1,i}, \quad k \geqslant 1. \end{cases}$$

where x_0 is the stationary probability vector of the censored Markov chain Ψ_0 to level 0 and the scalar τ is determined by $\sum_{k=0}^{\infty} \pi_k e = 1$ uniquely.

For a level-independent Markov chain of *GI*/*M*/1 type, it is easy to check that $R_{k-1,k} = R$ for $k \geqslant 2,$ thus we have

$$\begin{cases} \pi_0 = \tau x_0, \\ \pi_k = \pi_0 R_{0,1} R^{k-1}, \quad k \geqslant 1, \end{cases}$$

where x_0 is the stationary probability vector of the censored Markov chain Ψ_0 to level 0 and the scalar τ is determined by $\sum_{k=0}^{\infty} \pi_k e = 1$ uniquely. This is the same as the matrix-geometric solution in Neuts [26].

2.5 *A*- and *B*-measures

In this section, we define *A*- and *B*-measures for discrete-time block-structured Markov chains, and also construct their expressions by means of the *R*- and *G*-measures, respectively. Based on the *A*- and *B*-measures, we provide conditions for the state classification of the Markov chains.

We now define the *A*- and *B*-measures as follows:

Definition 2.2 (1) *For $i, j \geqslant 0$ with $i \neq j$, we define $A_{i,j}$ as a matrix of size $m_i \times m_j$ whose (r, s)th entry is the expected number of visits to state (j, s) before hitting any state in level i, given that the process starts in state (i, r). That is*

$$A_{i,j}(r,s) = E\ [\textit{number of visits to state } (j,\ s) \textit{ before hitting } L_i \mid x_0 = (i,r)].$$

(2) *For $i, j \geqslant 0$ with $i \neq j$, we define $B_{i,j}$ as a matrix of size $m_i \times m_j$ whose (r, s)th entry is the probability of visiting state (j, s) for the first time before hitting any state in level j, given that the process starts in state (i, r). That is*

$$B_{i,j}(r,s) = P\ \{\textit{hitting state } (j,\ s) \textit{ upon entering } L_j \textit{ for the first time} \mid x_0 = (i,r)\}.$$

Furthermore, the matrices $A_{i,i}$ and $B_{i,i}$ for $i \geqslant 0$ are explained as follows:

(3) *The (r, s)th entry of $A_{i,i}$ is the expected number of returning to state (i, s) before hitting any state in level i, given that the process starts in state (i, r).*

(4) *The (r, s)th entry of $B_{i,i}$ is the probability of returning to state (i, s) for the first time before hitting any state in level i, given that the process starts in state (i, r).*

We first derive expressions for $A_{0,j}$ and $B_{i,0}$ for $i, j \geqslant 0$. To do this, we write

$$P = \begin{pmatrix} P_{0,0} & U \\ V & W \end{pmatrix}. \tag{2.41}$$

Then

$$P^{[0]} = P_{0,0} + U\widehat{W}V, \tag{2.42}$$

where $\widehat{W} = \sum_{k=0}^{\infty} W^n = (I - W)^{-1}_{\min}$.

It is clear from the definitions for the A- and B-measures that

$$(A_{0,1}, A_{0,2}, A_{0,3}, \dots) = U\widehat{W}, \tag{2.43}$$

and

$$(B_{1,0}^{\mathrm{T}}, B_{2,0}^{\mathrm{T}}, B_{3,0}^{\mathrm{T}}, \dots)^{\mathrm{T}} = \widehat{W}V. \tag{2.44}$$

Note that

$$\widehat{W} = \sum_{k=0}^{\infty} W^k = I + W\widehat{W} = I + \widehat{W}W, \tag{2.45}$$

it follows from Eq. (2.43) that

$$(A_{0,1}, A_{0,2}, A_{0,3}, \dots) = U + (A_{0,1}, A_{0,2}, A_{0,3}, \dots)W, \tag{2.46}$$

which leads to the block-entry form

$$A_{0,j} = P_{0,j} + \sum_{k=1}^{\infty} A_{0,k} P_{k,j}, \quad j \geqslant 1; \tag{2.47}$$

and from Eq. (2.44) that

$$(B_{1,0}^{\mathrm{T}}, B_{2,0}^{\mathrm{T}}, B_{3,0}^{\mathrm{T}}, \dots)^{\mathrm{T}} = V + W(B_{1,0}^{\mathrm{T}}, B_{2,0}^{\mathrm{T}}, B_{3,0}^{\mathrm{T}}, \dots)^{\mathrm{T}}, \tag{2.48}$$

which leads to the block-entry form

$$B_{i,0} = P_{i,0} + \sum_{k=1}^{\infty} P_{i,k} B_{k,0}, \quad i \geqslant 1.$$

Remark 2.3 *Now, we provide comparisons between A-measure and R-measure, and between B-measure and G-measure.*

(1) *It is easy to see from Eq.* (2.43) *and Corollary 2.2 that*

$$A_{0,k} = (P_{0,1}, P_{0,2}, P_{0,3}, \dots)\, \widehat{W}_1^{(\bullet,k)}$$

and

$$R_{0,k} = (P_{0,k}, P_{0,k+1}, P_{0,k+2}, \dots)\, \widehat{W}_k^{(\bullet,1)};$$

similarly,

$$B_{k,0} = \widehat{W}_1^{(k,\bullet)} (P_{1,0}^{\mathrm{T}}, P_{2,0}^{\mathrm{T}}, P_{3,0}^{\mathrm{T}}, \dots)^{\mathrm{T}}$$

and

$$G_{0,k} = \widehat{W}_k^{(1,\bullet)} (P_{k,0}^{\mathrm{T}}, P_{k+1,0}^{\mathrm{T}}, P_{k+2,0}^{\mathrm{T}}, \dots)^{\mathrm{T}}.$$

(2) *Since* $(I-W)\widehat{W}=\widehat{W}(I-W)=I$, *it follows from Eq.* (2.42) *that*

$$P^{[0]}=P_{0,0}+(A_{0,1},A_{0,2},A_{0,3},\ldots)\,(I-W)\,(B_{1,0}^{\mathrm{T}},B_{2,0}^{\mathrm{T}},B_{3,0}^{\mathrm{T}},\ldots)^{\mathrm{T}}$$

and

$$P^{[0]}=P_{0,0}+(R_{0,1},R_{0,2},R_{0,3},\ldots)\,(I-\varDelta)\,(G_{1,0}^{\mathrm{T}},G_{2,0}^{\mathrm{T}},G_{3,0}^{\mathrm{T}},\ldots)^{\mathrm{T}},$$

where $\varDelta=\operatorname{diag}(\varPsi_1,\varPsi_2,\varPsi_3,\ldots)$.

Remark 2.4 *For the truncated transition matrix W, we have the UL-type RG-factorization as follows:*

$$I-W=(I-\tilde{R}_U)\,(I-\tilde{\varPsi}_D)\,(I-\tilde{G}_L).$$

Thus, we obtain

$$(A_{0,1},A_{0,2},A_{0,3},\ldots)=U(I-\tilde{G}_L)^{-1}\,(I-\tilde{\varPsi}_D)^{-1}\,(I-\tilde{R}_U)^{-1}$$

and

$$(B_{1,0}^{\mathrm{T}},B_{2,0}^{\mathrm{T}},B_{3,0}^{\mathrm{T}},\ldots)^{\mathrm{T}}=(I-\tilde{G}_L)^{-1}\,(I-\tilde{\varPsi}_D)^{-1}\,(I-\tilde{R}_U)^{-1}V.$$

Let $A_{0,j}^{[\leqslant n]}$ and $B_{i,0}^{[\leqslant n]}$ be the *A*- and *B*-measures for the censored chain $P^{[\leqslant n]}$ to the censored set $L_{\leqslant n}$, respectively. The following lemma provides an important censoring invariance for the *A*- and *B*-measures.

Lemma 2.8 *For* $n\geqslant 1$ *and* $1\leqslant i,j\leqslant n$,

$$A_{0,j}^{[\leqslant n]}=A_{0,j} \tag{2.49}$$

and

$$B_{i,0}^{[\leqslant n]}=B_{i,0}. \tag{2.50}$$

Proof We only prove Eq. (2.49), while the proof of Eq. (2.50) is similar.

To prove Eq. (2.49), we write

$$P=\begin{array}{c c} & \begin{array}{ccc} L_0 & L_{1\to n} & L_{\geqslant n+1} \end{array} \\ \begin{array}{c} L_0 \\ L_{1\to n} \\ L_{\geqslant n+1} \end{array} & \begin{pmatrix} P_{0,0} & U_1 & U_2 \\ V_1 & W_{11} & W_{12} \\ V_2 & W_{21} & W_{22} \end{pmatrix} \end{array}.$$

Then

$$P^{[\leqslant n]}=\begin{pmatrix} P_{0,0}+U_2\widehat{W}_{22}V_2 & U_1+U_2\widehat{W}_{22}W_{21} \\ V_1+W_{12}\widehat{W}_{22}V_2 & W_{11}+W_{12}\widehat{W}_{22}W_{21} \end{pmatrix}. \tag{2.51}$$

It follows from Eq. (2.51) and Eq. (2.46) that

$$\left(A_{0,1}^{[\leqslant n]}, A_{0,2}^{[\leqslant n]}, \ldots, A_{0,n-1}^{[\leqslant n]}, A_{0,n}^{[\leqslant n]}\right) = \left(U_1 + U_2 \widehat{W}_{22} W_{21}\right) \cdot \left(I - W_{11} - W_{12} \widehat{W}_{22} W_{21}\right)^{-1}.$$

Let

$$\widehat{W} = \begin{pmatrix} \widehat{W}(1,1) & \widehat{W}(1,2) \\ \widehat{W}(2,1) & \widehat{W}(2,2) \end{pmatrix} = \left[I - \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \right]^{-1}.$$

Then we obtain

$$\widehat{W}(1,1) = I - W_{11} - W_{12} \widehat{W}_{22} W_{21}$$

and

$$\widehat{W}(2,1) = \widehat{W}_{22} W_{21} (I - W_{11} - W_{12} \widehat{W}_{22} W_{21})^{-1}.$$

It follows from Eq. (2.46) that

$$\begin{aligned}(A_{0,1}, A_{0,2}, \ldots, A_{0,n-1}, A_{0,n}) &= (U_1, U_2) \begin{pmatrix} \widehat{W}(1,1) \\ \widehat{W}(2,1) \end{pmatrix} \\ &= \left(U_1 + U_2 \widehat{W}_{22} W_{21}\right) \left(I - W_{11} - W_{12} \widehat{W}_{22} W_{21}\right)^{-1}.\end{aligned}$$

Thus, we obtain that for $n \geqslant 1$

$$\left(A_{0,1}^{[\leqslant n]}, A_{0,2}^{[\leqslant n]}, \ldots, A_{0,n-1}^{[\leqslant n]}, A_{0,n}^{[\leqslant n]}\right) = (A_{0,1}, A_{0,2}, \ldots, A_{0,n-1}, A_{0,n}).$$

This completes the proof.

Let

$$\widetilde{R}_U = \begin{pmatrix} 0 & R_{1,2} & R_{1,3} & R_{1,4} & \cdots \\ & 0 & R_{2,3} & R_{2,4} & \cdots \\ & & 0 & R_{3,4} & \cdots \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}$$

and

$$\widetilde{G}_L = \begin{pmatrix} 0 & & & & \\ G_{2,1} & 0 & & & \\ G_{3,1} & G_{3,2} & 0 & & \\ G_{4,1} & G_{4,2} & G_{4,3} & 0 & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Note that the R-measure $\{R_{i,j}, 1 \leqslant i < j\}$ and G-measure $\{G_{i,j}, 1 \leqslant j < i\}$ are given in Theorem 2.5.

The following theorem constructs useful relationships between $A_{0,j}$ and the R-measure, and between $B_{i,0}$ and the G-measure.

Theorem 2.10 (1) *The matrix* $A_{0,j}$ *and the R-measure satisfy*

$$A_{0,j} = \begin{cases} R_{0,1}, & \text{if } j = 1, \\ R_{0,j} + \sum_{k=1}^{j-1} A_{0,k} R_{k,j}, & \text{if } j \geqslant 2, \end{cases}$$

or the matrix expression is

$$(A_{0,1}, A_{0,2}, A_{0,3}, \ldots)\,(I - \tilde{R}_U) = (R_{0,1}, R_{0,2}, R_{0,3}, \ldots).$$

(2) *The matrix* $B_{i,0}$ *and the G-measure satisfy*

$$B_{i,0} = \begin{cases} G_{1,0}, & \text{if } i = 1, \\ G_{i,0} + \sum_{k=1}^{i-1} G_{i,k} B_{k,0}, & \text{if } i \geqslant 2, \end{cases}$$

or the matrix expression is

$$(I - \tilde{G}_L)\,(B_{1,0}^{\mathrm{T}}, B_{2,0}^{\mathrm{T}}, B_{3,0}^{\mathrm{T}}, \ldots)^{\mathrm{T}} = (G_{1,0}^{\mathrm{T}}, G_{2,0}^{\mathrm{T}}, G_{3,0}^{\mathrm{T}}, \ldots)^{\mathrm{T}}.$$

Proof Let $P_{i,j}^{[\leqslant n]}$ be the (i, j)th block-entry of the censored chain to the censored set $L_{\leqslant n}$. Then using Eq. (2.47) we obtain

$$A_{0,n}^{[\leqslant n]} = P_{0,n}^{[\leqslant n]} + \sum_{k=1}^{n} A_{0,k}^{[\leqslant n]} P_{k,n}^{[\leqslant n]},$$

hence

$$A_{0,n}^{[\leqslant n]}\left[I - P_{n,n}^{[\leqslant n]}\right] = P_{0,n}^{[\leqslant n]} + \sum_{k=1}^{n-1} A_{0,k}^{[\leqslant n]} P_{k,n}^{[\leqslant n]},$$

which leads to

$$A_{0,n}^{[\leqslant n]} = P_{0,n}^{[\leqslant n]}\left[I - P_{n,n}^{[\leqslant n]}\right]^{-1} + \sum_{k=1}^{n-1} A_{0,k}^{[\leqslant n]} P_{k,n}^{[\leqslant n]}\left[I - P_{n,n}^{[\leqslant n]}\right]^{-1}.$$

Note that the R-measure has the following expression

$$R_{k,n}^{[\leqslant n]} = P_{k,n}^{[\leqslant n]}\left[I - P_{n,n}^{[\leqslant n]}\right]^{-1},$$

we get

$$A_{0,n}^{[\leqslant n]} = R_{0,n}^{[\leqslant n]} + \sum_{k=1}^{n-1} A_{0,k}^{[\leqslant n]} R_{k,n}^{[\leqslant n]}.$$

Applying the censoring invariance for the A- and R-measures, it is easy to see the desired result. This completes the proof.

The following corollary provides expressions for the A- and R-measures in terms of the R- and G-measures, respectively. The proof is easy by using the iterative computations.

Corollary 2.4 (1) *For* $k \geqslant 1$,

$$A_{0,k} = R_{0,k} + \sum_{1 \leqslant i_1 \leqslant k-1} R_{0,i_1} R_{i_1,k} + \sum_{1 \leqslant i_2 < i_1 \leqslant 1-k} R_{0,i_1} R_{i_1,i_2} R_{i_2,k} + \ldots + R_{0,1} R_{1,2} \ldots R_{k-2,k-1} R_{k-1,k}.$$

(2) *For* $k \geqslant 1$,

$$B_{k,0} = G_{k,0} + \sum_{1 \leqslant i_1 \leqslant k-1} G_{k,i_1} G_{i_1,0} + \sum_{1 \leqslant i_1 < i_2 \leqslant k-1} G_{k,i_1} G_{i_1,i_2} G_{i_2,0} + \ldots + G_{k,k-1} G_{k-1,k-2} \ldots G_{2,1} G_{1,0}.$$

Now, we further provide expressions for the matrices $A_{i,0}$ and $B_{0,j}$ for $i, j \geqslant 1$ interms of the LU-type R- and G- measures, respectively. To do this, we shall use the LU-type R- and G-measures. Note that

$$P^{[\geqslant 1]} = W + V(I - P_{0,0})^{-1} U.$$

Thus, we obtain

$$(A_{1,0}^{\mathrm{T}}, A_{2,0}^{\mathrm{T}}, A_{3,0}^{\mathrm{T}}, \ldots)^{\mathrm{T}} = V(I - P_{0,0})^{-1}$$

and

$$(B_{0,1}, B_{0,2}, B_{0,3}, \ldots) = (I - P_{0,0})^{-1} U.$$

It is easy to see that

$$A_{i,0} = P_{i,0}(I - P_{0,0})^{-1}$$

and

$$B_{0,j} = (I - P_{0,0})^{-1} P_{0,j}.$$

Using a probability argumentation, it is obvious from Theorem 2.10 that

$$A_{i,0} = \begin{cases} \overline{R}_{1,0}, & \text{if } i = 1, \\ \overline{R}_{i,0} + \sum_{k=1}^{i-1} \overline{R}_{i,k} A_{k,0}, & \text{if } i \geqslant 2, \end{cases}$$

and

$$B_{0,j} = \begin{cases} \overline{G}_{0,j}, & \text{if } j = 1, \\ \overline{G}_{0,j} + \sum_{k=1}^{j-1} B_{0,k} \overline{G}_{k,j}, & \text{if } j \geqslant 2. \end{cases}$$

Therefore, for $k \geqslant 1$ we obtain

$$A_{0,k} = \overline{R}_{k,0} + \sum_{1 \leqslant i_1 \leqslant k-1} \overline{R}_{k,i_1} \overline{R}_{i_1,0} + \sum_{1 \leqslant i_2 < i_1 \leqslant k-1} \overline{R}_{k,i_1} \overline{R}_{i_1,i_2} \overline{R}_{i_2,0} + \ldots + \overline{R}_{k,k-1} \overline{R}_{k-1,k-2} \ldots \overline{R}_{2,1} \overline{R}_{1,0}$$

and

$$B_{0,k} = \overline{G}_{0,k} + \sum_{1 \leqslant i_1 \leqslant k-1} \overline{G}_{0,i_1} \overline{G}_{i_1,k} + \sum_{1 \leqslant i_1 < i_2 \leqslant k-1} \overline{G}_{0,i_1} \overline{G}_{i_1,i_2} \overline{G}_{i_2,k} + \ldots + \overline{G}_{0,1} \overline{G}_{1,2} \ldots \overline{G}_{k-2,k-1} \overline{G}_{k-1,k}.$$

In what follows we express the matrices $A_{i,j}$ and $B_{i,j}$ for $i, j \geqslant 1$, and construct useful relations between A-measure and R-measure, and between B-measure and G-measure.

Let $P[i]$ be the matrix obtained by deleting the ith block-row and the ith block-column in the matrix P. Then it is easy to see that for $i \geqslant 1$,

$$(A_{i,0}, \ldots, A_{i,i-1}, A_{i,i+1}, \ldots) = (P_{i,0}, \ldots, P_{i,i-1}, P_{i,i+1}, \ldots)\, \widehat{P[i]}$$

and for $j \geqslant 1$,

$$(B_{0,j}^{\mathrm{T}}, \ldots, B_{j-1,j}^{\mathrm{T}}, B_{j+1,j}^{\mathrm{T}}, \ldots)^{\mathrm{T}} = \widehat{P[i]}\, (P_{0,j}^{\mathrm{T}}, \ldots, P_{j-1,j}^{\mathrm{T}}, P_{j+1,j}^{\mathrm{T}}, \ldots)^{\mathrm{T}}.$$

Let $A_{i,j}^{[\leqslant n]}$ and $B_{i,j}^{[\leqslant n]}$ for $1 \leqslant i, j \leqslant n$ be the A- and B-measures for the censored chain $P^{[\leqslant n]}$ to the censored set $L_{\leqslant n}$, respectively; and $A_{i,j}^{[\geqslant n]}$ and $B_{i,j}^{[\geqslant n]}$ for $i, j \geqslant n$, the A- and B-measures for the censored chain $P^{[\geqslant n]}$ to the censored set $L_{\geqslant n}$, respectively.

The following corollary provides the censoring invariance for the A- and B-measures. The proof is easy and is omitted here.

Corollary 2.5 (1) *For $n \geqslant 1$ and $1 \leqslant i, j \leqslant n$ with $i \neq j$,*

$$A_{i,j}^{[\leqslant n]} = A_{i,j}$$

and

$$B_{i,j}^{[\leqslant n]} = B_{i,j}.$$

(2) *For $n \geqslant 1$ and $i, j \geqslant n$ with $i \neq j$*

$$A_{i,j}^{[\geqslant n]} = A_{i,j}$$

and

$$B_{i,j}^{[\geqslant n]} = B_{i,j}.$$

Using a probability analysis, we obtain the following corollary which indicates useful relations between A-measure and R-measure, and between B-measure and G-measure.

Corollary 2.6 (1) *If* $1 \leqslant i < j$, *then*

$$A_{i,j} = \begin{cases} R_{i,i+1}, & \text{if } j = i+1, \\ R_{i,j} + \sum_{k=i+1}^{j-1} A_{i,k} R_{k,j}, & \text{if } j \geqslant i+2, \end{cases}$$

and

$$B_{i,j} = \begin{cases} \overline{G}_{i,j}, & \text{if } j = i+1, \\ \overline{G}_{i,j} + \sum_{k=1}^{j-1} B_{i,k} \overline{G}_{k,j}, & \text{if } j \geqslant i+2. \end{cases}$$

(2) *If* $i > j \geqslant 1$, *then*

$$A_{i,j} = \begin{cases} \overline{R}_{i,j}, & \text{if } j = i-1, \\ \overline{R}_{i,j} + \sum_{k=1}^{i-1} \overline{R}_{i,k} A_{k,j}, & \text{if } j \leqslant i-2, \end{cases}$$

and

$$B_{i,j} = \begin{cases} G_{i,j}, & \text{if } j = i-1, \\ G_{i,j} + \sum_{k=1}^{i-1} G_{i,k} B_{k,j}, & \text{if } j \leqslant i-2. \end{cases}$$

Now, we discuss the probability of the returning time and the expected returning number, which correspond to the matrices $B_{i,i}$ and $A_{i,i}$ for $i \geqslant 0$, respectively.

It is clear that

$$A_{i,i} = P_{i,i} + \sum_{k \neq i} A_{i,k} P_{k,i}$$

and

$$\begin{aligned} A_{i,i} &= \sum_{k<i} \overline{R}_{i,k} A_{k,i} + \sum_{k>i} A_{i,k} \overline{R}_{k,i} \\ &= \sum_{k<i} A_{i,k} R_{k,i} + \sum_{k>i} R_{i,k} A_{k,i}. \end{aligned}$$

Similarly, we have

$$B_{i,i} = P_{i,i} + \sum_{k \neq i}^{\infty} P_{i,k} B_{k,i}$$

and

$$\begin{aligned} B_{i,i} &= \sum_{k<i} B_{i,k} \overline{G}_{k,j} + \sum_{k>i} \overline{G}_{i,k} B_{k,i} \\ &= \sum_{k<i} G_{i,k} B_{k,i} + \sum_{k>i} B_{i,k} G_{k,i}. \end{aligned}$$

Lemma 2.9 *$B_{i,0} e = e$ for all $i \geqslant 1$ if and only if $\sum_{k=0}^{i-1} G_{i,k} e = e$ for all $i \geqslant 1$.*

Proof Suppose first that $B_{i,0} e = e$ for all $i \geqslant 1$. Note that

$$B_{i,0} = \begin{cases} G_{1,0}, & \text{if } i = 1, \\ G_{i,0} + \sum_{k=1}^{i-1} G_{i,k} B_{k,0}, & \text{if } i \geqslant 2, \end{cases}$$

it is clear that $\sum_{k=0}^{i-1} G_{i-k} e = e$ for all $i \geqslant 1$.

Suppose now that $\sum_{k=0}^{i-1} G_{i-k} e = e$ for all $i \geqslant 1$. Then $B_{1,0} e = G_{1,0} e = e$,

$$B_{2,0} e = G_{2,0} e + G_{2,1} B_{1,0} e = (G_{2,0} + G_{2,1})\, e = e,$$

by induction, we can obtain that $B_{i,0} e = e$ for all $i \geqslant 1$. This completes the proof.

Theorem 2.11 *Suppose that the Markov chain P given in Eq.* (2.1) *is irreducible and stochastic. P is recurrent if and only if $B_{i,0} e = e$ for all $i \geqslant 1$.*

Proof Suppose first that P is recurrent. Let $f_{(i,r),(j,s)}$ be the probability that the Markov chain ever makes a transition into state (j, s), given that the Markov chain starts in state (i, r). Then $f_{(i,r),(j,s)} = 1$ for all $i, j \geqslant 0$, $1 \leqslant r \leqslant m_i$ and $1 \leqslant s \leqslant m_j$. Note that

$$f_{(i,r),(0,s)} = B_{i,0}(r,s) + \sum_{w \neq s} B_{i,0}(r,w) f_{(0,w),(0,s)},$$

we obtain $\sum_{k=1}^{m_0} B_{i,0}(r,k) = 1$, that is, $B_{i,0} e = e$ for all $i \geqslant 1$.

Suppose now that $B_{i,0} e = e$ for all $i \geqslant 1$. Note that

$$P^{[0]} = P_{0,0} + \sum_{k=1}^{\infty} P_{0,k} B_{k,0},$$

we obtain that $P^{[0]} e = P_{0,0} e + \sum_{k=1}^{\infty} P_{0,k} B_{k,0} e = P_{0,0} e + \sum_{k=1}^{\infty} P_{0,k} e = e.$ Thus $P^{[0]}$ is stochastic.

Since P is irreducible, it is clear that P is recurrent. This completes the proof.

Corollary 2.7 *Suppose that the Markov chain P given in Eq.* (2.1) *is irreducible and stochastic. P is recurrent if and only if* $\sum_{k=1}^{i-1} G_{i,k} e = e$ *for all* $i \geqslant 1$.

Corollary 2.8 *Suppose that the Markov chain P given in Eq.* (2.1) *is irreducible and stochastic. P is recurrent if and only if* $B_{0,0}$ *is stochastic.*

Proof Suppose P is recurrent. It follows from Theorem 2.11 that $B_{i,0} e = e$ for all $i \geqslant 1$. Note that

$$B_{0,0} = P_{0,0} + \sum_{k=1}^{\infty} P_{0,k} B_{k,0},$$

we obtain that

$$B_{0,0} e = P_{0,0} e + \sum_{k=1}^{\infty} P_{0,k} B_{k,0} e = \sum_{k=0}^{\infty} P_{0,k} e = e$$

which indicates that $B_{0,0}$ is stochastic.

If $B_{0,0}$ is stochastic. Suppose that P would be transient. Then there would exist at least i_0 such that $B_{i_0,0} e \lneqq e$. In this case,

$$B_{0,0} e = P_{0,0} e + \sum_{k=1}^{\infty} P_{0,k} B_{k,0} e \lneqq \sum_{k=0}^{\infty} P_{0,k} e = e.$$

Hence $B_{0,0}$ would be strictly substochastic. This leads to a contradiction. This completes the proof.

For $i, j \geqslant 0$ with $i \neq j$, we define $M_{i,j}$ as a matrix of size $m_i \times m_j$ whose (r, s)th entry is the expected number of transitions needed to enter level j for the first time by hitting state (j, s), given that the process starts in state (i, r). For $i, j \geqslant 0$ with $i = j$, the (r, s)th entry of $M_{i,i}$ is the expected number of transitions needed to return to level i by hitting state (i, s), given that the process starts in state (i, r).

Lemma 2.10 $M_{0,0} e = e + \sum_{k=1}^{\infty} A_{0,k} e.$

Proof Let the rth row of $A_{0,n}$ be $A_{0,n}(r)$. Then it is easy to see that

$$\begin{aligned}\sum_{s=1}^{m_0} M_{0,0}(r,s) &= 1 + \sum_{n=1}^{\infty} \sum_{w=1}^{m_n} E[\text{number of visite to } (n,w) \\ &\qquad \text{before hitting level } 0 \mid X_0 = (0,r)] \\ &= 1 + \sum_{n=1}^{\infty} A_{0,n}(r).\end{aligned}$$

This completes the proof.

Lemma 2.11 *If the Markov chain P given in Eq.* (2.1) *is irreducible and stochastic, then P is positive recurrent if and only if the matrix $M_{0,0}$ is finite.*

We now provide some discussion on how to relate the matrices $A_{0,j}$ for $j \geqslant 1$ to the stationary probability vector π. Note that

$$(A_{0,1}, A_{0,2}, A_{0,3}, \ldots)(I - \tilde{R}_U) = (R_{0,1}, R_{0,2}, R_{0,3}, \ldots)$$

and

$$(\pi_1, \pi_2, \pi_3, \ldots)(I - \tilde{R}_U) = \tau x_0 (R_{0,1}, R_{0,2}, R_{0,3}, \ldots)$$

obtained by Theorem 2.9. Therefore, it is obvious that

$$\pi_k = \tau x_0 A_{0,k}, \quad k \geqslant 1.$$

Theorem 2.12 *If the Markov chain P given in Eq.* (2.1) *is irreducible and recurrent, then P is positive recurrent if and only if the vector* $\mathcal{A}_0 e = \sum_{k=1}^{\infty} A_{0,k} e$ *is finite.*

Proof Suppose first that P is positive recurrent. It is clear that the Markov chain P exists the stationary probability vector $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$. Note that $\pi_k = \tau x_0 A_{0,k}$ for $k \geqslant 1$, where x_0 is the stationary probability vector of the censored chain $P^{[0]}$ to level 0 and the constant τ makes $\sum_{k=0}^{\infty} \pi_k e = 1$, hence we have,

$$\tau x_0 \sum_{k=1}^{\infty} A_{0,k} e = \sum_{k=1}^{\infty} \pi_k e < \sum_{k=0}^{\infty} \pi_k e = 1.$$

Since $x_0 > 0$ and $e > 0$, it is easy to see that $\mathcal{A}_0 e = \sum_{k=1}^{\infty} A_{0,n} e$ is finite.

Suppose now that the vector $\mathcal{A}_0 e = \sum_{k=1}^{\infty} A_{0,n} e$ is finite. Since the Markov chain P is irreducible and recurrent, the censored chain $P^{[0]}$ to level 0 is positive recurrent with the stationary probability vector x_0. Let $\pi_0 = c x_0$ and $\pi_k = c x_0 A_{0,k}$ for $k \geqslant 1$. Then

$$\sum_{k=0}^{\infty} \pi_k e = c x_0 \left(e + \sum_{k=1}^{\infty} A_{0,k} e \right) < +\infty.$$

Taking $c = \left[x_0 \left(e + \sum_{k=1}^{\infty} A_{0,k} e \right) \right]^{-1} = \tau$, it is clear that $\{\pi_k\}$ is the stationar probability vector of P, which leads to that P is positive recurrent. This completes the proof.

2.6 Markov Chains with Finitely-Many Levels

In this section, we study block-structured Markov chains with finitely-many levels, and provide the RG-factorizations. Note that the results of this section should be regarded as a special case of the infinite-level case based on the condition that $P_{i,j} = 0$ for $i \geqslant M+1$ or $j \geqslant M+1$, thus we only give a simple discussion here.

We consider an irreducible discrete-time block-structured Markov chain with finitely-many levels whose transition probability matrix is given by

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,M} \\ P_{1,0} & P_{1,1} & \cdots & P_{1,M} \\ \vdots & \vdots & & \vdots \\ P_{M,0} & P_{M,1} & \cdots & P_{M,M} \end{pmatrix}, \tag{2.52}$$

where $P_{i,i}$ is a matrix of size $m_i \times m_i$ for all $0 \leqslant i \leqslant M$, and the sizes of the other blocks are determined accordingly.

2.6.1 The UL-Type *RG*-Factorization

For $0 \leqslant i, j \leqslant k$ and $0 \leqslant k \leqslant M$, it is clear from Section 2.2 that

$$P_{i,j}^{[\leqslant k]} = P_{i,j} + \sum_{n=k+1}^{M} P_{i,n}^{[\leqslant n]} \left\{ I - P_{n,n}^{[\leqslant n]} \right\}^{-1} P_{n,j}^{[\leqslant n]}.$$

Note that $P_{i,j}^{[\leqslant M]} = P_{i,j}$ and $P_{i,j}^{[\leqslant 0]} = P_{i,j}^{[0]}$.

Let

$$\Psi_n = P_{n,n}^{[\leqslant n]}, \quad 0 \leqslant n \leqslant M,$$
$$R_{i,j} = P_{i,j}^{[\leqslant j]} (I - \Psi_j)^{-1}, \quad 0 \leqslant i \leqslant j \leqslant M,$$

and

$$G_{i,j} = (I - \Psi_i)^{-1} P_{i,j}^{[\leqslant i]}, \quad 0 \leqslant j \leqslant i \leqslant M.$$

Then the UL-type RG-factorization is given by

$$I - P = (I - R_U)(I - \Psi_D)(I - G_L),$$

where

$$R_U = \begin{pmatrix} 0 & R_{0,1} & R_{0,2} & R_{0,3} & \cdots & R_{0,M-1} & R_{0,M} \\ & 0 & R_{1,2} & R_{1,3} & \cdots & R_{1,M-1} & R_{1,M} \\ & & 0 & R_{2,3} & \cdots & R_{2,M-1} & R_{2,M} \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & R_{M-2,M-1} & R_{M-2,M} \\ & & & & & 0 & R_{M-1,M} \\ & & & & & & 0 \end{pmatrix},$$

$$\Psi_D = \operatorname{diag}(\Psi_0, \Psi_1, \Psi_2, \Psi_3, ..., \Psi_{M-1}, \Psi_M)$$

and

$$G_L = \begin{pmatrix} 0 & & & & & & \\ G_{1,0} & 0 & & & & & \\ G_{2,0} & G_{2,1} & 0 & & & & \\ G_{3,0} & G_{3,1} & G_{3,2} & 0 & & & \\ G_{4,0} & G_{4,1} & G_{4,2} & G_{4,3} & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\ G_{M,0} & G_{M,1} & G_{M,2} & G_{M,3} & \cdots & G_{M,M-1} & 0 \end{pmatrix}.$$

2.6.2 The LU-Type *RG*-Factorization

For $k \leqslant i, j \leqslant M$ and $0 \leqslant k \leqslant M$, it is clear from Section 2.3 that

$$P_{i,j}^{[\geqslant k+1]} = P_{i,j} + \sum_{n=0}^{k} P_{i,n}^{[\geqslant n]} \left\{ I - P_{n,n}^{[\geqslant n]} \right\}^{-1} P_{n,j}^{[\geqslant n]}.$$

Note that $P_{i,j}^{[\geqslant M]} = P_{i,j}^{[M]}$ and $P_{i,j}^{[\geqslant 0]} = P_{i,j}$.

Let

$$\Phi_n = P_{n,n}^{[\geqslant n]}, \quad 0 \leqslant n \leqslant M,$$
$$\overline{R}_{i,j} = P_{i,j}^{[\geqslant j]} (I - \Phi_j)^{-1}, \quad 0 \leqslant j < i \leqslant M,$$

and

$$\overline{G}_{i,j} = (I - \Phi_i)^{-1} P_{i,j}^{[\geqslant i]}, \quad 0 \leqslant i < j \leqslant M.$$

Then the UL-type *RG*-factorization is given by

$$I - P = (I - \overline{R}_L)(I - \Phi_D)(I - \overline{G}_U),$$

where

$$\overline{R}_L = \begin{pmatrix} 0 & & & & & & \\ \overline{R}_{1,0} & 0 & & & & & \\ \overline{R}_{2,0} & \overline{R}_{2,1} & 0 & & & & \\ \overline{R}_{3,0} & \overline{R}_{3,1} & \overline{R}_{3,2} & 0 & & & \\ \overline{R}_{4,0} & \overline{R}_{4,1} & \overline{R}_{4,2} & \overline{R}_{4,3} & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ \overline{R}_{M,0} & \overline{R}_{M,1} & \overline{R}_{M,2} & \overline{R}_{M,3} & \cdots & \overline{R}_{M,M-1} & 0 \end{pmatrix}$$

$$\Psi_D = \text{diag}\,(\Phi_0, \Phi_1, \Phi_2, \Phi_3, \ldots, \Phi_{M-1}, \Phi_M)$$

and

$$\overline{G}_U = \begin{pmatrix} 0 & \overline{G}_{0,1} & \overline{G}_{0,2} & \overline{G}_{0,3} & \cdots & \overline{G}_{0,M-1} & \overline{G}_{0,M} \\ & 0 & \overline{G}_{1,2} & \overline{G}_{1,3} & \cdots & \overline{G}_{1,M-1} & \overline{G}_{1,M} \\ & & 0 & \overline{G}_{2,3} & \cdots & \overline{G}_{2,M-1} & \overline{G}_{2,M} \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & \overline{G}_{M-2,M-1} & \overline{G}_{M-2,M} \\ & & & & & 0 & \overline{G}_{M-1,M} \\ & & & & & & 0 \end{pmatrix}.$$

Now, we analyze an interesting Markov chain with finitely-many levels whose transition probability matrix is given by

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} & P_{0,4} & \cdots & P_{0,M-1} & P_{0,M} \\ P_{1,0} & P_{1,1} & P_{1,2} & & & & & \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} & & & & \\ P_{3,0} & & P_{3,2} & P_{3,3} & P_{3,4} & & & \\ \vdots & & & \ddots & \ddots & \ddots & & \\ P_{M-2,0} & & & & P_{M-2,M-3} & P_{M-2,M-2} & P_{M-2,M-1} & \\ P_{M-1,0} & & & & & P_{M-1,M-2} & P_{M-1,M-1} & P_{M-1,M} \\ P_{M,0} & & & & & & P_{M,M-1} & P_{M,M} \end{pmatrix}.$$

Note the special block structure of this Markov chain, it is easy to see that using the UL-type censoring computation, we can derive a simpler expression for the UL-type *RG*-factorization. However, the LU-type *RG*-factorization can not be simplified more effectively.

It is clear that

$$P_{M,M}^{[\leqslant M]} = P_{M,M}.$$

Then

$$P^{[\leqslant M-1]} = (P_{i,j})_{0\leqslant i,j\leqslant M-1} + \begin{pmatrix} P_{0,M} \\ 0 \\ \vdots \\ 0 \\ P_{M-1,M} \end{pmatrix} \left\{ I - P_{M,M}^{[\leqslant M]} \right\}^{-1} \begin{pmatrix} P_{M,0}^{\mathrm{T}} \\ 0^{\mathrm{T}} \\ \vdots \\ 0^{\mathrm{T}} \\ P_{M,M-1}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$$

and for $0\leqslant k\leqslant M-2$,

$$P^{[\leqslant k]} = (P_{i,j})_{0\leqslant i,j\leqslant k} + \begin{pmatrix} P_{0,k+1}^{[\leqslant k+1]} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \left\{ I - P_{k+1,k+1}^{[\leqslant k+1]} \right\}^{-1} \begin{pmatrix} P_{k+1,0}^{[\leqslant k+1]} \\ 0^{\mathrm{T}} \\ \vdots \\ 0^{\mathrm{T}} \\ 0^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}.$$

Therefore, we can obtain the UL-type RG-factorization as follows:

$$I - P = (I - R_U)\,(I - \varPsi_D)\,(I - G_L),$$

where

$$R_U = \begin{pmatrix} 0 & R_{0,1} & R_{0,2} & R_{0,3} & \cdots & R_{0,M-1} & R_{0,M} \\ & 0 & R_{1,2} & & & & \\ & & 0 & R_{2,3} & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & R_{M-2,M-1} & \\ & & & & & 0 & R_{M-1,M} \\ & & & & & & 0 \end{pmatrix},$$

$$\varPsi_D = \operatorname{diag}(\varPsi_0, \varPsi_1, \varPsi_2, \varPsi_3, \ldots, \varPsi_{M-1}, \varPsi_M)$$

and

$$G_L = \begin{pmatrix} 0 & & & & & & \\ G_{1,0} & 0 & & & & & \\ G_{2,0} & G_{2,1} & 0 & & & & \\ G_{3,0} & & G_{3,2} & 0 & & & \\ G_{4,0} & & & G_{4,3} & 0 & & \\ \vdots & & & & \ddots & \ddots & \\ G_{M,0} & & & & & G_{M,M-1} & 0 \end{pmatrix}.$$

Consider another interesting Markov chain whose transition probability matrix is given by

$$P=\begin{pmatrix} P_{0,0} & P_{0,1} & & & & P_{0,M} \\ P_{1,0} & P_{1,1} & P_{1,2} & & & P_{1,M} \\ & P_{2,1} & P_{2,2} & P_{2,3} & & P_{2,M} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & P_{M-1,M-2} & P_{M-1,M-1} & P_{M-1,M} \\ P_{M,1} & P_{M,2} & \cdots & P_{M,M-2} & P_{M,M-1} & P_{M,M} \end{pmatrix},$$

the LU-type *RG*-factorization can be simplified more effectively by using the Lu-type censoring computation.

2.6.3 The Stationary Probability Vector

If the block-structured Markov chain P with finitely-many levels given in Eq. (1.9) is irreducible and stochastic, then it must be positive recurrent. Let $\pi=(\pi_0,\pi_1,\pi_2,\pi_3,...,\pi_M)$ be the stationary probability vector of the Markov chain. Then using the UL-type *RG*-factorization we obtain

$$\begin{cases} \pi_0=\tau x_0, \\ \pi_k=\sum_{i=0}^{k-1}\pi_i R_{i,k}, \quad 1\leqslant k\leqslant M, \end{cases} \tag{2.53}$$

where x_0 is the stationary probability vector of the censored Markov chain Ψ_0 to level 0 and the scalar τ is determined by $\sum_{k=0}^{M}\pi_k e=1$ uniquely.

In what follows we consider two special cases: Markov chains of *GI*/*M*/1 type and Markov chains of *M*/*G*/1 type, and derive the matrix-product solution for the two cases.

(1) For a Markov chain of *GI*/*M*/1 type with finitely-many levels, using the UL-type *RG*-factorization, the stationary probability vector is given as the matrix-product solution

$$\begin{cases} \pi_0=\tau x_0, \\ \pi_k=\pi_0\prod_{i=1}^{k}R_{i-1,i}, \quad 1\leqslant k\leqslant M, \end{cases}$$

where x_0 is the stationary probability vector of the censored Markov chain Ψ_0 to level 0 and the scalar τ is determined by $\sum_{k=0}^{M} \pi_k e = 1$ uniquely.

(2) For a Markov chain of $M/G/1$ type with finitely-many levels, using the LU-type RG-factorization, the stationary probability vector is given as the matrix-product solution

$$\begin{cases} \pi_M = \tau x_M, \\ \pi_k = \pi_M \prod_{i=M-1}^{k} \overline{R}_{i+1,i}, \quad k \geqslant 1, \end{cases}$$

where x_M is the stationary probability vector of the censored Markov chain Ψ_M to level M and the scalar τ is determined by $\sum_{k=0}^{M} \pi_k e = 1$ uniquely.

2.7 Continuous-Time Markov Chains

In this section, the censoring technique is similarly applied to deal with any irreducible continuous-time block-structured Markov chain. The UL-and LU-types RG-factorizations are derived by means of a similar approach to that in Sections of 2.2 and 2.3, respectively. Based on this, we also provide a simple analysis for the stationary probability vector.

Consider an irreducible continuous-time block-structured Markov chain $\{x_t, t \geqslant 0\}$ whose infinitesimal generator is given by

$$Q = \begin{pmatrix} D_{0,0} & D_{0,1} & D_{0,2} & \cdots \\ D_{1,0} & D_{1,1} & D_{1,2} & \cdots \\ D_{2,0} & D_{2,0} & D_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \tag{2.54}$$

where $D_{k,k}$, $k \geqslant 0$, are $m_k \times m_k$ matrices and the sizes of the other block-entries can be determined accordingly. The state space Ω of this Markov chain is partitioned as $\Omega = \bigcup_{i=0}^{\infty} L_i$, where $L_i = \{(i, j); j = 1, 2, ..., m_i\}$.

For the continuous-time Markov chain Q, we first describe the censored Markov chain Q^E with the censored set $E \in \Omega$, which is different from that for the discrete-time case analyzed in Sections 2.2 and 2.3.

If the successive visits of $\{X_t\}$ to the subset E take place in the time intervals $[t_0, t_1]$, $[t_2, t_3]$, $[t_4, t_5]$, ... Note that t_{2k} is the entering time of the Markov chain to the subset E, while t_{2k+1} is the leaving time of the Markov chain from the subset E for $k \geqslant 1$. For $n \geqslant 1$, we write

$$\tilde{t} = \begin{cases} t, & \text{for } t \in [t_0, t_1), \\ t_{2n-1}^{-} - \sum_{i=1}^{n-1} (t_{2i} - t_{2i-1}), & \text{for } t \in [t_{2n-1}, t_{2n}), \\ t - \sum_{i=1}^{n} (t_{2i} - t_{2i-1}), & \text{for } t \in [t_{2n}, t_{2n+1}). \end{cases}$$

Let $X_t^E = X_{\tilde{t}}$ for $t \geqslant 0$. Then $\{X_t^E\}$ is the censored Markov chain with censoring set E. To study the continuous-time censored Markov chains, we partition the matrix Q as

$$Q = \begin{pmatrix} T & U \\ V & W \end{pmatrix}$$

according to the sets E and E^c. Then the infinitesimal generators of the censored chains $\{X_t^E\}$ and $\{X_t^{E^c}\}$ are respectively given by

$$Q^E = T + U(-W)_{\min}^{-1} V$$

and

$$Q^{E^c} = W + V(-T)^{-1} U.$$

Since the Markov chain defined by Eq. (2.54) is irreducible, the two truncated chains with infinitesimal generators T and W are all transient, and hence the matrices T and W are all invertible. Note that the inverse of the matrix T is ordinary while the invertibility of the matrix W is under an infinite-dimensional meaning. Although the matrix W of infinite size may have multiple inverses, we in general are interested in the maximal non-positive inverse $W_{\max}^{-1}$ of W, i.e., $W^{-1} \leqslant W_{\max}^{-1} \leqslant 0$ for every non-positive inverse W^{-1} *of* W. Of course, $0 \leqslant (-W)_{\min}^{-1} \leqslant (-W)^{-1}$ for every inverse $(-W)^{-1}$ of $-W$.

Here, we omit the detailed discussion for constructing the R-, U- and G-measures, including the censoring invariance; while we directly provide the following RG-factorizations for the irreducible continuous-time block-structured Markov chain.

2.7.1 The UL-type *RG*-factorization

Let

$$Q^{[\leqslant n]} = T + U(-W)_{\min}^{-1} V. \tag{2.55}$$

The block-entry expression of the matrix $Q^{[\leqslant n]}$ is written as

$$Q^{[\leqslant n]} = \begin{pmatrix} f_{0,0}^{(n)} & f_{0,1}^{(n)} & \cdots & f_{0,n}^{(n)} \\ f_{1,0}^{(n)} & f_{1,1}^{(n)} & \cdots & f_{1,n}^{(n)} \\ \vdots & \vdots & & \vdots \\ f_{n,0}^{(n)} & f_{n,1}^{(n)} & \cdots & f_{n,n}^{(n)} \end{pmatrix}$$

Lemma 2.12 *For $i, j \leqslant n-1$, we have*

$$f_{i,j}^{(n-1)} = D_{i,j} + \sum_{k=n}^{\infty} f_{i,k}^{(k)} \left[-f_{k,k}^{(k)} \right]^{-1} f_{k,j}^{(k)}. \tag{2.56}$$

Proof Since

$$Q^{[\leqslant (n-1)]} = \begin{pmatrix} f_{0,0}^{(n)} & f_{0,1}^{(n)} & \cdots & f_{0,n-1}^{(n)} \\ f_{1,0}^{(n)} & f_{1,1}^{(n)} & \cdots & f_{1,n-1}^{(n)} \\ \vdots & \vdots & & \vdots \\ f_{n-1,0}^{(n)} & f_{n-1,1}^{(n)} & \cdots & f_{n-1,n-1}^{(n)} \end{pmatrix} + \begin{pmatrix} f_{0,n}^{(n)} \\ f_{1,n}^{(n)} \\ \vdots \\ f_{n-1,n}^{(n)} \end{pmatrix} \left[-f_{n,n}^{(n)} \right]^{-1} \left(f_{n,0}^{(n)} f_{n,1}^{(n)} \cdots f_{n,n-1}^{(n)} \right),$$

we obtain

$$\begin{aligned} f_{i,j}^{(n-1)} &= f_{i,j}^{(n)} + f_{i,n}^{(n)} \left[-f_{n,n}^{(n)} \right]^{-1} f_{n,j}^{(n)} \\ &= f_{i,j}^{(n+1)} + f_{i,n+1}^{(n+1)} \left[-f_{n+1,n+1}^{(n+1)} \right]^{-1} f_{n+1,j}^{(n+1)} + f_{i,n}^{(n)} \left[-f_{n,n}^{(n)} \right]^{-1} f_{n,j}^{(n)} \\ &\;\vdots \\ &= f_{i,j}^{(\infty)} + \sum_{k=n}^{\infty} f_{i,k}^{(k)} \left[-f_{k,k}^{(k)} \right]^{-1} f_{k,j}^{(k)} \\ &= D_{i,j} + \sum_{k=n}^{\infty} f_{i,k}^{(k)} \left[-f_{k,k}^{(k)} \right]^{-1} f_{k,j}^{(k)}, \end{aligned}$$

note that $f_{i,j}^{(\infty)} = D_{i,j}$ for all $i, j \geqslant 0$. This completes the proof.

We define the U-measure as

$$U_n = f_{n,n}^{(n)}, \quad n \geqslant 0, \tag{2.57}$$

the R-measure as

$$R_{i,j} = f_{i,j}^{(j)} \left[-f_{j,j}^{(j)} \right]^{-1}, \quad 0 \leqslant i < j,$$

and the G-measure as

$$G_{i,j} = \left[-f_{i,i}^{(i)}\right]^{-1} f_{i,j}^{(i)}, \quad 0 \leqslant j < i.$$

It is obvious that

$$R_{i,j} = f_{i,j}^{(j)} (-U_j)^{-1}, \quad 0 \leqslant i < j, \tag{2.58}$$

and

$$G_{i,j} = (-U_i)^{-1} f_{i,j}^{(i)}, \quad 0 \leqslant j < i. \tag{2.59}$$

The following theorem provides the Wiener-Hopf equations for the UL-type R-, U- and G-measures.

Theorem 2.13 *The R-, U- and G-measures defined above satisfy the following Wiener-Hopf equations:*

$$R_{i,j}(-U_j) = D_{i,j} + \sum_{k=j}^{\infty} R_{i,k}(-U_k)\, G_{k,j}, \quad 0 \leqslant i < j, \tag{2.60}$$

$$(-U_i)\, G_{i,j} = D_{i,j} + \sum_{k=i}^{\infty} R_{i,k}(-U_k)\, G_{k,j}, \quad 0 \leqslant j < i, \tag{2.61}$$

and

$$U_n = D_{n,n} + \sum_{k=n}^{\infty} R_{n,k}(-U_k)\, G_{k,n}, \quad n \geqslant 0. \tag{2.62}$$

Proof We only prove Eq. (2.60), while Eq. (2.61) and Eq. (2.62) can be proved similarly. It follows from Eq. (2.58) that

$$R_{i,j}(-U_j) = f_{i,j}^{(j)}. \tag{2.63}$$

By Lemma 2.12, we have

$$\begin{aligned} f_{i,j}^{(j)} &= D_{i,j} + \sum_{k=j}^{\infty} f_{i,k}^{(k)} \left[-f_{k,k}^{(k)}\right]^{-1} f_{k,j}^{(k)} \\ &= D_{i,j} + \sum_{k=0}^{j-1} R_{i,k}(-U_k) G_{k,j}, \end{aligned} \tag{2.64}$$

which, together with Eq. (2.63), leads to the stated result.

By the Wiener-Hopf equations Eq. (2.60), Eq. (2.61) and Eq. (2.62), we construct the UL-type RG-factorization in the following theorem. The proof is clear and is omitted here.

Theorem 2.14 *The infinitesimal generator Q defined in Eq.* (2.54) *can be factorized as follows,*

$$Q = (I - R_U)\, U_D (I - G_U),$$

where

$$R_U = \begin{pmatrix} 0 & R_{0,1} & R_{0,2} & R_{0,3} & \cdots \\ & 0 & R_{1,2} & R_{1,3} & \cdots \\ & & 0 & R_{2,3} & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$U_D = \text{diag}\,(U_0, U_1, U_2, U_3, \ldots)$$

and

$$G_L = \begin{pmatrix} 0 & & & & \\ G_{1,0} & 0 & & & \\ G_{2,0} & G_{2,1} & 0 & & \\ G_{3,0} & G_{3,1} & G_{3,2} & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We consider an irreducible continuous-time block-structured Markov chain with finitely-many levels whose transition probability matrix is given by

$$Q = \begin{pmatrix} D_{0,0} & D_{0,1} & \cdots & D_{0,M} \\ D_{1,0} & D_{1,1} & \cdots & D_{1,M} \\ \vdots & \vdots & & \vdots \\ D_{M,0} & D_{M,1} & \cdots & D_{M,M} \end{pmatrix},$$

For $0 \leqslant i, j \leqslant k$ and $0 \leqslant k \leqslant M$, it is clear from Section 2.4 that

$$D_{i,j}^{[\leqslant k]} = D_{i,j} + \sum_{n=k+1}^{M} D_{i,n}^{[\leqslant n]} \left\{ -D_{n,n}^{[\leqslant n]} \right\}^{-1} D_{n,j}^{[\leqslant n]}.$$

Note that $D_{i,j}^{[\leqslant M]} = D_{i,j}$ and $D_{i,j}^{[\leqslant 0]} = D_{i,j}^{[0]}$.

Let

$$U_n = D_{n,n}^{[\leqslant n]}, \quad 0 \leqslant n \leqslant M,$$

$$R_{i,j} = D_{i,j}^{[\leqslant j]} (-U_j)^{-1}, \quad 0 \leqslant i < j \leqslant M,$$

and

$$G_{i,j} = (-U_i)^{-1} D_{i,j}^{[\leqslant i]}, \quad 0 \leqslant j < i \leqslant M.$$

Then the UL-type RG-factorization is given by

$$Q = (I - R_U) U_D (I - G_L),$$

where

$$R_U = \begin{pmatrix} 0 & R_{0,1} & R_{0,2} & R_{0,3} & \cdots & R_{0,M-1} & R_{0,M} \\ & 0 & R_{1,2} & R_{1,3} & \cdots & R_{1,M-1} & R_{1,M} \\ & & 0 & R_{2,3} & \cdots & R_{2,M-1} & R_{2,M} \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & R_{M-2,M-1} & R_{M-2,M} \\ & & & & & 0 & R_{M-1,M} \\ & & & & & & 0 \end{pmatrix},$$

$$U_D = (U_0, U_1, U_2, U_3, \ldots, U_{M-1}, U_M)$$

and

$$G_L = \begin{pmatrix} 0 & & & & & & \\ G_{1,0} & 0 & & & & & \\ G_{2,0} & G_{2,1} & 0 & & & & \\ G_{3,0} & G_{3,1} & G_{3,2} & 0 & & & \\ G_{4,0} & G_{4,1} & G_{4,2} & G_{4,3} & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ G_{M,0} & G_{M,1} & G_{M,2} & G_{M,3} & \cdots & G_{M,M-1} & 0 \end{pmatrix}.$$

2.7.2 The LU-Type *RG*-Factorization

Let

$$Q^{[\geqslant n]} = W + V(-T)^{-1} U. \tag{2.65}$$

The block-entry expression of the matrix $Q^{[\geqslant n]}$ is written as

$$Q^{[\geqslant n]} = \begin{pmatrix} h_{n,n}^{(n)} & h_{n,n+1}^{(n)} & h_{n,n+2}^{(n)} & \cdots \\ h_{n+1,n}^{(n)} & h_{n+1,n+1}^{(n)} & h_{n+1,n+2}^{(n)} & \cdots \\ h_{n+2,n}^{(n)} & h_{n+2,n+1}^{(n)} & h_{n+2,n+2}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Since

$$Q^{[\geqslant(n+1)]}=\begin{pmatrix} h_{n+1,n+1}^{(n)} & h_{n+1,n+2}^{(n)} & h_{n+1,n+3}^{(n)} & \cdots \\ h_{n+2,n+1}^{(n)} & h_{n+2,n+2}^{(n)} & h_{n+2,n+3}^{(n)} & \cdots \\ h_{n+3,n+1}^{(n)} & h_{n+3,n+2}^{(n)} & h_{n+3,n+3}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} + \begin{pmatrix} h_{n+1,n}^{(n)} \\ h_{n+2,n}^{(n)} \\ h_{n+3,n}^{(n)} \\ \vdots \end{pmatrix} \left[-h_{n,n}^{(n)}\right]^{-1} \left(h_{n,n+1}^{(n)} \quad h_{n,n+2}^{(n)} \quad h_{n,n+3}^{(n)} \quad \cdots \right),$$

we obtain

$$\begin{aligned} h_{i,j}^{(n+1)} &= h_{i,j}^{(n)} + h_{i,n}^{(n)}\left[-h_{n,n}^{(n)}\right]^{-1} h_{n,j}^{(n)} \\ &= h_{i,j}^{(n-1)} + h_{i,n-1}^{(n-1)}\left[-h_{n-1,n-1}^{(n-1)}\right]^{-1} h_{n-1,j}^{(n-1)} + h_{i,n}^{(n)}\left[-h_{n,n}^{(n)}\right]^{-1} h_{n,j}^{(n)} \\ &\ \ \vdots \\ &= h_{i,j}^{(0)} + \sum_{k=0}^{n} h_{i,k}^{(k)}\left[-h_{k,k}^{(k)}\right]^{-1} h_{k,j}^{(k)}, \end{aligned}$$

note that $h_{i,j}^{(0)}=D_{i,j}$ for all $i,j\geqslant 0$. Therefore, for $i,j\geqslant n+1$ we have

$$h_{i,j}^{(n+1)} = D_{i,j} + \sum_{k=0}^{n} h_{i,k}^{(k)}\left[-h_{k,k}^{(k)}\right]^{-1} h_{k,j}^{(k)}. \tag{2.66}$$

We define the U-measure as

$$\overline{U}_n = h_{n,n}^{(n)}, \quad n\geqslant 0, \tag{2.67}$$

the R-measure as

$$\overline{R}_{i,j} = h_{i,j}^{(j)}\left[-h_{j,j}^{(j)}\right]^{-1}, \quad 0\leqslant j<i,$$

and the G-measure as

$$\overline{G}_{i,j} = \left[-h_{i,i}^{(i)}\right]^{-1} h_{i,j}^{(i)}, \quad 0\leqslant i<j.$$

It is obvious that

$$\overline{R}_{i,j} = h_{i,j}^{(j)}(-\overline{U}_j)^{-1}, \quad 0\leqslant j<i, \tag{2.68}$$

and

$$\overline{G}_{i,j} = (-\overline{U}_i)^{-1} h_{i,j}^{(i)}, \quad 0 \leqslant i < j. \tag{2.69}$$

The following theorem provides the Wiener-Hopf equations for the LU-type *R*-, *U*-and *G*-measures.

Theorem 2.15 *The R-, U- and G-measures defined above satisfy the following Wiener-Hopf equations,*

$$\overline{R}_{i,j}(-\overline{U}_j) = D_{i,j} + \sum_{k=0}^{j-1} \overline{R}_{i,k}(-\overline{U}_k)\,\overline{G}_{k,j}, \quad 0 \leqslant j < i, \tag{2.70}$$

$$(-\overline{U}_i)\,\overline{G}_{i,j} = D_{i,j} + \sum_{k=0}^{i-1} \overline{R}_{i,k}(-\overline{U}_k)\,\overline{G}_{k,j}, \quad 0 \leqslant i < j, \tag{2.71}$$

and

$$\overline{U}_n = D_{n,n} + \sum_{k=0}^{n-1} \overline{R}_{n,k}(-\overline{U}_k)\,\overline{G}_{k,n}, \quad n \geqslant 0. \tag{2.72}$$

Proof We only prove Eq. (2.70), while Eq. (2.71) and Eq. (2.72) can be proved similarly. It follows from Eq. (2.68) that

$$\overline{R}_{i,j}(-\overline{U}_j) = h_{i,j}^{(j)}. \tag{2.73}$$

We obtain

$$h_{i,j}^{(j)} = D_{i,j} + \sum_{k=0}^{j-1} h_{i,k}^{(k)} \left[-h_{k,k}^{(k)}\right]^{-1} h_{k,j}^{(k)}. \tag{2.74}$$

From Eq. (2.68), Eq. (2.69) and Eq. (2.74) we obtain

$$h_{i,j}^{(j)} = D_{i,j} + \sum_{k=0}^{j-1} \overline{R}_{i,k}(-\overline{U}_k)\,\overline{G}_{k,j},$$

which, together with Eq. (2.73), leads to the stated result.

By the Wiener-Hopf equations Eq. (2.70), Eq. (2.71) and Eq. (2.72), the following theorem constructs the LU-type *RG*-factorization. The proof is obvious and is omitted here.

Theorem 2.16 *The infinitesimal generator Q defined in Eq.* (2.54) *can be factorized as follows,*

$$Q = (I - \overline{R}_L)\,\overline{U}_D(I - \overline{G}_U), \tag{2.75}$$

where

$$\overline{R}_L = \begin{pmatrix} 0 & & & & \\ \overline{R}_{1,0} & 0 & & & \\ \overline{R}_{2,0} & \overline{R}_{2,1} & 0 & & \\ \overline{R}_{3,0} & \overline{R}_{3,1} & \overline{R}_{3,2} & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\overline{U}_D = \text{diag}\,(\overline{U}_0, \overline{U}_1, \overline{U}_2, \overline{U}_3, ...)$$

and

$$\overline{G}_U = \begin{pmatrix} 0 & \overline{G}_{0,1} & \overline{G}_{0,2} & \overline{G}_{0,3} & \cdots \\ & 0 & \overline{G}_{1,2} & \overline{G}_{1,3} & \cdots \\ & & 0 & \overline{G}_{2,3} & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

We consider an irreducible continuous-time block-structured Markov chain with finitely-many levels whose transition probability matrix is given by

$$Q = \begin{pmatrix} D_{0,0} & D_{0,1} & \cdots & D_{0,M} \\ D_{1,0} & D_{1,1} & \cdots & D_{1,M} \\ \vdots & \vdots & & \vdots \\ D_{M,0} & D_{M,1} & \cdots & D_{M,M} \end{pmatrix}.$$

For $k \leqslant i, j \leqslant M$ and $0 \leqslant k \leqslant M$, it is clear from Section 2.3 that

$$D_{i,j}^{[\geqslant k+1]} = D_{i,j} + \sum_{n=0}^{k} D_{i,n}^{[\geqslant n]} \left\{-D_{n,n}^{[\geqslant n]}\right\}^{-1} D_{n,j}^{[\geqslant n]}.$$

Note that $D_{i,j}^{[\geqslant M]} = D_{i,j}^{[M]}$ and $D_{i,j}^{[\geqslant 0]} = D_{i,j}$.

Let

$$\overline{U}_n = D_{n,n}^{[\geqslant n]}, \quad 0 \leqslant n \leqslant M,$$

$$\overline{R}_{i,j} = D_{i,j}^{[\geqslant j]}(-\overline{U}_j)^{-1}, \quad 0 \leqslant j < i \leqslant M,$$

and

$$\overline{G}_{i,j} = (-\overline{U}_i)^{-1} D_{i,j}^{[\geqslant i]}, \quad 0 \leqslant i < j \leqslant M.$$

Then the UL-type *RG*-factorization is given by

$$Q = (I - \overline{R}_L)\, \overline{U}_D (I - \overline{G}_U),$$

where

$$\overline{R}_L = \begin{pmatrix} 0 & & & & & & \\ \overline{R}_{1,0} & 0 & & & & & \\ \overline{R}_{2,0} & \overline{R}_{2,1} & 0 & & & & \\ \overline{R}_{3,0} & \overline{R}_{3,1} & \overline{R}_{3,2} & 0 & & & \\ \overline{R}_{4,0} & \overline{R}_{4,1} & \overline{R}_{4,2} & \overline{R}_{4,3} & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ \overline{R}_{M,0} & \overline{R}_{M,1} & \overline{R}_{M,2} & \overline{R}_{M,3} & \cdots & \overline{R}_{M,M-1} & 0 \end{pmatrix},$$

$$\overline{U}_D = (\overline{U}_0, \overline{U}_1, \overline{U}_2, \overline{U}_3, \dots, \overline{U}_{M-1}, \overline{U}_M)$$

and

$$\overline{G}_U = \begin{pmatrix} 0 & \overline{G}_{0,1} & \overline{G}_{0,2} & \overline{G}_{0,3} & \cdots & \overline{G}_{0,M-1} & \overline{G}_{0,M} \\ & 0 & \overline{G}_{1,2} & \overline{G}_{1,3} & \cdots & \overline{G}_{1,M-1} & \overline{G}_{1,M} \\ & & 0 & \overline{G}_{2,3} & \cdots & \overline{G}_{2,M-1} & \overline{G}_{2,M} \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & \overline{G}_{M-2,M-1} & \overline{G}_{M-2,M} \\ & & & & & 0 & \overline{G}_{M-1,M} \\ & & & & & & 0 \end{pmatrix}.$$

2.7.3 The Stationary Probability Vector

Let $\pi = (\pi_0, \pi_1, \pi_2, ...)$ be the stationary probability vector of the continuous-time Markov chain Q. Then $\pi Q = 0$ and $\pi e = 1$. Thus we have

$$\pi Q = \pi (I - R_U) U_D (I - G_L).$$

We write $x = (x_0, x_1, x_2, ...)$ and

$$x = \pi (I - R_U).$$

Then

$$x U_D (I - G_L) = 0.$$

Let $(\tau x_0, 0, 0, ...)$ be a non-zero nonnegative solution to the equation $x U_D (I - G_L) = 0$, where x_0 is the stationary probability vector of the censored Markov chain U_0 to level 0. Then $\pi (I - R_U) = (\tau x_0, 0, 0, ...)$, where τ is a constant. Therefore, the stationary probability vector of the continuous-time block-structured Markov chain is given by

$$\begin{cases} \pi_0 = \tau x_0, \\ \pi_k = \sum_{i=0}^{k-1} \pi_i R_{i,k}, \quad k \geqslant 1, \end{cases}$$

where the scalar τ is uniquely determined by $\sum_{k=0}^{\infty} \pi_k e = 1$.

For a Markov chain Q of $GI/M/1$ type, the stationary probability vector is given by

$$\begin{cases} \pi_0 = \tau x_0, \\ \pi_k = \pi_0 \prod_{i=1}^{k} R_{i-1,i}, \quad k \geqslant 1, \end{cases}$$

where x_0 is the stationary probability vector of the censored Markov chain U_0 to level 0 and the scalar τ is determined by $\sum_{k=0}^{M} \pi_k e = 1$ uniquely.

For a Markov chain Q of $GI/M/1$ type with finitely-many levels, using the UL-type RG-factorization, the stationary probability vector is given by

$$\begin{cases} \pi_0 = \tau x_0, \\ \pi_k = \pi_0 \prod_{i=1}^{k} R_{i-1,i}, \quad 1 \leqslant k \leqslant M, \end{cases}$$

where x_0 is the stationary probability vector of the censored Markov chain U_0 to level 0 and the scalar τ is determined by $\sum_{k=0}^{M} \pi_k e = 1$ uniquely.

For a Markov chain of $M/G/1$ type with finitely-many levels, the stationary probability vector of is given by

$$\begin{cases} \pi_M = \tau x_M, \\ \pi_k = \pi_M \prod_{i=M-1}^{k} \overline{R}_{i+1,i}, \quad k \geqslant 1, \end{cases}$$

where x_M is the stationary probability vector of the censored Markov chain U_M to level M and the scalar τ is determined by $\sum_{k=0}^{M} \pi_k e = 1$ uniquely.

2.8 Notes in the Literature

The censored Markov chain, also called watched Markov chain, was first considered by Lévy [15 − 17]. Since then, the censored Markov chains have been very useful in the study of Markov chains. Kemeny, Snell and Knapp [11] applied the censoring technique to show that each recurrent Markov chain has a positive regular measure unique to multiplication by a scalar. Freedman [4] used the censoring technique to approximate countable Markov chains for the limiting behavior and also for more general issues.

The censoring technique has been successfully applied to block-structured Markov chains and Markov renewal processes. Examples include Grassmann and Heyman [5], Latouche [13], Heyman [8], Zhao and Liu [29], Zhao, Li and Braun [31, 32], Zhao, Li and Alfa [30], Latouche and Ramaswami [14], Zhao [28], Dudin and Klimenok [2], Li and Zhao [21 − 25], Li and Cao [18], Li and Liu [19], Klimenok and Dudin [12] and Dudin, Kim and Klimenok [3].

It is well known now how significant the matrices R and G are in the study of Markov chains of $GI/M/1$ type and Markov chains of $M/G/1$ type, respectively. Readers may refer to Neuts [26, 27], Latouche and Ramaswami [14] and Bini, Latouche and Meini [1] for more details. Grassmann and Heyman [5] first extended the two matrices R and G to the R- and G-measures $\{R_k\}$ and $\{G_k\}$ for Markov chains of $GI/G/1$ type in which $R_k = R_{n-k,n}$ and $G_k = G_{n,n-k}$ for $1 \leqslant k < n.$ The A-measure was first introduced in Karlin and Taylor [9, 10]. After then, Zhao, Li and Braun [31, 32] and Zhao, Li and Alfa [30] provided a detailed analysis on the A- and B-measures.

In the study of Markov chains and stochastic models, we always encounter some systems of linear equations which may be either finitely or infinitely dimensional. Gaussian elimination is very useful in solving the systems of linear equations. Grassmann and Heyman [5] provided a detailed interpretation on Gaussian elimination by means of the censored technique, and they used Gaussian elimination to derive the Wiener-Hopf equations, which is a key for analyzing Markov chains of $GI/G/1$ type. Li and Zhao [21, 23] and Li and Liu [19] indicated that the Wiener-Hopf equations can lead to the RG-factorizations for any irreducible Markov chain. As indicated in Li and Cao [18], the RG-factorizations can be applied in dealing with the matrix equations $x(I-P)=b$ or $(I-P)x=b$ for discrete-time Markov chains and $xQ=b$ or $Qx=b$ for continuous-time Markov chains. Therefore, the RG-factorizations have established a new theoretic and algorithmic framework in the study of Markov chains and stochastic models for solving systems of linear equations. An important applied example is Li and Zhao [20, 22] for discussing quasi-stationary distributions of Markov chains of $GI/M/1$ type and Markov chains of $M/G/1$ type, respectively; while more general examples will be discussed in Chapter 9 of this book.

Grassmann [6] established a UL-type two-matrix factorization for the matrix $I-P$, where P is an irreducible transition probability matrix of finite size. Grassmann and Heyman [7] gave the same factorization for an irreducible Markov chain of $GI/G/1$ type, while Heyman [8] extended the result to an irreducible and positive recurrent Markov chain with infinitely-many states. Based on the Wiener-Hopf equations for the transition probability matrix of $GI/G/1$ type, Zhao [28] obtained a UL-type RG-factorization for the matrix $I-P$ for the first time. For a level-dependent QBD process, Li and Cao [18] first provided two types: UL- and LU-types of RG-factorizations. Li and Zhao [21, 23] generalized the UL-type RG-factorization to an irreducible block-structured Markov renewal process with infinitely-many states, which can immediately lead to the UL-type RG-factorization for any irreducible block-structured Markov chain. Li and Liu [19] constructed the LU-type RG-factorization for any irreducible block-structured Markov chain, which is parallel to but different from that of Li and Zhao [21]. It is worthwhile to note that the UL-type RG-factorization is very useful for computing the stationary probability vector and more generally, analyzing the stationary performance measures; while the LU-type RG-factorization is a key for calculating the transient performance measures such as the first passage time and the sojourn time, as illustrated in Chapters 6 to 11 of this book.

In this chapter, we mainly refer to Zhao [28], Zhao, Li and Braun [31, 32], Li and Zhao [21, 23], Li and Cao [18] and Li and Liu [19]. At the same time, we also add some new results without publication for a more systemical organization of this chapter.

Problems

2.1 Provide a unified definition for the censored Markov chains of an irreducible Markov chain which may be either discrete-time or continuous-time.

2.2 For the Markov chain P and two state subsets E_1 and E_2 with $E_1 \subset E_2$, prove that $(P^{E_2})^{E_1} = P^{E_1}$.

2.3 Let Ω be the state space of the Markov chain P. Prove that P is irreducible if and only if P^E is irreducible for each subset $E \in \Omega$.

2.4 Note that $R_{i,j} = \sum_{k=1}^{\infty} R_{i,j}(k)$ and $G_{i,j} = \sum_{k=1}^{\infty} G_{i,j}(k)$ given in Section 2.2, please explain the reason why the two matrices $R_{i,j}$ and $G_{i,j}$ have different probabilistic meaning.

2.5 Prove the censoring invariant property:

(1) $R_{i,j}^{[\leqslant n]} = R_{i,j}$ for $0 \leqslant i < j \leqslant n$.

(2) $R_{i,j}^{[\geqslant n]} = R_{i,j}$ for $n+1 \leqslant j < i$.

(3) $A_{i,j}^{[\leqslant n]} = A_{i,j}$ for $0 \leqslant i, j \leqslant n$.

(4) $A_{i,j}^{[\geqslant n]} = A_{i,j}$ for $i, j \geqslant n+1$.

2.6 For a Markov chain of $M/G/1$ type, prove that for $k \geqslant 1$ and $j \geqslant 2$,

$$\widehat{W}_{j,1}^{(k)} = G_{k+j} G_{k+j-1} \dots G_{k+1} \widehat{W}_{1,1}^{(k)}.$$

2.7 Let

$$A=\begin{pmatrix} 0 & C_1 & & & \\ & 0 & C_2 & & \\ & & 0 & C_3 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}$$

and

$$B=\begin{pmatrix} 0 & & & & \\ C_1 & 0 & & & \\ & C_2 & 0 & & \\ & & C_3 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Please compute the inverses of the two matrices $I-A$ and $I-B$.

2.8 If an irreducible discrete-time Markov chain whose transition probability matrix is given by

$$P=\begin{pmatrix} P_{0,0} & P_{0,1} & & & \\ P_{1,0} & P_{1,1} & P_{1,2} & & \\ P_{2,0} & & P_{2,2} & P_{2,3} & \\ \vdots & & & \ddots & \ddots \end{pmatrix},$$

please derive the UL- and LU-types of *RG*-factorizations of the matrix $I-P$.

2.9 For an irreducible level-independent Markov chain of *GI*/*M*/1 type, derive the UL- and LU-types of *RG*-factorizations.

2.10 For an irreducible level-independent Markov chain of *M*/*G*/1 type, derive the UL- and LU-types of *RG*-factorizations.

2.11 If an irreducible discrete-time Markov chain whose transition probability matrix is given by

$$P=\begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} & \cdots \\ P_{1,0} & P_{1,1} & & & \\ P_{2,0} & & P_{2,2} & & \\ P_{3,0} & & & P_{3,3} & \\ \vdots & & & & \ddots \end{pmatrix},$$

please derive the UL- and LU-types of *RG*-factorizations of the matrix $I-P$.

2.12 Let

$$A = \begin{pmatrix} 0 & & & & \\ C_1 & 0 & & & \\ & C_2 & 0 & & \\ & & C_3 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

and C_i is nonzero matrix of size m for $i \geqslant 1$. Please provide a nonzero solution to the matrix equation $x(I - A) = 0$.

2.13 For a $PH/M/1$ queue, apply the UL-type RG-factorization to compute the queue length distribution at time $t > 0$ and the probability distribution of the busy period.

2.14 In Theorem 2.10, please compute

$$(R_{0,1}, R_{0,2}, R_{0,3}, \ldots)\left(I - \widetilde{R_U}\right)^{-1}$$

2.15 Give some concrete examples and indicate useful difference between $\sum_{k=1}^{\infty} A_{0,k}$ and $\sum_{k=1}^{\infty} B_{k,0}$.

2.16 For a QBD process whose transition probability matrix is given by

$$P = \begin{pmatrix} B_1 & B_0 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

compute $A_{i,i}$ and $B_{i,i}$ for $i \geqslant 0$.

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3 Markov Chains of *GI*/*G*/1 Type

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Abstract In this chapter, we simplify the *R*-, *U*- and *G*-measures and the *RG*-factorizations for Markov chains of *GI/G/*1 type. Also, we derive a new *RG*-factorization for the repeated blocks and the four basic inequalities for the boundary blocks, which are useful in spectral analysis of Markov chains of *GI/G/*1 type. We analyze the dual Markov chain of any irreducible block-structured Markov chain, and specifically discuss the dual chain of a Markov chain of *GI/G/*1 type. Furthermore, we simplify the *A*- and *B*-measures for Markov chains of *GI/G/1* type, and also express the *A*- and *B*-measures by means of the *R*- and *G*-measures, respectively. Based on the *A*- and *B*-measures, we provide spectral analysis for the *R*- and *G*-measures, and provide conditions for the state classification of Markov chains of *GI/G/*1 type.

Keywords stochastic model, block-structured Markov chain, Markov chain of *GI*/*G*/1 type, *RG*-factorization, dual Markov chain, spectral analysis, state classification.

In this chapter, we simplify the *R*-, *U*- and *G*-measures and the *RG*-factorizations given in Chapter 2, for Markov chains of $GI/G/1$ type. Two important examples: Markov chains of $GI/M/1$ type and Markov chains of $M/G/1$ type (e.g., see Neuts [13, 14]), are given a detailed analysis under the *RG*-factorization framework. Also, we derive a new *RG*-factorization for the repeated blocks and the four basic inequalities for the boundary blocks, which are useful in spectral analysis of Markov chains of $GI/G/1$ type. We analyze the dual chain of a block-structured Markov chain, and provide a detailed discussion for the dual chain of a Markov chain of $GI/G/1$ type. Furthermore, we simplify the *A*- and *B*-measures for Markov chains of $GI/G/1$ type, and also express the *A*- and *B*-measures by means of the *R*- and *G*-measures, respectively. Based on the *A*- and *B*-measures, we provide spectral analysis for the *R*- and *G*-measures, and provide conditions for the state classification of Markov chains of $GI/G/1$ type.

This chapter is organized as follows. Section 3.1 simplifies the UL- and LU-types of *RG*-factorizations, and derives a new *RG*-factorization for the repeated blocks and the four inequalities for the boundary blocks. Section 3.2 introduces the dual Markov chains, which are used to derive dual measures for the *R*- and *G*-measures. Section 3.3 simplifies expressions for the *A*- and *B*-measures by means of the *R*- and *G*-measures. Section 3.4 gives spectral analysis for the *R*- and *G*-measures, which lead to conditions for the state classification. Section 3.5 studies the minimal positive solution to the matrix generating function equations $\det(I - R^*(z)) = 0$ and $\det(I - G^*(z)) = 0.$ On a similar line, Section 3.6 provides a simple introduction to continuous-time Markov chains of $GI/G/1$ type, which are necessary for analyzing practical systems in many applied areas. Finally, Section 3.7 provides some notes for the references related to the results of this chapter.

3.1 Markov Chains of *GI*/*G*/1 Type

This section considers Markov chains of $GI/G/1$ type, simplifies the *R*- and *G*-measures and the *RG*-factorizations given in Chapter 2, and derives a new *RG*-factorization for the repeated blocks and the four inequalities for the boundary blocks.

Consider a Markov chain of $GI/G/1$ type whose transition probability matrix is given by

$$P = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ D_{-1} & A_0 & A_1 & A_2 & \cdots \\ D_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ D_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3.1}$$

where the sizes of the matrices D_0, D_i, D_{-i} for $i \geqslant 1$ and A_j for $-\infty < j < \infty$ are $m_0 \times m_0$, $m_0 \times m$, $m \times m_0$ and $m \times m$, respectively.

For the transition probability matrices, comparing Eq. (3.1) with Eq. (2.1), it is easy to see that $P_{0,j} = D_j$ for $j \geqslant 0$, $P_{i,0} = D_{-i}$ for $i \geqslant 1$ and $P_{i,j} = A_{j-i}$ for $i, j \geqslant 1$.

Let W_n be the southeast corner of P begainning from level *n*. It is clear that $W_n = W$ for all $n \geqslant 1$. Thus, we write $\widehat{W}^{(\cdot,1)} = \widehat{W}_n^{(\cdot,1)}$ and $\widehat{W}^{(1,\cdot)} = \widehat{W}_n^{(1,\cdot)}$ for all $n \geqslant 1$. Therefore,

$$R_{0,j} = (D_j, D_{j+1}, D_{j+2}, \ldots)\,\widehat{W}^{(\cdot,1)}, \quad j \geqslant 1,$$

$$R_{i,j} = (A_{j-i}, A_{j-i+1}, A_{j-i+2}, \ldots)\,\widehat{W}^{(\cdot,1)}, \quad 1 \leqslant i < j.$$

Obviously, the matrices $R_{i,j}$ for $1 \leqslant i < j$ only depend on the difference $j-i$. We write $R_{i,j}$ as R_{j-i} for all $1 \leqslant i < j$. Therefore, for $k \geqslant 1$,

$$R_k = (A_k, A_{k+1}, A_{k+2}, \ldots)\, \widehat{W}^{(\cdot,1)}. \tag{3.2}$$

Similarly,

$$G_{i,0} = \widehat{W}^{(1,\cdot)}\left(D_{-i}^{\mathrm{T}}, D_{-(i+1)}^{\mathrm{T}}, D_{-(i+2)}^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}, \quad i \geqslant 1,$$

$$G_{i,j} = \widehat{W}^{(1,\cdot)}\left(A_{-(i-j)}^{\mathrm{T}}, A_{-(i-j+1)}^{\mathrm{T}}, A_{-(i-j+2)}^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}, \quad 1 \leqslant j < i.$$

The matrices $G_{i,j}$ for $1 \leqslant j < i$ only depend on the difference $i-j$. We write $G_{i,j}$ as G_{i-j} for all $1 \leqslant j < i$. Therefore, for $k \geqslant 1$,

$$G_k = \widehat{W}^{(1,\cdot)}\left(A_{-k}^{\mathrm{T}}, A_{-(k+1)}^{\mathrm{T}}, A_{-(k+2)}^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}. \tag{3.3}$$

The following lemma is a consequence of the repeating blocks in the Markov chain of $GI/G/1$ type. In fact, this corresponds to the censoring invariance given in Theorem 2.3.

Lemma 3.1 *For* $n \geqslant 1$, i, $j = 1, 2, 3, \ldots, n$,

$$P_{n-i,n-j}^{[\leqslant n]} = P_{n+1-i,n+1-j}^{[\leqslant (n+1)]} = P_{n+2-i,n+2-j}^{[\leqslant (n+2)]} = \ldots.$$

Proof For $n \geqslant 1$, $1 \leqslant i$, $j \leqslant n$, it is easy to see that

$$P_{n-i,n-j}^{[\leqslant n]} = A_{i-j} + (A_{i+1}, A_{i+2}, \ldots)\, \widehat{W}\left(A_{-(j+1)}^{\mathrm{T}}, A_{-(j+2)}^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}, \tag{3.4}$$

which is independent of $n \geqslant 1$. Thus

$$P_{n-i,n-j}^{[\leqslant n]} = P_{n+1-i,n+1-j}^{[\leqslant (n+1)]} = P_{n+2-i,n+2-j}^{[\leqslant (n+2)]} = \ldots.$$

This completes the proof.

Based on the censoring invariance in Lemma 3.1, we define that for $1 \leqslant i$, $j \leqslant n$,

$$\varPhi_0 = P_{n,n}^{[\leqslant n]}, \tag{3.5}$$

$$\varPhi_i = P_{n-i,n}^{[\leqslant n]}, \tag{3.6}$$

$$\varPhi_{-j} = P_{n,n-j}^{[\leqslant n]}. \tag{3.7}$$

It is easy to see that the (r,s)th entry of $\varPhi_i$ is the transition probability of the censored chain $P^{[\leqslant n]}$ from state $(n-i,r)$ to state (n,s), while the (r,s)th entry of $\varPhi_{-j}$ is the transition probability of the censored chain $P^{[\leqslant n]}$ from state (n,r) to state $(n-j,s)$.

The following theorem explicitly expresses the R- and G-measures in terms of the matrices $\varPhi_i$ for $-\infty < i < +\infty$.

Theorem 3.1 (1) *For* $i \geqslant 1$,

$$R_i = \varPhi_i (I - \varPhi_0)^{-1}. \tag{3.8}$$

(2) *For* $j > 1$,

$$G_j = (I - \varPhi_0)^{-1} \varPhi_{-j}. \tag{3.9}$$

Proof We only prove Eq. (3.8), while Eq. (3.9) can be proved similarly.

It follows from Lemmas 3.1 and 2.5 that

$$R_i = R_i^{[\leqslant n]} = R_{n-i,n}^{[\leqslant n]} = P_{n-i,n}^{[\leqslant n]} \sum_{l=0}^{\infty} \left[P_{n,n}^{[\leqslant n]} \right]^l = \varPhi_i (I - \varPhi_0)^{-1}.$$

This completes the proof.

Remark 3.1 *Using Lemma 2.5, for the Markov chain of* $GI/G/1$ *type we can provide expressions for the matrices* $R_{0,j}$ *and* $G_{i,0}$ *for* $i, j \geqslant 1$ *as follows:*

$$R_{0,j} = \phi_{0,j}^{(j)} (I - \varPhi_0)^{-1}$$

and

$$G_{i,0} = (I - \varPhi_0)^{-1} \phi_{i,0}^{(i)}.$$

Theorem 3.2 *If the matrix* $A = \sum_{k=-\infty}^{\infty} A_k$ *is stochastic, then* $\lim_{i \to \infty} G_{i,0} = 0$.

Proof Since

$$D_{-i} e + \sum_{k=-i+1}^{\infty} A_k e \leqslant e$$

and

$$Ae = \sum_{k=-\infty}^{\infty} A_k e = e,$$

where e is a column vector of ones with switable size, it is easy to see that $\lim_{i \to \infty} D_{-i} e = 0$. Note that

$$\begin{aligned} G_{i,0} &= \widehat{W}^{(1,\cdot)} \left(D_{-i}^{\mathrm{T}}, D_{-(i+1)}^{\mathrm{T}}, D_{-(i+2)}^{\mathrm{T}}, \ldots \right)^{\mathrm{T}} \\ &= \sum_{k=1}^{\infty} \widehat{W}^{(1,k)} D_{-(i+k-1)}, \end{aligned}$$

We obtain

$$\lim_{i\to\infty} G_{i,0} = \lim_{i\to\infty}\sum_{k=1}^{\infty}\widehat{W}^{(1,k)}D_{-(i+k-1)} = \sum_{k=1}^{\infty}\widehat{W}^{(1,k)}\lim_{i\to\infty}D_{-(i+k-1)} = 0$$

by means of the dominated convergence theorem due to the fact that for all $k \geqslant 1$ and $i \geqslant 1$, $\widehat{W}^{(1,k)}D_{-(i+k-1)}e \leqslant e$. This completes the proof.

Remark 3.2 (1) *In the matrix P given in Eq.* (3.1)*, if* $\sum_{k=-\infty}^{\infty} A_k$ *is finite, then* $\lim_{k\to\infty} A_{-k} = \lim_{k\to\infty} A_k = 0$*. By using a similar analysis to the proof for Theorem 3.2, it follows from Eq. (3.2) and Eq. (3.3) that* $\lim_{k\to\infty} R_k = \lim_{k\to\infty} G_k = 0$*.*

(2) *In the matrix P given in Eq.* (3.1)*,* $\sum_{k=0}^{\infty} D_k$ *is finite, so* $\lim_{k\to\infty} D_k = 0$*. Thus we obtain* $\lim_{k\to\infty} R_{0,k} = 0$*.*

Let

$$R_0 = \sum_{k=1}^{\infty} R_{0,k}, \quad R = \sum_{k=1}^{\infty} R_k;$$

and

$$G_0 = \sum_{k=1}^{\infty} G_{k,0}, \quad G = \sum_{k=1}^{\infty} G_k.$$

The following theorem provides an important property for the entry structure of the four matrices R_0, R, G_0 and G. Note that two important examples of this theorem were discussed in Lemma 1.2.4 of Neuts [13] for Markov chains of $GI/M/1$ type and in Lemma 2.3.6 of Neuts [14] for Markov chains of $M/G/1$ type.

Theorem 3.3 *If the Markov chain* P *is irreducible, then*

(1) *each column vector of the matrix* R *or* R_0 *is not zero, and*

(2) *each row vector of the matrix* G *or* G_0 *is not zero.*

Proof We only prove R_0, while R, G and G_0 can be proved similarly.

Suppose that the j_0 th column of R_0 was zero. Since $0 \leqslant R_{0,1} \leqslant R_0$, the j_0 th column of $R_{0,1}$ would be zero too. Let $P^{[\leqslant 1]}$ be the censored matrix of the Markov chain P given in Eq. (3.1) with censoring levels 0 and 1, partition $P^{[\leqslant 1]}$ according to levels 0 and 1 as

$$P^{[\leqslant 1]} = \begin{pmatrix} \Psi_{00} & \Psi_{01} \\ \Psi_{10} & \Psi_{11} \end{pmatrix}.$$

Then

$$R_{0,1}^{[\leqslant 1]} = \Psi_{01}(I - \Psi_{11})^{-1}, \tag{3.10}$$

and we define

$$R_{1,0}^{[\leqslant 1]} = \Psi_{10}(I - \Psi_{00})^{-1}, \tag{3.11}$$

noting that $R_{0,1}^{[\leqslant 1]} = R_{0,1}$ according to the censoring invariance, the j_0th column of $R_{0,1}^{[\leqslant 1]}$ would be zero.

Let $N((1, j_0); (1, j_0))$ be the number of state transitions for the censored Markov chain $P^{[\leqslant 1]}$ to eventually return to state $(1, j_0)$ by going through level 0 in intermediate transitions, given that the chain starts in state $(1, j_0)$. Then $P\{N((1, j_0); (1, j_0)) < +\infty\} > 0$, since $P^{[\leqslant 1]}$ is irreducible.

Let $\hat{i}$ be the number of state transitions for the censored Markov chain $P^{[\leqslant 1]}$ to travel from level 1 to level 0. Then

$$P\left\{N((1, j_0); (1, j_0)) < \infty\right\} = \lim_{M \to \infty} P\{N((1, j_0); (1, j_0)) < \infty, 0 \leqslant \hat{i} \leqslant M\}.$$

To evaluate $P\{N((1, j_0); (1, j_0)) < \infty, 0 \leqslant \hat{i} \leqslant M\}$, we consider

$$E\left[z^{N((1,j_0);(1,j_0))} 1_{(0 \leqslant \hat{i} \leqslant M)} \right] \leqslant \alpha\langle j_0 \rangle (I - z\Psi_{11})^{-1} \Psi_{10} (I - z\Psi_{00})^{-1}$$
$$\cdot \sum_{i=0}^{M} \left[\Psi_{01}(I - z\Psi_{11})^{-1} \Psi_{10}(I - z\Psi_{00})^{-1} \right]^i \Psi_{01}(I - z\Psi_{11})^{-1} \left\lceil j_0 \right\rceil,$$

where $\alpha\langle j_0 \rangle = \left(\underbrace{0, 0, ..., 0}_{j_0 - 1 \text{ zeros}}, 1, \underbrace{0, 0, ..., 0}_{m - j_0 \text{ zeros}} \right)$, $(I - \Psi_{11})^{-1} \left\lceil j_0 \right\rceil$ is the j_0th column of $(I - \Psi_{11})^{-1}$ and $1_{(\cdot)}$ is an indicator function. It follows from Eq. (3.10) and Eq. (3.11) that

$$P\{N((1, j_0); (1, j_0)) < \infty, \ 0 \leqslant \hat{i} \leqslant M\} = E\left[z^{N((1,j_0);(1,j_0))} 1_{(0 \leqslant \hat{i} \leqslant M)} \right]_{|z=1}$$
$$= \alpha\langle j_0 \rangle (I - \Psi_{11})^{-1} R_{1,0}^{[\leqslant 1]} \sum_{i=0}^{M} \left[R_{0,1}^{[\leqslant 1]} R_{1,0}^{[\leqslant 1]} \right]^i R_{0,1}^{[\leqslant 1]} \left\lceil j_0 \right\rceil,$$

where $R_{0,1}^{[\leqslant 1]} \left\lceil j_0 \right\rceil$ is the j_0th column of $R_{0,1}^{[\leqslant 1]}$. Since $R_{0,1}^{[\leqslant 1]} \left\lceil j_0 \right\rceil = 0$, we would obtain

$$P\{N((1, j_0); (1, j_0)) < \infty, \ 0 \leqslant \hat{i} \leqslant M\} = 0, \quad \text{for all } M \geqslant 0,$$

therefore,

$$P\left\{N((1, j_0); (1, j_0)) < \infty\right\} = \lim_{M \to \infty} P\{N((1, j_0); (1, j_0)) < \infty, \quad 0 \leqslant \hat{i} \leqslant M\} = 0.$$

This leads to a contradiction to the above result with

$$P\{N((1,j_0);(1,j_0))<+\infty\}>0.$$

This completes the proof.

The following theorem provides Wiener-Hopf equations for Markov chains of $GI/G/1$ type. Note that the Wiener-Hopf equations for any irreducible Markov chain have been constructed in Theorem 2.4.

Theorem 3.4 (1) *For the repeated row and* $i \geqslant 1$,

$$R_i(I-\Phi_0)=A_i+\sum_{k=1}^{\infty}R_{i+k}(I-\Phi_0)\,G_k, \tag{3.12}$$

$$(I-\Phi_0)\,G_i=A_{-i}+\sum_{k=1}^{\infty}R_k(I-\Phi_0)\,G_{i+k} \tag{3.13}$$

and

$$\Phi_0=A_0+\sum_{k=1}^{\infty}R_k(I-\Phi_0)G_k. \tag{3.14}$$

(2) *For the boundary blocks and* $i \geqslant 1$,

$$R_{0,i}(I-\Phi_0)=D_i+\sum_{k=1}^{\infty}R_{0,i+k}(I-\Phi_0)\,G_k, \tag{3.15}$$

$$(I-\Phi_0)\,G_{i,0}=D_{-i}+\sum_{k=1}^{\infty}R_k(I-\Phi_0)\,G_{i+k,0} \tag{3.16}$$

and

$$\Psi_0=P^{[0]}=D_0+\sum_{k=1}^{\infty}R_{0,k}(I-\Phi_0)\,G_{k,0}. \tag{3.17}$$

Proof We only prove Eq. (3.12), while Eq. (3.13) to Eq. (3.17) can be proved similarly.

When n is big enough, it follows from Theorem 3.1 that

$$R_i(I-\Phi_0)=\Phi_i(I-\Phi_0)^{-1}(I-\Phi_0)=\Phi_i=P^{[\leqslant n]}_{n-i,n},$$

and

$$\begin{aligned}
P^{[\leqslant n]}_{n-i,n} &= P^{[\leqslant(n+1)]}_{n-i,n}+R^{[\leqslant(n+1)]}_{i+1}P^{[\leqslant(n+1)]}_{n+1,n}\\
&= P^{[\leqslant(n+1)]}_{n-i,n}+R^{[\leqslant(n+1)]}_{i+1}\left[I-P^{[\leqslant(n+1)]}_{n+1,n+1}\right]G^{[\leqslant(n+1)]}_1\\
&= P^{[\leqslant(n+1)]}_{n-i,n}+R_{i+1}(I-\Phi_0)G_1\\
&= \ldots = P^{[\leqslant(n+N)]}_{n-i,n}+\sum_{k=1}^{N}R_{i+k}(I-\Phi_0)G_k\\
&= A_i+\sum_{k=1}^{\infty}R_{i+k}(I-\Phi_0)G_k.
\end{aligned}$$

This completes the proof.

For any irreducible Markov chain of $GI/G/1$ type, the UL-type RG-factorization given in Eq. (3.1) can be simplified as

$$I - P = (I - R_U)(I - \Phi_D)(I - G_L), \tag{3.18}$$

where

$$R_U = \begin{pmatrix} 0 & R_{0,1} & R_{0,2} & R_{0,3} & \cdots \\ & 0 & R_1 & R_2 & \cdots \\ & & 0 & R_1 & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$\Phi_D = \operatorname{diag}\left(\Psi_0, \Phi_0, \Phi_0, \Phi_0, \ldots\right)$$

and

$$G_L = \begin{pmatrix} 0 & & & & \\ G_{1,0} & 0 & & & \\ G_{2,0} & G_1 & 0 & & \\ G_{3,0} & G_2 & G_1 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let

$$A^*(z) = \sum_{i=-\infty}^{\infty} z^i A_i, \quad R^*(z) = \sum_{i=1}^{\infty} z^i R_i, \quad G^*(z) = \sum_{j=1}^{\infty} z^{-j} G_j.$$

For the Markov chain of $GI/G/1$ type, the following theorem provides a new RG-factorization for the repeated row based on the z-transformation of the repeating blocks. Note that this RG-factorization is necessary in spectral analysis of Markov chains of $GI/G/1$ type which will be organized in Sections 3.4 and 3.5 of this chapter.

Theorem 3.5

$$I - A^*(z) = [I - R^*(z)]\,(I - \Phi_0)\,[I - G^*(z)]. \tag{3.19}$$

Proof It follows from Eq. (3.12), Eq. (3.13) and Eq. (3.14) that

$$\begin{aligned} & R^*(z)(I - \Phi_0) + (I - \Phi_0)G^*(z) + \Phi_0 \\ & \qquad = I_1 + I_2 + \sum_{k=1}^{\infty} z^k R(I - \Phi_0) z^{-k} G_k, \end{aligned}$$

where

$$I_1 = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} z^{i+k} R_{i+k} (I-\Phi_0) z^{-k} G_k$$
$$= R^*(z)(I-\Phi_0)[I-G^*(z)] - \sum_{k=1}^{\infty}\sum_{i=1}^{k} z^i R_i (I-\Phi_0) z^{-k} G_k$$

and

$$I_2 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} z^k R_k (I-\Phi_0) z^{-(j+k)} G_{j+k}$$
$$= R^*(z)(I-\Phi_0)[I-G^*(z)] - \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} z^j R_j (I-\Phi_0) z^{-k} G_k.$$

Since

$$I_1 + I_2 + \sum_{k=1}^{\infty} z^k R (I-\Phi_0) z^{-k} G_k = A^*(z) + R^*(z)(I-\Phi_0)G^*(z),$$

we get

$$R^*(z)(I-\Phi_0) + (I-\Phi_0)G^*(z) + \Phi_0 = A^*(z) + R^*(z)(I-\Phi_0)G^*(z),$$

which is equivalent to Eq. (3.19). This completes the proof.

From Theorem 3.5, we have an important relation as follows:

$$I - A = (I-R)(I-\Phi_0)(I-G). \tag{3.20}$$

Now, we consider the boundary matrix sequence $\{D_k, -\infty < k < \infty\}$. However, such an *RG*-factorization given in Theorem 3.5 does not exist; but we can find a close and useful relationship for the boundary blocks.

Let

$$D_+^*(z) = \sum_{i=1}^{\infty} z^i D_i, \quad D_-^*(z) = \sum_{i=1}^{\infty} z^{-i} D_{-i},$$

and

$$R_0^*(z) = \sum_{i=1}^{\infty} z^i R_{0,i}, \quad G_0^*(z) = \sum_{j=1}^{\infty} z^{-j} G_{j,0}. \tag{3.21}$$

The following theorem provides upper or lower bounds for the *R*- and *G*-measures with respect to the boundary blocks.

Theorem 3.6 *For* $z \geqslant 0$,

$$R_0^*(z) \geqslant D_+^*(z)(I-\Phi_0)^{-1}, \tag{3.22}$$

$$R_0^*(z)(I-\Phi_0)[I-G^*(z)] \leqslant D_+^*(z), \tag{3.23}$$

$$G_0^*(z) \geqslant (I-\Phi_0)^{-1} D_-^*(z) \tag{3.24}$$

and

$$[I-R^*(z)](I-\Phi_0)G_0^*(z) \leqslant D_-^*(z). \tag{3.25}$$

Proof We only prove Eq. (3.22) and Eq. (3.23), while Eq. (3.24) and Eq. (3.25) can be proved similarly.

It follows from Eq. (3.15) that

$$R_{0,i}(I-\Phi_0) = D_i + \sum_{k=1}^{\infty} R_{0,i+k}(I-\Phi_0)G_k, \tag{3.26}$$

and from Eq. (3.9) that

$$G_k = (I-\Phi_0)^{-1}\Phi_{-k}.$$

It is obvious that

$$\sum_{k=1}^{\infty} R_{0,i+k}(I-\Phi_0)G_k = \sum_{k=1}^{\infty} R_{0,i+k}\Phi_{-k} \geqslant 0.$$

Hence it follows from Eq. (3.26) that

$$R_{0,i}(I-\Phi_0) \geqslant D_i$$

and for $z > 0$,

$$R_0^*(z)(I-\Phi_0) \geqslant D_+^*(z). \tag{3.27}$$

Since the Markov chain is irreducible, the spectral radius of the matrix Φ_0 is

$$sp(\Phi_0) < 1.$$

Hence, the matrix $I-\Phi_0$ is invertible and $(I-\Phi_0)^{-1} = \sum_{k=0}^{\infty}(\Phi_0)^k \geqslant 0$. It follows from Eq. (3.27) that

$$R_0^*(z) \geqslant D_+^*(z)(I-\Phi_0)^{-1}.$$

Note that

$$\begin{aligned}\sum_{i=1}^{\infty} z^i \sum_{k=1}^{\infty} R_{0,i+k}(I-\Phi_0)G_k &= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} z^{i+k} R_{0,i+k}(I-\Phi_0)z^{-k}G_k \\ &\leqslant R_0^*(z)(I-\Phi_0)G^*(z),\end{aligned}$$

it follows from Eq. (3.26) that

$$R_0^*(z)(I-\Phi_0) \leqslant D_+^*(z) + R_0^*(z)(I-\Phi_0)G^*(z),$$

simple computations lead to

$$R_0^*(z)(I-\Phi_0)[I-G^*(z)] \leqslant D_+^*(z).$$

This completes the proof.

Applying Theorem 3.6, we have the following important inequalities

$$R_0 \geqslant D_+(I-\Phi_0)^{-1},$$

$$R_0(I-\Phi_0)(I-G) \leqslant D_+,$$

$$G_0 \geqslant (I-\Phi_0)^{-1}D_-$$

and

$$(I-R)(I-\Phi_0)G_0 \leqslant D_-.$$

Although the matrix sequence $\{R_{0,k}\}$ has not such an RG-fatorization given in Theorem 3.5, we can provide an expression for the matrix $R_{0,k}$ for $k \geqslant 1$. The following lemma is useful for this purpose.

Lemma 3.2 *Let* $B_i = \Phi_{-i}(I-\Phi_0)^{-1}$ *for* $i \geqslant 1$ *and*

$$\Lambda = \begin{pmatrix} I & & & & \\ -B_1 & I & & & \\ -B_2 & -B_1 & I & & \\ -B_3 & -B_2 & -B_1 & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$\Lambda^{-1} = \begin{pmatrix} I & & & & \\ X_1 & I & & & \\ X_2 & X_1 & I & & \\ X_3 & X_2 & X_1 & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$X_l = \sum_{i=1}^{\infty} \sum_{\substack{n_1+n_2+\ldots+n_i=l \\ n_j \geqslant 1, 1 \leqslant j \leqslant i}} B_{n_1} B_{n_2} \ldots B_{n_i}, \quad l \geqslant 1.$$

Proof Since $\Lambda^{-1}\Lambda = I$, we obtain

$$-B_k - \sum_{i=1}^{k-1} B_{k-i} X_i + X_k = 0, \quad \text{for all } k \geqslant 1.$$

Let $X^*(z) = \sum_{k=1}^{\infty} z^k X_k$ and $B^*(z) = \sum_{k=1}^{\infty} z^k B_k$. Then

$$X^*(z) = \left[I - B^*(z) \right]^{-1} B^*(z) = \sum_{i=1}^{\infty} \left[B^*(z) \right]^i$$
$$= \sum_{l=1}^{\infty} z^l \sum_{i=1}^{\infty} \sum_{\substack{n_1+n_2+\ldots+n_i=l \\ n_j \geqslant 1, 1 \leqslant j \leqslant i}} B_{n_1} B_{n_2} \ldots B_{n_i}.$$

Thus we have

$$X_l = \sum_{i=1}^{\infty} \sum_{\substack{n_1+n_2+\ldots+n_i=l \\ n_j \geqslant 1, 1 \leqslant j \leqslant i}} B_{n_1} B_{n_2} \ldots B_{n_i}, \quad l \geqslant 1.$$

This completes the proof.

The following theorem characterizes the expression for the matrix $R_{0,k}$ for $k \geqslant 1$.

Theorem 3.7 *For the Markov chain of $GI/G/1$ type, for each $k \geqslant 1$ we have*

$$R_{0,k} = D_k (I - \Phi_0)^{-1} + \sum_{i=1}^{\infty} D_{k+i} (I - \Phi_0)^{-1} X_i. \tag{3.28}$$

Proof It follows from Eq. (3.15) and Eq. (3.9) that for all $k \geqslant 1$,

$$R_{0,k} - \sum_{i=1}^{\infty} R_{0,k+i} \Phi_{-i} (I - \Phi_0)^{-1} = D_k (I - \Phi_0)^{-1},$$

or

$$(R_{0,1}, R_{0,2}, R_{0,3}, R_{0,4}, \ldots) \begin{pmatrix} I & & & & \\ -B_1 & I & & & \\ -B_2 & -B_1 & I & & \\ -B_3 & -B_2 & -B_1 & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} F_1^{\mathrm{T}} \\ F_2^{\mathrm{T}} \\ F_3^{\mathrm{T}} \\ F_4^{\mathrm{T}} \\ \vdots \end{pmatrix}^{\mathrm{T}},$$

where $F_k = D_k (I - \Phi_0)^{-1}$ for $k \geqslant 1$. Using Corollary 3.10 yields that for all $k \geqslant 1$,

$$R_{0,k} = F_k + \sum_{i=1}^{\infty} F_{k+i} X_i$$
$$= D_k (I - \Phi_0)^{-1} + \sum_{i=1}^{\infty} D_{k+i} (I - \Phi_0)^{-1} X_i.$$

This completes the proof.

Let ϕ_R, ϕ_{R_0}, ϕ_G, ϕ_{G_0}, ϕ_{A+}, ϕ_{A-}, ϕ_{D+} and ϕ_{D-} denote the radii of convergence for $R^*(z)$, $R_0^*(z)$, $G^*(z)$, $G_0^*(z)$, $A_+^*(z)$, $A_-^*(z)$, $D_+^*(z)$ and $D_-^*(z)$, respectively, where

$$A_+^*(z) = \sum_{k=1}^{\infty} z^k A_k, \quad A_-^*(z) = \sum_{k=1}^{\infty} z^{-k} A_{-k}.$$

The following theorem provides an important relation among the radii of convergence.

Theorem 3.8 (1) $\phi_R = \phi_{A+}$ and $\phi_G = \phi_{A-}$.

(2) $\phi_{R_0} = \phi_{D+}$ and $\phi_{G_0} = \phi_{D-}$.

Proof We only prove $\phi_R = \phi_{A+}$ and $\phi_{R_0} = \phi_{D+}$; while the other two can be proved similarly.

We first provide $\phi_R = \phi_{A+}$. Since $\sum_{k=1}^{\infty} A_{-k}$ and $\sum_{k=1}^{\infty} A_k$ are substochastic, $0 \leqslant \phi_{A_-} \leqslant 1 \leqslant \phi_{A_+} \leqslant +\infty$. At the same time, it is seen that $0 \leqslant \phi_G \leqslant 1 \leqslant \phi_R \leqslant +\infty$. It follows from Eq. (3.19) that

$$A^*(z) = \Phi_0 + R^*(z) + G^*(z) + R^*(z)\Phi_0 G^*(z) - R^*(z)\Phi_0 - \Phi_0 G^*(z) - R^*(z)G^*(z)$$

Since $R^*(z)$ is a power series and $G^*(z)$ is a Laurent series with only negative powers of z, $R^*(z)$ is analytic in $|z| < \phi_R$ and $G^*(z)$ is analytic in $|z| > \phi_G$. Noting that

$$A^*(z) = R^*(z)\left[I - \Phi_0 - G^*(z) + \Phi_0 G^*(z)\right] + I + \Phi_0 - \Phi_0 G^*(z)$$

and

$$A^*(z) = \left[I - \Phi_0 - R^*(z) + R^*(z)\Phi_0\right]G^*(z) + I + \Phi_0 - R^*(z)\Phi_0,$$

it is clear that $\phi_R = \phi_{A+}$ and $\phi_G = \phi_{A_-}$ by using $0 \leqslant \phi_{A_-} \leqslant 1 \leqslant \phi_{A+} \leqslant +\infty$ and $0 \leqslant \phi_G \leqslant 1 \leqslant \phi_R \leqslant +\infty$.

Now, we prove $\phi_{R_0} = \phi_{D+}$.

Note that for $z > 0$,

$$R_0^*(z) \geqslant D^*(z)(I - \Phi_0)^{-1} \tag{3.29}$$

and

$$R_0^*(z)(I - \Phi_0)[I - G^*(z)] \leqslant D^*(z).$$

Note that either $\phi_{D+}=1$ or $\phi_{D+}>1$, thus we need to discuss the following two possible cases:

Case Ⅰ $\phi_{D+}=1$. In this case, it is easy to see from Eq. (3.29) that $\phi_{R_0}\leqslant\phi_{D+}=1$. Noting that $R_0(1)$ is finite, we have $\phi_{R_0}\geqslant 1$. Thus $\phi_{R_0}=\phi_{D+}=1$.

Case Ⅱ $\phi_{D+}>1$. In this case, it is easy to see from Corollary 3.10 that $I-G^*(z)$ is invertible for any $z>1$ when P is irreducible and positive recurrent. Since $G^*(z)$ is non-increasing for $z\in(1,+\infty)$, we can consider $\delta>0$ to be small enough such that for all $z>1+\delta$,

$$R_0^*(z)(I-\Phi_0)[I-G^*(1+\delta)]\leqslant R_0^*(z)(I-\Phi_0)[I-G^*(z)]\leqslant D^*(z),$$

hence

$$R_0^*(z)\leqslant D^*(z)[I-G^*(1+\delta)]^{-1}(I-\Phi_0)^{-1},$$

which, together with Eq. (3.29), illustrates that for $z>1+\delta$,

$$D^*(z)(I-\Phi_0)^{-1}\leqslant R_0^*(z)\leqslant D^*(z)[I-G^*(1+\delta)]^{-1}(I-\Phi_0)^{-1}, \tag{3.30}$$

which indicates that $\phi_{R_0}=\phi_{D+}>1$.

This completes the proof.

In the rest of this section, we discuss the LU-type R-measure $\{\overline{R}_{i,j},0\leqslant j<i\}$ and G-measure $\{\overline{G}_{i,j},0\leqslant i<j\}$, and provide expressions for the LU-type R- and G-measures.

For the Markov chain of $GI/G/1$ type, it is easy to see that the sizes of the matrices $\overline{R}_{i,0}$ for $i\geqslant 1$ and $\overline{G}_{0,j}$ for $j\geqslant 1$ are $m\times m_0$ and $m_0\times m$, respectively; while the sizes of all the matrices $\overline{R}_{i,j}$ for $1\leqslant j<i$ and $\overline{G}_{i,j}$ for $1\leqslant i<j$ are $m\times m$. Let

$$\overline{R}^*_{\bullet,j}(z)=\sum_{i=j+1}^{\infty}z^{-i}\overline{R}_{i,j},\quad j\geqslant 0,$$

and

$$\overline{G}^*_{i,\bullet}(z)=\sum_{j=i+1}^{\infty}z^{j}\overline{G}_{i,j},\quad i\geqslant 0.$$

The following theorem and corollary provide expressions for $\overline{R}^*_{\bullet,j}(z)$ for $j\geqslant 0$ and $\overline{G}^*_{i,\bullet}(z)$ for $i\geqslant 0$. The proof is clear and is omitted here.

Theorem 3.9 (1)

$$\overline{R}^*_{\bullet,0}(z)(I-\Phi_0)=D^*_{-}(z)$$

and for $j\geqslant 1$,

$$\overline{R}^*_{\bullet,j}(z)(I-\Phi_j)=z^{-j}A^*_-(z)+\sum_{k=0}^{j-1}\overline{R}^*_{\bullet,k}(z)(I-\Phi_k)\overline{G}_{k,j}.$$

(2)

$$(I-\Phi_0)\overline{G}^*_{0,\bullet}(z)=D^*_+(z)$$

and for $i\geqslant 1$,

$$(I-\Phi_i)\overline{G}^*_{i,\bullet}(z)=z^iA^*_+(z)+\sum_{k=0}^{i-1}\overline{R}_{i,k}(I-\Phi_k)\overline{G}^*_{k,\bullet}(z).$$

For $0\leqslant l<k$, we write

$$S_{l,k}=\overline{G}_{l,k}+\sum_{1\leqslant i_1\leqslant k-1}\overline{G}_{l,i_1}\overline{G}_{i_1,k}+\sum_{1\leqslant i_1<i_2\leqslant k-1}\overline{G}_{l,i_1}\overline{G}_{i_1,i_2}\overline{G}_{i_2,k}$$
$$+\ldots+\overline{G}_{l,l+1}\overline{G}_{l+1,l+2}\ldots\overline{G}_{k-1,k}$$

and

$$T_{k,l}=\overline{R}_{k,l}+\sum_{1\leqslant i_1\leqslant k-1}\overline{R}_{k,i_1}\overline{R}_{i_1,l}+\sum_{1\leqslant i_2<i_1\leqslant k-1}\overline{R}_{k,i_1}\overline{R}_{i_1,i_2}\overline{R}_{i_2,l}$$
$$+\ldots+\overline{R}_{k,k-1}\overline{R}_{k-1,k-2}\ldots\overline{R}_{l+1,l}.$$

Corollary 3.1 *For* $k\geqslant 1$,

$$\overline{R}^*_{\bullet,k}(z)(I-\Phi_k)=z^{-k}A^*_-(z)+D^*_-(z)S_{0,k}+A^*_-(z)\sum_{l=1}^{k-1}z^{-l}S_{l,k}$$

and

$$(I-\Phi_k)\overline{G}^*_{k,\bullet}(z)=z^kA^*_+(z)+T_{k,0}D^*_+(z)+\sum_{l=1}^{k-1}z^lT_{k,l}A^*_+(z).$$

3.2 Dual Markov Chains

In this section, we define a dual Markov chain for any irreducible Markov chain, and provide a useful relationship for the *R*- and *G*-measures between the dual Markov chain and the original Markov chain. For the Markov chain of $GI/G/1$ type, we further analyze the *R*- and *G*-measures, and obtain useful properties based on the dual chain.

Definition 3.1 *For a Markov chain P, if there exists a positive left super-regular vector* $\alpha=(\alpha_0,\alpha_1,\alpha_2,\ldots)$ *such that* $\alpha\geqslant\alpha P$*, we write*

$$\widetilde{P}=\mathrm{diag}^{-1}(\alpha)P^{\mathrm{T}}\mathrm{diag}(\alpha),$$

where $diag(\alpha) = diag(\alpha_0, \alpha_1, \alpha_2, ...)$, *then we call* $\widetilde{P}$ *a dual Markov chain of the Markov chain P.*

It is clear that if $\widetilde{P} = (\widetilde{P}_{i,j})$ is a dual chain of the Markov chain $P = (P_{i,j})$, then

$$\widetilde{P}_{i,j} = \text{diag}^{-1}(\alpha_i) P_{j,i}^{\mathrm{T}} \text{ diag}(\alpha_j).$$

In general, the dual chain $\widetilde{P}$ can be probabilistically interpreted in terms of the time reversibility of the original chain *P*, e.g., see Kemeny, Snell and Knapp [5].

Let $\{R_{i,j}\}$ and $\{G_{i,j}\}$, $\{\widetilde{R}_{i,j}\}$ and $\{\widetilde{G}_{i,j}\}$ be the UL-type *R*- and *G*-measures for the Markov chain *P* and the dual chain $\widetilde{P}$, respectively. It is easy to check from Corollary 2.2 that

$$\widetilde{R}_{i,j} = \text{diag}^{-1}(\alpha_i)\, G_{j,i}^{\mathrm{T}} \text{diag}(\alpha_j)$$

and

$$\widetilde{G}_{i,j} = \text{diag}^{-1}(\alpha_i) R_{j,i}^{\mathrm{T}} \text{diag}(\alpha_j).$$

At the same time, we have

$$\widetilde{\varPsi}_n = \text{diag}^{-1}(\alpha_n)\, \varPsi_n^{\mathrm{T}} \text{diag}(\alpha_n).$$

On the other hand, for the LU-type *R*-, *U*- and *G*-measures we have

$$\hat{R}_{i,j} = \text{diag}^{-1}(\alpha_i)\, \overline{G}_{j,i}^{\mathrm{T}} \text{diag}(\alpha_j),$$

$$\hat{G}_{i,j} = \text{diag}^{-1}(\alpha_i)\, \overline{R}_{j,i}^{\mathrm{T}} \text{diag}(\alpha_j)$$

and

$$\widetilde{\varPhi}_n = \text{diag}^{-1}(\alpha_n)\, \varPhi_n^{\mathrm{T}} \text{diag}(\alpha_n).$$

Let $P[0]$ be a matrix obtained from the matrix *P* by deleting the first row and the first column. The following proposition provides a relationship of the positive left super-regular vectors between *P* and $P[0]$. The proof is clear and is omitted here.

Proposition 3.1 *If* $\alpha = (\alpha_0, \alpha_1, \alpha_2, ...)$ *is a positive left super-regular vector of P, then* $\beta = (\alpha_1, \alpha_2, \alpha_3, ...)$ *is also a positive left super-regular vector of* $P[0]$.

Applying the dual chain, the following theorem gives an important property for the R-measure based on $G_{i,j}e \leqslant e$ and $\overline{G}_{i,j}e \leqslant e$.

Theorem 3.10 *If* $\alpha = (\alpha_0, \alpha_1, \alpha_2, ...)$ *is a positive left super-regular vector of P, then*

(1) $\alpha_i R_{i,j} \leqslant \alpha_j$ *for* $0 \leqslant i < j$, *and*

(2) $\alpha_i \overline{R}_{i,j} \leqslant \alpha_j$ *for* $0 \leqslant j < i$.

Proof We only provide (1), while (2) can be proved similarly.

According to the definition of the G-measure, it is easy to see that $\widetilde{G}_{j,i} e \leqslant e$. Applying $\widetilde{G}_{j,i} = \text{diag}^{-1}(\alpha_j) R_{i,j}^{\mathrm{T}} \text{diag}\,(\alpha_i)$, we obtain

$$\text{diag}^{-1}(\alpha_j) R_{i,j}^{\mathrm{T}} \text{diag}(\alpha_i) e = \widetilde{G}_{j,i} e \leqslant e,$$

Thus, $\alpha_i R_{i,j} \leqslant \alpha_j$ for $0 \leqslant j < i$. This completes the proof.

In what follows we study the dual chain for the the Markov chain P of $GI/G/1$ type.

Let

$$P[0] = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 & \cdots \\ A_{-1} & A_0 & A_1 & A_2 & \cdots \\ A_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ A_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The following lemma provides a relation of the positive left super-regular vectors between P and A. The proof is clear and is omitted here.

Lemma 3.3 *If θ is a positive left super-regular vector of the Markov chain $A = \sum_{k=-\infty}^{\infty} A_k$, then $(\theta, \theta, \theta, \ldots)$ is also a positive left super-regular vector of the Markov chain P[0].*

Remark 3.3 *If the Markov chain A with finite states is irreducible and stochastic, then the Markov chain A is positive recurrent, and θ is its stationary probability vector.*

By using Lemma 3.3, it is easy to see that

$$\widetilde{R}_k = \text{diag}^{-1}(\theta)\, G_k^{\mathrm{T}} \text{diag}\,(\theta)$$

and

$$\widetilde{G}_k = \text{diag}^{-1}(\theta) R_k^{\mathrm{T}} \text{diag}\,(\theta).$$

Definition 3.2 *For the Markov chain P of $GI/G/1$ type, we write*

$$P = \begin{pmatrix} D_0 & U \\ V & P[0] \end{pmatrix}$$

and

$$\widetilde{P_{[0]}} = \begin{pmatrix} D_0 & U \\ V & \widetilde{P[0]} \end{pmatrix},$$

where

$$\widetilde{P[0]} = \mathrm{diag}^{-1}(\theta, \theta, \theta, ...)P[0]^{\mathrm{T}}\mathrm{diag}\,(\theta, \theta, \theta, ...)$$

Then we call $\widetilde{P_{[0]}}$ *the dual generating chain of P.*

Theorem 3.11 *If θ is a positive left super-regular vector of the Markov chain A, then* $\theta R \leqslant \theta$.

Proof In the dual generating chain $\widetilde{P_{[0]}}$, $\widetilde{G} = \sum_{k=1}^{\infty} \widetilde{G}_k$ is stochastic or substochastic, that is, $\widetilde{G}e \leqslant e$. Since $\widetilde{G} = \mathrm{diag}^{-1}(\theta)R^{\mathrm{T}}\mathrm{diag}\,(\theta)$, we obtain

$$e \geqslant \widetilde{G}e = \mathrm{diag}^{-1}(\theta)R^{\mathrm{T}}\mathrm{diag}(\theta)e,$$

which leads to

$$\theta^{\mathrm{T}} = \mathrm{diag}(\theta)e \geqslant R^{\mathrm{T}}\mathrm{diag}(\theta)e = R^{\mathrm{T}}\theta^{\mathrm{T}},$$

clearly, $\theta R \leqslant \theta$. This completes the proof.

3.3 The *A*- and *B*-Measures

In this section, we simplify the *A*- and *B*-measures, given in Section 2.5, for Markov chains of $GI/G/1$ type, and construct new expressions for the *A*- and *B*-measures by means of the *R*- and *G*-measures. Based on the *A*- and *B*-measures, we provide conditions for the state classification of Markov chains of $GI/G/1$ type.

For the Markov chain of $GI/G/1$ type, we write

$$\widetilde{R}_U = \begin{pmatrix} 0 & R_1 & R_2 & R_3 & \cdots \\ & 0 & R_1 & R_2 & \cdots \\ & & 0 & R_1 & \cdots \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}$$

and

$$\widetilde{G_L} = \begin{pmatrix} 0 & & & & \\ G_1 & 0 & & & \\ G_2 & G_1 & 0 & & \\ G_3 & G_2 & G_1 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

From (1) in Theorem 2.10, we have

$$A_{0,j} = \begin{cases} R_{0,1}, & \text{if } j = 1, \\ R_{0,j} + \sum_{k=1}^{j-1} A_{0,k} R_{k-j}, & \text{if } j \geqslant 2, \end{cases} \tag{3.31}$$

or the matrix expression

$$(A_{0,1}, A_{0,2}, A_{0,3}, \ldots)\left(I - \widetilde{R_U}\right) = (R_{0,1}, R_{0,2}, R_{0,3}, \ldots).$$

Using (2) in Theorem 2.10, the matrix $B_{i,0}$ and the G-measure satisfy

$$B_{i,0} = \begin{cases} G_{1,0}, & \text{if } i = 1, \\ G_{i,0} + \sum_{k=1}^{i-1} G_{i-k} B_{k,0}, & \text{if } i \geqslant 2, \end{cases} \tag{3.32}$$

or the matrix expression

$$\left(I - \widetilde{G_L}\right)\left(B_{1,0}^{\mathrm{T}}, B_{2,0}^{\mathrm{T}}, B_{3,0}^{\mathrm{T}}, \ldots\right)^{\mathrm{T}} = \left(G_{1,0}^{\mathrm{T}}, G_{2,0}^{\mathrm{T}}, G_{3,0}^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}.$$

If $1 \leqslant i < j$, then using Corollary 2.6, we have

$$A_{i,j} = \begin{cases} R_1, & \text{if } j = i+1, \\ R_{j-i} + \sum_{k=i+1}^{j-1} A_{i,k} R_{j-k}, & \text{if } j \geqslant i+2, \end{cases} \tag{3.33}$$

and if $1 \leqslant j < i$, then

$$B_{i,j} = \begin{cases} G_1, & \text{if } j = i-1, \\ G_{i-j} + \sum_{k=1}^{i-1} G_{i-k} B_{k,j}, & \text{if } j \leqslant i-2. \end{cases} \tag{3.34}$$

At the same time, it is clear that

$$A_{i,i} = A_{i,0} R_{0,i} + \sum_{k=1}^{i-1} A_{i,k} R_{i-k} + \sum_{k=i+1}^{\infty} R_{k-i} A_{k,i}$$

and

$$B_{i,i} = G_{i,0} B_{0,i} + \sum_{k=1}^{i-1} G_{i-k} B_{k,i} + \sum_{k=i+1}^{\infty} B_{i,k} G_{k-i}.$$

Let

$$\mathcal{A}_0^*(z) = \sum_{j=1}^{\infty} z^j A_{0,j}$$

and

$$\mathcal{A}_i^*(z) = \sum_{j=i+1}^{\infty} z^j A_{i,j};$$

$$\mathcal{B}_0^*(z) = \sum_{i=1}^{\infty} z^{-i} B_{i,0}$$

and

$$\mathcal{B}_j^*(z) = \sum_{i=j+1}^{\infty} z^{-i} B_{i,j}.$$

Then it respectively follows from Eq. (3.31) to Eq. (3.34) that

$$\mathcal{A}_0^*(z)[I - R^*(z)] = R_0^*(z),$$

$$\mathcal{A}_i^*(z)[I - R^*(z)] = R^*(z), \quad i \geqslant 1,$$

$$[I - G^*(z)]\mathcal{B}_0^*(z) = G_0^*(z)$$

and

$$[I - G^*(z)]\mathcal{B}_j^*(z) = G^*(z), \quad j \geqslant 1.$$

Let

$$\mathcal{A}_i = \sum_{j=i+1}^{\infty} A_{i.j}, \quad i \geqslant 0$$

and

$$\mathcal{B}_j = \sum_{i=j+1}^{\infty} B_{i,j}, \quad j \geqslant 0.$$

Then

$$\mathcal{A}_0 (I - R) = R_0,$$

$$\mathcal{A}_i (I - R) = R, \quad i \geqslant 1;$$

$$(I - G)\mathcal{B}_0 = G_0$$

and

$$(I - G)\mathcal{B}_j = G, \quad j \geqslant 1.$$

We now provide conditions for the state classification of Markov chains of $GI/G/1$ type based on the A- and B-measures. For convenience of description, we assume that the Markov chain P given in Eq. (3.1) and the matrix A are irreducible and stochastic, unless stated.

Lemma 3.4 *The Markov chain P is recurrent if and only if for all* $i \geqslant 1$

$$G_{i,0}e + \sum_{k=1}^{i-1} G_{i-k}e = e.$$

Proof This proof may directly follow Lemma 2.9 and Theorem 2.11.

It is seen from Theorem 3.2 that if the matrix A is stochastic, then $\lim_{i\to\infty} G_{i,0} = 0.$ Therefore, we have the following corollary.

Corollary 3.2 (1) *The Markov chain P is recurrent if and only if G is stochastic.*

(2) *The Markov chain P is transient if and only if and G is strictly substochastic.*

Theorem 3.12 *If the Markov chain P is recurrent, then P is positive recurrent if and only if* R_0 *is finite and* $\lim_{k\to\infty} R^k = 0.$

Proof Suppose first that P is positive recurrent. It follows from Theorem 3.8 that the matrix A_0 is finite. Note that $\mathcal{A}_0(I-R) = R_0,$ we obtain $R_0 \leqslant \mathcal{A}_0,$ which leads to the conclusion that R_0 is finite. Since

$$\begin{aligned}
\mathcal{A}_0 &= R_0 + \mathcal{A}_0 R \\
&= R_0 + \left(R_0 + \mathcal{A}_0 R\right)R = R_0 + R_0 R + \mathcal{A}_0 R^2 \\
&= R_0 + R_0 R + \left(R_0 + \mathcal{A}_0 R\right)R^2 = R_0 + R_0 R + R_0 R^2 + \mathcal{A}_0 R^3 \\
&= \dots \\
&= R_0 \sum_{k=0}^{N-1} R^k + \mathcal{A}_0 R^N \\
&\geqslant R_0 \sum_{k=0}^{N} R^k
\end{aligned}$$

for all $N \geqslant 1,$ hence we obtain

$$\lim_{k\to\infty} R^k = 0.$$

Suppose now that R_0 is finite and $\lim_{k\to\infty} R^k = 0.$ In this case, the matrix $I-R$ is irreducible. Since $\mathcal{A}_0(I-R) = R_0,$ we obtain

$$\mathcal{A}_0 = R_0 (I-R)^{-1} < +\infty.$$

Therefore, P is positive recurrent by means of Theorem 2.12. This completes the proof.

Let

$$W = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 & \cdots \\ A_{-1} & A_0 & A_1 & A_2 & \cdots \\ A_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ A_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

For the Markov chain W, we introduce the A- and B-measures for the first block-row and the first block-column as follows: $\overline{A}_0 = \overline{B}_0 = I$, $\overline{A}_k = A_{1,k+1}$ and $\overline{B}_k = B_{k+1,1}$ for $k \geqslant 1$.

The following corollary provides two simple and useful relations between the A- and R-measures and the B- and G-measures, respectively. The proof is easy by using Eq. (3.33) and Eq. (3.34).

Corollary 3.3

$$\overline{A}_k = \sum_{i=0}^{k-1} \overline{A}_i R_{k-i} \tag{3.35}$$

and

$$\overline{B}_k = \sum_{i=0}^{k-1} G_{k-i} \overline{B}_i. \tag{3.36}$$

Let

$$\overline{A}^*(z) = \sum_{k=0}^{\infty} z^k \overline{A}_k$$

and

$$\overline{B}^*(z) = \sum_{k=0}^{\infty} z^k \overline{B}_k.$$

Then it follows from Eq. (3.35) that

$$\overline{A}^*(z)[I - R^*(z)] = I \tag{3.37}$$

and from Eq. (3.36) that

$$[I - G^*(z)]\overline{B}^*(z) = I. \tag{3.38}$$

Now, we provide a useful relation between Φ_n and the A- and B-measures. This relation leads to a new expression for the R- and G-measures.

Corollary 3.4 (1) *For* $n \geqslant 0$,

$$\Phi_n = \sum_{i=0}^{\infty} A_{n+i} \overline{B}_i$$

and

$$\Phi_{-n} = \sum_{i=0}^{\infty} \overline{A}_i A_{-n-i}.$$

(2) *For* $n \geqslant 0$,

$$R_n(I - \Phi_0) = \sum_{i=0}^{\infty} A_{n+i} \overline{B}_i$$

and

$$(I - \Phi_0) G_n = \sum_{i=0}^{\infty} \overline{A}_i A_{-n-i}.$$

Proof It follows from Eq. (3.5) and Eq. (3.4) that

$$\Phi_0 = W_{n,n}^{[\leqslant n]} = A_0 + (A_1, A_2, A_3, \ldots) \widehat{W} \left(A_{-1}^{\mathrm{T}}, A_{-2}^{\mathrm{T}}, A_{-3}^{\mathrm{T}}, \ldots \right)^{\mathrm{T}},$$

where $\widehat{W} = (I - W)_{\min}^{-1}$. Using Eq. (2.44) we obtain

$$\begin{aligned} \widehat{W}\left(A_{-1}^{\mathrm{T}}, A_{-2}^{\mathrm{T}}, A_{-3}^{\mathrm{T}}, \ldots \right)^{\mathrm{T}} &= \left(B_{2,1}^{\mathrm{T}}, B_{3,1}^{\mathrm{T}}, B_{4,1}^{\mathrm{T}}, \ldots \right)^{\mathrm{T}} \\ &= \left(\overline{B}_1^{\mathrm{T}}, \overline{B}_2^{\mathrm{T}}, \overline{B}_3^{\mathrm{T}}, \ldots \right)^{\mathrm{T}}, \end{aligned}$$

thus it is easy to see that

$$\Phi_0 = \sum_{i=0}^{\infty} A_i \overline{B}_i.$$

On the other hand, note that

$$\begin{aligned} (A_1, A_2, A_3, \ldots) \widehat{W} &= (A_{1,2}, A_{1,3}, A_{1,4}, \ldots) \\ &= (\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots), \end{aligned}$$

thus we obatin

$$\Phi_0 = \sum_{i=0}^{\infty} \overline{A}_i A_{-i}.$$

Similarly, we can easily prove the other equations by means of $R_n(I - \Phi_0) = \Phi_n$ and $(I - \Phi_0) G_n = \Phi_{-n}$ for $n \geqslant 1$. This completes the proof.

The following lemma expresses the fundamental matrix $\widehat{W} = (\widehat{W}_{i,j})$ by means of the *R*-, *U*- and *G*-measures, where $\widehat{W}_{i,j}$ is a matrix of size $m \times m$.

Lemma 3.5 (1)

$$\widehat{W}_{1,1} = (I - \Phi_0)^{-1}.$$

(2) *For* $i \geqslant 2$ *or* $j \geqslant 2$,

$$\widehat{W}_{i,j} = \begin{cases} \sum_{k=1}^{i-1} G_{i-k} \widehat{W}_{k,j}, & \text{if } i \geqslant j, \\ \sum_{k=1}^{j-1} \widehat{W}_{i,k} R_{j-k} & \text{if } i \leqslant j. \end{cases}$$

Proof (1) is clear. We only need to prove the first part of (2) with $i \geqslant j$; while the other part for $i \leqslant j$ can be proved similarly.

The *RG*-factorization of the matrix $I - W$ is given by

$$I - W = (I - R_U)(I - \varPhi_D)(I - G_L),$$

which leads to

$$(I - G_L)\widehat{W} = (I - \varPhi_D)^{-1}(I - R_U)^{-1}. \tag{3.39}$$

Hence we obtain

$$\begin{pmatrix} I & & & & \\ -G_1 & I & & & \\ -G_2 & -G_1 & I & & \\ -G_3 & -G_2 & -G_1 & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} (\widehat{W}_{i,j}) = \begin{pmatrix} (I - \varPhi_0)^{-1} & * & * & \cdots \\ & (I - \varPhi_0)^{-1} & * & \cdots \\ & & (I - \varPhi_0)^{-1} & \cdots \\ & & & \ddots \end{pmatrix}, \tag{3.40}$$

where $*$ denotes the corresponding block-entry of the matrix $(I - \varPhi_D)^{-1}(I - R_U)^{-1}$. It follows from Eq. (3.40) that

$$\begin{cases} \widehat{W}_{1,1} = (I - \varPhi_0)^{-1}, \\ \widehat{W}_{i,j} - \sum_{k=1}^{i-1} G_{i-k} \widehat{W}_{k,j} = 0, \quad i > j \geqslant 1. \end{cases}$$

Simple matrix computation can easily lead to the desired results.

A useful relation between the fundamental matrix and the *A*- and *B*-measures is given in the following theorem.

Theorem 3.13

$$\widehat{W}_{i,j} = \widehat{W}_{i-1,j-1} + \overline{B}_{i-1} \widehat{W}_{1,1} \overline{A}_{j-1}$$

or

$$\widehat{W}_{i,j} = \begin{cases} \sum_{k=0}^{i-1} \overline{B}_k \widehat{W}_{1,1} \overline{A}_{k+j-i}, & \text{if } 1 \leqslant i \leqslant j, \\ \sum_{k=0}^{j-1} \overline{B}_{k+i-j} \widehat{W}_{1,1} \overline{A}_k, & \text{if } 1 \leqslant j \leqslant i. \end{cases}$$

Proof Let

$$W = \begin{pmatrix} A_0 & U \\ V & W \end{pmatrix}.$$

Then

$$\begin{aligned}\widehat{W} &= \begin{pmatrix} * & * \\ * & \widehat{W} + \widehat{W}V(I - A_0 - U\widehat{W}V)^{-1}U\widehat{W} \end{pmatrix} \\ &= \begin{pmatrix} * & * \\ * & \widehat{W} + \widehat{W}V(I - \Phi_0)^{-1}U\widehat{W} \end{pmatrix}. \end{aligned} \tag{3.41}$$

Note that

$$\widehat{W}_{1,1} = (I - \Phi_0)^{-1},$$

$$\widehat{W}V = \left(\overline{B}_1^{\mathrm{T}}, \overline{B}_2^{\mathrm{T}}, \overline{B}_3^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}$$

and

$$U\widehat{W} = (\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots),$$

it follows from Eq. (3.41) that

$$\widehat{W}_{i,j} = \widehat{W}_{i-1,j-1} + \overline{B}_{i-1}\widehat{W}_{1,1}\overline{A}_{j-1}.$$

Some matrix computations can lead to the desired result.

Corollary 3.5 (1)

$$\sum_{i=1}^{\infty} \widehat{W}_{i,1} = \overline{B}\widehat{W}_{1,1}$$

(2)

$$\sum_{j=1}^{\infty} \widehat{W}_{1,j} = \widehat{W}_{1,1}\overline{A}.$$

Proof The proof only needs to use the two relations as follows:

$$\widehat{W}_{i,1} = \overline{B}_{i-1}\widehat{W}_{1,1}$$

and

$$\widehat{W}_{1,j} = \widehat{W}_{1,1}\overline{A}_{j-1}.$$

This completes the proof.

The following theorem provides a useful relation between $\{R_{0,k}\}$ and $\{D_k\}$.

Theorem 3.14 *If the Markov chain P is irreducible and stochastic, then the condition that* $\sum_{k=1}^{\infty} kD_k$ *is finite implies that* R_0 *is finite.*

Proof Since $\widehat{W}_{1,1}$ is finite and nonnegative, there is always a constant $b > 0$ such that $\widehat{W}_{1,1} e \leqslant be$. Therefore,

$$\sum_{i=k}^{\infty} \widehat{W}_{i,1} e \leqslant \sum_{i=1}^{\infty} \widehat{W}_{i,1} e = \overline{B} \widehat{W}_{1,1} e \leqslant be$$

due to $\overline{B}e \leqslant e$. Note that

$$\begin{aligned} R_0 e &= \sum_{k=1}^{\infty} R_{0,k} e = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} D_i \widehat{W}_{i,1} e \\ &\leqslant \sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} D_i \right) \left(\sum_{i=k}^{\infty} \widehat{W}_{i,1} \right) e \\ &\leqslant b \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} D_i e = b \sum_{i=1}^{\infty} \sum_{k=1}^{i} D_i e \\ &= b \sum_{i=1}^{\infty} i D_i e, \end{aligned}$$

it is easy to see that the condition that $\sum_{k=1}^{\infty} kD_k$ is finite implies that R_0 is finite.

In general, the two matrices R and G are always finite for any irreducible Markov chain. However, the matrices $\overline{A}^*(1) = \overline{A}$ and $\overline{B}^*(1) = \overline{B}$ may be infinite for some irreducible Markov chain. In this case, we need sufficient conditions under which $\overline{A}$ or $\overline{B}$ is finite. Therefore, if $\overline{A}$ is finite, then

$$\overline{A}(I - R) = I; \tag{3.42}$$

if $\overline{B}$ is finite, then

$$(I - G)\overline{B} = I. \tag{3.43}$$

The following lemma provides conditions under which the matrix $\overline{A}$ or $\overline{B}$ is finite.

Lemma 3.6 *Suppose that the Markov chain P of GI/G/1 type is irreducible and stochastic, and the matrix* $\sum_{k=1}^{\infty} kD_k$ *is finite.*

(1) *P is positive recurrent if and only if* $\overline{A} < +\infty$,

(2) *P is transient if and only if* $\overline{B} < +\infty$, *and*

(3) *P is null recurrent if and only if* $\overline{A} = +\infty$ *and* $\overline{B} = +\infty$.

Proof We only prove (1), while (2) and (3) can be proved similarly.

Suppose first that P is positive recurrent. It follows from Theorem 3.12 that $\lim_{k\to\infty} R^k = 0$, and $(I-R)^{-1} = \sum_{k=0}^{\infty} R^k < \infty$. Thus, from Eq. (3.37) we obtain

$$\overline{A} = \lim_{z\to 1^-} \overline{A}^*(z) = \lim_{z\to 1^-} [I - R^*(z)]^{-1} = (I-R)^{-1} < \infty.$$

Suppose now that $\overline{A} < +\infty$. It follows from Eq. (3.42) that

$$\sum_{k=0}^{\infty} R^k = \overline{A} < \infty,$$

which leads to

$$\lim_{k\to\infty} R^k = 0,$$

therefore, P is positive recurrent by means of Theorem 3.12. This completes the proof.

The following theorem provides a necessary and sufficient condition for the state classification of Markov chains of $GI/G/1$ type.

Theorem 3.15 *Suppose that the Markov chain P of $GI/G/1$ type is irreducible and stochastic, and the matrix $\sum_{k=1}^{\infty} kD_k$ is finite.*

(1) *P is positive recurrent if and only if for each $i \geqslant 1$, $\sum_{j=1}^{\infty} \widehat{W}_{i,j}$ is finite,*

(2) *P is transient if and only if for each $j \geqslant 1$, $\sum_{i=1}^{\infty} \widehat{W}_{i,j}$ is finite, and*

(3) *P is null recurrent if and only if there exists an $i \geqslant 1$ such that $\sum_{j=1}^{\infty} \widehat{W}_{i,j}$ is infinite, and there exists a $j \geqslant 1$ such that $\sum_{i=1}^{\infty} \widehat{W}_{i,j}$ is infinite.*

Proof We only prove (1), while (2) and (3) can be proved similarly.

Using Theorem 3.13, we obtain

$$\sum_{j=1}^{\infty} \widehat{W}_{i,j} = \left(\sum_{k=0}^{i-1} \overline{B}_k \right) \widehat{W}_{1,1} \sum_{j=1}^{\infty} \overline{A}_{k+j-i} = \left(\sum_{k=0}^{i-1} \overline{B}_k \right) \widehat{W}_{1,1} \overline{A},$$

we obtain that for each $i \geqslant 1$, $\sum_{j=1}^{\infty} \widehat{W}_{i,j}$ is finite if and only if $\overline{A} < +\infty$. Since $\overline{A}(I-R) = I$, P is positive recurrent if and only if $\overline{A} < +\infty$. Hence it is easy to see the desired result.

The following theorem provides a sufficient condition under which the Markov chain P of $GI/G/1$ type is positive recurrent.

Theorem 3.16 *Suppose that P is irreducible and stochastic, and the matrix $\sum_{k=1}^{\infty} kD_k$ is finite. If the matrix A is strictly substochastic, then P is positive recurrent.*

Proof It is easy to check that if the matrix A is strictly substochastic, then the matrix $(I-W)^{-1}$ is finite, and for each $i \geqslant 1$, $\sum_{j=1}^{\infty} \widehat{W}_{i,j}$ is finite. The desired result follows from Theorems 3.15 and 3.14. This completes the proof.

The following corollary provides some important relations among the A- and B-measures and the R- and G-measures.

Corollary 3.6 *For the Markov chain W,*

(1) $\overline{A} < +\infty$ *if and only if* $\theta R < \theta$,

(2) $\overline{B} < +\infty$ *if and only if* $Ge < e$, *and*

(3) $\overline{A} = +\infty$ *and* $\overline{B} = +\infty$ *if and only if* $\theta R = \theta$ *and* $Ge = e$.

Proof We only prove (1), while (2) and (3) can be proved similarly.

We construct a Markov chain of $GI/G/1$ type whose transition probability matrix is given by

$$P = \begin{pmatrix} D_0 & D_+ \\ D_- & W \end{pmatrix},$$

where

$$D_+ = (D_1, D_2, D_3, \ldots)$$

with $\sum_{k=1}^{\infty} kD_k < +\infty$, and

$$D_- = \left(D_{-1}^{\mathrm{T}}, D_{-2}^{\mathrm{T}}, D_{-3}^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}.$$

Since $\sum_{k=1}^{\infty} kD_k < +\infty$, it is clear from Lemma 3.6 that Markov chain P is positive recurrent if and only if $\overline{A} < +\infty$, and if and only if $\lim_{k\to\infty} R^k = 0$, which leads to $\theta R < \theta$. This completes the proof.

3.4 Spectral Analysis

In this section, we provide spectral analysis for the R- and G-measures of Markov chains of $GI/G/1$ type. Note that the spectral analysis is a standard technique used in dealing with stochastic models, e.g., see Neuts [13, 14] for Markov chains of $GI/M/1$ type and Markov chains of $M/G/1$ type, respectively.

Let H be a nonnegative matrix of size m, and μ_i the ith eigenvalue of H for $1 \leqslant i \leqslant m$. we write

$$sp(H) = \max\{\mu_1, \mu_2, ..., \mu_m\}.$$

The following lemma first provides a spectral property for the matrices R and G.

Lemma 3.7 (1) *Let A be irreducible and stochastic. Then $sp(R) \leqslant 1$ and $sp(G) \leqslant 1$.*

(2) *Let A be irreducible and strictly substochastic. Then $sp(R)<1$ and $sp(G)<1$.*

Proof We first prove (1). Let A be irreducible and stochastic. Then $sp(A)=1$. It follows from Eq. (3.20) that

$$I - A = (I - R)(I - \Phi_0)(I - G)$$

and

$$\det(I - A) = \det(I - R)\det(I - \Phi_0)\det(I - G).$$

Since

$$\theta(I - A) = 0, \quad (I - A)e = 0,$$

it follows from $sp(A)=1$ that $\theta R \leqslant \theta$ and $Ge \leqslant e$, thus it is clear that $sp(R) \leqslant 1$ and $sp(G) \leqslant 1$.

We now prove (2). If A is irreducible and strictly substochastic, then $\det(I - A) \neq 0$, and hence $\det(I - R) \neq 0$ and $\det(I - G) \neq 0$. Again, applying the condition that $sp(R) \leqslant 1$ and $sp(G) \leqslant 1$, it is easy to see that $sp(R) < 1$ and $sp(G) < 1$.

This completes the proof.

It is seen from (2) of Lemma 3.7 that if A is strictly substochastic, then $sp(R) < 1$ and $sp(G) < 1$. Thus, in the rest of this section we always assume that A is irreducible and stochastic. Let θ be the stationary probability vector of the Markov chain A.

The following theorem provides a necessary and sufficient condition on the state classification of the Markov chain P of $GI/G/1$ type.

Theorem 3.17 *Suppose that P is irreducible and stochastic, and the matrix $\sum_{k=1}^{\infty} kD_k$ is finite.*

(1) *P is positive recurrent if and only if $sp(R) < 1$. In this case, $sp(G) = 1$.*

(2) *P is null recurrent if and only if $sp(R) = 1$ and $sp(G) = 1$.*

(3) *P is transient if and only if $sp(G) < 1$. In this case, $sp(R) = 1$.*

Proof We only prove (1), while (2) and (3) can be proved similarly.

If P is positive recurrent, then it is seen from Lemma 3.6 that $\overline{A} < +\infty$. Since

$\overline{A}(I-R)=I$, we obtain that $\sum_{k=0}^{\infty} R^k = (I-R)^{-1} = \overline{A} < +\infty$, which leads to $sp(R)<1$.

On the other hand, if P is positive recurrent, then $\overline{B}=+\infty$ by using Lemma 3.6. Suppose that $sp(G)<1$. Since $(I-G)\overline{B}=I$, we would obtain that $\overline{B}=(I-G)^{-1}<+\infty$. This leads to a contradiction that $\overline{B}=+\infty$. Thus $sp(G)=1$.

If $sp(R)<1$, then the matrix $I-R$ is invertible. Since $\overline{A}(I-R)=I$, we obatin that $\overline{A}=(I-R)^{-1}<+\infty$. Hence P is positive recurrent.

This completes the proof.

The following corollary provides conditions for the state classification of the Markov chain P of $GI/G/1$ type. The proof follows Theorem 3.17 directly.

Corollary 3.7 *Suppose that P is irreducible and stochastic, and the matrix* $\sum_{k=1}^{\infty} kD_k$ *is finite.*

(1) P is positive recurrent if and only if $\det(I-R)\neq 0$,

(2) P is null recurrent if and only if $\det(I-R)=0$ and $\det(I-G)=0$, and

(3) P is transient if and only if $\det(I-G)\neq 0$.

The following theorem provides important properties for the matrices R and G, which are related to the state classification.

Theorem 3.18 *If P is irreducible and stochastic, and the matrix* $\sum_{k=1}^{\infty} kD_k$ *is finite.*

(1) *P is positive recurrent if and only if* $\theta R<\theta$. *In this case,* $Ge=e$.

(2) *P is null recurrent if and only if* $\theta R=\theta$ *and* $Ge=e$.

(3) *P is transient if and only if* $Ge<e$. *In this case,* $\theta R=\theta$.

Proof We only prove (1), while (2) and (3) can be proved similarly.

It follows from Eq. (3.20) that the RG-factorization for the repeated row

$$I-A=(I-R)(I-\Phi_0)(I-G).$$

Since A is irreducible and stochastic, it is clear that $\theta A=\theta$ and $Ae=e$. Thus we obtain

$$\theta(I-R)(I-\Phi_0)(I-G)=0$$

and

$$(I-R)(I-\Phi_0)(I-G)e=0.$$

Since P is positive recurrent if and only if $sp(R)<1$. In this case, $sp(G)=1$. Thus it is clear that $\theta R<\theta$ and $Ge=e$. This completes the proof.

Corollary 3.8 *If P is irreducible and stochastic, and the matrix* $\sum_{k=1}^{\infty} kD_k$ *is finite.*

(1) *P is positive recurrent if and only if the matrix* $I-\sum_{k=0}^{\infty}\Phi_k$ *is invertible. In*

this case, the matrix $I-\sum_{k=0}^{\infty}\Phi_{-k}$ *is singular.*

(2) *P is null recurrent if and only if the two matrices* $I-\sum_{k=0}^{\infty}\Phi_{-k}$ *and* $I-\sum_{k=0}^{\infty}\Phi_{k}$ *are both singular.*

(3) *P is transient if and only if the matrix* $I-\sum_{k=0}^{\infty}\Phi_{-k}$ *is invertible. In this case, the matrix* $I-\sum_{k=0}^{\infty}\Phi_{k}$ *is singular.*

Proof In the proof, we only need to note that

$$I-R=\left(I-\sum_{k=0}^{\infty}\Phi_{k}\right)(I-\Phi_0)^{-1}$$

and

$$I-G=(I-\Phi_0)^{-1}\left(I-\sum_{k=0}^{\infty}\Phi_{-k}\right),$$

thus the results are clear according to Theorem 3.18. This completes the proof.

The following theorem provides conditions for the state classification of the Markov chain P of $GI/G/1$ type. This is the same as Chapter $X\,3-4$ in Asmussen [2] for a necessary and sufficient condition based on the mean drift.

Theorem 3.19 *Suppose that P is irreducible and stochastic, and the matrix* $\sum_{k=1}^{\infty}kD_k$ *is finite.*

(1) *P is positive recurrent if and only if* $\theta\sum_{k=1}^{\infty}kA_{-k}e>\theta\sum_{k=1}^{\infty}kA_{k}e,$

(2) *P is null recurrent if and only if* $\theta\sum_{k=1}^{\infty}kA_{-k}e=\theta\sum_{k=1}^{\infty}kA_{k}e,$ *and*

(3) *P is transient if and only if* $\theta\sum_{k=1}^{\infty}kA_{-k}e<\theta\sum_{k=1}^{\infty}kA_{k}e.$

Proof We only prove (1), while (2) and (3) can be proved similarly.

Taking the derivative of the *RG*-factorization:

$$I-A^*(z)=[I-R^*(z)](I-\Phi_0)[I-G^*(z)]$$

at $z=1$, we obtain

$$\theta\sum_{k=1}^{\infty}k(A_k-A_{-k})e=\theta\left(\sum_{k=1}^{\infty}k\Phi_k\right)(e-Ge)-(\theta-\theta R)\left(\sum_{k=1}^{\infty}k\Phi_{-k}\right)e. \qquad (3.44)$$

Since P is irreducible, the matrix $I - \Phi_0$ is invertible. Note that

$$R = \sum_{k=1}^{\infty} \Phi_k (I - \Phi_0)^{-1}$$

and

$$G = (I - \Phi_0)^{-1} \sum_{k=1}^{\infty} \Phi_{-k},$$

it follows from Theorem 3.3 that there is no zero column in $\sum_{k=1}^{\infty} \Phi_k \left(\text{thus} \sum_{k=1}^{\infty} k\Phi_k \right)$ and there is no zero row in $\sum_{k=1}^{\infty} \Phi_{-k} \left(\text{thus} \sum_{k=1}^{\infty} k\Phi_{-k} \right)$.

Suppose firstly that P is positive recurrent. It follows from Theorem 3.18 that $\theta R < \theta$ and $Ge = e$. It follows from Eq. (3.44) that

$$\theta \sum_{k=1}^{\infty} k(A_k - A_{-k})e = -(\theta - \theta R) \left(\sum_{k=1}^{\infty} k\Phi_{-k} \right) e < 0$$

or

$$\theta \sum_{k=1}^{\infty} kA_k e < \theta \sum_{k=1}^{\infty} kA_{-k} e.$$

Suppose now that $\theta \sum_{k=1}^{\infty} kA_{-k} e > \theta \sum_{k=1}^{\infty} kA_k e$. It is clear that $\theta \sum_{k=1}^{\infty} k(A_k - A_{-k})e < 0$. Note that $\theta \left(\sum_{k=1}^{\infty} k\Phi_k \right)(e - Ge) \geqslant 0$, it follows from Eq. (3.44) that $(\theta - \theta R) \cdot \left(\sum_{k=1}^{\infty} k\Phi_{-k} \right) e > 0$, which leads to $\theta R < \theta$, hence P is positive recurrent, and $Ge = e$. This completes the proof.

We denote by $\chi(z)$, $r(z)$ and $g(z)$ the maximal eigenvalues of the matrices $A^*(z)$, $R^*(z)$ and $G^*(z)$, respectively.

Corollary 3.9 *Suppose that P is irreducible and stochastic, and the matrices $\sum_{k=1}^{\infty} kD_k$ and $\sum_{k=1}^{\infty} kA_k$ are all finite.*

(1) *P is positive recurrent if and only if $\chi'(1) < 0$. In this case, $g'(1) < 0$ and*

$$\chi'(1) = \theta(I - R)(I - \Phi_0)e \cdot g'(1).$$

(2) *P is null recurrent if and only if $\chi'(1) = 0$.*

(3) *P is transient if and only if* $\chi'(1)>0$. *In this case,* $r'(1)>0$ *and*

$$\chi'(1)=\theta(I-\Phi_0)(I-G)e\cdot r'(1).$$

Proof We only prove (1), while (2) and (3) can be proved similarly.

For $z\geqslant 0$, we denote by $\chi^*(z)$, $\theta^*(z)$ and $e^*(z)$ the maximal eigenvalue of the matrix $A^*(z)$ and the associated left and right eigenvectors, respectively. It is clear that $\theta^*(1)=\theta$ and $e^*(1)=e$. It is clear that

$$1-\chi^*(z)=\theta^*(z)[I-A^*(z)]e^*(z)$$

which leads to

$$\chi'(1)=\theta\sum_{k=1}^{\infty}kA_k e-\theta\sum_{k=1}^{\infty}kA_{-k}e.$$

Using Theorem 3.19, it is easy to see that $\theta\sum_{k=1}^{\infty}kA_k e<\theta\sum_{k=1}^{\infty}kA_{-k}e$, that is, $\chi'(1)<0$.

It follows from Eq. (3.19) that

$$1-\chi^*(z)=\theta^*(z)[I-R^*(z)](I-\Phi_0)[I-G^*(z)]e^*(z). \tag{3.45}$$

Since $\theta^*(z)[I-R^*(z)](I-\Phi_0)$ is the left eigenvector of the matrix $G^*(z)$, we have

$$\theta^*(z)[I-R^*(z)](I-\Phi_0)G^*(z)=g(z)\theta^*(z)[I-R^*(z)](I-\Phi_0)$$

or

$$1-\chi^*(z)=[1-g^*(z)]\theta^*(z)[I-R^*(z)](I-\Phi_0)e^*(z). \tag{3.46}$$

When *P* is positive recurrent, taking the derivative of Eq. (3.46) and using $(I-G)e=0$, we obtain

$$\chi'(1)=g'(1)\theta(I-R)(I-\Phi_0)e,$$

thus we get

$$g'(1)=[\theta(I-R)(I-\Phi_0)e]^{-1}\chi'(1).$$

Note that $\theta(I-R)(I-\Phi_0)e>0$ and $\chi'(1)<0$, it is clear that $g'(1)<0$. This completes the proof.

The following theorem provides a useful relationship between the Markov chain *P* and its dual generating chain $\widetilde{P_{[0]}}$.

Theorem 3.20 *If P is irreducible and stochastic, and the matrix* $\sum_{k=1}^{\infty}kD_k$ *is finite.*

(1) *P is positive recurrent if and only if* $\widetilde{P_{[0]}}$ *is transient,*
(2) *P is null recurrent if and only if* $\widetilde{P_{[0]}}$ *is null recurrent, and*

(3) *P is transient if and only if* $\widetilde{P_{[0]}}$ *is positive recurrent.*

Proof We only prove (1), while (2) and (3) can be proved similarly.

If A is irreducible and stochastic, it is clear that θ is the stationary probability vector of $\widetilde{A} = \text{diag}^{-1}(\theta) A^{\text{T}} \text{diag}(\theta)$. Note that P and $\widetilde{P_{[0]}}$ have the same block-entries in both the first block-row and the first block-column, and the matrix $\sum_{k=1}^{\infty} kD_k$ is finite, thus we only need to check the mean drift condition as follows:

$$\begin{aligned}\theta \sum_{k=1}^{\infty} k\left(\widetilde{A}_k - \widetilde{A}_{-k}\right) e &= \theta \text{diag}^{-1}(\theta) \sum_{k=1}^{\infty} k(A_{-k} - A_k)^{\text{T}} \text{diag}(\theta) e \\ &= e^{\text{T}} \sum_{k=1}^{\infty} k(A_{-k} - A_k)^{\text{T}} \theta^{\text{T}} \\ &= -\theta \sum_{k=1}^{\infty} k(A_k - A_{-k}) e,\end{aligned}$$

which leads to the conclusion that P is positive recurrent if and only if $\widetilde{P_{[0]}}$ is transient. This completes the proof.

The following theorem characterizes the zeros of the two equations $\det(I - R^*(z)) = 0$ and $\det(I - G^*(z)) = 0$.

Lemma 3.8 (1) *All the zeros of the equation* $\det(I - R^*(z)) = 0$ *lie in the region* $|z| \geqslant 1$.

(2) *All the zeros of the equation* $\det(I - G^*(z)) = 0$ *lie in the region* $|z| \leqslant 1$.

Proof We only prove (1), while (2) can be proved similarly.

We denote by z_0 a zero of the equation $\det(I - R^*(z)) = 0$ and assume $|z_0| < 1$. There exists a non-zero column vector $x = (x_1, x_2, ..., x_m)^{\text{T}}$ such that $x = R^*(z_0)x$. Let $|x| = (|x_1|, |x_2|, \dots, |x_m|)^{\text{T}}$. Then $|x| \leqslant R^*(|z_0|)|x| < R|x|$, which leads to

$$\theta |x| < \theta R |x| \leqslant \theta R |x| \leqslant \theta |x|.$$

This is a contradiction. This completes the proof.

The following corollary provides a necessary and sufficient condition for the state classification of the Markov chain P of $GI/G/1$ type by means of the root distributions for the equations $\det(I - R^*(z)) = 0$ and $\det(I - G^*(z)) = 0$.

Corollary 3.10 *Suppose that P is irreducible and stochastic, and the matrix* $\sum_{k=1}^{\infty} kD_k$ *is finite.*

(1) *P is positive recurrent if and only if all the zeros of the equation* $\det(I - R^*(z)) = 0$ *reside outside the unit circle* $|z| > 1$.

(2) *P is null recurrent if and only if all the zeros of the equations* $\det(I - R^*(z)) = 0$ *and* $\det(I - G^*(z)) = 0$ *reside on the unit circle* $|z| = 1$.

(3) *P is transient if and only if all the zeros of the equation* $\det(I - G^*(z)) = 0$ *reside inside the unit circle* $|z| < 1$.

3.5 Distribution of the Minimal Positive Root

In this section, we discuss distribution of the minimal positive root of the equation: $\det(I - R^*(z)) = 0$ or $\det(I - G^*(z)) = 0$.

We first construct some useful relations between a matrix equation and its associate spectral equation. Based on the RG-factorization for the repeated row and the state classification, we analyze the following three cases.

3.5.1 The Positive Recurrence

If the Markov chain P of $GI/G/1$ type is irreducible and positive recurrent, then it follows from Eq. (3.19) that

$$\{0 < z < \phi_{A+} : \det(I - A^*(z)) = 0\} = \{0 < z < \phi_{A+} : \det(I - R^*(z)) = 0\} \\ \bigcup \{0 < z < \phi_{A+} : \det(I - G^*(z)) = 0\}, \quad (3.47)$$

since $I - \Phi_0$ is invertible. Hence it follows from Eq. (3.47) and Corollary 3.10 that

$$\{1 < z < \phi_{A+} : \det(I - A^*(z)) = 0\} = \{1 < z < \phi_{A+} : \det(I - R^*(z)) = 0\} \quad (3.48)$$

and

$$\{0 < z \leqslant 1 : \det(I - A^*(z)) = 0\} = \{0 < z \leqslant 1 : \det(I - G^*(z)) = 0\}.$$

Let $r_i(z)$, $2 \leqslant i \leqslant m$, be all other eigenvalues of $R^*(z)$ whose maximal eigenvalue is $r(z)$.

The following theorem provides a useful relation between the maximal eigenvalue equation and the matrix equation $\det(I - R^*(z)) = 0$.

Theorem 3.21 *Suppose that the Markov chain of* $GI/G/1$ *type is irreducible and positive recurrent.*

(1) $\min\{1 < z < \phi_{A+} : \det(I - R^*(z)) = 0\} = \min\{1 < z < \phi_{A+} : r(z) = 1\}$. (3.49)

(2) $\min\{1 < z < \phi_{A+} : \det(I - A^*(z)) = 0\} = \min\{1 < z < \phi_{A+} : \chi(z) = 1\}$.

(3) $\min\{1 < z < \phi_{A+} : r(z) = 1\} = \min\{1 < z < \phi_{A+} : \chi(z) = 1\}$.

Proof (1) Let

$$\Omega_0 = \{1 < z < \phi_{A+} : r(z) = 1\}$$

and

$$\Omega = \{1 < z < \phi_{A+} : \det(I - R^*(z)) = 0\}.$$

It is clear that $\Omega_0 \subset \Omega$, since $\det(I - R^*(z)) = [1 - r(z)]\prod_{i=2}^{\infty}[1 - r_i(z)]$ for $z > 1$.

There are two possible cases for the set Ω as follows:

Case Ⅰ Ω is not empty. In this case, it follows from Theorem 3.23 that there exists an η such that

$$\eta = \min\{1 < z < \phi_{A+} : \det(I - R^*(z)) = 0\} = \min \Omega.$$

Suppose that $\eta \notin \Omega_0$, i.e., $r(\eta) \neq 1$. Since $\det(I - R^*(\eta)) = 0$, there must exist a $r_{i_0}(\eta)$, $2 \leqslant i_0 \leqslant m$, such that $r_{i_0}(\eta) = 1$. Noting that $r(\eta)$ is the maximal eigenvalue of $R^*(\eta)$, therefore, $r(\eta) \geqslant r_{i_0}(\eta) = 1$. By the assumption that $r(\eta) \neq 1$, we should have $r(\eta) > r_{i_0}(\eta) = 1$.

Let $f(z) = 1 - r(z)$. Then $f(z)$ is continuous for $z \in [1, \eta]$, $f(1) > 0$ and $f(\eta) < 0$, hence there exists a point $\xi \in (1, \eta)$ such that $f(\xi) = 0$, i.e., $r(\xi) = 1$. Clearly, $\xi \in \Omega$. This leads to a contradiction that η is the minimum in Ω. Therefore, $r(\eta) = 1$ or $\eta \in \Omega_0$.

Since $\Omega_0 \subset \Omega$, $\min \Omega \leqslant \min \Omega_0$. Clearly, $\eta = \min \Omega_0$ follows from $\eta = \min \Omega$ and $\eta \in \Omega_0$. Therefore, $\eta = \min \Omega = \min \Omega_0$ implies that Eq. (3.49) is true.

Case Ⅱ Ω is empty. In this case, Ω_0 also is empty, since $\Omega_0 \subset \Omega$. Hence Eq. (3.49) is true in the sense that both sides are infinite.

(2) The proof is slightly different from that in (1). In this case, let $f(z) = 1 - \chi(z)$. Since $\chi(1) = 1$, we need to find a $z_0 > 1$ such that $\chi(z_0) < 1$ to make the argument used in (1).

Since the Markov chain P of $GI/G/1$ type is assumed to be irreducible and positive recurrent, and $\phi_{A+} > 1$, it follows from Proposition 4.6 in Asmussen [1] that $\theta \sum_{k=1}^{\infty} k(A_k - A_{-k})e < 0$, which leads to

$$\chi'(1) = \theta \sum_{k=1}^{\infty} k(A_k - A_{-k})e < 0.$$

Hence there exists a $\delta > 0$ small enough such that

$$\chi(1 + \delta) < \chi(1) = 1.$$

Let $z_0 = 1 + \delta$. Then $f(z_0) > 0$. The rest of the proof is similar to that in (1).

(3) It follows from Eq. (3.47) that

$$\min\{1 < z < \phi_{A+} : \det(I - R^*(z)) = 0\} = \min\{1 < z < \phi_{A+} : \det(I - A^*(z)) = 0\}.$$

The proof follows from (1) and (2). This completes the proof.

3.5.2 The Null Recurrence

If the Markov chain P of $GI/G/1$ type is irreducible and null recurrent, then

$$\{z=1:\det(I-A^*(z))=0\}=\{z=1:\det(I-R^*(z))=0\}$$
$$\bigcup\{z=1:\det(I-G^*(z))=0\}.$$

Furthermore, we have

$$\{z=1:\chi(z)=1\}=\{z=1:r(z)=1\}\bigcup\{z=1:g(z)=1\}.$$

3.5.3 The Transience

If the Markov chain P of $GI/G/1$ type is irreducible and transient, then

$$\{1\leqslant z<\phi_{A+}:\det(I-A^*(z))=0\}=\{1\leqslant z<\phi_{A+}:\det(I-R^*(z))=0\}$$

and

$$\{0<z<1:\det(I-A^*(z))=0\}=\{0<z<1:\det(I-G^*(z))=0\}.$$

Therefore, we obtain

$$\{1\leqslant z<\phi_{A+}:\chi(z)=1\}=\{1\leqslant z<\phi_{A+}:r(z)=1\}$$

and

$$\{0<z<1:\chi(z)=1\}=\{0<z<1:g(z)=1\}.$$

3.5.4 The Minimal Positive Root

Now, we consider an irreducible and positive recurrent Markov chain of $GI/G/1$ type. In this case, we analyze the minimal positive root of $\det(I-R^*(z))=0$. To do this, we first provide expression for the generating function, $\Pi^*(z)=\sum_{k=1}^{\infty}z^k\pi_k$, of the stationary probability vector $\pi=\{\pi_k\}$.

Since the stationary probability vector $\pi=\{\pi_k\}$ can be expressed in terms of the R-measure as

$$\pi_k=\pi_0R_{0,k}+\sum_{i=1}^{k-1}\pi_iR_{k-i},\quad k\geqslant 1,$$

we get

$$\Pi^*(z) = \pi_0 R_0^*(z) + \Pi^*(z) R^*(z), \tag{3.50}$$

which leads to

$$\Pi^*(z) = \pi_0 R_0^*(z)[I - R^*(z)]^{-1} \tag{3.51}$$

whenever both sides are defined.

Theorem 3.22 *Suppose that the Markov chain of $GI/G/1$ type is irreducible and positive recurrent, and the matrix $A = \sum_{k=-\infty}^{\infty} A_k$ is irreducible.*

(1) *If the set*

$$\Omega = \{z : 1 < |z| < \phi_{A+}, \det(I - R^*(z)) = 0\}$$

is not empty, then there must exist a solution $\eta > 1$ to $\det(I - R^*(z)) = 0$ such that $\eta \leqslant |z_0|$, for any $z_0 \in \Omega$.

(2) *If the set*

$$\Omega = \{z : \phi_{A-} < |z| < 1, \det(I - G^*(z)) = 0\}$$

is not empty, then there must exist a solution $0 < \xi < 1$ to $\det(I - G^*(z)) = 0$ such that $\xi \geqslant |z_0|$, for any $z_0 \in \Omega$.

Proof We only prove (1), while (2) can be proved similarly.

To prove this theorem, we need construct a new Markov chain based on $\{A_k\}$:

$$P_\xi = \begin{pmatrix} \widetilde{D}_0 & \widetilde{D}_1 & \widetilde{D}_2 & \widetilde{D}_3 & \cdots \\ D_{-1} & A_0 & A_1 & A_2 & \cdots \\ D_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ D_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3.52}$$

where the matrices D_{-k}, $k \geqslant 1$, and A_l, $-\infty < l < +\infty$, are the same as that of P given in Eq. (3.1) and the matrices $\widetilde{D}_k$, $k \geqslant 0$, are constructed in a way such that

(a) $\phi_{\widetilde{D}} \geqslant \phi_{A+}$, and

(b) the new Markov chain P_ξ is irreducible and positive recurrent.

Such a construction is always possible, for example,

$$\widetilde{D}_0 = \frac{1}{m} \mathrm{e}^{-\tau} E, \quad \widetilde{D}_k = \frac{\tau^k}{k!} \mathrm{e}^{-\tau} E, \quad k \geqslant 1,$$

where E is the matrix of ones of size m and τ is a positive scalar.

It is easy to check that $\phi_D = +\infty$, therefore, $\sum_{k=1}^{\infty} k D_k = \frac{\mathrm{d}}{\mathrm{d}z} D^*(z)_{|z=1}$ is finite, which, together with $\phi_{A+} > 1$ and the face that the Markov chain P in Eq. (3.1) is positive recurrent, implies that P_ξ is positive recurrent, see Proposition 4.6 in Asmussen [1]. Since every entry of $\widetilde{D}_k$ is positive, $k \geqslant 0$, and the Markov chain P in Eq. (3.1) is irreducible, it is obvious that P_ξ is irreducible.

Let $\{\tilde{\pi}_k\}$ be the stationary probability vector of P_ξ, which can be explicitly expressed in terms of the R-measure of P_ξ. Let $\{\widetilde{R}_{0,k}\}$ and $\{\widetilde{R}_k\}$ be the R-measure of P_ξ. It is easy to see from Eq. (3.52) and Eq. (3.1) that $\widetilde{R}_k = R_k$, for all $k \geqslant 1$. Let $\widetilde{\Pi}^*(z)$ and $\widetilde{R}_0^*(z)$ be the generating functions of $\{\tilde{\pi}_k\}$ and $\{\widetilde{R}_{0,k}\}$, respectively. It follows from Eq. (3.51) that

$$\widetilde{\Pi}^*(z) = \pi_0 \widetilde{R}_0^*(z)[I - R^*(z)]^{-1} = \frac{1}{\det(I - R^*(z))} \pi_0 \widetilde{R}_0^*(z) \operatorname{adj}(I - R^*(z)),$$

where $\operatorname{adj}(I - R^*(z))$ is the adjoint of $I - R^*(z)$. It follows from Theorem 3.8 that $\phi_{\widetilde{R}_0} = \phi_{\widetilde{D}} \geqslant \phi_{A+}$ and $\phi_R = \phi_{A+}$. Let ϕ_Π be the radius of convergence of the vector series $\Pi^*(z) = \sum_{k=1}^{\infty} z^k \pi_k$. Then, it follows from a standard result in complex analysis (see Theorem 17.13 in Markushevich [12]) that $z = \phi_\Pi$ is a singular point of $\Pi^*(z)$. Since Ω is not empty, there exists a solution z_1 with $1 < |z_1| < \phi_{A+}$ to equation $\det(I - R^*(z)) = 0$. This shows that $z = z_1$ is a singular point of the vector complex function $\pi_0 R_0^*(z) \operatorname{adj}(I - R^*(z)) / \det(I - R^*(z))$. Therefore, $1 < \phi_\Pi \leqslant |z_1| < \phi_{A+}$, according to

$$\phi_\Pi = \min\{\phi_{R_0}, \phi_R, \eta\}.$$

Suppose that $\det(I - R^*(\phi_\Pi)) \neq 0$, then $\pi_0 R_0^*(z) \operatorname{adj}(I - R^*(z)) / \det(I - R^*(z))$ would be analytic at $z = \phi_\Pi$, since $1 < \phi_\Pi < \min\{\phi_{A+}, \phi_{\widetilde{D}}\}$. Hence $\Pi^*(z)$ also would be analytic at $z = \phi_\Pi$. This leads to a contradiction that $z = \phi_\Pi$ is a singular point of $\Pi^*(z)$. This completes the proof.

Now, we provide a discussion on existence for a positive solution to $\det(I - R^*(\phi_\Pi)) = 0$, including the case where ϕ_{A+} itself can be such a solution.

Let $r(z)$ be the maximal eigenvalue of $R^*(z)$ for $z \in [1, \phi_{A+})$. We can extend the definition of $r(z)$ to $z = \phi_{A+}$ by defining $r(\phi_{A+}) = \lim_{z \nearrow \phi_{A+}} r(z)$. Clearly, $r(z) \geqslant 0$ for $1 \leqslant z < \phi_{A+}$, since $R^*(z) \geqslant 0$.

Theorem 3.23 *Suppose that the Markov chain of* $GI/G/1$ *type is irreducible and positive recurrent. To equation* $\det[I - R(z)] = 0$*,*

(1) *if* $r(\phi_{A+}) > 1$*, then there exists a positive solution* η *satisfying* $1 < \eta < \phi_{A+}$*;*

(2) *if* $r(\phi_{A+}) = 1$ *and* $\phi_{A+} < +\infty$*, then* ϕ_{A+} *is a positive solution; and*

(3) *if* $r(\phi_{A+}) < 1$, *then there does not exist any positive solution.*

Proof Let $f(z) = 1 - r(z)$. It is clear that $r(1) < 1$, hence $f(1) = 1 - r(1) > 0$.

(1) If $r(\phi_{A+}) > 1$, there are two possible cases:

Case Ⅰ $r(\phi_{A+}) = +\infty$. In this case, there always exists ξ with $1 < \xi < \phi_{A+}$ such that $1 < r(\xi) < +\infty$, since $r(z)$ is continuous at z for $1 < z < \phi_{A+}$. Thus, we have $f(\xi) = 1 - r(\xi) < 0$. Since $f(z)$ is continuous on $[1, \xi]$, $f(1) > 0$ and $f(\xi) < 0$, there must exist a point $\eta \in (1, \xi)$ such that $f(\eta) = 0$.

Case Ⅱ $1 < r(\phi_{A+}) < +\infty$. In this case, an analysis similar to that of case Ⅰ shows that there always exists a point $\eta \in (1, \phi_{A+})$ satisfying $f(\eta) = 0$.

(2) If $r(\phi_{A+}) = 1$ and $\phi_{A+} < +\infty$, it is obvious that $z = \phi_{A+}$ is a solution to equation $f(z) = 0$.

(3) If $r(\phi_{A+}) < 1$, since $f(1) \geqslant f(z) \geqslant f(\phi_{A+}) > 0$, there does not exist any positive solution to equation $\det[I - R(z)] = 0$ for $z > 1$.

This completes the proof.

Remark 3.4 *Suppose that the Markov chain of* $GI/G/1$ *type is irreducible and positive recurrent.* (4) *If* $r(\phi_{A+}) = 1$ *and* $\phi_{A+} = +\infty$, *then it is possible that there does not exist any positive solution to equation* $\det[I - R(z)] = 0$ *for* $z > 1$, *for example, the Markov chain of* $GI/G/1$ *type satisfies a condition that* $r(z)$ *is strictly increasing for* $z \in (a, +\infty)$, *where* $a > 1$.

3.6 Continuous-time Chains

In this section, it is necessary for practical applications that we simply list some crucial results for continuous-time Markov chains of $GI/G/1$ type.

Consider a continuous-time Markov chain of $GI/G/1$ type whose transition probability matrix is given by

$$Q = \begin{pmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ B_{-1} & C_0 & C_1 & C_2 & \cdots \\ B_{-2} & C_{-1} & C_0 & C_1 & \cdots \\ B_{-3} & C_{-2} & C_{-1} & C_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3.53}$$

where the sizes of the matrices B_0, B_i, B_{-i} for $i \geqslant 1$ and C_j for $-\infty < j < \infty$ are $m_0 \times m_0$, $m_0 \times m$, $m \times m_0$ and $m \times m$, respectively.

Let W_n be the southeast corner of Q beginning from level n. Then it is clear that $W_n = W$ for all $n \geqslant 1$ and $\widehat{W} = W_{\max}^{-1}$. For $n \geqslant 1$, i, $j = 1, 2, 3, \ldots, n$,

$$Q_{n-i,n-j}^{[\leqslant n]} = Q_{n+1-i,n+1-j}^{[\leqslant (n+1)]} = Q_{n+2-i,n+2-j}^{[\leqslant (n+2)]} = \cdots.$$

We can define that for $1 \leqslant i,\ \ j \leqslant n$,

$$\Phi_0 = Q_{n,n}^{[\leqslant n]},$$

$$\Phi_i = Q_{n-i,n}^{[\leqslant n]},$$

$$\Phi_{-j} = Q_{n,n-j}^{[\leqslant n]}.$$

It is easy to see that the (r,s)th entry of Φ_i is the transition rate of the censored chain $Q^{[\leqslant n]}$ from state $(n-i,r)$ to state (n,s), while the (r,s)th entry of Φ_{-j} is the transition rate of the censored chain $Q^{[\leqslant n]}$ from state (n,r) to state $(n-j,s)$. The *R*- and *G*-measures can be given in terms of the matrices Φ_i for $-\infty < i < +\infty$ as follows:

$$R_i = \Phi_i(-\Phi_0)^{-1}, \quad i \geqslant 1,$$

and

$$G_j = (-\Phi_0)^{-1}\Phi_{-j}, \quad j > 1.$$

At the same time, we can provide expressions for the matrices $R_{0,j}$ and $G_{i,0}$ for $i,j \geqslant 1$ as follows:

$$R_{0,j} = \phi_{0,j}^{(j)}(-\Phi_0)^{-1}$$

and

$$G_{i,0} = (-\Phi_0)^{-1}\phi_{i,0}^{(i)}.$$

The censored chain of Q to level 0 is given by

$$\Psi_0 = Q^{[0]} = B_0 + \sum_{k=1}^{\infty} R_{0,k}(-\Phi_0)G_{k,0}.$$

For the continuous-time Markov chain of $GI/G/1$ type, the UL-type *RG*-factorization is given as

$$Q = (I - R_U)\Phi_D(I - G_L),$$

where

$$R_U = \begin{pmatrix} 0 & R_{0,1} & R_{0,2} & R_{0,3} & \cdots \\ & 0 & R_1 & R_2 & \cdots \\ & & 0 & R_1 & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$\Phi_D = \text{diag}\,(\Psi_0, \Phi_0, \Phi_0, \Phi_0, \ldots)$$

and

$$G_L = \begin{pmatrix} 0 & & & & \\ G_{1,0} & 0 & & & \\ G_{2,0} & G_1 & 0 & & \\ G_{3,0} & G_2 & G_1 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The RG-factorization for the repeated row based on the z-transformation of the repeating blocks. Let $C^*(z) = \sum_{k=-\infty}^{\infty} z^k C_k$. Then

$$C^*(z) = [I - R^*(z)](-\Phi_0)[I - G^*(z)].$$

We now consider conditions for the state classification. Suppose that Q and $C = \sum_{k=-\infty}^{\infty} C_k$ are irreducible, $Qe = Ce = 0$, and the matrix $\sum_{k=1}^{\infty} kB_k$ is finite. Let θ be the stationary probability vector of the Markov chain C. Then

(1) Q is positive recurrent if and only if $\theta \sum_{k=1}^{\infty} kC_{-k}e > \theta \sum_{k=1}^{\infty} kC_k e.$

(2) Q is null recurrent if and only if $\theta \sum_{k=1}^{\infty} kC_{-k}e = \theta \sum_{k=1}^{\infty} kC_k e.$

(3) Q is transient if and only if $\theta \sum_{k=1}^{\infty} kC_{-k}e < \theta \sum_{k=1}^{\infty} kC_k e.$

On the other hand, we have the following useful conditions for the state classification:

(1) Q is positive recurrent if and only if $\theta R < \theta.$ In this case, $Ge = e.$

(2) Q is null recurrent if and only if $\theta R = \theta$ and $Ge = e.$

(3) Q is transient if and only if $Ge < e.$ In this case, $\theta R = \theta.$

When the Markov chain Q is positive recurrent, the stationary probability vector $\pi = \{\pi_k\}$ can be expressed in terms of the R-measure as

$$\pi_k = \pi_0 R_{0,k} + \sum_{i=1}^{k-1} \pi_i R_{k-i}, \quad k \geqslant 1,$$

and

$$\Pi^*(z) = \pi_0 R_0^*(z)[I - R^*(z)]^{-1}.$$

3.7 Notes in the Literature

Grassmann and Heyman [4] first analyzed Markov chains of $GI/G/1$ type by means of the censored technique that generalized Markov chains of $GI/M/1$ type in Neuts [13] and Markov chains of $M/G/1$ type in Neuts [14]. Zhao, Li, and Braun [20] and Zhao [17] gave a detailed discussion for the R- and G-measures and the A- and B-measures for the Markov chain of $GI/G/1$ type. Ramaswami [15, 16] provided a duality theorem between the Markov chain of $GI/M/1$ type and the Markov chain of $M/G/1$ type. Asmussen and Ramaswami [3] gave probabilistic interpretations for the duality theorem. Zhao, Li and Alfa [18] extended the duality theorem to Markov chains of $GI/G/1$ type. Zhao, Li, and Braun [20], Zhao, Li and Alfa [18], Zhao and Li, and Braun [21] gave a detailed discussion for spectral analysis for the R- and G-measures, which leads to conditions for the state classification of Markov chains of $GI/G/1$ type. Spectral analysis for the R- and G-measures plays a key role in the study of stochastic models. Important examples include Chapter 1 of Neuts [13] for spectral analysis of the matrix R in Markov chains of $GI/M/1$ type, and Chapters 2 and 3 of Neuts [14] for spectral analysis of the matrix G in Markov chains of $M/G/1$ type. Zhao, Li, and Braun [19] provided the RG-factorization for the repeated blocks in a Markov chain of $GI/G/1$ type. Li and Cao [6] provided the UL-type RG-factorization for any irreducible QBD process with either finitely-many levels or infinitely-many levels. Li and Zhao [8, 9] provided the UL-type RG-factorization for any irreducible Markov chain, while Li and Liu [7] gave the LU-type RG-factorization for any irreducible Markov chain.

In this chapter, we mainly refer to Zhao, Li, and Braun [19 − 21], Zhao, Li and Alfa [18], Zhao [17], Li and Zhao [8 − 11], Li and Cao [6] and Li and Liu [7].

Problems

3.1 If the Markov chain P given in Eq. (3.1) is irreducible and stochastic, and the matrix A is substochastic, prove that the Markov chain P is positive recurrent.

3.2 Prove that if the Markov chain P given in Eq. (3.1) is irreducible, then each row of the matrix $G=\sum_{k=1}^{\infty} G_k$ is not zero.

3.3 Prove that $\phi_{G_0}=\phi_{D_-}$ and $\phi_G=\phi_{A_-}$.

3.4 For the Markov chain P given in Eq. (3.1), compare the expression of the R-measure with that of the A-measure by means of the following partition

$$P=\begin{pmatrix} D_0 & U \\ V & W \end{pmatrix}.$$

Similarly, compare the G-measure with the B-measure.

3.5 Suppose that the Markov chains P and A are irreducible and stochastic, and the matrix $\sum_{k=1}^{\infty} kD_k$ is finite. Prove that

(1) P is positive recurrent if and only if $\theta R < \theta$.
(2) P is null recurrent if and only if $\theta R = \theta$ and $Ge = e$.
(3) P is transient if and only if $Ge < e$.

3.6 For the minimal positive root η to the equation $\det(I - R^*(z)) = 0$, please provide examples to indicate the four cases of η given in Section 3.5.

3.7 For an irreducible QBD process with the repeated blocks A_{-1}, A_0 and A_1 of size m, we assume that the matrix $A = A_{-1} + A_0 + A_1$ is stochastic, prove that
(1) $(I - \Phi)^{-1} \geqslant 0$.
(2) $R = A_1 (I - \Phi)^{-1}$ and $G = (I - \Phi)^{-1} A_{-1}$.
(3) Use the structures of the matrices A_{-1}, A_0 and A_1 to analyze the structures of the two matrices R and G, respectively.

3.8 For an irreducible QBD process with the repeated blocks A_{-1}, A_0 and A_1 of size m, if the matrix $A = A_{-1} + A_0 + A_1$ is irreducible and d-periodic, please discuss the period of the two matrices R and G.

3.9 For the $MAP/G/1$ queue, compute the A- and B-measures $\{A_{0,k}\}$ and $\{B_{k,0}\}$, respectively.

3.10 For the $BMAP/PH^X/1$ queue, provide conditions for the state classification and expression for the stationary queue length distribution.

3.11 For the $PH/PH/1$ queue, write the R-, G-, A- and B-measures, and construct their useful relations.

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4 Asymptotic Analysis

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Abstract In this chapter, we consider asymptotic behavior for the stationary probability vector of any ergodic Markov chain of *GI*/*G*/1 type, and provide conditions under which the stationary probability vector is either light-tailed or heavy-tailed by means of the *RG*-factorization. At the same time, we provide expressions for both the light tail and the heavy tail. Note that the conditions and expressions can be completely determined by the repeating row and the boundary row.

Keywords stochastic model, Markov chain of *GI*/*G*/1 type, *RG*-factorization, asymptotic analysis, light tail, geometric tail, semi- geometric tail, heavy tail, long tail, subexponential, regularly varying.

This chapter considers asymptotic behavior for the stationary probability vector, $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ or $\{\pi_k\}$, of any ergodic Markov chain of $GI/G/1$ type given in (3.1), including heavy tail and light tail. We indicate that this asymptotic behavior can be completely determined by the repeating row with the *RG*-factorization and the boundary row with the Wiener-Hopf equations.

This chapter is organized as follows. Section 4.1 derives a necessary and sufficient condition under which the stationary probability vector of the Markov chain of $GI/G/1$ type is either light-tailed or heavy-tailed. Sections 4.2 discusses three singularity classes of the stationary probability vector, where the first two classes are light-tailed while the third one is for the heavy tail. For the light tail, Sections 4.3 and 4.4 provide two classes of explicit expressions: The geometric tail and the semi-geometric tail. For the heavy tail, Section 4.5 derives long-tailed asymptotic for the matrix sequence $\{R_k\}$ in terms of the *RG*-factorization for the repeating matrix sequence. Section 4.6 gives a detailed analysis for subexponential asymptotics of the stationary probability vector. Finally, Section 4.7 provides some notes to the references related to the results of this chapter. Note that a simple introduction to the light and heavy tails of the matrix sequences is given in Appendix B.

4.1 A Necessary and Sufficient Condition

In this section, we provide a necessary and sufficient condition under which the stationary probability vector $\{\pi_k\}$ is either light-tailed or heavy-tailed. As expected, the condition depends on asymptotic behavior of both the repeating blocks and the boundary blocks.

We recall the R-measure expressions of the stationary probability vector $\{\pi_k\}$ as follows:

$$\pi_0 = \tau x_0,$$
$$\pi_k = \pi_0 R_{0,k} + \sum_{i=1}^{k-1} \pi_i R_{k-i}, \quad k \geqslant 1,$$
$$\Pi^*(z) = \pi_0 R_0^*(z) + \Pi^*(z) R^*(z),$$

or

$$\Pi^*(z) = \pi_0 R_0^*(z)[I - R^*(z)]^{-1}. \tag{4.1}$$

Obviously, the two radii ϕ_{R_0} and ϕ_R of convergence determine tailed classification of the stationary probability vector $\{\pi_k\}$ and the associated asymptotic expressions. Note that the Markov chain P of $GI/G/1$ type is determined by the two matrix sequences $\{A_k\}$ and $\{D_k\}$, thus it is necessary to show that the two radii ϕ_{R_0} and ϕ_R of convergence can be related to the two radii ϕ_{A+} and ϕ_{D+} of convergence, respectively.

The following lemma provides useful asymptotic relations between $\{A_k\}$ and $\{R_k\}$, and between $\{D_k\}$ and $\{R_{0,k}\}$, respectively. Note that $\{A_k, k \geqslant 1\}$ is light-tailed if and only if $\phi_{A+} > 1$, and $\{D_k, k \geqslant 1\}$ is light-tailed if and only if $\phi_{D+} > 1$.

Lemma 4.1 (1) $\{R_k\}$ *is light-tailed if and only if* $\phi_{A+} > 1$.

(2) $\{R_{0,k}\}$ *is light-tailed if and only if* $\phi_{D+} > 1$.

Proof We only prove (1), while (2) can similarly be proved by means of $\phi_{R_0} = \phi_{D+}$ by Theorem 3.8.

It is easy to check from Definition B.1 in Appendix B that the nonnegative matrix sequence $\{C_k\}$ of size $m \times n$ is light-tailed if and only if $\phi_C > 1$, where ϕ_C is the radius of convergence of $\sum_{k=1}^{\infty} z^k C_k$. Thus, the conclusion in this lemma follows from $\phi_R = \phi_{A+}$ by Theorem 3.8. This completes the proof.

Let ϕ_Π be the radius of convergence of $\Pi^*(z)$. The Eq. (4.1) implies that $\phi_\Pi \leqslant \min\{\phi_{R_0}, \phi_R\}$. Since ϕ_Π is equal to the radius of convergence of the expansion of the function in a power series on the right hand side, (4.1) shows that ϕ_Π will depend on ϕ_{R_0}, ϕ_R and the roots of the equation $\det(I - R^*(z)) = 0$. If det $(I - R^*(z)) = 0$ has a solution z such that $|z| > 1$, then, by the continuity of an implicit function, there exists a solution z_0 such that $\eta = |z_0| = \min\{|z|: \det(I - R^*(z)) = 0\} > 1$, since $\det(I - R^*(1)) \neq 0$ in terms of Corollary 3.7. Thus, we obtain

$$\phi_{\Pi} = \min\{\phi_{R_0}, \phi_R, \eta\}. \tag{4.2}$$

The following theorem provides a necessary and sufficient condition under which the stationary probability vector $\{\pi_k\}$ is light-tailed.

Theorem 4.1 *Suppose that the Markov chain of $GI/G/1$ type is irreducible and positive recurrent. $\{\pi_k\}$ is light-tailed if and only if both $\{D_k\}$ and $\{A_k\}$ are light-tailed (that is, $\min\{\phi_{A+}, \phi_{D+}\} > 1$).*

Proof Suppose first that both $\{D_k\}$ and $\{A_k\}$ are light-tailed. Hence $\min\{\phi_{A+}, \phi_{D+}\} > 1$. Note that if $\det(I - R^*(z)) = 0$ has a solution, then it follows from Theorems 3.22 and 3.23 and Remark 3.4 that

$$\eta = |z_0| = \min\{|z|, 1 < |z| \leqslant \phi_{A+}, \det(I - R^*(z)) = 0\}$$

implies that either $1 < \eta < \infty$ or $\eta = \infty$. Therefore, using $\min\{\phi_{A+}, \phi_{D+}\} > 1$ and $1 < \eta \leqslant \infty$, we obtain $\phi_{\Pi} = \min\{\phi_{R_0}, \phi_R, \eta\} > 1$, hence $\{\pi_k\}$ is light-tailed.

Now, suppose that $\{\pi_k\}$ is light-tailed. We assume that at least one of the two matrix sequences $\{D_k\}$ and $\{A_k\}$ is heavy-tailed, for example, $\{A_k\}$ is heavy-tailed. Obviously, we would have that $\phi_{A+} = 1$, which would lead to $\phi_{\Pi} = \min\{\phi_{R_0}, \phi_R, \eta\} = 1$, thus $\{\pi_k\}$ is heavy-tailed. This leads to a contradiction that $\{\pi_k\}$ is light-tailed. This completes the proof.

In what follows we provide a necessary and sufficient condition under which the stationary probability vector $\{\pi_k\}$ is heavy-tailed. To do this, we first need to introduce two types of convolutions of nonnegative matrix sequences.

For a sequence of nonnegative scalars $\{g_n\}$ with $\sum_{n=0}^{\infty} g_n < +\infty$, two associative functions are defined as $g_{\leqslant x} = \sum_{0 \leqslant k \leqslant x} g_k$ and $g_{>x} = \sum_{k>x} g_k$ for an arbitrary real number $x \geqslant 0$. Specifically, for an integer $n \geqslant 0$, $g_{\leqslant n} = \sum_{k=0}^{n} g_k$ and $g_{>n} = \sum_{k=n+1}^{\infty} g_k$. For convenience, we also write $g_{>n}$ as $g_{\geqslant n+1}$.

For the real function $g_{\leqslant x}$ associated with the sequence $\{g_n\}$, the tail of $g_{\leqslant x}$ is defined and expressed as $\overline{g_{\leqslant x}} = g_{<+\infty} - g_{\leqslant x} = g_{>x}$ for $x \geqslant 0$. Specifically, for an integer $n \geqslant 0$, $\overline{g_{\leqslant n}} = g_{\geqslant n+1}$. It is clear that if $\{g_n\}$ is a probability sequence, then $g_{\leqslant x}$ is its distribution function and $\overline{g_{\leqslant x}}$ is the tail of this distribution function.

In terms of the Riemann-Stieltjes integral, the convolution of two functions $F(x)$ and $G(x)$ is defined as

$$F(x) * G(x) = \int_0^x F(x-y)\mathrm{d}G(y). \tag{4.3}$$

We denote by $[x]$ the maximum integer part of a real number x. For two sequences $\{c_n\}$ and $\{d_n\}$, it follows from Eq. (4.3) that

$$c_{\leqslant x} * d_{\leqslant x} = \int_0^x F(x-y)\mathrm{d}G(y) = \sum_{k=0}^{[x]} \left(\sum_{i=0}^{[x]-k} c_i \right) d_k = \sum_{k=0}^{[x]} c_{\leqslant [x]-k} d_k.$$

Specifically, for an integer $n \geqslant 0$,

$$c_{\leqslant n} * d_{\leqslant n} = \sum_{k=0}^{n} c_{\leqslant n-k} d_k, \tag{4.4}$$

which is referred to as the convolution associated with the two sequences $\{c_n\}$ and $\{d_n\}$. Furthermore, for a sequence $\{c_n\}$ we define $c_{\leqslant n}^{r*} = c_{\leqslant n}^{(r-1)*} * c_{\leqslant n}$ for $r \geqslant 2$ with $c_{\leqslant n}^{1*} = c_{\leqslant n}$.

It should be noted that the usual convolution of two sequences $\{c_n\}$ and $\{d_n\}$ is denoted by $c_n \circledast d_n$, defined as

$$c_n \circledast d_n = \sum_{k=0}^{n} c_{n-k} d_k. \tag{4.5}$$

We further define $c_n^{r\circledast} = c_n^{(r-1)\circledast} \circledast c_n$ for $r \geqslant 2$ with $c_n^{1\circledast} = c_n$.

According to Eq. (4.4) and Eq. (4.5), it is worthwhile to note a useful relationship between the usual convolution and the convolution associated with the sequences:

$$c_{\leqslant n} * d_{\leqslant n} = c_{\leqslant n} \circledast d_n = \sum_{k=0}^{n} c_k \circledast d_k \tag{4.6}$$

and

$$c_n \circledast d_n = c_{\leqslant n} * d_{\leqslant n} - c_{\leqslant n-1} * d_{\leqslant n-1}.$$

Also, it is clear from Eq. (4.6) that

$$\overline{c_{\leqslant n} * d_{\leqslant n}} = \sum_{k=n+1}^{\infty} c_k \circledast d_k. \tag{4.7}$$

Remark 4.1 *The above two convolutions can be extended to sequences* $\{c_n : n = 0, \pm 1, \pm 2, \ldots\}$ *and* $\{d_n : n = 0, \pm 1, \pm 2, \ldots\}$ *by*

$$c_{\leqslant n} * d_{\leqslant n} = \sum_{i+j=n} c_{\leqslant i} d_j$$

and

$$c_n \circledast d_n = \sum_{i+j=n} c_i d_j,$$

respectively. We then obtain that for an arbitrary integer n,

$$c_{\leqslant n} * d_{\leqslant n} = \sum_{k=-\infty}^{n} c_k \circledast d_k$$

and

$$\overline{c_{\leqslant n} * d_{\leqslant n}} = \sum_{k=n+1}^{\infty} c_k \circledast d_k .$$

For a sequence $\{c_n, n \geqslant 0\}$, if we set $c_{-n} = 0$ for all $n \geqslant 1$, then

$$c_{\leqslant n}^{2*} = \sum_{k=0}^{n} c_{\leqslant k} c_{n-k} = \sum_{k=0}^{\infty} c_{\leqslant k} c_{n-k} .$$

Specifically, if $\{c_n, n \geqslant 0\}$ is a probability sequence, simple computations lead to

$$\overline{c_{\leqslant n}^{2*}} = 1 - c_{\leqslant n}^{2*} = \sum_{k=0}^{\infty} c_k \left[1 - c_{\leqslant n-k}\right] = \sum_{k=0}^{\infty} c_k \overline{c_{\leqslant n-k}} . \tag{4.8}$$

Now, we analyze the heavy-tailed asymptotics of the stationary probability vector $\{\pi_k\}$. Since the Markov chain is positive recurrent, Corollary 3.10 shows that all solutions to the equation $\det (I - R^*(z)) = 0$, if any, reside outside the unit circle $|z| > 1$. This means that $I - R^*(z)$ is always invertible for all $|z| \leqslant 1$. Therefore,

$$\Pi^*(z) = x_0 R_0^*(z)[I - R^*(z)]^{-1}, \tag{4.9}$$

which implies

$$\pi_k = x_0 R_{0,k} \circledast \sum_{n=0}^{\infty} R_k^{n\circledast} . \tag{4.10}$$

Therefore,

$$\overline{\pi_{\leqslant k}} = \sum_{l=k+1}^{\infty} x_0 R_{0,l} \circledast \sum_{n=0}^{\infty} R_l^{n\circledast} . \tag{4.11}$$

The following lemma provides an expression for the tail of the stationary probability vector $\{\pi_k\}$.

Lemma 4.2 *For all* $k \geqslant 1$,

$$\overline{\pi_{\leqslant k}} = x_0 \overline{R_{0,\leqslant k} * \sum_{n=0}^{\infty} R_{\leqslant k}^{n\circledast}} , \tag{4.12}$$

where $R_{0,\leqslant k} = \sum_{l=1}^{k} R_{0,l}$, $R_{\leqslant k}^{n\circledast} = \sum_{l=1}^{k} R_l^{n\circledast}$ *and*

$$\overline{R_{0,\leqslant k} * \sum_{n=0}^{\infty} R_{\leqslant k}^{n\circledast}} = R_0 (I-R)^{-1} - R_{0,\leqslant k} * \sum_{n=0}^{\infty} R_{\leqslant k}^{n\circledast}.$$

Proof Note that

$$\sum_{l=0}^{k} x_0 R_{0,l} \circledast \sum_{n=0}^{\infty} R_l^{n\circledast} + \sum_{l=k+1}^{\infty} x_0 R_{0,l} \circledast \sum_{n=0}^{\infty} R_l^{n\circledast} = R_0 (I-R)^{-1},$$

it follows from Eq. (4.10) and Eq. (4.7) that

$$\begin{aligned}\overline{\pi_{\leqslant k}} &= x_0 R_0 (I-R)^{-1} - \sum_{l=1}^{k} x_0 R_{0,l} \circledast \sum_{n=0}^{\infty} R_l^{n\circledast} \\ &= R_0 (I-R)^{-1} - R_{0,\leqslant k} * \sum_{n=0}^{\infty} R_{\leqslant k}^{n\circledast} \\ &= x_0 \overline{R_{0,\leqslant k} * \sum_{n=0}^{\infty} R_{\leqslant k}^{n\circledast}}.\end{aligned}$$

This completes the proof.

The following lemma provides a useful heavy-tailed property for the matrix sequences $\{A_k\}$ and $\{D_k\}$.

Lemma 4.3 (1) *If* $\phi_{A+} = 1$, *then the matrix sequence* $\{R_k\}$ *is heavy-tailed.*

(2) *If* $\phi_{D+} = 1$, *then the matrix sequence* $\{R_{0,k}\}$ *is heavy-tailed.*

Proof We only prove (1), while (2) can be proved similarly.

Since $A_k \geqslant 0$ for each integer k and $A = \sum_{k=-\infty}^{\infty} A_k$ is stochastic, we obtain

$$\phi_{A+} = \sup \left\{ z \geqslant 1 : \sum_{k=1}^{\infty} z^k A_k < +\infty \right\} \geqslant 1.$$

If $\phi_{A+} = 1$, then $\sum_{k=1}^{\infty} (1+\delta)^k A_k$ is infinite for any $\delta > 0$. Hence, there exists at least a pair (i_0, j_0) such that

$$\sum_{k=1}^{\infty} \mathrm{e}^{k \ln(1+\delta)} a_k (i_0, j_0) = \sum_{k=1}^{\infty} (1+\delta)^k a_k (i_0, j_0) = +\infty.$$

Let $\varepsilon = \ln(1+\delta)$. Then $\varepsilon > 0$ due to $\delta > 0$. Therefore, it is clear from (1) of definition B.3 that $\{A_k\}$ is heavy-tailed. Note that $\phi_R = \phi_{A+} = 1$, thus $\{R_k\}$ is heavy-tailed. This completes the proof.

The following theorem provides a necessary and sufficient condition under which the stationary probability vector $\{\pi_k\}$ is heavy-tailed.

Theorem 4.2 *If the Markov chain of $GI/G/1$ type is positive recurrent, then the stationary probability vector $\{\pi_k\}$ is heavy-tailed if and only if* $\min\{\phi_{A+}, \phi_{D+}\} = 1$.

Proof We first prove the necessity of the condition. Suppose that $\min\{\phi_{A+}, \phi_{D+}\} > 1$. Then both $\phi_{A+} > 1$ and $\phi_{D+} > 1$. It follows from (4.1) that

$$\Pi^*(z) = \pi_0 R_0^*(z)[I - R^*(z)]^{-1}$$

would yield that $\phi_\pi = \min\{\phi_{A+}, \phi_{D+}, \eta\} > 1$. Hence, $\{\pi_k\}$ would be light-tailed. This is a contradiction to the assumption that $\{\pi_k\}$ is heavy-tailed.

We now prove the sufficiency of the condition. Note that when $\phi_{A+} \geqslant 1$ and $\phi_{D+} \geqslant 1$, the assumption $\min\{\phi_{A+}, \phi_{D+}\} = 1$ implies $\phi_{A+} = 1$ or $\phi_{D+} = 1$.

Case Ⅰ $\phi_{D+} = 1$. In this case, since

$$\Pi^*(z) = \pi_0 R_0^*(z)[I - R^*(z)]^{-1} \geqslant \pi_0 R_0^*(z),$$

$\pi_k \geqslant \pi_0 R_{0,k}$ for $k \geqslant 1$. Note that the Markov chain of $GI/G/1$ type is irreducible and positive recurrent, the censored chain Ψ_0 to level 0 is irreducible and positive recurrent, which leads to $\pi_0 > 0$ because $\pi_0 = \tau x_0$ with $\tau > 0$ and $x_0 > 0$. Under the assumption that $\phi_{D+} = 1$, $\{R_{0,k}\}$ is heavy-tailed according to Lemma 4.3. Thus, there always exists at least a pair (i, j) such that the sequence $\{r_{0,k}(i, j)\}$ is heavy-tailed, where $r_{0,k}(i, j)$ is the (i, j)th entry of the matrix $R_{0,k}$ for each $k \geqslant 1$. It is clear that $\pi_k \geqslant \pi_0 R_{0,k}$ implies

$$\pi_k \geqslant (\underbrace{0,\ldots,0}_{j-1 \text{ zeros}}, \pi_0(i) r_{0,k}(i, j), \underbrace{0,\ldots,0}_{m-j \text{ zeros}}),$$

where $\pi_0(i)$ is the ith entry of the positive row vector π_0. Therefore, $\{\pi_k\}$ is heavy-tailed.

Case Ⅱ $\phi_{D+} > 1$ and $\phi_{A+} = 1$. In this case, since

$$\Pi^*(z) = \pi_0 R_0^*(z)[I - R^*(z)]^{-1} \geqslant \pi_0 R_0^*(z) R^*(z),$$

which implies

$$\pi_k \geqslant \pi_0 R_{0,k} \circledast R_k, \quad \text{for all } k \geqslant 1. \tag{4.13}$$

Under the assumption that $\phi_{A+} = 1$, $\{R_k\}$ is heavy-tailed according to Lemma 4.3. Thus there exists at least a pair $\{i_0, j_0\}$ such that $\{r_k(i_0, j_0)\}$ is heavy-tailed. The assumption that the Markov chain is irreducible and positive recurrent leads to $\pi_0 > 0$, and Theorem 3.3 leads to $\pi_0 R_0 > 0$. Note that $R_0 = \sum_{k=1}^{\infty} R_{0,k}$. Therefore, there always exists an $l_0 \geqslant 1$ such that the (i^*, i_0)th element $r_{0,l_0}(i^*, i_0)$ of R_{0,l_0} is positive. Thus, it follows from Eq. (4.13) that for $k \geqslant N$,

$$\pi_k \geqslant (\underbrace{0,\ldots,0}_{i_0-1 \text{ zeros}}, \pi_0(i^*) r_{0,l_0}(i^*, i_0) r_{k-l_0}(i_0, j_0), \underbrace{0,\ldots,0}_{m-i_0 \text{ zeros}}). \tag{4.14}$$

Since $\pi_0(i^*) r_{0,l_0}(i^*, i_0) > 0$ and $\{r_{k-l_0}(i_0, j_0)\}$ is heavy-tailed, Eq. (4.14) implies that $\{\pi_k\}$ is heavy-tailed. This completes this proof.

4.2 Three Asymptotic Classes of $\{\boldsymbol{\pi}_k\}$

In this section, we discuss three asymptotic classes of the stationary probability vector $\{\pi_k\}$, and provide conditions of classification based on the two matrix sequences $\{D_k\}$ and $\{A_k\}$ in the Markov chain of $GI/G/1$ type.

We define a collection of matrix sequences consisting of all nonnegative matrix sequences (not necessarily square ones), whose sum is convergent. Let

$$\Re = \left\{ \{B_k\} : B_k \geqslant 0, \sum_{k=0}^{\infty} B_k < \infty \right\}.$$

We classify the matrix sequences in $\Re$ into three classes as follows.

Definition 4.1 *Let ϕ_B be the radius of convergence of $B^*(z) = \sum_{k=0}^{\infty} z^k B_k$ for $\{B_k\} \in \Re$. Then,*

(1) *$\{B_k\}$ is called class Ⅰ if $\phi_B > 1$ and $B^*(\phi_B)$ is infinite;*
(2) *$\{B_k\}$ is called class Ⅱ if $\phi_B > 1$ and $B^*(\phi_B)$ is finite;*
(3) *$\{B_k\}$ is called class Ⅲ if $\phi_B = 1$.*

It is easy to see from $\phi_B > 1$ or $\phi_B = 1$ that $\{B_k\}$ in class Ⅰ or class Ⅱ is light-tailed, while $\{B_k\}$ in class Ⅲ is heavy-tailed.

When $\min\{\phi_{A+}, \phi_{D+}\} > 1$, both $\{D_k\}$ and $\{A_k\}$ are either class Ⅰ or class Ⅱ. On the other hand, the condition $\min\{\phi_{A+}, \phi_{D+}\} = 1$ implies that either $\{A_k\}$ or $\{D_k\}$ is class Ⅲ, and therefore, $\{\pi_k\}$ is class Ⅲ.

Recall that $\eta = +\infty$ if $\det(I - R^*(z)) \neq 0$ for all $|z| \leqslant \phi_A$.

Lemma 4.4 *Suppose that the Markov chain of $GI/G/1$ type is irreducible and positive recurrent.*

(1) *If $1 < \eta < +\infty$ and $\eta \leqslant \phi_{A+}$, then $\left\{\sum_{n=0}^{\infty} R_k^{n*}\right\}$ is class Ⅰ.*

(2) *If $\eta = +\infty$, then $\left\{\sum_{n=0}^{\infty} R_k^{n*}\right\}$ is class Ⅰ when $\{A_k\}$ is class Ⅰ; $\left\{\sum_{n=0}^{\infty} R_k^{n*}\right\}$ is*

class II *when* $\{A_k\}$ *is class* II.

(3) *If* $\phi_{D+} > 1$, *then* $\{R_{0,k}\}$ *is class* I *when* $\{D_k\}$ *is class* I; $\{R_{0,k}\}$ *is class* II *when* $\{D_k\}$ *is class* II.

Proof We prove (1) and (2), while (3) can be proved similarly.

(1) Assume that $1 < \eta < +\infty$ and $\eta \leqslant \phi_{A+}$. In this case, the radius of convergence of

$$\sum_{k=1}^{\infty} z^k \sum_{n=0}^{\infty} R_k^{n*} = \frac{1}{\det(I - R^*(z))} \operatorname{adj}(I - R^*(z))$$

is η. When $z = \eta$, $\det(I - R^*(\eta)) = 0$ and $\operatorname{adj}(I - R^*(\eta)) \neq 0$ according to Lemma 4.6. Thus, $\left\{\sum_{n=0}^{\infty} R_k^{n*}\right\}$ is class I.

(2) Since $\phi_R = \phi_{A+}$ from Theorem 3.8, it follows from Eq. (3.19) that

$$\begin{aligned}\sum_{k=1}^{\infty} z^k \sum_{n=0}^{\infty} R_k^{n*} &= [I - R^*(z)]^{-1} = (I - \Phi_0)[I - G^*(z)][I - A^*(z)]^{-1} \\ &= (I - \Phi_0)[I - G^*(z)]\sum_{n=0}^{\infty} [A^*(z)]^n.\end{aligned}$$

Thus, the radius of convergence of $\sum_{k=1}^{\infty} z^k \sum_{n=0}^{\infty} R_k^{n*}$ is ϕ_{A+}.

If $\{A_k\}$ is class I, then $A^*\{\phi_{A+}\}$ is infinite, hence $\sum_{n=0}^{\infty} [A^*(\phi_{A+})]^n$ is infinite. Since $I - \Phi_0$ and $I - G^*(\phi_{A+})$ are invertible, $\sum_{k=1}^{\infty} \phi_{A+}^k \sum_{n=0}^{\infty} R_k^{n*}$ is infinite. Therefore, $\left\{\sum_{n=0}^{\infty} R_k^{n*}\right\}$ is class I.

If $\{A_k\}$ is class II, then $\phi_{A+} < +\infty$ and $A^*(\phi_{A+}) < +\infty$. Since $\eta = +\infty$, it follows from Eq. (3.19) that $I - A^*\{\phi_{A+}\}$ is invertible. Thus,

$$\sum_{k=1}^{\infty} \phi_{A+}^k \sum_{n=0}^{\infty} R_k^{n*} = (I - \Phi_0)[I - G^*(\phi_{A+})][I - A^*(\phi_{A+})]^{-1} < +\infty.$$

Therefore, $\left\{\sum_{n=0}^{\infty} R_k^{n*}\right\}$ is class II.

This completes the proof.

Theorem 4.3 *Suppose that the Markov chain of* $GI/G/1$ *type is irreducible and positive recurrent.*

(1) *If* $1<\eta<+\infty$ *and* $\eta \leqslant \min\{\phi_{A+}, \phi_{D+}\}$, *then* $\{\pi_k\}$ *is class* Ⅰ.

(2) *If* $\eta=+\infty$ *and* $1<\phi_{A+}<\phi_{D+}$, *then* $\{\pi_k\}$ *is class* Ⅰ *when* $\{A_k\}$ *is class* Ⅰ; $\{\pi_k\}$ *is class* Ⅱ *when* $\{A_k\}$ *is class* Ⅱ.

(3) *If* $\eta=+\infty$ *and* $\phi_{A+}=\phi_{D+}$, *then* $\{\pi_k\}$ *is class* Ⅰ *when at least one of* $\{A_k\}$ *and* $\{D_k\}$ *is class* Ⅰ; $\{\pi_k\}$ *is class* Ⅱ *when both* $\{A_k\}$ *and* $\{D_k\}$ *are class* Ⅱ.

(4) *If* $1<\eta<+\infty$ *and* $1<\phi_{D+}<\eta\leqslant\phi_{A+}$, *or* $\eta=+\infty$ *and* $1<\phi_{D+}<\phi_{A+}$, *then* $\{\pi_k\}$ *is class* Ⅰ *when* $\{D_k\}$ *is class* Ⅰ; $\{\pi_k\}$ *is class* Ⅱ *when* $\{D_k\}$ *is class* Ⅱ.

Proof We only prove (1), while (2), (3) and (4) can be similarly proved.

Assume that $1<\eta<+\infty$ and $\eta\leqslant\min\{\phi_{A+},\phi_{D+}\}$. Since the radius of convergence of

$$\Pi^*(z)=\frac{1}{\det(I-R^*(z))}\pi_0 R_0^*(z)\operatorname{adj}(I-R^*(z))$$

is η, $\det(I-R^*(\eta))=0$ and

$$\pi_0 R_0^*(\eta)\operatorname{adj}(I-R^*(\eta))=\pi_0 R_0^*(\eta)v(\eta)u(\eta)\prod_{i=2}^{m}[1-r_i(\eta)]\gneq 0$$

according to Lemma 4.6 and Theorem 3.3, $\Pi^*(\eta)$ is infinite. Therefore, $\{\pi_k\}$ is class Ⅰ. This completes the proof.

Note that Sections 4.3 and 4.4 will analyze the light-tailed behavior of the stationary probability vector $\{\pi_k\}$, which is in classes Ⅰ and Ⅱ; while Sections 4.5 to 4.7 will discuss the heavy-tailed case of $\{\pi_k\}$, which is in class Ⅲ.

4.3 The Asymptotics Based on the Solution η

In this section, we assume that the minimal positive solution $\eta>1$ to $\det(I-R^*(z))=0$ satisfies that $\eta<\phi_{A+}$ and $\eta<\phi_{D+}$, which together with $\phi_\Pi=\min\{\phi_{R_0},\phi_R,\eta\}$, implies that $\{\pi_k\}$ is light-tailed and is determined by η only. Based on this, we can derive explicit asymptotic expressions for the light tail of the stationary probability vector $\{\pi_k\}$.

4.3.1 *A* is Irreducible

Note that $\phi_R=\phi_{A+}$, $\phi_{R_0}=\phi_{D+}$, $\eta<\phi_{A+}$ and $\eta<\phi_{D+}$, it is clear that

$$\phi_\Pi=\min\{\phi_{R_0},\phi_R,\eta\}=\eta>1.$$

Thus it follows from Eq.(4.1) that for any z with $1<|z|<\eta$,

$$\Pi^*(z) = \frac{1}{\det(I - R^*(z))} \pi_0 R_0^*(z) \cdot \text{adj}(I - R^*(z)). \tag{4.15}$$

The analysis on the singularity of $\Pi^*(z)$ depends on that of the three functions: $1/\det(I - R^*(z))$, $R_0^*(z)$ and $\text{adj}(I - R^*(z))$. Since $R_0^*(z)$ and $\text{adj}(I - R^*(z))$ are analytic in $|z| < \min\{\phi_{A+}, \phi_{D+}\}$, they are analytic at $z = \eta$. Noting that $z = \eta$ is a singular point of $1/\det(I - R^*(z))$, thus it also is a singular point of $\Pi^*(z)$.

In the following, we show that when A is irreducible, $z = \eta$ is a pole of order 1 of $\Pi^*(z)$.

For $z > 0$, $r(z)$ is the maximal eigenvalue of the matrix $R^*(z)$, let $r_i(z)$ for $2 \leqslant i \leqslant m$ be all the other eigenvalues of $R^*(z)$. It is clear that $r(\eta) = 1$ and $r_i(\eta) \neq 1$ for all $2 \leqslant i \leqslant m$. We denote by $u(z)$ and $v(z)$ the Perron-Frobenius left and right eigenvectors of $R^*(z)$ for $z \geqslant 0$, respectively. It is useful that $u(z)$ and $v(z)$ can be expressed by the Perron-Frobenius left and right eigenvectors of $A^*(z)$, respectively. The following lemma characterizes $u(\eta)$ and $v(\eta)$ based on that of $A^*(\eta)$.

Lemma 4.5 *If $s(\eta)$ and $t(\eta)$ are the Perron-Frobenius left and right eigenvectors of $A^*(\eta)$, respectively, then*

(1) $u(\eta) = a \cdot s(\eta)$ *and* $v(\eta) = b \cdot (I - \Phi_0)[I - G^*(\eta)] \cdot t(\eta)$, *where a and b are two free positive factors.*

(2) $u(\eta) > 0$ *if and only if* $s(\eta) > 0$; $v(\eta) \geqslant 0$ *but* $v(\eta) \neq 0$ *if and only if* $t(\eta) > 0$; *and* $u(z)e = u(z)v(z) = 1$.

Proof In this proof, (1) is obvious by using Eq. (3.19), and (2) can be obtained by noting that $(I - \Phi_0)$ and $I - G^*(\eta)$ are all invertible, where the invertibility of $I - G^*(\eta)$ follows from $\eta > 1$ and (1) of Corollary 3.10. This completes the proof.

To describe $\Pi^*(z)$ in Eq. (4.1) for details, we need to express the adjoint matrix of $I - R^*(\eta)$ more explicitly, which is given in the following lemma.

Lemma 4.6 *If the Markov chain of $GI/G/1$ type is irreducible and positive recurrent, then*

$$\text{adj}(I - R^*(\eta)) = \prod_{i=2}^{\infty} [1 - r_i(\eta)] \cdot v(\eta) u(\eta).$$

Proof Noting that

$$\det(I - A^*(\eta)) = \det(I - R^*(\eta)) \det(I - \Phi_0) \det(I - G^*(\eta)) = 0,$$

and rank $(I - A^*(\eta)) \leqslant m - 1$. Since $A^*(\eta)$ is irreducible, $1 - \chi(\eta) = 0$ is a simple eigenvalue of $I - A^*(\eta)$, and for any other eigenvalue $1 - y(\eta)$ of $I - A^*(\eta)$, $|1 - y(\eta)| > |1 - \chi(\eta)| = 0$. Thus, rank$(I - A^*(\eta)) = m - 1$. Since the Markov chain of $GI/G/1$ type is positive recurrent, it follows from Corollary 3.10 that for

$\eta > 1$, $I - G^*(\eta)$ is invertible. Thus, from (3.19),

$$I - R^*(\eta) = [I - A^*(\eta)][I - G^*(\eta)]^{-1}(I - \Phi_0)^{-1}.$$

Therefore,

$$\operatorname{rank}(I - R^*(\eta)) = \operatorname{rank}(I - A^*(\eta)) = m - 1$$

and rank $(\operatorname{adj}(I - R^*(\eta))) = 1$. Noting that $\operatorname{adj}(I - \alpha R^*(\eta))$ is continuous for $\alpha \in (0,1]$, we obtain

$$\operatorname{adj}(I - R^*(\eta)) = \lim_{\alpha \nearrow 1} \operatorname{adj}(I - \alpha R^*(\eta)).$$

For $\alpha \in (0,1)$, we have

$$\begin{aligned}\operatorname{adj}(I - \alpha R^*(\eta)) &= \det(I - \alpha R^*(\eta)) \cdot [I - \alpha R^*(\eta)]^{-1} \\ &= \prod_{i=2}^{m} [1 - \alpha r_i(\eta)] \cdot (1 - \alpha)[I - \alpha R^*(\eta)]^{-1}.\end{aligned}$$

Let $T(\alpha)$ be an invertible matrix such that

$$T(\alpha)^{-1}[I - \alpha R^*(\eta)]T(\alpha) = \begin{pmatrix} 1 - \alpha & \\ & J(\alpha) \end{pmatrix}$$

is the Jordan canonical form of $I - \alpha R^*(\eta)$, where the modules of all the diagonal entries of $J(\alpha)$ are all greater than $1 - \alpha$. Since rank $(I - R^*(\eta)) = m - 1$,

$$T(1)^{-1}\left[I - R^*(\eta)\right]T(1) = \begin{pmatrix} 0 & \\ & J(1) \end{pmatrix}$$

and $J(1)$ is invertible. Therefore, we obtain

$$(1 - \alpha)[I - \alpha R^*(\eta)]^{-1} = T(\alpha)\begin{pmatrix} 1 & \\ & (1 - \alpha)J(\alpha)^{-1} \end{pmatrix}T(\alpha)^{-1}.$$

Noting that $\lim_{\alpha \nearrow 1}(1 - \alpha)J(\alpha)^{-1} = 0$, we get

$$\begin{aligned}\operatorname{adj}(I - R^*(\eta)) &= \lim_{\alpha \nearrow 1} \operatorname{adj}(I - \alpha R^*(\eta)) \\ &= \lim_{\alpha \nearrow 1} \prod_{i=2}^{m} [1 - \alpha r_i(\eta)] \cdot (1 - \alpha)[I - \alpha R^*(\eta)]^{-1} \\ &= \prod_{i=2}^{m} [1 - r_i(\eta)] \cdot T(1)\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}T(1)^{-1} \\ &= \prod_{i=2}^{m} [1 - r_i(\eta)] \cdot v(\eta)u(\eta).\end{aligned}$$

This completes the proof.

It is clear from the proof of Lemma 4.6 that $z=\eta$ is the only root of $\det(I-R^*(\eta))=0$ on $|z|=\eta$. Therefore, $z=\eta$ is the only singular point of $\Pi^*(z)$ on $|z|=\eta$. The following lemma further illustrates that the singular point $z=\eta$ is a pole of order 1 of $\Pi^*(z)$.

Lemma 4.7 *Let*

$$1-r(z)=(\eta-z)h(z). \tag{4.16}$$

Then, $h(\eta)>0$.

Proof It follows from Eq. (4.16) that

$$-r'(z)=-h(z)+\left(\eta-z\right)h'(z),$$

which follows that

$$h(\eta)=r'(\eta). \tag{4.17}$$

Noting that

$$u(z)R^*(z)=r(z)u(z),$$

we have

$$u'(z)R^*(z)+u(z)\cdot\frac{\mathrm{d}}{\mathrm{d}z}R^*(z)=r'(z)u(z)+r(z)u'(z).$$

This equation, multiplied by $v(z)$ from the right and using $u(z)\cdot v(z)=1$, yields

$$r'(z)=u(z)\cdot\frac{\mathrm{d}}{\mathrm{d}z}R^*(z)\cdot v(z).$$

Thus, we obtain

$$\begin{aligned} r'(\eta) &= u(\eta)\cdot\sum_{k=1}^{\infty}k\eta^{k-1}R_k\cdot v(\eta)=\frac{1}{\eta}u(\eta)\cdot\sum_{k=1}^{\infty}k\eta^{k}R_k\cdot v(\eta)\\ &\geqslant\frac{1}{\eta}u(\eta)R^*(\eta)v(\eta)=\frac{1}{\eta}r(\eta)=\frac{1}{\eta}>0. \end{aligned}$$

It follows from Eq. (4.17) that $h(\eta)=r'(\eta)>0$. This completes the proof.

We are now ready to provide an expression for $\Pi^*(z)$ by means of Lemmas 4.6 and 4.7.

According to Lemma 4.6, we write

$$\operatorname{adj}(I-R^*(z))=\prod_{i=2}^{m}[1-r_i(\eta)]\cdot v(\eta)u(\eta)+(\eta-z)H(z).$$

It follows from Eq. (4.1) and Eq. (4.16) that

$$\varPi^*(z) = \frac{1}{\eta - z} \cdot \frac{1}{h(z)\prod_{i=2}^{m}[1-r_i(z)]} \cdot \pi_0 R_0^*(z) \cdot \left[\prod_{i=2}^{m}[1-r_i(\eta)]v(\eta)u(\eta) + (\eta - z)H(z)\right]. \tag{4.18}$$

Let

$$L(z) = \frac{\pi_0 R_0^*(z) v(\eta) \prod_{i=2}^{m}[1-r_i(\eta)]}{h(z)\prod_{i=2}^{m}[1-r_i(z)]}.$$

Since $h(z)$ is continuous at $z = \eta$ and $h(\eta) > 0$, there exists a $\sigma > 0$ small enough such that $h(z) > 0$, for all $z \in (\eta - \sigma, \eta + \sigma)$. Note that $h(z)$, $\prod_{i=2}^{m}[1-r_i(z)]$ and $R_0^*(z)$ are all analytic at $z = \eta$, it is obvious that $L(z)$ is analytic at $z = \eta$. Thus, an expansion of $L(z)$ in a power series at $z = \eta$ is given by

$$L(z) = \frac{\pi_0 R_0^*(\eta) v(\eta)}{h(\eta)} + \sum_{k=1}^{\infty} \frac{L^{(k)}(\eta)}{k!}(\eta - z)^k, \tag{4.19}$$

where $L^{(k)}(\eta) = \frac{\mathrm{d}^k}{\mathrm{d}z^k} L(z)|_{z=\eta}$. It follows from Eq. (4.18) and Eq. (4.19) that

$$\varPi^*(z) = \frac{K}{\eta - z} u(\eta) + g(z), \tag{4.20}$$

where

$$K = \frac{\pi_0 R_0^*(\eta) v(\eta)}{h(\eta)}$$

is a constant, and

$$g(z) = \frac{\pi_0 R_0^*(z)}{h(z)\prod_{i=2}^{m}[1-r_i(z)]} \cdot H(z) + \sum_{k=1}^{\infty} \frac{L^{(k)}(\eta)}{k!}(\eta - z)^{k-1} u(\eta)$$

is analytic at $z = \eta$, since $\frac{\pi_0 R_0^*(z)}{h(z)\prod_{i=2}^{m}[1-r_i(z)]}$, $H(z)$ and $\sum_{k=1}^{\infty} \frac{L^{(k)}(\eta)}{k!}(\eta - z)^k$ are all analytic at $z = \eta$.

Theorem 4.4 *Suppose that the Markov chain of $GI/G/1$ type is irreducible and positive recurrent. If $1<\eta<\min\{\phi_{A+},\phi_{D+}\}$, and the matrix A is irreducible, then the asymptotics of $\{\pi_k\}$ is geometric, i.e.,*

$$\pi_k = K\eta^{-(k+1)}u(\eta)+O\left((\eta+\varepsilon)^{-k}\right)e^T, \tag{4.21}$$

where ε is a small positive number.

Proof Based on Eq. (4.20), we first need to check $K>0$. Then Eq. (4.21) is obviously true according to a standard result on asymptotics of complex functions (for example, Theorem 5.2.1 in Wilf [37]).

According to the assumption made on the Markov chain of $GI/G/1$ type, every π_k is positive. It is clear that $\pi_0 R_0^*(\eta)\geqslant \pi_0 R_0^*(1)>0$, since $\pi_0>0$ and each column vector of $R_0^*(1)$ is not identically zero according to Theorem 3.3. Noting that $v(\eta)\geqslant 0$ and $v(\eta)\neq 0$, we obtain $\pi_0 R_0^*(\eta)v(\eta)\geqslant \pi_0 R_0^*(1)v(\eta)>0$. Now, $K>0$ follows from $h(\eta)>0$. This completes the proof.

4.3.2 Markov Chains of *GI*/*M*/1 Type

As an illustrating example, we consider a Markov chain of $GI/M/1$ type. In this case, $R_k=0$ and $R_{0,k}=0$ for $k\geqslant 2$. Therefore, $R_0^*(z)=zR_{0,1}$, $R^*(z)=zR_1$, $\phi_{R_0}=\phi_{D+}=+\infty$ and $\phi_R=\phi_{A+}=+\infty$. It follows from (4.1) and $\pi_1=\pi_0 R_{0,1}$ that

$$\Pi^*(z)=z\pi_1(I-zR_1)^{-1}=\frac{1}{1-zr}\cdot\frac{z\pi_1}{\prod_{i=2}^{m}(1-zr_i)}\cdot \operatorname{adj}(I-zR_1),$$

where r is the maximal eigenvalue of R_1, which is smaller than one when the Markov chain is positive recurrent, and r_i, $2\leqslant i\leqslant m$, are the other eigenvalues of R_1. It is clear that $\eta=1/r$. Therefore, $1<\eta<\min\{\phi_{A+},\phi_{D+}\}=+\infty$, and the asymptotics of $\{\pi_k\}$ is geometric.

4.3.3 Markov Chains of *M*/*G*/1 Type

The method of generating function is one of the common methods used to study the light-tailed behavior of the stationary probability vector $\{\pi_k\}$. (3.3.2) in Neuts [29] (or (2.1) in Falkenberg [11]) expresses $\Pi^*(z)$ in terms of $\sum_{k=1}^{\infty} z^k A_k$ and another function $D^*(z)$. Note that our expression Eq. (4.1) is different from those in the literature, that is, $\Pi^*(z)$ is expressed in terms of the two R-measure generating functions $R_0^*(z)$ and $R^*(z)$. Furthermore, the relationship between $R^*(z)$ and $A^*(z)$, and between $R_0^*(z)$ and $D^*(z)$ established in Theorems 3.5 and 3.6, enables us to obtain sharper results than those in the literature.

Following the idea in Falkenberg [11], the generating function for Markov chains of $GI/G/1$ type can be written as

$$\Pi^*(z)[I - A^*(z)] = \pi_0 D^*(z) - \sum_{i=1}^{\infty} z^i \pi_i \sum_{j=-i}^{\infty} z^{-j} A_{-j}.$$

For a Markov chain of $M/G/1$ type, it becomes

$$\sum_{i=1}^{\infty} z^i \pi_i \sum_{j=-i}^{\infty} z^{-j} A_{-j} = \pi_1 A_{-1}$$

and

$$\Pi^*(z) = [\pi_0 D^*(z) - \pi_1 A_{-1}][I - A^*(z)]^{-1},$$

which was used in the literature to study the asymptotic behavior, including Falkenberg [11] among others. However, for the Markov chain of $GI/G/1$ type, it is very inconvenient or difficult to explicitly express $\sum_{i=1}^{\infty} z^i \pi_i \sum_{j=-i}^{\infty} z^{-j} A_{-j}$. This illustrates the reason why the R-measure is effective for explicitly expressing the generating function $\Pi^*(z)$.

4.3.4 A is Reducible

After reordering the states, we assume that $A^*(z)$ is written in the normal form

$$A^*(z) = \begin{pmatrix} a_1(z) & & & & & & \\ & \ddots & & & & & \\ & & a_p(z) & & & & \\ & & & a_{p+1}(z) & & & \\ & & & & \ddots & & \\ & & & & & a_{p+q} & \\ r_1(z) & \dots & r_p(z) & r_{p+1}(z) & \dots & r_{p+q}(z) & r_{p+q+1}(z) \end{pmatrix},$$

where $p+q \geqslant 1$, $a_i(z)$ for $1 \leqslant i \leqslant p+q$ are irreducible and stochastic, $\eta > 1$ is a solution to equation $\det(I - a_i(z)) = 0$ for $1 \leqslant i \leqslant p$, while $I - r_{p+q+1}(\eta)$ and $I - a_{p+j}(\eta)$ for $1 \leqslant j \leqslant q$ are all invertible. It follows from Eq. (3.19) and Corollary 3.10 that for $1 < z < \eta$

$$I - R^*(z) = [I - A^*(z)][I - G^*(z)]^{-1}(I - \Phi_0)^{-1}.$$

Thus we obtain

$$\det(I - R^*(z)) = (\eta - z)^p \prod_{i=1}^{p} b_i(z) \cdot \prod_{j=1}^{q} \det[I - a_{p+j}(z)] \cdot \det[I - r_{p+q+1}(z)] \\ \cdot [\det(I - G^*(z))]^{-1} \cdot [\det(I - \Phi_0)]^{-1},$$

where $\det(I - a_i(z)) = (\eta - z) b_i(z)$ for $1 \leqslant i \leqslant p$. It follows from Lemma 4.7 that $b_i(\eta) \neq 0$ for $1 \leqslant i \leqslant p$. Thus, it follows from Eq. (4.1) and Eq. (4.16) that

$$\Pi^*(z) = \frac{1}{(\eta - z)^p} K(z),$$

where

$$K(z) = \left\{ \prod_{i=1}^{p} b_i(z) \prod_{j=1}^{q} \det[I - a_{p+j}(z)] \cdot \det[I - r_{p+q+1}(z)] \right\}^{-1} \\ \cdot [\det\,(I - G^*(z))][\det(I - \Phi_0)] \cdot \pi_0 R_0^*(z) \cdot \mathrm{adj}(I - R^*(z))$$

is analytic at $z = \eta$. Let $K(z) = \sum_{k=0}^{\infty} \frac{K^{(k)}(\eta)}{k!} (\eta - z)^k$, where $K^{(k)}(\eta) = \frac{\mathrm{d}^k}{\mathrm{d}z^k} K(z)_{|z=\eta}$.

Then

$$\Pi^*(z) = \sum_{i=1}^{p} \frac{K^{(p-i)}(\eta)}{(\eta - z)^i} + \sum_{j=0}^{\infty} K^{(p+j)}(\eta)(\eta - z)^j.$$

By a standard result on asymptotics of complex functions (for example, Theorem 5.2.1 in Wilf [37]) we obtain

$$\pi_k = \sum_{l=1}^{p} (-1)^l \eta^{-(k+l)} \binom{k+l-1}{l-1} K^{(p-l)}(\eta) + O((\eta + \varepsilon)^{-k}) e^{\mathrm{T}}. \tag{4.22}$$

Remark 4.2 *If* $1 < \eta < \min\{\phi_{A+}, \phi_{D+}\}$, *then Eq.* (4.22) *provides a more general result on the light tail of* $\{\pi_k\}$. *Let* $\det(I - A^*(z)) = (\eta - z)^p T(z)$, *where* $T(\eta) \neq 0$. *Then, the tail of* $\{\pi_k\}$ *is geometric if* $p = 1$; *the tail of* $\{\pi_k\}$ *is non-geometric if* $p \geqslant 2$.

4.4 The Asymptotics Based on the Boundary Matrices

In this section, we consider the role of the boundary matrices $\{D_k\}$ played in analyzing the light-tailed asymptotics of $\{\pi_k\}$. We assume that $1 < \phi_{D+} \leqslant \eta < \phi_{A+}$.

It is easy to see from Theorem 3.8 that $z=\phi_{D+}$ is a singular point of $\Pi^*(z)$ so that the boundary matrices $\{D_k\}$ determine the asymptotics of $\{\pi_k\}$. For simplicity of description, we only discuss the cases where the singular point is either a pole of order d or an algebraic singular point.

4.4.1 ϕ_{D+} is a Pole

We assume that ϕ_{D+} is a pole of order d for the generating function matrix $D^*(z)$. The following lemma describes a singular point of the matrix $R_0^*(z)$.

Lemma 4.8 *If $z=\phi_{D+}$ is a pole of order d of $D^*(z)$ of size $m_0\times m$, then $z=\phi_{D+}$ is also a pole of order d of $R_0^*(z)$, i.e.,*

$$R_0^*(z)=\frac{1}{(\phi_{D+}-z)^d}S(z), \tag{4.23}$$

where $S(z)$ is analytic in $|z|<\phi_{D+}+\sigma$ for some $\sigma>0$, and $S(\phi_{D+})\neq 0$.

Proof If $z=\phi_{D+}$ is a pole of order d of $D^*(z)$ of size $m_0\times m$, then there always exists a matrix $B(z)$ such that

$$D^*(z)=\frac{1}{(\phi_{D+}-z)^d}B(z),$$

where $B(z)$ is analytic in $|z|<\phi_{D+}+\sigma$ for some $\sigma>0$, and $B(\phi_{D+})\neq 0$. It follows from Eq. (3.30) that when $\delta>0$ is small enough and $z>1+\delta$,

$$\frac{1}{(\phi_{D+}-z)^d}B(z)(I-\Phi_0)^{-1}\leqslant R_0^*(z)\leqslant\frac{1}{(\phi_{D+}-z)^d}B(z)[I-G^*(1+\delta)](I-\Phi_0)^{-1},$$

which illustrates that $z=\phi_{D+}$ is a pole of order d of $R_0^*(z)$ with Eq. (4.23). This completes the proof.

Theorem 4.5 *Assume that $1<\phi_{D+}<\eta<\phi_{A+}$, and $z=\phi_{D+}$ is a pole of order d of $D^*(z)$. Let $R_0^*(z)$ be given by Eq. (4.23). Then, for some small $\varepsilon>0$,*

$$\pi_k=\sum_{j=1}^{d}(-1)^j\phi_{D+}^{-(k+j)}\binom{k+j-1}{j-1}L_{d-j}+O((\phi_{D+}+\varepsilon)^{-k})e^{\mathrm{T}},$$

where

$$L_j=\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\{\pi_0 S(z)[I-R^*(z)]^{-1}\}_{|z=\phi_{D+}}.$$

Proof It follows from Lemma 4.8 and Eq. (4.1) that

$$\Pi^*(z) = \frac{1}{(\phi_{D+} - z)^d} \cdot \pi_0 S(z)[I - R^*(z)]^{-1},$$

where $S(z)[I - R^*(z)]^{-1}$ is analytic in $|z| < \phi_{D+} + \sigma$ for $0 < \sigma < \phi_{A+} - \phi_{D+}$. An expansion of $\pi_0 S(z)[I - R^*(z)]^{-1}$ in a power series at $z = \phi_{D+}$ is

$$\pi_0 S(z)[I - R^*(z)]^{-1} = \sum_{j=0}^{\infty} L_j (\phi_{D+} - z)^j.$$

Hence we obtain

$$\Pi^*(z) = \sum_{j=1}^{d} \frac{1}{(\phi_{D+} - z)^j} L_{d-j} + \sum_{i=0}^{\infty} L_{d+i} (\phi_{D+} - z)^i.$$

A standard result on asymptotics of complex functions (for example, Theorem 5.2.1 in Wilf [37]) leads to the desired result. This completes the proof.

Corollary 4.1 *Assume that* $1 < \phi_{D+} = \eta < \phi_{A+}$, $z = \phi_{D+}$ *is a pole of order d of* $D^*(z)$ *and the matrix A is irreducible. Let* $R_0^*(z)$ *be given by Eq.* (4.23). *Then, for some small* $\varepsilon > 0$,

$$\pi_k = \sum_{j=1}^{d+1} (-1)^j \phi_{D+}^{-(k+j)} \binom{k+j-1}{j-1} L_{d+1-j} + O((\phi_{D+} + \varepsilon)^{-k}) e^{\mathrm{T}}.$$

Proof Under the irreducible assumption on the matrix A, η is a simple root of the equation $\det(I - R^*(z)) = 0$. Therefore, $z = \phi_{D+}$ is a pole of order $d+1$ of $\Pi^*(z)$. The rest of the proof is similar to that for Theorem 4.5. This completes the proof.

It is easy to see from Theorem 4.5 that the asymptotics of $\{\pi_k\}$ is geometric if $d = 1$ and $1 < \phi_{D+} \leqslant \eta < \phi_{A+}$; otherwise it is not geometric, though it is still light-tailed. Clearly, if $1 < \phi_{D+} = \eta < \phi_{A+}$, then the asymptotics of $\{\pi_k\}$ is not geometric.

4.4.2 ϕ_{D+} is an Algebraic Singular Point

We assume that $z = \phi_{D+}$ is an algebraic singular point of $D^*(z)$ such that

$$D^*(z) = \widetilde{D} + (\phi_{D+} - z)^{\alpha} \widetilde{D}^*(z), \tag{4.24}$$

where $\alpha > 1$ is not an integer, $\widetilde{D} = D^*(1) - (\phi_{D+} - 1)^{\alpha} \widetilde{D}^*(1)$, $\widetilde{D}^*(z)$ is analytic in $|z| < \phi_{D+} + \sigma$ for some $\sigma > 0$, and $\widetilde{D}^*(\phi_{D+}) \neq 0$. The restriction of $\alpha > 1$ imposed here is to guarantee that $\sum_{k=1}^{\infty} k D_k < +\infty$.

To provide the asymptotics of $\{\pi_k\}$ in this case, we need to introduce an operational principle. This principle for the continuous case was discussed in Abate and Whitt [2] (p.186).

Heaviside Operational Principle Suppose that $f^*(z)=\sum_{k=1}^{\infty} z^k f_k$ is the generating function of a nonnegative sequence $\{f_k\}$. If z^* is the radius of convergence of $f^*(z)$, and the asymptotic expansion is

$$f^*(z)=\sum_{k=1}^{\infty} a_k (z^*-z)^k+\theta^*(z),$$

where $\theta^*(z)$ is the generating function of a nonnegative sequence $\{\theta_k\}$, then $f_k \sim \theta_k$ as $k \to \infty$.

It follows from Eq. (3.30) that

$$W(z) \leqslant R_0^*(z) \leqslant V(z), \tag{4.25}$$

where

$$W(z)=[\widetilde{D}+\left(\phi_{D+}-z\right)^{\alpha}\widetilde{D}^*(z)](I-\Phi_0)^{-1} \tag{4.26}$$

and

$$V(z)=\left[\widetilde{D}+(\phi_{D+}-z)^{\alpha}\widetilde{D}^*(z)\right][I-G^*(1+\delta)]^{-1}(I-\Phi_0)^{-1}.$$

It follows from Eq. (4.25) and Eq. (4.1) that for $0<z<\phi_{D+}$,

$$\pi_0 W(z)[I-R^*(z)]^{-1} \leqslant \Pi^*(z) \leqslant \pi_0 V(z)[I-R^*(z)]^{-1}, \tag{4.27}$$

since $\pi_0>0$ and $[I-R^*(z)]^{-1} \geqslant 0$.

There are two possible cases:

Case Ⅰ $1<\phi_{D+}<\eta<\phi_{A+}$. In this case, $[I-R^*(z)]^{-1}$ is analytic at $z=\phi_{D+}$, hence an expansion of $[I-R^*(z)]^{-1}$ in a power series at $z=\phi_{D+}$ is given by

$$[I-R^*(z)]^{-1}=\sum_{n=0}^{\infty}\Gamma_n(\phi_{D+}-z)^n, \tag{4.28}$$

where $\Gamma_n=\dfrac{1}{n!}\cdot\dfrac{\mathrm{d}^n}{\mathrm{d}z^n}[I-R^*(z)]^{-1}|_{z=\phi_{D+}}$. Eq. (4.28), together with Eq. (4.27), yields

$$\begin{aligned}\pi_0 W(z)[I-R^*(z)]^{-1}=&\,\pi_0\widetilde{D}(I-\Phi_0)^{-1}\sum_{n=0}^{\infty}\Gamma_n(\phi_{D+}-z)^n\\&+\sum_{n=0}^{\infty}(\phi_{D+}-z)^{\alpha+n}\pi_0\widetilde{D}^*(z)(I-\Phi_0)^{-1}\Gamma_n.\end{aligned}$$

Let

$$\Re^{(1)}(z)=\pi_0 W(z)[I-R^*(z)]^{-1}=\sum_{k=1}^{\infty} z^k \Re_k^{(1)}$$

and

$$\widetilde{D}^*(z)=\sum_{j=0}^{\infty}\widetilde{D}_j(\phi_{D+}-z)^j.$$

By Heaviside Operational Principle and Theorem 5.3.1 in Wilf [37] we obtain that for any given $1\leqslant m\leqslant k$ and as $k\to\infty$

$$\begin{aligned}\Re_k^{(1)}&=\sum_{n=0}^{\infty}\phi_{D+}^{-(k-\alpha-n)}\sum_{j=0}^{m}\binom{k-\alpha-n-j-1}{k}\pi_0\widetilde{D}_j(I-\Phi_0)^{-1}\Gamma_n\\&\quad+O\left(\sum_{n=0}^{\infty}\phi_{D+}^{-(k-\alpha-n)}k^{-m-n-\alpha-2}\pi_0\tilde{D}^*(1)(I-\Phi_0)^{-1}\Gamma_n\right)\\&=\phi_{D+}^{-(k-\alpha)}\sum_{j=0}^{m}\pi_0\widetilde{D}_j(I-\Phi_0)^{-1}\sum_{n=0}^{\infty}\phi_{D+}^{n}\binom{k-\alpha-n-j-1}{k}\Gamma_n\\&\quad+O\left(\phi_{D+}^{-(k-\alpha)}k^{-m-\alpha-2}\cdot\pi_0\tilde{D}^*(1)(I-\Phi_0)^{-1}\cdot\sum_{n=0}^{\infty}\phi_{D+}^{n}k^{-n}\Gamma_n\right).\end{aligned}$$

Let

$$\zeta_{k,j,\alpha}=\sum_{n=0}^{\infty}\phi_{D+}^{n}\binom{k-\alpha-n-j-1}{k}\Gamma_n,\quad \varphi_k=\sum_{n=0}^{\infty}\phi_{D+}^{n}k^{-n}\Gamma_n.$$

Then

$$\begin{aligned}\Re_k^{(1)}&=\phi_{D+}^{-(k-\alpha)}\sum_{j=0}^{m}\pi_0\widetilde{D}_j(I-\Phi_0)^{-1}\zeta_{k,j,\alpha}\\&\quad+O\left(\phi_{D+}^{-(k-\alpha)}k^{-m-\alpha-2}\mathrm{e}^T\varphi_k\right).\end{aligned}\tag{4.29}$$

Let

$$\Re^{(2)}(z)=\pi_0 V(z)[I-R^*(z)]^{-1}=\sum_{k=1}^{\infty} z^k \Re_k^{(2)}.$$

A similar analysis to Eq. (4.29) gives that for any given $1\leqslant m\leqslant k$ and as $k\to\infty$,

$$\begin{aligned}\Re_k^{(2)}&=\phi_{D+}^{-(k-\alpha)}\sum_{j=0}^{m}\pi_0\widetilde{D}_j[I-G^*(1+\delta)]^{-1}(I-\Phi_0)^{-1}\zeta_{k,j,\alpha}\\&\quad+O\left(\phi_{D+}^{-(k-\alpha)}k^{-m-\alpha-2}\mathrm{e}^T\varphi_k\right).\end{aligned}\tag{4.30}$$

It follows from Eq. (4.27) that for $k \geqslant 1$,

$$\Re_k^{(1)} \leqslant \pi_k \leqslant \Re_k^{(2)}, \tag{4.31}$$

It follows from Eq. (4.29), Eq. (4.30) and Eq. (4.31) that there exists a bounded matrix sequence $\{H_k(m,\alpha)\}$ with

$$(I-\Phi_0)^{-1} \leqslant H_k(m,\alpha) \leqslant [I-G^*(1+\delta)]^{-1}(I-\Phi_0)^{-1}$$

such that for any given $1 \leqslant m \leqslant k$ and as $k \to \infty$,

$$\begin{aligned}\pi_k = {} & \phi_{D+}^{-(k-\alpha)} \sum_{j=0}^{m} \pi_0 \widetilde{D}_j H_k(m,\alpha) \zeta_{k,j,\alpha} \\ & + O(\phi_{D+}^{-(k-\alpha)} k^{-m-\alpha-2} e^{\mathrm{T}} \varphi_k).\end{aligned} \tag{4.32}$$

Case Ⅱ $1 < \phi_{D+} = \eta < \phi_{A+}$, and the matrix A is irreducible. In this case, we have

$$[I - R^*(z)]^{-1} = \frac{1}{\phi_{D+} - z} S(z),$$

where $S(z)$ is analytic at $z = \phi_{D+}$, and $S(\phi_{D+}) \neq 0$. A similar analysis to Eq. (4.32) can be used to obtain that for any given $1 \leqslant m \leqslant k$ and as $k \to \infty$,

$$\begin{aligned}\pi_k = {} & \phi_{D+}^{-(k-\alpha+1)} \sum_{j=0}^{m} \pi_0 \widetilde{D}_j H_k(m,\alpha-1) \zeta_{k,j,\alpha-1} \\ & + O\left(\phi_{D+}^{-(k-\alpha+1)} k^{-m-\alpha-1} e^{\mathrm{T}} \varphi_k\right).\end{aligned}$$

We summarize the above results into the following theorem.

Theorem 4.6 *Suppose that* $z = \phi_{D+}$ *is an algebraic singular point of* $D^*(z)$ *given by Eq.* (4.23).

(1) *If* $1 < \phi_{D+} < \eta < \phi_{A+}$, *then for any given* $1 \leqslant m \leqslant k$ *and as* $k \to \infty$,

$$\begin{aligned}\pi_k = {} & \phi_{D+}^{-(k-\alpha)} \sum_{j=0}^{m} \pi_0 \widetilde{D}_j H_k(m,\alpha) \zeta_{k,j,\alpha} \\ & + O\left(\phi_{D+}^{-(k-\alpha)} k^{-m-\alpha-2} e^{\mathrm{T}} \varphi_k\right).\end{aligned}$$

(2) *If* $1 < \phi_{D+} = \eta < \phi_{A+}$, *and* A *is irreducible, then for any given* $1 \leqslant m \leqslant k$ *and as* $k \to \infty$,

$$\begin{aligned}\pi_k = {} & \phi_{D+}^{-(k-\alpha+1)} \sum_{j=0}^{m} \pi_0 \widetilde{D}_j H_k(m,\alpha-1) \zeta_{k,j,\alpha-1} \\ & + O\left(\phi_{D+}^{-(k-\alpha+1)} k^{-m-\alpha-1} e^{\mathrm{T}} \varphi_k\right).\end{aligned}$$

4.5 Long-Tailed Asymptotics of the Sequence $\{R_k\}$

In this section, we provide long-tailed asymptotics for the matrix sequence $\{R_k\}$ if the matrix sequence $\{A_k\}$ is long-tailed. The results in this section are key to deriving subexponential and regularly varying asymptotics of the stationary probability vector $\{\pi_k\}$ in subsequent sections.

For a matrix sequence $\{B_k\}$, if there exists a scalar sequence $\{\beta_k\}$ and a finite, non-zero nonnegative matrix W such that $\lim_{k\to\infty}\frac{\overline{B}_{\leqslant k}}{\overline{\beta}_{\leqslant k}}=W$, then $\{\beta_k\}$ and W are called a uniformly dominant squence and the associate ratio matrix, respectively.

The following two lemmas are useful in determining a uniformly dominant sequence of the matrix sequence $\{R_k\}$ and the associated ratio matrix if the matrix sequence $\{A_k\}$ is long-tailed.

Lemma 4.9 *If the Markov chain of $GI/G/1$ type is positive recurrent and $\sum_{k=-\infty}^{\infty}|k|A_k$ is finite, then $\sum_{k=1}^{\infty}kG_k$ is finite.*

Proof It follows from Eq. (3.20) that

$$\sum_{k=1}^{\infty}kA_{-k}-\sum_{k=1}^{\infty}kA_k=(I-R)(I-\Phi_0)\sum_{k=1}^{\infty}kG_k-\sum_{k=1}^{\infty}kR_k(I-\Phi_0)(I-G).$$

Since the Markov chain is positive recurrent, it follows from (1) in Corollary 3.7 that $I-R$ is invertible and $(I-G)e=0$. It is clear that $(I-\Phi_0)^{-1}(I-R)^{-1}\gneqq 0$ is finite. Since $\sum_{k=-\infty}^{\infty}|k|A_k$ is finite,

$$\sum_{k=1}^{\infty}kG_ke=(I-\Phi_0)^{-1}(I-R)^{-1}\left(\sum_{k=1}^{\infty}kA_{-k}-\sum_{k=1}^{\infty}kA_k\right)e$$

is finite. Therefore, $\sum_{k=1}^{\infty}kG_k$ is finite. This completes the proof.

When the Markov chain of $GI/G/1$ type is positive recurrent, the matrix $I-R$ is invertible according to Corollary 3.7. It follows from Eq. (3.20) that $I-A=(I-R)(I-\Phi_0)(I-G)$. When A is irreducible and stochastic, the maximal eigenvalue of A is simple and equal to one. Hence, rank $(I-A)=m-1$. Since the matrix $I-\Phi_0$ is invertible, we obtain that rank $(I-G)=m-1$, hence the maximal eigenvalue of G is simple and equal to one. Letting $g_1=1$ and g_i, for $1\leqslant i\leqslant m$, be the m eigenvalues of the nonnegative matrix G, we have the following lemma.

Lemma 4.10 *If the Markov chain of $GI/G/1$ type is positive recurrent and* *the matrix A is irreducible and stochastic, then the adjoint matrix of $I-G$ can be expressed as*

$$\operatorname{adj}(I-G)=\kappa_G e\frac{\theta(I-R)(I-\Phi_0)}{\theta(I-R)(I-\Phi_0)\,e}, \tag{4.33}$$

where $\kappa_G=\prod_{k=2}^{m}(1-g_i)\neq 0.$

Proof Note that when the maximal eigenvalue of the matrix G is simple and is equal to one, we obtain for $\alpha\in(0,1)$,

$$\begin{aligned}\operatorname{adj}(I-\alpha G)&=\det(I-\alpha G)\cdot(I-\alpha G)^{-1}\\&=\prod_{i=2}^{m}(1-\alpha g_i)\cdot(1-\alpha)(I-\alpha G)^{-1}.\end{aligned}$$

Let $T(\alpha)$ be an invertible matrix such that

$$T(\alpha)^{-1}(I-\alpha G)T(\alpha)=\begin{pmatrix}1-\alpha & \\ & J(\alpha)\end{pmatrix},$$

which is the Jordan canonical form of the matrix $I-\alpha G.$ Then

$$(1-\alpha)(I-\alpha G)^{-1}=T(\alpha)\begin{pmatrix}1 & \\ & (1-\alpha)J(\alpha)^{-1}\end{pmatrix}T(\alpha)^{-1}.$$

Since in the matrix

$$T(1)^{-1}(I-G)T(1)=\begin{pmatrix}0 & \\ & J(1)\end{pmatrix},$$

$J(1)$ is invertible due to rank $(I-G)=m-1,$ this implies $\lim_{\alpha\nearrow 1}(1-\alpha)J(\alpha)^{-1}=0.$ Note that adj $(I-\alpha G)$ is continuous for $\alpha\in(0,1],$ we get

$$\begin{aligned}\operatorname{adj}(I-G)&=\lim_{\alpha\nearrow 1}\operatorname{adj}(I-\alpha G)\\&=\lim_{\alpha\nearrow 1}\prod_{i=2}^{m}(1-\alpha g_i)\cdot(1-\alpha)(I-\alpha G)^{-1}\\&=\kappa_G T(1)\begin{pmatrix}1 & \\ & 0\end{pmatrix}T(1)^{-1}\\&=\kappa_G e\frac{\theta(I-R)(I-\Phi_0)}{\theta(I-R)(I-\Phi_0)e}.\end{aligned}$$

since the vectors e and $\dfrac{\theta(I-R)(I-\Phi_0)}{\theta(I-R)(I-\Phi_0)e}$ are the right and left Perron-Frobenius eigenvectors of G, respectively. Since rank $(I-G)=m-1$, it is clear that adj $(I-G)\neq 0$, which implies $\kappa_G \neq 0$. This completes the proof.

To study long-tailed asymptotics of the matrix sequence $\{R_k\}$, we need to extend the results in Lemma 4 and Proposition 1 of Jelenković and Lazar [17] to a matrix setting, which are described in the following two lemmas. All the measures involved in the following can be signed measures.

Let $\mathfrak{B}(\mathbb{R})$ be the σ-algebra of Borel sets on $\mathbb{R}=(-\infty,+\infty)$. The convolution of two measures μ_1 and μ_2 is defined as

$$(\mu_1 * \mu_2)(B)=\int_{(-\infty,+\infty)} \mu_1(B-x)\mu_2(\mathrm{d}x),$$
$$B\in\mathfrak{B}(\mathbb{R}), B-x=\{y: y+x\in B\}.$$

For $B\in\mathfrak{B}(\mathbb{R})$, let $U(B)$ and $V(B)$ be two matrices of size $m\times m$ whose entries are finite measures, given as

$$U(B)=(u_{ij}(B))_{1\leqslant i,j\leqslant m} \text{ and } V(B)=(v_{ij}(B))_{1\leqslant i,j\leqslant m}.$$

The convolution of the two matrices U and V of finite measures is defined as

$$(U*V)(B)=\left(\sum_{k=1}^{m}(u_{ik}*v_{kj})(B)\right)_{1\leqslant i,j\leqslant m}$$

and the convolution of a matrix U of finite measures and a finite scalar measure v is defined as

$$(U*v)(B)=((u_{ij}*v)(B))_{1\leqslant i,j\leqslant m}.$$

Remark 4.3 *It should be noted that when B is a singleton, the convolution for measures coincides with the ordinary convolution for sequences.*

Lemma 4.11 *Let U and U_- be two matrices of finite measures of size m on* $(\mathbb{R},\mathfrak{B}(\mathbb{R}))$. *If* (1) $\lim\limits_{x\to+\infty}\dfrac{U([x,+\infty))}{\overline{F}_{(x)}}=C$, *where $F(x)$ is a long-tailed distribution function and the matrix C is finite, and* (2) *U_- has a support on* $(-\infty,0]$, *then the matrix $\varGamma=U_- * U$ satisfies*

$$\lim_{x\to+\infty}\frac{\varGamma([x,+\infty))}{\overline{F}(x)}=U_-((-\infty,0])C,$$

and the matrix $\tilde{\varGamma}=U*U_-$ satisfies

$$\lim_{x\to+\infty}\frac{\tilde{\varGamma}([x,+\infty))}{\overline{F}(x)}=CU_-((-\infty,0]).$$

Proof By using Lemma 4 in Jelenković and Lazar [17], we obtain

$$\begin{aligned}\lim_{x\to+\infty}\frac{\Gamma([x,+\infty))}{\overline{F}(x)} &= \lim_{x\to+\infty}\frac{1}{\overline{F}(x)}\left(\sum_{k=1}^{m}(u_{-ik}*u_{kj})([x,+\infty))\right)_{1\leqslant i,j\leqslant m}\\ &=\left(\sum_{k=1}^{m}u_{-ik}((-\infty,0])\,c_{kj}\right)_{1\leqslant i,j\leqslant m}\\ &=U_{-}((-\infty,0])C.\end{aligned}$$

This completes the proof.

Lemma 4.12 *Let* U *and* U_+ *be two matrices of finite measures of size* m *on* $(\mathbb{R},\mathfrak{B}(\mathbb{R}))$. *Assume that*

(1) μ_- *is a finite scalar measure with a support on* $(-\infty,0]$ *such that* $\mu_-((-\infty,0])=0$ *and* $0<\left|\int_{(-\infty,0]}x\mu_-(\mathrm{d}x)\right|<+\infty$;

(2) U_+ *has a support on* $[0,+\infty)$ *with at least one non-zero element, and all the non-zero elements* U_+ *of are strictly positive on* $[a,+\infty)$ *for* $a>0$; *and*

(3) $\lim_{x\to+\infty}\dfrac{U([x,+\infty))}{\overline{F}_{(x)}}=C$, *where* $F(x)$ *is a long-tailed distribution function and the matrix* C *is finite. If* $U=\mu_- * U_+$, *then*

$$\lim_{x\to+\infty}\frac{U_+([x,+\infty))}{\int_{[x,+\infty]}\overline{F}_{(y)}\mathrm{d}y}=\frac{1}{\int_{(-\infty,0]}x\mu_-\left(\mathrm{d}x\right)}C.$$

Proof The proof is obvious according to (2) of Proposition 1 in Jelenković and Lazar [17].

The following lemma provides a structural property for the matrix sequence $\{R_k\}$ if the matrix sequence $\{A_k\}$ is long-tailed.

Lemma 4.13 *Suppose that the Markov chain of* $GI/G/1$ *type is positive recurrent. If* $\{A_k\}$ *is long-tailed with a uniformly dominant sequence* $\{p_k\}$ *and the associated ratio matrix W, then*

$$\underline{\lim}_{k\to+\infty}\frac{\overline{R_{\leqslant k}}}{\overline{p_{\leqslant k}}}\geqslant W(I-\Phi_0)^{-1}.$$

Proof Note that the Markov chain of $GI/G/1$ type is positive recurrent, it follows from Eq. (3.12) and Eq. (3.9) that for all $k\geqslant 1$,

$$\begin{aligned}R_k &= A_k(I-\Phi_0)^{-1}+\sum_{l=1}^{\infty}R_{k+l}\Phi_{-l}(I-\Phi_0)^{-1}\\ &\geqslant A_k(I-\Phi_0)^{-1},\end{aligned}$$

since $(I-\Phi_0)^{-1} \geqslant 0$, $R_k \geqslant 0$ and $\Phi_{-k} \geqslant 0$ for $k \geqslant 1$. Hence, for all $k \geqslant 1$,

$$\overline{R_{\leqslant k}} = \sum_{l=k+1}^{\infty} R_l \geqslant \sum_{l=k+1}^{\infty} A_k (I-\Phi_0)^{-1} = \overline{A_{\leqslant k}} (I-\Phi_0)^{-1}.$$

Since $\{A_k\}$ is long-tailed and $\lim_{k\to\infty} = \dfrac{\overline{A}_{\leqslant k}}{\overline{p}_{\leqslant k}} = W$, it is clear that

$$\lim_{k\to+\infty} \frac{\overline{R_{\leqslant k}}}{\overline{\overline{p_{\leqslant k}}}} \geqslant \lim_{k\to+\infty} \frac{\overline{A_{\leqslant k}}}{\overline{\overline{p_{\leqslant k}}}} (I-\Phi_0)^{-1} = W(I-\Phi_0)^{-1}.$$

Using Lemmas 4.11 and 4.12, we are able to prove the following theorem, which characterizes long-tailed asymptotics of the matrix sequence $\{R_k\}$.

Theorem 4.7 *Suppose that the Markov chain of $GI/G/1$ type is positive recurrent, $\phi_{A-} < 1$ and $\sum_{k=-\infty}^{\infty} |k| A_k$ is finite. If $\{A_k\}$ is long-tailed with a uniformly dominant probability sequence $\{p_k\}$ and the associated ratio matrix W, then*

$$\lim_{k\to+\infty} \frac{\overline{R_{\leqslant k}}}{\overline{\overline{p_{\leqslant k}}}} = \frac{We\theta(I-R)}{\theta(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} jG_j e}, \tag{4.34}$$

where $\overline{\overline{p_{\leqslant k}}} = \sum_{n=k+1}^{\infty} \overline{p_{\leqslant n}}$.

Proof It follows from Eq. (3.19) that

$$[R^*(z)-I]\det(I-G^*(z)) = [A^*(z)-I]\operatorname{adj}(I-G^*(z))(I-\Phi_0)^{-1}, \tag{4.35}$$

when the matrix $I-G^*(z)$ is invertible. To evaluate the asymptotics of the coefficient matrix sequence in the generating function $R^*(z)-I$, we first analyze the asymptotics of the coefficient matrix sequence in the generating function $[A^*(z)-I].\operatorname{adj}(I-G^*(z))$ according to Lemma 4.11. Since $A^*(z)-I$ and $\operatorname{adj}(I-G^*(z))$ are analytic for $\phi_{A-} < |z| < \phi_{A+} = 1$, where $\phi_{A+} = 1$ results from the fact that $\{A_k\}$ is long-tailed, we can write

$$A^*(z) - I = \sum_{k=-\infty}^{\infty} z^k \widehat{A}_k,$$

where

$$\widehat{A}_k = \begin{cases} A_k, & \text{if } k \neq 0, \\ A_0 - I, & \text{if } k = 0. \end{cases}$$

Since $G^*(z) = \sum_{k=1}^{\infty} z^{-k} G_k$, according to the definition of the adjoint matrix we can write

$$\mathrm{adj}(I - G^*(z)) = \sum_{k=-\infty}^{0} z^k S_k \tag{4.36}$$

and define $S_k = 0$ for all $k \geqslant 1$. Let

$$\sum_{k=-\infty}^{\infty} z^k Q_k = [A^*(z) - I] \cdot \mathrm{adj}(I - G^*(z)).$$

Then

$$\sum_{k=-\infty}^{\infty} z^k Q_k = \sum_{k=-\infty}^{\infty} z^k \widehat{A}_k \cdot \sum_{k=-\infty}^{\infty} z^k S_k$$

with $\sum_{k=1}^{\infty} z^k S_k = 0$ implies

$$Q_k = \sum_{i+j=k} \widehat{A}_i S_j = \widehat{A}_k \circledast S_k \tag{4.37}$$

with $S_j = 0$ for $j \geqslant 1$. Therefore, we obtain that for $k \geqslant 1$

$$\overline{Q_{\leqslant k}} = \overline{A_{\leqslant k} * S_{\leqslant k}}.$$

If for a matrix sequence $\{C_k\}$, we define the matrix of measures by $\mu_C(B) = \sum_{k \in B} C_k$, then it follows from Remark 4.3, Eq. (4.37) and Lemma 4.11 that

$$\lim_{k \to +\infty} \frac{\overline{Q_{\leqslant k}}}{\overline{\overline{p_{\leqslant k}}}} = W \cdot \sum_{k=-\infty}^{0} S_k = W \cdot \mathrm{adj}\big(I - G\big). \tag{4.38}$$

We now evaluate asymptotics of the coefficient matrix sequence in the generating function $R^*(z) - I$ according to Eq. (3.19), Eq. (4.38) and Lemma 4.12. Let

$$\det(I - G^*(z)) = \sum_{k=-\infty}^{0} z^k g_k.$$

Define $\mu_-(B) = \sum_{k \in B} g_k$. Since the Markov chain of $GI/G/1$ type is positive recurrent, then applying (1) of Corollary 3.7 we have

$$\sum_{k=-\infty}^{0} g_k = \det(I - G^*(1)) = \det(I - G) = 0.$$

It is clear that $\sum_{k=-\infty}^{\infty} kg_k = \frac{\mathrm{d}}{\mathrm{d}z}\{\det(I - G^*(z))\}_{|z=1}$. To compute $\sum_{k=-\infty}^{0} kg_k$, taking the derivatives, elementwise, of both sides of the equation

$$\det(I - G^*(z)) \cdot I = \operatorname{adj}(I - G^*(z)) \cdot [I - G^*(z)]$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}z}\{\det(I - G^*(z))\}_{|z=1} \cdot I = \operatorname{adj}(I - G) \cdot \sum_{k=1}^{\infty} kG_k$$
$$+ \frac{\mathrm{d}}{\mathrm{d}z}\{\operatorname{adj}(I - G^*(z))\}_{|z=1} \cdot (I - G). \qquad (4.39)$$

Multiplying by θ and e on the both sides of Eq. (4.39), and using the fact that $\theta e = 1$ and $(I - G)e = 0$, it follows from Lemma 4.10 that

$$\begin{aligned}\sum_{k=-\infty}^{0} kg_k &= \frac{\mathrm{d}}{\mathrm{d}z}\{\det(I - G^*(z))\}_{|z=1} \\ &= \theta \cdot \operatorname{adj}(I - G)\sum_{k=1}^{\infty} kG_k \cdot e \\ &= \kappa_G \frac{\theta(I - R)(I - \Phi_0)}{\theta(I - R)(I - \Phi_0)e} \cdot \sum_{k=1}^{\infty} kG_k e. \qquad (4.40)\end{aligned}$$

Note that $\sum_{k=-\infty}^{\infty} |k| A_k$ is finite, Lemma 4.9 illustrat es that $\sum_{k=1}^{\infty} kG_k$ is finite. Thus $\sum_{k=-\infty}^{\infty} kg_k$ is finite and non-zero according to Lemma 4.10 and $\sum_{k=1}^{\infty} kG_k e \geqslant Ge = e$. Therefore, $0 < \left|\sum_{k=-\infty}^{0} kg_k\right| < +\infty$. Since $\{A_k\}$ is long-tailed with a uniformly dominant probabicity sequence $\{p_k\}$ and the associated ratio matrix W, Lemma 4.13 implies

$$\varliminf_{k \to +\infty} \frac{\overline{R_{\leqslant k}}}{p_{\geqslant k}} \geqslant W(I - \Phi_0)^{-1}.$$

Note the fact that $W \gneqq 0$ and $(I - \Phi_0)^{-1} \gneqq 0$, it is clear that $W(I - \Phi_0)^{-1} \gneqq 0$, since the matrix $I - \Phi_0$ is invertible. Hence, there exists at least a pair (i_0, j_0) such that the (i_0, j_0) th element of the matrix $W(I - \Phi_0)^{-1}$ is positive. Therefore, Lemma 4.13 implies $\overline{r_{\leqslant k}}(i_0, j_0) > 0$ for all $k \geqslant N$, where N is a large enough positive integer. Similarly, for each positive element of the matrix $W(I - \Phi_0)^{-1}$, denoted as the (i^*, j^*) th element, we have $\overline{r_{\leqslant k}}(i^*, j^*) > 0$ for all $k \geqslant N$. Define $U_+(B) = \sum_{k \in B} R_k$ and $U(B) = \sum_{k \in B} Q_k (I - \Phi_0)^{-1}$. It follows from Eq. (3.19), Eq. (4.38) and Lemma 4.12 that

$$\lim_{k\to+\infty}\frac{\overline{\overline{R_{\leqslant k}}}}{\overline{\overline{p_{\leqslant k}}}}=\frac{W\cdot \mathrm{adj}(I-G)(I-\Phi_0)^{-1}}{\sum_{k=-\infty}^{0} kg_k}. \tag{4.41}$$

Substituting Eq. (4.33) and Eq. (4.40) into Eq. (4.41) leads to the expression in Eq. (4.34). This completes the proof.

4.6 Subexponential Asymptotics of $\{\pi_k\}$

In this section, we assume that $\phi_{A+}=1$ and $\phi_{D+}>1$. Under the condition that $\{A_k\}\in S^*$ and $\{D_k\}$ is light-tailed, we characterize subexponential asymptotics of the stationary probability vector $\{\pi_k\}$. At the same time, we explicitly express a uniformly dominant sequence of $\{\pi_k\}$ and the associated ratio vector.

Since

$$\overline{\pi_{\leqslant k}}=x_0\overline{R_{0,\leqslant k}*\sum_{n=0}^{\infty}R_{\leqslant k}^{n\circledast}}$$

for $k\geqslant 1$ according to Lemma 4.2, the tail of the stationary probability vector $\{\pi_k\}$ can be expressed as a tail of convolution of the two matrix sequences $\{R_{0,k}\}$ and $\left\{\sum_{n=0}^{\infty}R_k^{n\circledast}\right\}$. It is well-known that the convolution of two long-tailed matrix sequences may not be long-tailed. Therefore, it is possible that the vector sequence $\{\pi_k\}$ is not long-tailed, even though the matrix sequence $\{A_k\}$ is long-tailed. Based on this, the matrix sequence $\{A_k\}$ is restricted to the subexponential class (including regularly varying class).

According to Lemma 4.2, it is crucial to characterize subexponential asymptotics of the matrix sequence $\left\{\sum_{n=0}^{\infty}R_k^{n\circledast}\right\}$. To do this, we need Lemma 4.3 in Asmussen, Henriksen and Klüppelberg [3], which is restated in the following lemma.

Lemma 4.14 *Let $H(x)$ be a matrix of nonnegative function such that $H=H(+\infty)-H(0)$ is strictly substochastic (therefore, the spectral radius of H is strictly less than one). If there exists a probability distribution $F(x)\in S$ and a finite matrix L such that $\lim_{x\to+\infty}\frac{\overline{H}(x)}{\overline{F}(x)}=L$, then*

$$\lim_{k\to\infty}\frac{\sum_{n=0}^{\infty}\overline{H^{n\circledast}}(x)}{\overline{F}(x)}=(I-H)^{-1}L(I-H)^{-1}.$$

The following lemma characterizes subexponential asymptotics for the matrix sequence $\left\{\sum_{n=0}^{\infty} R_k^{n\circledast}\right\}$.

Lemma 4.15 *Suppose that the Markov chain of GI/G/1 type is positive recurrent,* $\phi_{A-} < 1$ *and* $\sum\limits_{k=-\infty}^{\infty} |k| A_k$ *is finite. If* $\{A_k\} \in S^*$ *with a uniformly dominant probability sequence* $\{p_k\}$ *and the associated ratio matrix W, then* $\left\{\sum\limits_{n=0}^{\infty} R_k^{n\circledast}\right\} \in S$ *and*

$$\lim_{k\to\infty} \frac{\overline{\overline{\sum\limits_{n=0}^{\infty} R_{\leqslant k}^{n\circledast}}}}{\overline{\overline{p_{\leqslant k}}}} = \frac{(I-R)^{-1} W e \theta}{\theta (I-R)(I-\Phi_0)\sum\limits_{j=1}^{\infty} j G_j e}. \tag{4.42}$$

Proof Since $\{A_k\} \in S^* \subset \mathcal{L}$ and $p_k^{(I)} = \dfrac{1}{\mu_p}\overline{p_{\leqslant k}}$ for $k \geqslant 1$, where $\mu_p = \sum\limits_{k=1}^{\infty} k p_k = \sum\limits_{k=1}^{\infty} \overline{p_{\leqslant k}} < +\infty$ according to the assumption that $\sum\limits_{k=-\infty}^{\infty} |k| A_k$ is finite. It follows from Theorem 4.7 that

$$\lim_{k\to\infty} \frac{\overline{R_{\leqslant k}}}{\overline{\overline{p_{\leqslant k}^{(I)}}}} = \lim_{k\to\infty} \frac{\overline{R_{\leqslant k}}}{\dfrac{1}{\mu_p}\overline{\overline{p_{\leqslant k}}}} = \mu_p L, \tag{4.43}$$

where

$$L = \frac{W e \theta (I-R)}{\theta (I-R)(I-\Phi_0)\sum\limits_{j=1}^{\infty} j G_j e}.$$

Let $\widetilde{G}_k = \Delta^{-1} R_k^{\mathrm{T}} \Delta$ for $k \geqslant 1$, where $\Delta = \mathrm{diag}\,(\theta_1, \theta_2, \ldots, \theta_m)$. Since $\theta R \lneqq \theta$, it is clear that $\widetilde{G} = \sum\limits_{k=1}^{\infty} \widetilde{G}_k$ is strictly substochastic, since

$$\widetilde{G} e = \sum_{k=1}^{\infty} \widetilde{G}_k e = \sum_{k=1}^{\infty} \Delta^{-1} R_k^{\mathrm{T}} \Delta e = \Delta^{-1} (\theta R)^{\mathrm{T}} \lneqq \Delta^{-1} \theta^{\mathrm{T}} = e.$$

It follows from Eq. (4.42) that $\lim\limits_{k\to\infty} \dfrac{\overline{\widetilde{G}_{\leqslant k}}}{\overline{\overline{p_{\leqslant k}^{(I)}}}} = \mu_p \Delta^{-1} L^{\mathrm{T}} \Delta$ and from Proposition B.1 that $\left\{p_k^{(I)}\right\}$ is a probability sequence in $\mathcal{S}$. Therefore, using Lemma 4.14 we yield

$$\lim_{k\to\infty}\frac{\sum_{n=0}^{\infty}\overline{\widetilde{G}_{\leqslant k}^{n\circledast}}}{\overline{\overline{p_{\leqslant k}^{(I)}}}}=(I-\widetilde{G})^{-1}\cdot\mu_p\Delta^{-1}L^{\mathrm{T}}\Delta\cdot(I-\widetilde{G})^{-1}. \tag{4.44}$$

Since the Markov chain of $GI/G/1$ type is positive recurrent, $I-R$ is invertible. It is easy to see that $(I-R)^{-1}L(I-R)^{-1}\neq 0$ due to $L\neq 0$. Therefore, it follows from Eq. (4.44) that

$$\begin{aligned}\lim_{k\to\infty}\frac{\sum_{n=0}^{\infty}\overline{R_{\leqslant k}^{n\circledast}}}{\overline{\overline{p_{\leqslant k}^{(I)}}}}&=\mu_p(I-R)^{-1}L(I-R)^{-1}\\&=\frac{\mu_p(I-R)^{-1}We\theta}{\theta(I-R)(I-\Phi_0)\sum_{j=1}^{\infty}jG_je}.\end{aligned} \tag{4.45}$$

Notice that

$$\overline{\sum_{n=0}^{\infty}R_{\leqslant k}^{n\circledast}}=\sum_{n=0}^{\infty}\overline{R_{\leqslant k}^{n\circledast}},$$

it follows from Eq. (4.45) that

$$\begin{aligned}\lim_{k\to\infty}\frac{\overline{\sum_{n=0}^{\infty}R_{\leqslant k}^{n\circledast}}}{\overline{\overline{p_{\leqslant k}}}}&=\lim_{k\to\infty}\frac{\frac{1}{\mu_p}\overline{\sum_{n=0}^{\infty}R_{\leqslant k}^{n\circledast}}}{\overline{\overline{p_{\leqslant k}^{(I)}}}}\\&=\frac{(I-R)^{-1}We\theta}{\theta(I-R)(I-\Phi_0)\sum_{j=1}^{\infty}jG_je}.\end{aligned}$$

Therefore, $\left\{\sum_{n=0}^{\infty}R_k^{n\circledast}\right\}\in\mathcal{S}$ according to Proposition B.4. This completes the proof.

For simplicity of description, we need the asymptotic assumption: If $\phi_{D+}>1$, then the matrix sequence $\{R_{0,k}\}$ is light-tailed. We only consider the light-tailed case in which there exists a uniformly dominant sequence $\{d_k\}$ and the associated ratio matrix D such that

$$\lim_{k\to\infty}\frac{\overline{R_{0,\leqslant k}}}{\overline{\overline{d_{\leqslant k}}}}=D. \tag{4.46}$$

The following theorem characterizes subexponential asymptotics of $\{\pi_k\}$ for the case: $\{A_k\}\in\mathcal{S}^*$ and $\{D_k\}$ is light-tailed.

Theorem 4.8 *Suppose that the Markov chain of* $GI/G/1$ *type is positive recurrent, and* $\sum_{k=-\infty}^{\infty} |k| A_k$ *is finite. If* $\phi_{D+} > 1$, *the light-tailed* $\{R_{0,k}\}$ *has a uniformly dominant sequence* $\{d_k\}$ *with associated ratio matrix D, and if* $\{A_k\} \in \mathcal{S}^*$ *with a uniformly dominant sequence* $\{p_k\}$ *and associated ratio matrix W, then* $\{\pi_k\} \in \mathcal{S}$ *and*

$$\lim_{k\to\infty} \frac{\overline{\pi_{\leqslant k}}}{\overline{\overline{p_{\leqslant k}}}} = \frac{\pi_0 D (I-R)^{-1} We\theta}{\theta(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} jG_j e}.$$

Proof If $\{A_k\} \in \mathcal{S}^*$ with a uniformly dominant sequence $\{p_k\}$ and associated ratio matrix W, then

$$\lim_{k\to\infty} \frac{\overline{\sum_{n=0}^{\infty} R_{\leqslant k}^{n\circledast}}}{\overline{\overline{p_{\leqslant k}}}} = \frac{(I-R)^{-1} We\theta}{\theta(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} jG_j e}$$

according to Lemma 4.15. If $\phi_{D+} > 1$, then $\{R_{0,k}\}$ is light-tailed. When $\{R_{0,k}\}$ has a uniformly dominant sequence $\{d_k\}$ and associated ratio matrix D, it is clear that $\overline{d_{\leqslant k}} = o\left(\overline{p_{\leqslant k}^{(I)}}\right)$, since $p_k^{(I)} \in \mathcal{S}$ according to $\{A_k\} \in \mathcal{S}^*$. Therefore, it follows from Lemma 4.2 and Proposition B.7 that

$$\lim_{k\to\infty} \frac{\overline{\pi_{\leqslant k}}}{\overline{\overline{p_{\leqslant k}}}} = \frac{\pi_0 D (I-R)^{-1} We\theta}{\theta(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} jG_j e}.$$

Therefore, $\{\pi_k\} \in \mathcal{S}$. This completes the proof.

4.6.1 Markov Chains of *M/G/*1 Type

We consider a Markov chain of $M/G/1$ type with $\{A_k\} \in \mathcal{S}^*$ and $\phi_{D+} > 1$. It is clear that $G_j = 0$ and $\Phi_{-j} = 0$ for $j \geqslant 2$. In this case, we have

$$\lim_{k\to\infty} \frac{\overline{\pi_{\leqslant k}}}{\overline{\overline{p_{\leqslant k}}}} = \theta \frac{\pi_0 V (I-\Phi_0)^{-1} (I-R)^{-1} We}{\theta(I-R)(I-\Phi_0) G_1 e}.$$

4.6.2 Regularly Varying Asymptotics of $\{\pi_k\}$

If the matrix sequence $\{A_k\}$ is regularly varying and the matrix sequence $\{D_k\}$ is light-tailed, then the stationary probability vector $\{\pi_k\}$ is regularly varying. At the same time, we provide explicit expressions for a uniformly dominant sequence of $\{\pi_k\}$ and the associated ratio vector.

Theorem 4.9 *Suppose that the Markov chain of $GI/G/1$ type is positive recurrent, and $\sum_{k=1}^{\infty} kD_k$ and $\sum_{k=-\infty}^{\infty} |k| A_k$ are both finite. If $\phi_{D+} > 1$ and $\{A_k\} \in \Re_{-\alpha}$ for $\alpha \geqslant 2$, then $\{\pi_k\} \in \Re_{-(\alpha-1)}$.*

Proof If $\{A_k\} \in \Re_{-\alpha}$ with a uniformly dominant sequence $\{p_k\}$ and associated ratio matrix W, then $\{p_k\} \in \Re_{-\alpha}$. Thus, $\overline{p_{\leqslant k}} = k^{-\alpha}\overline{l_{\leqslant k}}$ for all $k \geqslant 1$, where $\{l_k\} \in \Re_0$. Since $\sum_{k=-\infty}^{\infty} |k| A_k$ is finite, it is clear that $\mu_p = \sum_{k=0}^{\infty} kp_k < +\infty$ and $\alpha \geqslant 2$ according to Proposition 1.3.6 in Bingham, Goldie and Teugels [7]. Let $p_k^{(I)} = \frac{1}{\mu_p}\sum_{k=0}^{k}\overline{p_{\leqslant k}}$. It follows from Proposition 1.5.10 in Bingham, Goldie and Teugels [7] that as $k \to \infty$,

$$\begin{aligned}\overline{p_{\leqslant k}^{(I)}} &= \frac{1}{\mu_p}\sum_{l=k+1}^{\infty}\overline{p_{\leqslant k}} = \frac{1}{\mu_p}\sum_{l=k+1}^{\infty} k^{-\alpha}\overline{l_{\leqslant k}} \\ &\sim \frac{1}{\mu_p(\alpha-1)}(k+1)^{-(\alpha-1)}\overline{l_{\leqslant k+1}}.\end{aligned}$$

Hence $\{p_k^{(I)}\} \in \Re_{-(\alpha-1)}$. It follows from Lemma 4.15 that

$$\lim_{k\to\infty}\frac{\overline{\sum_{n=0}^{\infty} R_{\leqslant k}^{n\circledast}}}{\overline{p_{\leqslant k}^{(I)}}} = \frac{\mu_p(\alpha-1)(I-R)^{-1}We\theta}{\theta(I-R)(I-\varPhi_0)\sum_{j=1}^{\infty} jG_j e}.$$

Therefore, $\left\{\sum_{n=0}^{\infty} R_k^{n\circledast}\right\} \in \Re_{-(\alpha-1)}$. According to (2) of Theorem 4.8, we obtain that $\overline{\pi_{\leqslant k}} \sim k^{-(\alpha-1)}\overline{l_{\leqslant k}^{(1)}}\pi_0\varPsi_2\varPsi_1$ as $k \to \infty$. This completes the proof.

4.7 Notes in the Literature

Asymptotic analysis of block-structured Markov chains is an interesting topic and has been studied for many years. For light-tailed asymptotic behavior of stationary probability vectors, it might have been inspired by Takahashi [35] and Neuts and Takahashi [30]. Neuts [27] provided an excellent overview for asymptotic behavior of Markov chains of $GI/M/1$ type. To establish useful relations between the light-tailed asymptotics and the parameters of a concrete queueing model, Neuts [28] discussed the caudal characteristic curves for some queues by means of the matrix-geometric solution. Bean, Li and Taylor [6] analyzed the caudal characteristics for QBD processes and also for Markov chains of $GI/M/1$ type. Fujimoto, Takahashi and Makimoto [13] obtained an interesting result on the asymptotics of QBD processes with both infinite levels and infinite phases. Subsequent papers have been published on this theme, e.g., see Miyazawa and Zhao [25], Kroese, Scheinhardt and Taylor [20], Haque, Zhao and Liu [15], Li, Miyazawa and Zhao [21]. In contrast to Markov chains of $GI/M/1$ type, it is more difficult to analyze asymptotic behavior of stationary probability vectors of Markov chains of $M/G/1$ type. This difficulty is due to two basic facts: The matrix-iterative solution makes such an asymptotic analysis more difficult; and the stationary probability vectors of Markov chains of $M/G/1$ type can be either light-tailed or heavy-tailed. The light-tailed asymptotics of stationary probability vectors of Markov chains of $M/G/1$ type was studied by Falkenberg [11], Abate, Choudhury and Whitt [1], Choudhury and Whitt [9], MØller [26], Takine [23], Li and Zhao[23].

For subexponential asymptotics of stationary queue lengths, Resnick and Samorodnitsky [31] analyzed heavy-tailed asymptotic behavior for the stationary queue length of a $G/M/1$ queue in terms of stochastic comparison when the arrival process is long range dependent. Based on a property on the generating functions of regularly varying sequences, Roughan, Veitch and Rumsewicz [32] derived power law asymptotics for the stationary queue length of an $M/G/1$ queue with power law service times. According to the distributional version of Little's law, Asmussen, Klüppelberg and Sigman [4] studied subexponential asymptotics for the stationary queue length of a $GI/G/1$ queue with subexponential service times. Using the Mellin transform, Jacquet [16] provided results on polynomial tails for the stationary queue length of a single-server queue when the arrival process contains a finite or infinite number of on-off input sources. Shuang, Liu and Li [33] studied the subexponential asymptotics of stationary queue length for a $M/G/1$ retrial queue. Li, Liu and Shuang [22] analyzed the regularly varying tail of the stationary buffer content for an infinite-buffer fluid queue driven by an $M/G/1$ queue. Borovkov and Korshunov [8], Jelenković and Lazar [17], Foss and Zachary [12] and Zachary [38] studied heavy-tailed asymptotics for random walks. Some researchers have studied heavy-tailed asymptotics of the stationary probability vectors. For a Markov chain of $GI/G/1$ type with subexponential increments and with the repeating and boundary matrix sequences being tail-equivalent, Asmussen and MØller [5] discussed subexponential asymptotics for the stationary level process. Takine [36] and Li and Zhao [24] discussed heavy-tailed asymptotics for the stationary probability vectors of Markov chains of

$GI/G/1$ type. Kim and Sohraby [18] studied tail behavior of the queue size and waiting time in a queue with discrete autoregressive arrivals, while Kim and Kim [19] analyzed regularly varying tails in a queue with discrete autoregressive arrivals of order p.

This chapter mainly refers to Li and Zhao [23,24] and Jelenković and Lazar [17].

Problems

4.1 For the $MAP/G/1$ queue, prove that

(1) the stationary queue length is light-tailed if and only if the service time distribution is light-tailed;

(2) the stationary queue length is subexponential if and only if the service time distribution is subexponential.

4.2 For the $BMAP/M/1$ queue with the BMAP expression $\{D_k\}$, if the matrix sequence $\{D_k\}$ is regularly varying $\Re_{-\alpha}$ for $\alpha \geqslant 2$, then compute the tails of the stationary queue length and the stationary waiting time.

4.3 For the $M^X/M^X/1$ queue, if the bulk arrival distribution and the bulk service distribution are regularly varying $\Re_{-\alpha}$ and $\Re_{-\beta}$ for $\alpha, \beta \geqslant 2$, then compute the tails of the stationary queue length and the stationary waiting time.

4.4 For the $M/G/1$ retrial queue, if the service time distribution is regularly varying $\Re_{-\alpha}$ for $\alpha \geqslant 2$, then compute the tails of the stationary queue length and the stationary waiting time.

4.5 For the $M/G(M/G)/1$ queue with a repairable server, if the service time distribution and the repair time distribution are regularly varying $\Re_{-\alpha}$ and $\Re_{-\beta}$ for $\alpha, \beta \geqslant 2$, respectively, then compute the tails of the stationary queue length and the stationary waiting time.

4.6 For the $M^X/G/1$ queue with server multiple vacations, if the arrival bulk size distribution, the service time distribution and the vacation time distribution are all regularly varying, then compute the tails of the stationary queue length and the stationary waiting time.

4.7 For the $BMAP/G/1$ queue with server single vacation, if the arrival bulk size distribution, the service time distribution and the vacation time distribution are all regularly varying, then compute the tails of the stationary queue length and the stationary waiting time.

4.8 Consider a double queue that arises when each arriving customer simultaneously place two demands handled by two servers independently. Customer arrivals form a Poisson process with rate $\lambda > 0$. Server one has regularly varying service times with probability distribution $G(x) \in \Re_{-\alpha}$ for $\alpha > 2$, while server two has exponential service times with rate $\mu > 0$. Compute the tails for the two-dimensional stationary joint queue length, the first stationary queue length and the second stationary queue length.

4.9 Consider a queueing system consisting of two parallel servers, each of which has a queue of itself. Customer arrivals form a Poisson process with rate $\lambda > 0$. On arrival the customer joins the shorter queue. When both queues have equal length, he joins the first queue with probability θ and the second one with probability $1-\theta$. Server one has regularly varying service times with probability distribution $G(x) \in \Re_{-\alpha}$ for $\alpha > 2$, while server two has exponential service times with rate $\mu > 0$. Compute the tails for the two-dimensional stationary joint queue length, the first stationary queue length and the second stationary queue length.

4.10 A queueing system consists of a server and two queues formed by two types of customers, respectively. Arrivals of the two types of customers form two Poisson processes with rates $\lambda_1, \lambda_2 > 0$. The server serves a customer in the longer queue. When both queues have equal lengths, he serves the first queue with probability θ and the second one with probability $1-\theta$.

(1) If the service times for the first type of customers are regularly varying with probability distribution $G(x) \in \Re_{-\alpha}$ for $\alpha > 2$, while the service times for the second type are exponential with rate $\mu > 0$, compute the tails for the two-dimensional stationary joint queue length, the first stationary queue length and the second stationary queue length.

(2) If the service times for the two types of customers are all regularly varying with probability distributions $F(x) \in \Re_{-\alpha}$ and $G(x) \in \Re_{-\beta}$ for $\alpha, \beta > 2$, respectively, compute the tails for the two-dimensional stationary joint queue length, the first stationary queue length and the second stationary queue length.

4.11 A queueing system consists of a server and two queues formed by two types of customers respectively. Arrivals of the two types of customers form two Poisson processes with rates $\lambda_1, \lambda_2 > 0$. The server alternately serves customers between both queues, for example, queue one to queue two, queue two to queue one, and so on.

(1) If the service times for the first type of customers are regularly varying with probability distribution $G(x) \in \Re_{-\alpha}$ for $\alpha > 2$, while the service times for the second type are exponential with rate $\mu > 0$, compute the tails for the two-dimensional stationary joint queue length, the first stationary queue length and the second stationary queue length.

(2) If the service times for the two types of customers are all regularly varying with probability distributions $F(x) \in \Re_{-\alpha}$ and $G(x) \in \Re_{-\beta}$ for $\alpha, \beta > 2$, respectively, compute the tails for the two-dimensional stationary joint queue length, the first stationary queue length and the second stationary queue length.

4.12 We consider an irreducible QBD process with N levels whose infinitesimal generator is given by

$$Q = \begin{pmatrix} B_1 & B_0 & & & & & \\ B_2 & A_1 & A_0 & & & & \\ & A_2 & A_1 & A_0 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_2 & A_1 & A_0 & \\ & & & & A_2 & A_1 & C_0 \\ & & & & & C_2 & C_1 \end{pmatrix}.$$

Let $A = A_0 + A_1 + A_2$ be irreducible and stochastic, and θ the stationary probability vector of the Markov chain A with finite states. It is clear that the QBD process is positive recurrent, and its stationary probability vector is expressed as $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$. Please discuss the limit $\lim_{N\to\infty} \pi_N$ for each of the following three cases: (1) $\theta A_0 e < \theta A_2 e$, (2) $\theta A_0 e > \theta A_2 e$ and (3) $\theta A_0 e = \theta A_2 e$.

4.13 We consider a $BMAP/G/1/N$ queue. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ be the stationary distribution of the queue length. Please discuss the limit $\lim_{N\to\infty} \pi_N$ for each of the following three cases: (1) $\rho < 1$, (2) $\rho > 1$ and (3) $\rho = 1$, where $\rho = \lambda/\mu$.

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5 Markov Chains on Continuous State Space

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Abstract In this chapter, we discuss Markov chains on continuous state space. We first analyze a discrete-time Markov chain on continuous state space, and then discuss a discrete-time Markov chain on a bivariate state space. Applying the censoring technique, we provide expression for the *RG*-factorizations, which are used to derive the stationary probability of the Markov chain. Further, we consider a continuous-time Markov chain on continuous state space. Specifically, we deal with a continuous-time level-dependent QBD process with continuous phase variable, and provide orthonormal representations for the *R*-, *U*- and *G*-measures, which lead to the matrix-structured computation of the stationary probability. As an application, we introduce continuous-phase type (CPH) distribution and continuous-phase Markovian arrival process (CMAP), and then analyze a *CMAP*/*CPH*/1 queue. Finally, we study a piecewise deterministic Markov process, which is applied to deal with more general queues such as the *GI*/*G*/*c* queue.

Keywords Markov chains on continuous state space, QBD process with continuous phase variable, orthonormal representation, continuous-phase type distribution, continuous-phase Markovian arrival process, piecewise deterministic Markov process.

In this chapter, we discuss Markov chains on continuous state space, which are a useful mathematical tool in the study of stochastic models. We first analyze a discrete-time Markov chain on continuous state space, and then discuss a discrete-time Markov chain on a bivariate state space $\Omega = N \times R$, where $N = \{0, 1, 2, ...\}$ and $R_+ = [0, +\infty)$. Applying the censoring technique, we provide expression for the *RG*-factorizations, which are used to derive the stationary distribution. As an example, we study the $GI/G/1$ queue in terms of the discrete-time Markov chain on continuous state space. Further, we consider a continuous-time Markov chain on continuous state space, and specifically deal with a continuous-time level-dependent

QBD process with continuous phase variable, and provide the *RG*-factorizations. In $L_2([0,+\infty)^2)$, which is a space of square integrable bivariate real functions, we provide orthonormal representations for the *R*-, *U*- and *G*-measures, which lead to the matrix structure of the *RG*-factorizations. Based on this, we introduce continuous-phase type (CPH) distribution and continuous-phase Markovian arrival process (CMAP), and then analyze a $CMAP/CPH/1$ queue. Finally, we study a piecewise deterministic Markov process to be able to deal with more general queueing systems such as the $GI/G/c$ queue.

This chapter is organized as follows. Section 5.1 simply defines a discrete-time Markov chain on continuous state space, and discusses its irreducibility and aperiodicity. Section 5.2 extends the censoring technique to be able to deal with the Markov chain on bivariate state space $\{0,1,2,...\}\times[0,+\infty)$, derives the *RG*-factorizations and also expresses the stationary probability distribution by means of an algebraic algorithm with orthonormal representations. Section 5.3 applies the *RG*-factorizations to discuss the $GI/G/1$ queue in terms of the Markov chain of $GI/M/1$ type and the Markov chain of $M/G/1$ type, respectively. Section 5.4 defines a continuous-time Markov chain on continuous state space, and expresses the stationary probability distribution. Section 5.5 deals with the continuous-time QBD process with continuous phase variable, provides the UL-type *RG*-factorization and expresses the stationary probability distribution as an operator-multiplicative solution. In Section 5.6, if the matrix of generalized density functions of the continuous-time QBD process with continuous phase variable is in $L_2([0,+\infty)^2)$, then the integral equations given in Section 5.5 can be converted into the associated matrix equations. Section 5.7 introduces CPH distribution and continuous-phase Markovian arrival process (CMAP), and then analyzes a $CMAP/CPH/1$ queue. Section 5.8 studies a piecewise deterministic Markov process which leads to be able to deal with more general queueing systems such as the $GI/G/c$ queue. Finally, Section 5.9 provides some notes to the references on Markov chains on continuous state space.

5.1 Discrete-Time Markov Chains

In this section, we define a discrete-time Markov chain on continuous state space, and discuss its irreducibility and aperiodicity simply.

Let $\sigma(D)$ be a Borel σ-algebra on the interval *D*, for example, *D* is either $[0,+\infty)$, $[0,a)$ or $(b,+\infty)$.

We first introduce the identity kernel as follows. For a nonnegative kernel $F(x,A)$ for $x\geqslant 0$ and $A\in\sigma(\Omega)$, if the bivariate function matrix $I(x,y)$ satisfies that for all $x\geqslant 0$ and $A\in\sigma(\Omega)$,

$$\int_0^{+\infty} I(x,y)\,F(y,A)\mathrm{d}y = F(x,A)$$

and

$$\int_0^{+\infty} F(x, y)\, I(y, A)\mathrm{d}y = F(x, A),$$

then $I(x, A)$ is called the identity kernel of $F(x, A)$.

In this chapter, we always use the notation: $F^0(x, y) = I(x, y)$; and for $n \geqslant 1$,

$$F^n(x, y) = \int_0^{+\infty} F^{n-1}(x, z) F(z, y)\mathrm{d}z.$$

It is easy to check that

$$F^n(x, y) = \int_0^{+\infty} F(x, z) F^{n-1}(z, y)\mathrm{d}z, \ n \geqslant 1.$$

A Markov chain is a sequence of random variables $X_0, X_1, X_2, \dots$, taking values in the state space Ω. A basic property of Markov chains is that the past is conditionally independent of the future, given the present. In our previous chapters, we have discussed the Markov chains with Ω be discrete; while this chapter will study a Markov chain on state space Ω be either continuous or semi-continuous. In this case, the Markov chain is governed by a transition kernel $K(x, A)$ for $x \in \Omega$ and $A \subset \Omega$. Let $\sigma(\Omega)$ be a Borel σ-algebra on Ω. Then the transition kernel $K: \Omega \times \sigma(\Omega) \to [0,1]$ defines a Markov chain $\{X_k\}$ through the relation

$$P\{X_{k+1} \in A \mid X_k, X_{k-1}, \dots, X_0\} = K(X_k, A).$$

It is clear that $K(x, A)$ denotes the probability to move in one step from the state x into the state set A. The transition kernel $K(x, A)$ has two main properties as follows:

(1) $K(x, \cdot)$ is a probability measure for each $x \in \Omega$, and

(2) $K(\cdot, A)$ is measurable for each $A \subset \Omega$.

The two properties are explained as follows. The first property shows that $K(x, \cdot)$ defines a probability density for which the Markov chain will move to the next step, given that the Markov chain is currently at x. The second property indicates that we can always evaluate the probability that the Markov chain will jump into some state set A from all possible state x.

If there exists a function $K(x, y)$ such that for all $x \in \Omega$ and $A \subset \Omega$,

$$K(x, A) = \int_A K(x, y)\mathrm{d}y$$

then $K(x, y)$ is said to be a density of the transition kernel $K(x, A)$. We write

$$K^1(x, y) = K(x, y)$$

and for $n \geqslant 2$,

$$K^n(x,y) = \int_{\Omega} K^{n-1}(x,z)K(z,y)\mathrm{d}z$$
$$= \int_{\Omega} K(x,z)K^{n-1}(z,y)\mathrm{d}z.$$

If the kernel $K(x,A)$ is well behaved, then the Markov chain will have a stationary distribution $\pi(x)$ such that

$$\pi(y) = \int_{\Omega} \pi(x)K(x,y)\mathrm{d}x, \quad y \in \Omega, \tag{5.1}$$

or

$$\pi(A) = \int_{\Omega} \pi(x)K(x,A)\mathrm{d}x, \quad A \in \Omega. \tag{5.2}$$

It is clear that $\pi(y)$ is a density of the stationary probability $\pi(A)$.

The main requirement for the Markov chain to reach its stationary distribution is that the Markov chain is irreducible and aperiodic. The irreducibility is defined as, for any $x, y \in \Omega$, there always exists a positive integer n such that $K^n(x,y) > 0$. In other words, the Markov chain can jump into any state from any other state in a finite number of steps. If $K(x,y) > 0$ for any $x, y \in \Omega$, then the Markov chain is said to be strongly irreducible. The aperiodicity means that there exist no subsets of the state space Ω that can only be periodically visited by the Markov chain.

Now, we extend the Markov chain on continuous state space $[0,+\infty)$ to a more general Markov chain on semi-continuous state space $\Omega = \{0,1,2,...\} \times [0,+\infty)$. For such a Markov chain, we define the transition kernel as follows:

$$K(i,x;j,A) = P\{X_{k+1} \in (j,A) \mid X_k = (i,x)\}.$$

At the same time, the nth step iteration of $K(i,x;j,A)$ is given by

$$K^n(i,x;j,A) = \sum_{k=0}^{\infty} \int_0^{+\infty} K^{n-1}(i,x;k,y)K(k,y;j,A)\mathrm{d}y, \quad n \geqslant 1.$$

The following proposition describes some properties of the transition kernel $K(i,x;j,A)$. The proof is clear and is omitted here.

Proposition 5.1 (1) $0 \leqslant K(i,x;j,A) \leqslant 1$ *for* $i,j \geqslant 0$, $x \geqslant 0$ *and* $A \in \sigma(\Omega)$,

(2) $K(i,x;j,\varnothing) = 0$ *for* $i,j \geqslant 0$, $x \geqslant 0$ *and the state set* $\varnothing$ *is null, and*

(3) $\sum_{k=0}^{\infty} \int_0^{+\infty} K(i,x;k,y)\mathrm{d}y = 1$ *for* $i \geqslant 0$ *and* $x \geqslant 0$.

Let $P_{i,j}(x,A) = K(i,x;j,A)$ and

$$P(x,A) = \begin{pmatrix} P_{0,0}(x,A) & P_{0,1}(x,A) & P_{0,2}(x,A) & \cdots \\ P_{1,0}(x,A) & P_{1,1}(x,A) & P_{1,2}(x,A) & \cdots \\ P_{2,0}(x,A) & P_{2,1}(x,A) & P_{2,2}(x,A) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{5.3}$$

Then the Markov chain on semi-continuous state space $\{0, 1, 2, ...\} \times [0, +\infty)$ is determined by the transition kernel given in Eq. (5.3).

It is clear that

$$P^n(x, A) = \int_0^{+\infty} P^{n-1}(x, y) P(y, A) \mathrm{d}y, \quad n \geqslant 1,$$

which can be expressed by means of the transition sub-kernel $K^n(i, x; j, A)$ for $i, j \geqslant 0$, $x \geqslant 0$ and $A \in \sigma(\Omega)$.

In what follows we provide some important examples of the Markov chain on semi-continuous state space $\{0, 1, 2, ...\} \times [0, +\infty)$. It is worthwhile to note that these examples are all the corresponding generalized versions of the discrete-time Markov chains with discrete state space studied in the previous chapters.

5.1.1 Markov Chains of *GI*/*G*/1 Type

The transition kernel in Eq. (5.3) is simplified as

$$P(x, A) = \begin{pmatrix} D_0(x, A) & D_1(x, A) & D_2(x, A) & \cdots \\ D_{-1}(x, A) & \boldsymbol{A}_0(x, A) & \boldsymbol{A}_1(x, A) & \cdots \\ D_{-2}(x, A) & \boldsymbol{A}_{-1}(x, A) & \boldsymbol{A}_0(x, A) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

5.1.2 Markov Chains of *GI*/*M*/1 Type

The transition kernel in Eq. (5.3) is simplified as

$$P(x, A) = \begin{pmatrix} D_1(x, A) & D_0(x, A) & & & \\ D_2(x, A) & C_1(x, A) & C_0(x, A) & & \\ D_3(x, A) & C_2(x, A) & C_1(x, A) & C_0(x, A) & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

5.1.3 Markov Chains of *M*/*G*/1 Type

The transition kernel in Eq. (5.3) is simplified as

$$P(x, A) = \begin{pmatrix} D_1(x, A) & D_2(x, A) & D_3(x, A) & D_4(x, A) & \cdots \\ D_0(x, A) & C_1(x, A) & C_2(x, A) & C_3(x, A) & \cdots \\ & C_0(x, A) & C_1(x, A) & C_2(x, A) & \cdots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

5.1.4 QBD Processes

The transition kernel in Eq. (5.3) is simplified as

$$P(x,A)=\begin{pmatrix} A_1^{(0)}(x,A) & A_0^{(0)}(x,A) & & & \\ A_2^{(1)}(x,A) & A_1^{(1)}(x,A) & A_0^{(1)}(x,A) & & \\ & A_2^{(2)}(x,A) & A_1^{(2)}(x,A) & A_0^{(2)}(x,A) & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

5.2 The *RG*-Factorizations

In this section, we extend the censoring technique to be able to deal with the discrete-time Markov chain on semi-continuous state space $\Omega=\{0,1,2,...\}\times[0,+\infty)$, Based on this, we derive the *RG*-factorizations and express the stationary probability distributions.

Let $E=\{0,1,2,...,n\}$ and $E^c=\{n+1,n+2,n+3,...\}$. According to the subsets E and E^c, the transition kernel $P(x,A)$ is partitioned as

$$P(x,A)=\begin{array}{c c} & \begin{array}{cc} E & E^c \end{array} \\ \begin{array}{c} E \\ E^c \end{array} & \begin{pmatrix} T(x,A) & U(x,A) \\ V(x,A) & W(x,A) \end{pmatrix} \end{array}. \tag{5.4}$$

It is clear that if $P(x,A)$ is irreducible and $V(x,A)\gneq 0$, then each element of $\widehat{W}(x,A)=\sum_{n=0}^{\infty}W^n(x,A)$ is finite, where

$$W^0(x,A)=I(x,A),$$

$$W^1(x,A)=W(x,A)$$

and for $n\geqslant 2$,

$$W^n(x,A)=\int_{\Omega}W^{n-1}(x,z)W(z,A)\mathrm{d}z=\int_{\Omega}W(x,z)W^{n-1}(z,A)\mathrm{d}z.$$

The matrix $\widehat{W}(x,A)$ is referred to as the fundamental matrix of $W(x,A)$.

Suppose that $\{X_k,k\geqslant 0\}$ is an irreducible Markov chain on semi-continuous state space $\{0,1,2,...\}\times[0,+\infty)$. If the successive visits of X_n to the subset E take place at the n_kth step of state transition, we write $X_k^E=X_{n_k}$ for $k\geqslant 1$. Then the sequence $\{X_k^E,k\geqslant 1\}$ is called the censored chain with censoring set E. For convenience, we write $P^{[\leqslant n]}(x,A)$ for the censored transition kernel $P^E(x,A)$ if the censored set $E = L_{\leqslant n}$, in particular, $P^{[<+\infty]}(x, A) = P(x, A)$ and $P^{[0]}(x, A) = P^{[\leqslant 0]}(x, A)$. Similarly, $P^{[\geqslant n]}(x, A)$ is the censored transition kernel with the censored set $E = L_{\geqslant n}$, specifically, $P^{[\geqslant 0]}(x, A) = P(x, A)$.

The following lemma shows that the censored chain $\{X_k^E, k \geqslant 1\}$ is a Markov chain again. The proof is clear and is omitted here.

Lemma 5.1 (1) *The censored chain* $\{X_k^E, k \geqslant 1\}$ *is a Markov chain whose transition kernel is given by*

$$P^E(x, A) = T(x, A) + \int_0^{+\infty} \int_0^{+\infty} U(x, y) \widehat{W}(y, z) V(z, A) \mathrm{d}y \mathrm{d}z. \tag{5.5}$$

(2) *The censored chain* $\{X_k^{E^c}, k \geqslant 1\}$ *is a Markov chain whose transition kernel is given by*

$$P^{E^c}(x, A) = W(x, A) + \int_0^{+\infty} \int_0^{+\infty} V(x, y) \widehat{T}(y, z) U(z, A) \mathrm{d}y \mathrm{d}z.$$

Note that the two censored Markov chains $\{X_k^E, k \geqslant 1\}$ and $\{X_k^{E^c}, k \geqslant 1\}$ have different utilities, which can lead to two different types of *RG*-factorizations.

5.2.1 The UL-Type *RG*-Factorization

Let

$$P^{[\leqslant n]}(x, A) = \begin{pmatrix} \phi_{0,0}^{(n)}(x, A) & \phi_{0,1}^{(n)}(x, A) & \cdots & \phi_{0,n}^{(n)}(x, A) \\ \phi_{1,0}^{(n)}(x, A) & \phi_{1,1}^{(n)}(x, A) & \cdots & \phi_{1,n}^{(n)}(x, A) \\ \vdots & \vdots & & \vdots \\ \phi_{n,0}^{(n)}(x, A) & \phi_{n,1}^{(n)}(x, A) & \cdots & \phi_{n,n}^{(n)}(x, A) \end{pmatrix}, \quad n \geqslant 0,$$

be partitioned according to the levels.

The following lemma provides a useful relationship among the entries of the censored Markov chains, which are essentially the Wiener-Hopf equations for the Markov chain. The proof is clear and is omitted here.

Lemma 5.2 *For* $n \geqslant 0$, $0 \leqslant i, j \leqslant n$,

$$\phi_{i,j}^{(n)}(x, A) = P_{i,j}(x, A) + \sum_{k=n+1}^{\infty} \int_0^{+\infty} \int_0^{+\infty} \phi_{i,k}^{(k)}(x, y) \widehat{\phi_{k,k}^{(k)}}(y, z) \phi_{k,j}^{(k)}(z, A) \mathrm{d}y \mathrm{d}z.$$

Based on Lemma 5.2, we provide expressions for the *R*-, *U*- and *G*-measures. For $0 \leqslant i < j$,

$$R_{i,j}(x, A) = \int_0^{+\infty} \phi_{i,j}^{(j)}(x, y) \widehat{\phi_{j,j}^{(j)}}(y, A) \mathrm{d}y.$$

For $0 \leqslant j < i$,

$$G_{i,j}(x,A) = \int_0^{+\infty} \widehat{\phi_{i,i}^{(i)}}(x,y)\phi_{i,j}^{(i)}(y,A)\mathrm{d}y.$$

For $n \geqslant 0$,

$$\Psi_n(x,A) = \phi_{n,n}^{(n)}(x,A).$$

The following theorem provides the UL-type RG-factorization for the transition kernel $P(x,A)$ given in Eq. (5.3).

Theorem 5.1 *For the Markov chain with the transition kernel* $P(x,A)$ *given in Eq.* (5.3)*,*

$$\begin{aligned} I(x,A) - P(x,A) = \int_0^{+\infty}\int_0^{+\infty} &[I(x,y) - R_U(x,y)] \\ &\cdot [I(y,z) - \Psi_D(y,z)][I(z,A) - G_L(z,A)]\mathrm{d}y\mathrm{d}z, \end{aligned} \tag{5.6}$$

where

$$R_U(x,A) = \begin{pmatrix} 0 & R_{0,1}(x,A) & R_{0,2}(x,A) & R_{0,3}(x,A) & \cdots \\ & 0 & R_{1,2}(x,A) & R_{1,3}(x,A) & \cdots \\ & & 0 & R_{2,3}(x,A) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$\Psi_D(x,A) = \mathrm{diag}(\Psi_0(x,A), \Psi_1(x,A), \Psi_2(x,A), \Psi_3(x,A), \ldots)$$

and

$$G_L(x,A) = \begin{pmatrix} 0 & & & & \\ G_{1,0}(x,A) & 0 & & & \\ G_{2,0}(x,A) & G_{2,1}(x,A) & 0 & & \\ G_{3,0}(x,A) & G_{3,1}(x,A) & G_{3,2}(x,A) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

5.2.2 The LU-Type *RG*-Factorization

Let

$$P^{[\geqslant n]}(x,y) = \begin{pmatrix} \eta_{n,n}^{(n)}(x,y) & \eta_{n,n+1}^{(n)}(x,y) & \eta_{n,n+2}^{(n)}(x,y) & \cdots \\ \eta_{n+1,n}^{(n)}(x,y) & \eta_{n+1,n+1}^{(n)}(x,y) & \eta_{n+1,n+2}^{(n)}(x,y) & \cdots \\ \eta_{n+2,n}^{(n)}(x,y) & \eta_{n+2,n+1}^{(n)}(x,y) & \eta_{n+2,n+2}^{(n)}(x,y) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Then for $i, j \geqslant n+1$,

$$\eta_{i,j}^{(n+1)}(x,y) = P_{i,j}(x,y) + \sum_{k=0}^{n} \int_0^{+\infty} \int_0^{+\infty} \eta_{i,k}^{(k)}(x,u) \widehat{\eta_{k,k}^{(k)}}(u,v) \eta_{k,j}^{(k)}(v,y) \mathrm{d}u \mathrm{d}v.$$

We define the U-measure as

$$\Phi_n(x,y) = \eta_{n,n}^{(n)}(x,y), \quad n \geqslant 0,$$

the R-measure as

$$\overline{R}_{i,j}(x,y) = \int_0^{+\infty} \eta_{i,j}^{(j)}(x,z) \widehat{\Phi_j}(z,y) \mathrm{d}z, \quad 0 \leqslant j < i,$$

and the G-measure as

$$\overline{G}_{i,j}(x,y) = \int_0^{+\infty} \widehat{\Phi_i}(x,z) \eta_{i,j}^{(i)}(z,y) \mathrm{d}z, \quad 0 \leqslant i < j.$$

The LU-type RG-factorization for the Markov chain $P(x,y)$ given in Eq.(5.3) is given by

$$I(x,y) - P(x,y) = \int_0^{+\infty} \int_0^{+\infty} [I(x,u) - \overline{R}_L(x,u)][I(u,v) - \Phi_D(u,v)] \cdot [I(v,y) - \overline{G}_U(v,y)] \mathrm{d}u \mathrm{d}v,$$

where

$$\overline{R}_L(x,y) = \begin{pmatrix} 0 & & & & \\ \overline{R}_{1,0}(x,y) & 0 & & & \\ \overline{R}_{2,0}(x,y) & \overline{R}_{2,1}(x,y) & 0 & & \\ \overline{R}_{3,0}(x,y) & \overline{R}_{3,1}(x,y) & \overline{R}_{3,2}(x,y) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Phi_D(x,y) = \mathrm{diag}(\Phi_0(x,y), \Phi_1(x,y), \Phi_2(x,y), \Phi_3(x,y), \ldots)$$

and

$$\overline{G}_U(x,y) = \begin{pmatrix} 0 & \overline{G}_{0,1}(x,y) & \overline{G}_{0,2}(x,y) & \overline{G}_{0,3}(x,y) & \cdots \\ & 0 & \overline{G}_{1,2}(x,y) & \overline{G}_{1,3}(x,y) & \cdots \\ & & 0 & \overline{G}_{2,3}(x,y) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

5.2.3 The Stationary Probability Distribution

Based on the UL-type RG-factorization, the following theorem provides expression for the stationary probability distribution of the discrete-time Markov chain on a semi-continuous state space.

Theorem 5.2 *The stationary probability distribution of the Markov chain with the transition kernel* $P(x,A)$ *given in Eq.* (5.3) *is given by*

$$\begin{cases} \pi_0(A) = \tau x_0(A), \\ \pi_k(A) = \sum_{i=0}^{k-1} \int_0^{+\infty} \pi_i(x) R_{i,k}(x,A)\mathrm{d}x, \quad k \geqslant 1, \end{cases} \tag{5.7}$$

where $x_0(A)$ *is the stationary probability distribution of the censored Markov chain with the transition kernel* $\varPsi_0(x,A)$ *to level 0, and the scalar* τ *is uniquely determined by* $\sum_{k=0}^{\infty} \int_0^{+\infty} \pi_k(x)\mathrm{d}x = 1$.

Proof If the Markov chain $P(x,A)$ given in Eq. (5.3) exists the stationary probability vector $\pi(A) = (\pi_0(A), \pi_1(A), \pi_2(A), \ldots)$, then $\int_0^{+\infty} \pi(x)[I(x,A) - P(x,A)]\mathrm{d}x = 0$ and $\int_0^{+\infty} \pi(x)\mathrm{d}x e = 1$. Based on Theorem 5,1, we have

$$\begin{aligned} 0 &= \int_0^{+\infty} \pi(x)[I(x,A) - P(x,A)]\mathrm{d}y \\ &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \pi(x)[I(x,y) - R_U(x,y)][I(y,z) - \varPsi_D(y,z)] \\ &\quad \cdot [I(z,A) - G_L(z,A)]\mathrm{d}x\mathrm{d}y\mathrm{d}z. \end{aligned}$$

We write

$$x(A) = \int_0^{+\infty} \pi(x)[I(x,A) - R_U(x,A)]\mathrm{d}x$$

and

$$x(A) = (x_0(A), x_1(A), x_2(A), \ldots).$$

Then

$$\int_0^{+\infty}\int_0^{+\infty} x(y)[I(y,z) - \varPsi_D(y,z)][I(z,A) - G_L(z,A)]\mathrm{d}y\mathrm{d}z = 0,$$

which leads to

$$\begin{cases} \int_0^{+\infty} x_0(y)\left[I(y,A) - \varPsi_0(y,A)\right]\mathrm{d}y - \sum_{k=1}^{\infty} \int_0^{+\infty}\int_0^{+\infty} x_k(y) \\ \qquad \cdot \left[I(y,z) - \varPsi_k(y,z)\right] G_{k,0}(z,A)\mathrm{d}y\mathrm{d}z = 0, \\ \int_0^{+\infty} x_i(y)\left[I(y,A) - \varPsi_i(y,A)\right]\mathrm{d}y - \sum_{k=i+1}^{\infty} \int_0^{+\infty}\int_0^{+\infty} x_k(y) \\ \qquad \cdot \left[I(y,z) - \varPsi_k(y,z)\right] G_{k,i}(z,A)\mathrm{d}y\mathrm{d}z = 0, \quad \text{for } i \geqslant 1. \end{cases} \tag{5.8}$$

Note that $\Psi_0(y, A)$ is the transition kernel of the censored chain to level 0. If the Markov chain with the transition kernel $P(x, A)$ given in Eq. (5.3) is positive recurrent, then $\Psi_0(y, A)$ is also positive recurrent, thus it exists the stationary probability distribution $x_0(A)$. Hence there exists the stationary probability density function $x_0(y)$ such that $\int_0^{+\infty} x_0(y)[I(y, A) - \Psi_0(y, A)]\mathrm{d}y = 0$ and $\int_0^{+\infty} x_0(y)\mathrm{d}y = 1.$ It is easy to check that $(\tau x_0(A), 0, 0, ...)$ is a non-zero nonnegative solution to the systems of Eq. (5.8), where τ is uniquely determined by $\sum_{k=0}^{\infty} \int_0^{+\infty} \pi_k(x)\mathrm{d}x$=1. Solving the simplified system of linear equations $\int_0^{+\infty} \pi(x)[I(x, A) - R_U(x, A)]\mathrm{d}x =$ $(\tau x_0(A), 0, 0, ...)$, we obtain the desired result.

In what follows we derive expression for the stationary probability distribution of some important examples.

5.2.4 Markov Chains of *GI*/*G*/1 Type

The R-measure is given by $R_{0,k}(x, A)$ and $R_k(x, A)$ for $k \geqslant 1$, $x \geqslant 0$ and $A \in \sigma(\Omega)$. In this case, it follows from Theorem 5.2 that

$$\begin{cases} \pi_0(A) = \tau x_0(A), \\ \pi_k(A) = \int_0^{+\infty} \pi_0(x) R_{0,k}(x, A)\mathrm{d}x + \sum_{i=1}^{k-1} \int_0^{+\infty} \pi_i(x) R_{k-i}(x, A)\mathrm{d}x, \quad k \geqslant 1, \end{cases}$$

where $x_0(A)$ is the stationary probability distribution of the censored Markov chain with the transition kernel $\Psi_0(x, A)$ to level 0, and the scalar τ is uniquely determined by $\sum_{k=0}^{\infty} \int_0^{+\infty} \pi_k(x)\mathrm{d}x = 1.$

5.2.5 Markov Chains of *GI*/*M*/1 Type

The R-measure is given by $R(x, A)$ and $R_{0,1}(x, A)$, where $R(x, A)$ is the minimal nonnegative solution to the kernel equation

$$R(x, A) = \sum_{k=0}^{\infty} \int_0^{+\infty} R^k(x, y) C_k(y, A)\mathrm{d}y. \tag{5.9}$$

Applying the kernel $R(x, A),$ we can obtain another kernel $R_{0,1}(x, A)$ as follows:

$$R_{0,1}(x,A)=\int_0^{+\infty}D_0(x,y)\widehat{U}(y,A)\mathrm{d}y, \tag{5.10}$$

where

$$U(x,A)=\sum_{k=1}^{\infty}\int_0^{+\infty}R^{k-1}(x,y)C_k(y,A)\mathrm{d}y.$$

In this case, it follows from Theorem 5.2 and the R-measure given in Eq. (5.9) and Eq. (5.10) that

$$\begin{cases}\pi_0(A)=\tau x_0(A),\\ \pi_1(A)=\int_0^{+\infty}\pi_0(x)R_{0,1}(x,A)\mathrm{d}x\\ \pi_k(A)=\int_0^{+\infty}\pi_1(x)R^{k-1}(x,A)\mathrm{d}x, \quad k\geqslant 2.\end{cases} \tag{5.11}$$

Note that Eq. (5.11) is the same as Theorem 2 of Tweedie [27].

5.2.6 Markov Chains of *M*/*G*/1 Type

The R-measure is given by $R_{0,k}(x,A)$ and $R_k(x,A)$ for $k\geqslant 1$, $x\geqslant 0$ and $A\in\sigma(\Omega)$. Thus, the expression for the stationary probability distribution is the same as that for $GI/G/1$ type. However, for a Markov chain of $M/G/1$ type, the R-measure $R_{0,k}(x,A)$ and $R_k(x,A)$ for $k\geqslant 1$, $x\geqslant 0$ and $A\in\sigma(\Omega)$ can be determined by the kernel $G\left(x,A\right)$, which is the minimal nonnegative solution to the kernel equation

$$G(x,A)=\sum_{k=0}^{\infty}\int_0^{+\infty}C_k\left(x,y\right)G^k(y,A)\mathrm{d}y. \tag{5.12}$$

In this case, we have

$$R_{0,k}(x,A)=\sum_{i=1}^{\infty}\int_0^{+\infty}\int_0^{+\infty}D_{k+i}(x,y)G^{i-1}(y,z)\widehat{U}(z,A)\mathrm{d}y\mathrm{d}z \tag{5.13}$$

and

$$R_k(x,A)=\sum_{i=1}^{\infty}\int_0^{+\infty}\int_0^{+\infty}C_{k+i}(x,y)G^{i-1}(y,z)\widehat{U}(z,A)\mathrm{d}y\mathrm{d}z, \tag{5.14}$$

where

$$U(x,A)=\sum_{k=1}^{\infty}\int_0^{+\infty}C_k(x,y)G^{k-1}(y,A)\mathrm{d}y.$$

5.2.7 QBD Processes

The R- and G-measures are $\{R_k(x,y), k \geqslant 0\}$ and $\{G_l(x,y), l \geqslant 1\}$, respectively. The R-measure $\{R_k(x,y), k \geqslant 0\}$ is the minimal nonnegative solution to the system of nonlinear kernel equations

$$\begin{aligned} R_k(x,y) &= A_0^{(k)}(x,y) + \int_0^{+\infty} R_k(x,z) A_1^{(k+1)}(z,y)\mathrm{d}z \\ &\quad + \int_0^{+\infty}\int_0^{+\infty} R_k(x,u) R_{k+1}(u,v) A_2^{(k+2)}(v,y)\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{5.15}$$

The G-measure $\{G_k, k \geqslant 1\}$ is the minimal nonnegative solution to the system of nonlinear kernel equations

$$\begin{aligned} G_k(x,y) &= A_2^{(k)}(x,y) + \int_0^{+\infty} A_1^{(k)}(x,z) G_k(z,y)\mathrm{d}z \\ &\quad + \int_0^{+\infty}\int_0^{+\infty} A_0^{(k)}(x,u) G_{k+1}(u,v) G_k(v,y)\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{5.16}$$

The U-measure $\{U_k(x, y), k \geqslant 0\}$ is given by

$$\begin{aligned} U_k(x,y) &= A_1^{(k)}(x,y) + \int_0^{+\infty} R_k(x,z) A_2^{(k+1)}(z,y)\mathrm{d}z \\ &= A_1^{(k)}(x,y) + \int_0^{+\infty} A_0^{(k+1)}(x,z) G_{k+1}(z,y)\mathrm{d}z. \end{aligned} \tag{5.17}$$

5.2.8 An Algorithmic Framework

We provide an algorithmic framework for computing the stationary probability distribution. To do this, we need to introduce an orthogonal decomposition of functions in $L_2([0,+\infty)^2)$.

We denote by $\{\psi_k(x), k \geqslant 0\}$ an orthogonal basis in $L_2([0,+\infty))$. Then $\int_0^{+\infty} \psi_k(x)\psi_l(x)\mathrm{d}x = \delta_{kl}$, where δ_{kl} is 1 or 0 according as $k = l$ or $k \neq l$, respectively. Such an orthogonal basis in $L_2([0,+\infty))$ always exits. In the function space $L_2([0,+\infty))$, Li, Wang and Zhou [19] gave two different orthogonal bases, while Nielsen and Ramaswami [24] provided a finite-interval orthogonal basis. For the three orthogonal bases, we simply describe them as follows:

5.2.8.1 A Laguerre-Polynomial Orthogonal Base

Let

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

$$a_k^{(n)} = (-1)^k \binom{n}{k} \frac{1}{k!}, \quad n \geqslant 0, 0 \leqslant k \leqslant n, \tag{5.18}$$

and

$$l_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^k, \quad n \geqslant 0. \tag{5.19}$$

Then the function sequence $\left\{ \exp\left\{ -\frac{1}{2}x \right\} l_n(x), n \geqslant 0 \right\}$ is an orthogonal basis in $L_2([0, +\infty))$.

5.2.8.2 A Hermite-Polynomial Orthogonal Base

Let $t_n = \binom{n}{2}$ and

$$H_n(x) = \sum_{k=0}^{t_n} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}.$$

Then the function sequence $\left\{ \left(1/\sqrt[4]{\pi}\sqrt{2^{n-1} n!} \right) \exp\{-x^2/2\} H_n(x), n \geqslant 0 \right\}$ forms an orthogonal basis in $L_2([0, +\infty))$.

5.2.8.3 A Finite-Interval Orthogonal Base

Nielsen and Ramaswami [24] provided an orthogonal basis $\{\psi_n(x), n \geqslant 0\}$ in $L_2([0,1])$, where

$$\psi_n(x) = \frac{\sqrt{2n+1}}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} [x(1-x)]^n.$$

For simplicity of description, we consider the Markov chain of $GI/M/1$ type whose transition kernel is given by

$$P(x, A) = \begin{pmatrix} D_1(x,A) & D_0(x,A) & & & \\ D_2(x,A) & C_1(x,A) & C_0(x,A) & & \\ D_3(x,A) & C_2(x,A) & C_1(x,A) & C_0(x,A) & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

We assume that the two kernel sequences $\{D_k(x, y), k \geqslant 0\}$ and $\{C_k(x, y), k \geqslant 0\}$ are all in $L_2([0, +\infty))$. We write

$$C_k(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n}^{(k)} \psi_m(x) \psi_n(y) = \Psi(x) \mathbf{C}_k \Psi(y)^{\mathrm{T}} \tag{5.20}$$

and

$$D_k(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{m,n}^{(k)} \psi_m(x)\psi_n(y) = \Psi(x)\mathbf{D}_k\Psi(y)^{\mathrm{T}}, \tag{5.21}$$

where

$$\Psi(x) = (\psi_0(x), \psi_1(x), \psi_2(x), \ldots),$$

$$\mathbf{C}_k = \begin{pmatrix} c_{0,0}^{(k)} & c_{0,1}^{(k)} & c_{0,2}^{(k)} & \cdots \\ c_{1,0}^{(k)} & c_{1,1}^{(k)} & c_{1,2}^{(k)} & \cdots \\ c_{2,0}^{(k)} & c_{2,1}^{(k)} & c_{2,2}^{(k)} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

and

$$\mathbf{D}_k = \begin{pmatrix} d_{0,0}^{(k)} & d_{0,1}^{(k)} & d_{0,2}^{(k)} & \cdots \\ d_{1,0}^{(k)} & d_{1,1}^{(k)} & d_{1,2}^{(k)} & \cdots \\ d_{2,0}^{(k)} & d_{2,1}^{(k)} & d_{2,2}^{(k)} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

It is easy to check that

$$R(x,y) = \Psi(x)R\Psi(y)^{\mathrm{T}}, \tag{5.22}$$

where the matrix R is the suitable solution to the nonlinear matrix equation

$$R = \sum_{k=0}^{\infty} R^k \mathbf{C}_k. \tag{5.23}$$

The stationary probability distribution $\pi(x) = (\pi_0(x), \pi_1(x), \pi_2(x), \ldots)$ is given by

$$\pi_k(x) = \pi_k \Psi(x)^{\mathrm{T}}, \quad k \geqslant 0, \tag{5.24}$$

where

$$\pi_k = \tau x_0 R^k, \quad k \geqslant 0, \tag{5.25}$$

and the vector x_0 is a nonnegative non-zeroz solution to the system of equations $x_0 \sum_{k=1}^{\infty} R^k \mathbf{D}_k = 0$ and $x_0 e = 1$, and the scalar τ is uniquely determined by $\sum_{k=0}^{\infty} \int_0^{+\infty} \pi_k(x) \mathrm{d}x = 1.$

5.3 The *GI*/*G*/1 Queue

In this section, we apply the *RG*-factorizations to discuss the $GI/G/1$ queue, which is constructed as either a Markov chain of $GI/M/1$ type or a Markov chain of $M/G/1$ type under the semi-continuous state space.

We consider a $GI/G/1$ queue, where the service time and interarrival time distributions are denoted by the functions $G(t)$ and $F(t)$ with $\mu^{-1} = \int_0^{+\infty} t\mathrm{d}G(t)$ and $\lambda^{-1} = \int_0^{+\infty} t\mathrm{d}F(t)$, respectively.

5.3.1 Constructing a Markov Chain of *GI*/*M*/1 Type

We consider the Markov chain $\{X_n\}$, where $X_n = (N_n, S_n)$ for $n \geqslant 1$, N_n and S_n are the number of customers immediately before the *n*th interarrival time and the residual service time immediately after the *n*th interarrival time, respectively.

Let σ_1, σ_2, σ_3, ... denote a renewal process with $\sigma_n - \sigma_{n-1}$ having the service time distribution $G(t)$; and let R_t denote the residual lifetime at time *t* in this process, that is, $R_t = \sigma_{N(t)+1} - t$, where $N(t)$ is the renewal number in $[0, t)$. If $R_0 = x$, then $\sigma_1 = x$. We write

$$P_n^t(x, y) = P\{\sigma_n \leqslant t < \sigma_{n+1}, R_t \leqslant y \mid R_0 = x\}.$$

Clearly, $P_n^t(x, y)$ is the probability that *n* renewals occur in $[0, t)$ and the residual lifetime at time *t* is in $[0, y)$, given $R_0 = x$.

Let $Y_n = \sigma_n - \sigma_{n-1}$ with $\sigma_0 = 0$ for $n \geqslant 1$. Then $\{Y_n\}$ can be regarded as a Markov chain with continuous state space $[0, +\infty)$ whose transition kernel is given by

$$K(x, A) = P\{Y_{n+1} \in A \mid Y_n = x\}$$

which is independent of the number $n \geqslant 1$. We write

$$D_1(x, y) = P\{Y_{n+1} \leqslant y \mid Y_n = x\}$$

and

$$D_2(x, y) = P\{Y_{n+1} > y \mid Y_n = x\}.$$

The following lemma provides an iterative relationship for the kernel sequence $\{P_n^t(x, y)\}$.

Lemma 5.3 *For the kernel sequence* $\{P_n^t(x, y), n \geqslant 0\}$,

$$P_0^t(x, y) = I(x, y) \tag{5.26}$$

and for $n \geqslant 1$,

$$P_n^t(x,y) = \int_0^{+\infty} P_{n-1}^t(x,z)D_1(z,y)\mathrm{d}z + \int_0^{+\infty} P_n^t(x,z)D_2(z,y)\mathrm{d}z. \tag{5.27}$$

Proof Equation (5.26) is clear. Hence we only need to prove Eq. (5.27). This is completed by the following computations

$$\begin{aligned}
P_n^t(x,y) &= P\{\sigma_n \leqslant t < \sigma_{n+1}, R_t \leqslant y \mid R_0 = x\} \\
&= \int_0^{+\infty} P\{\sigma_n \leqslant t < \sigma_{n+1}, R_t \leqslant y \mid Y_2 = z, R_0 = x\}\mathrm{d}\{Y_2 \leqslant z \mid R_0 = x\} \\
&\quad + \int_0^{+\infty} P\{\sigma_n \leqslant t < \sigma_{n+1}, R_t \leqslant y \mid Y_2 = z, R_0 = x\}\mathrm{d}\{Y_2 > z \mid R_0 = x\} \\
&= \int_0^{+\infty} P\{\sigma_n \leqslant t < \sigma_{n+1}, R_t \leqslant y \mid Y_2 = z, R_0 = x\}\mathrm{d}\{Y_2 \leqslant z \mid Y_1 = x\} \\
&\quad + \int_0^{+\infty} P\{\sigma_n \leqslant t < \sigma_{n+1}, R_t \leqslant y \mid Y_2 = z, R_0 = x\}\mathrm{d}\{Y_2 > z \mid Y_1 = x\} \\
&= \int_0^{+\infty} P_{n-1}^t(x,z)D_1(z,y)\mathrm{d}z + \int_0^{+\infty} P_n^t(x,z)D_2(z,y)\mathrm{d}z.
\end{aligned}$$

This completes the proof.

When using the kernel sequence $\{P_n^t(x,y), n \geqslant 0\}$, the $GI/G/1$ queue can be described as a Markov chain of $GI/M/1$ type on continuous state space $[0, +\infty)$. In this case, we have

$$C_n(x,y) = \int_0^{+\infty} P_n^t(x,y)\mathrm{d}F(t), \quad n \geqslant 0,$$

and

$$D_n(x,y) = G(y)\sum_{k=n+1}^{\infty} C_k(x, +\infty), \quad n \geqslant 0.$$

Therefore, the Markov chain of $GI/M/1$ type is determined by the two kernel sequences $\{C_n(x,y)\}$ and $\{D_n(x,y)\}$.

Let

$$C(x,y) = \sum_{n=0}^{\infty} C_n(x,y).$$

Then

$$\begin{aligned}
C(x,y) &= \sum_{n=0}^{\infty} \int_0^{+\infty} P_n^t(x,y)\mathrm{d}F(t) \\
&= \int_0^{+\infty} P\{R_t \leqslant y \mid R_0 = x\}\mathrm{d}F(t).
\end{aligned}$$

The bivariate function $C(x,y)$ is the transition probability kernel of the residual lifetime process sampled at points of an independent renewal process generated by $F(t)$. If $\mu^{-1} = \int_0^{+\infty} t\mathrm{d}G(t) < +\infty$, then the residual lifetime process has the invariant measure $\theta(x)=\mu\int_0^x[1-G(t)]\mathrm{d}t$. This is summarized in the following lemma.

Lemma 5.4 $\theta(x)$ *is the invariant measure of the Markov chain with transition probability kernel* $C(x,y)$.

Proof We only need to check the equality $\int_0^{+\infty}\theta(x)C(x,y)\mathrm{d}x=\theta(y)$.

Note that

$$P\{R_0\leqslant x\}=\theta(x)$$

$$\begin{aligned}\int_0^{+\infty}\theta(x)C(x,y)\mathrm{d}x&=\int_0^{+\infty}P\{R_0\leqslant x\}\int_0^{+\infty}P\{R_t\leqslant y\mid R_0=x\}\mathrm{d}F(t)\mathrm{d}x\\&=\int_0^{+\infty}P\{R_t\leqslant y\}\mathrm{d}F(t)\\&=P\{R_0\leqslant y\}\\&=\theta(y).\end{aligned}$$

This completes the proof.

Let

$$\begin{aligned}\beta^t(x)&=\sum_{n=0}^{\infty}nP_n^t(x,+\infty)\\&=\sum_{n=0}^{\infty}nP\{\sigma_n\leqslant t<\sigma_{n+1}\mid R_0=x\}\\&=E[N(t)\mid R_0=x].\end{aligned}$$

Then

$$\begin{aligned}\rho^{-1}&=\int_0^{+\infty}\int_0^{+\infty}\beta^t(x)\mathrm{d}\theta(x)\mathrm{d}F(t)\\&=\int_0^{+\infty}\int_0^{+\infty}E[N(t)\mid R_0=x]\mathrm{d}P\{R_0\leqslant x\}\mathrm{d}F(t)\\&=\int_0^{+\infty}E[N(t)]\mathrm{d}F\left(t\right)=\mu\int_0^{+\infty}t\mathrm{d}F(t)\\&=\frac{\mu}{\lambda},\end{aligned}$$

where $E[N(t)]=\mu t$ and $\lambda^{-1}=\int_0^{+\infty}t\mathrm{d}F(t)$. Hence $\rho=\lambda/\mu$. Obviously, if $\rho<1$, then the $GI/G/1$ queue is stable.

Let $R(x,y)$ be the minimal nonnegative solution to the nonlinear kernel equation

$$R(x,y)=\sum_{n=0}^{\infty}\int_0^{+\infty}R^n(x,z)C_n(z,y)\mathrm{d}z.$$

We write

$$\Psi_0(x,y)=\sum_{n=1}^{\infty}\int_0^{+\infty}R^{n-1}(x,z)D_n(z,y)\mathrm{d}z.$$

It is easy to check that $\Psi_0(x,y)$ is the transition probability kernel of the censored chain to level 0.

Lemma 5.5 *Let $x_0(x)$ be the invariant measure of the censored Markov chain with transition probability kernel $\Psi_0(x,y)$ to level* 0. *Then $x_0(x)=G(x)$ for $x\geqslant 0$.*

Proof We only need to check the equality $\int_0^{+\infty}x_0(x)\Psi_0(x,y)\mathrm{d}x=x_0(y)$.

$$\begin{aligned}\int_0^{+\infty}x_0(x)\Psi_0(x,y)\mathrm{d}x&=\int_0^{+\infty}x_0(x)\sum_{n=1}^{\infty}\int_0^{+\infty}R^{n-1}(x,z)D_n(z,y)\mathrm{d}z\mathrm{d}x\\&=\int_0^{+\infty}x_0(x)\sum_{n=1}^{\infty}\int_0^{+\infty}R^{n-1}(x,z)\sum_{k=n+1}^{\infty}C_k(x,+\infty)\mathrm{d}zG(y)\mathrm{d}x.\end{aligned}$$

Let

$$c=\int_0^{+\infty}x_0(x)\sum_{n=1}^{\infty}\int_0^{+\infty}R^{n-1}(x,z)\sum_{k=n+1}^{\infty}C_k(x,+\infty)\mathrm{d}z\mathrm{d}x.$$

Then

$$\int_0^{+\infty}x_0(x)\Psi_0(x,y)\mathrm{d}x=cG(y).$$

Thus we can take $x_0(y)=cG(y)$. Note that $x_0(y)$ is the invariant measure of the censored Markov chain with transition probability kernel $\Psi_0(x,y)$, we obtain $\int_0^{+\infty}x_0(y)\mathrm{d}y=1$. Since $\int_0^{+\infty}x_0(y)\mathrm{d}y=c$, hence we have $c=1$, which leads to $x_0(y)=G(y)$ for $y\geqslant 0$. This completes the proof.

The following theorem provides the distribution of the stationary queue length for the $GI/G/1$ queue. The proof is clear and is omitted here.

Theorem 5.3 *If $\rho<1$, then the distribution of the stationary queue length is given by*

$$\pi_0(x)=\kappa G(x)$$

and

$$\pi_n(x)=\kappa\int_0^{+\infty}G(z)R^n(z,x)\mathrm{d}z,\quad n\geqslant 1,$$

where

$$\kappa = \frac{1}{1 + \sum_{n=1}^{\infty} \int_0^{+\infty} \int_0^{+\infty} G(z) R^n(z, x) \mathrm{d}z \mathrm{d}x}.$$

5.3.2 Constructing a Markov Chain of *M*/*G*/1 Type

We take the Markov chain $\{X_n\}$, where $X_n = (N_n, T_n)$ for $n \geqslant 1$, N_n and T_n are the number of customers immediately after the nth interarrival time and the residual interarrival time immediately before the nth service time, respectively.

Let τ_1, τ_2, τ_3, ... denote a renewal process with $\tau_n - \tau_{n-1}$ having the interarrival time distribution $F(t)$; and let R_t denote the residual lifetime at time t in this process, that is, $R_t = \tau_{N(t)+1} - t$, where $N(t)$ is the renewal number in $[0, t)$. If $R_0 = x$, then $\tau_1 = x$. We write

$$P_n^t(x, y) = P\{\tau_n \leqslant t < \tau_{n+1}, R_t \leqslant y \mid R_0 = x\}.$$

Clearly, $P_n^t(x, y)$ is the probability that n renewals occur in $[0, t)$ and the residual lifetime at time t is in $[0, y)$, given $R_0 = x$.

For the Markov chain of $M/G/1$ type, we write

$$D_0(x, y) = \int_0^{+\infty} P_0^t(x, y) \mathrm{d}G(t),$$

$$D_n(x, y) = \int_0^{+\infty} P_{n-1}^t(x, y) \mathrm{d}G(t), \quad n \geqslant 1,$$

and

$$C_n(x, y) = \int_0^{+\infty} P_n^t(x, y) \mathrm{d}G(t), \quad n \geqslant 0.$$

Let

$$C(x, y) = \sum_{n=0}^{\infty} C_n(x, y).$$

Then

$$\begin{aligned} C(x, y) &= \sum_{n=0}^{\infty} \int_0^{+\infty} P_n^t(x, y) \mathrm{d}G(t) \\ &= \int_0^{+\infty} P\{R_t \leqslant y \mid R_0 = x\} \mathrm{d}G(t). \end{aligned}$$

Hence $C(x, y)$ is the transition probability kernel of the residual lifetime process sampled at points of an independent renewal process generated by $G(t)$. If

$\lambda^{-1} = \int_0^{+\infty} t\mathrm{d}F(t) < +\infty$, then the residual lifetime process has the invariant measure $\theta(x) = \lambda \int_0^x [1 - F(t)]\mathrm{d}t$.

Let

$$\beta^t(x) = \sum_{n=0}^{\infty} nP_n^t(x, +\infty) = E[N(t) \mid R_0 = x].$$

Then

$$\begin{aligned}\rho &= \int_0^{+\infty} \int_0^{+\infty} \beta^t(x)\mathrm{d}\theta(x)\mathrm{d}G(t) \\ &= \int_0^{+\infty} E[N(t)]\mathrm{d}G(t) = \lambda \int_0^{+\infty} t\mathrm{d}G(t) \\ &= \frac{\lambda}{\mu},\end{aligned}$$

where $\mu^{-1} = \int_0^{+\infty} t\mathrm{d}G(t)$. Hence $\rho = \lambda/\mu$. Obviously, if $\rho < 1$, then the $GI/G/1$ queue is stable.

Let $G(x, y)$ be the minimal nonnegative solution to the nonlinear kernel equation

$$G(x, y) = \sum_{n=0}^{\infty} \int_0^{+\infty} C_n(z, y) G^n(x, z)\mathrm{d}z.$$

Let

$$R_{0,k}(x, y) = \sum_{i=1}^{\infty} \int_0^{+\infty} \int_0^{+\infty} D_{k+i}(x, u) G^{i-1}(u, v) \widehat{U}(v, y)\mathrm{d}u\mathrm{d}v$$

and

$$R_k(x, y) = \sum_{i=1}^{\infty} \int_0^{+\infty} \int_0^{+\infty} C_{k+i}(x, u) G^{i-1}(u, v) \widehat{U}(v, y)\mathrm{d}u\mathrm{d}v,$$

where

$$U(x, y) = \sum_{k=1}^{\infty} \int_0^{+\infty} C_k(x, z) G^{k-1}(z, y)\mathrm{d}z.$$

We write

$$\varPsi_0(x, y) = \sum_{n=1}^{\infty} \int_0^{+\infty} D_n(x, z) G^n(z, y)\mathrm{d}z.$$

It is easy to check that $\varPsi_0(x, y)$ is the transition probability kernel of the censored Markov chain to level 0. Let $x_0(x)$ be the invariant measure of the censored

Markov chain with transition probability kernel $\Psi_0(x,y)$.

If $\rho<1$, then the distribution of the stationary queue length is given by

$$\begin{cases}\pi_0(x)=\kappa x_0(x),\\ \pi_k(x)=\int_0^{+\infty}\pi_0(z)R_{0,k}(z,x)\mathrm{d}z+\sum_{i=1}^{k-1}\int_0^{+\infty}\pi_i(z)R_{k-i}(z,x)\mathrm{d}z, \quad k\geqslant 1,\end{cases}$$

where the scalar κ is uniquely determined by $\sum_{k=0}^{\infty}\int_0^{+\infty}\pi_k(x)\mathrm{d}x=1$.

5.4 Continuous-Time Markov Chains

In this section, we define and study continuous-time Markov chains on continuous state space. We provide expression for the stationary probability distribution of the Markov chain, which is Harris recurrent and ergodic.

For convenience of description, we always use calligraphic letters (for example, $\mathcal{A},\mathcal{B}$) to denote elements in $\sigma(D)$.

To define a continuous-time Markov chain on continuous state space $[0,+\infty)$, let real number $\lambda(x,\mathcal{B})$ denote the kernel of the Markov chain from an initial state x to a state set $\mathcal{B}\in\sigma([0,+\infty))$. Intuitively, we interpret $\lambda(x,\mathcal{B})$ in two possible cases:

(1) If $x\notin\mathcal{B}$, then $\lambda(x,\mathcal{B})$ is the transition rate of the Markov chain from x to B.

(2) If $x\in\mathcal{B}$, then $\lambda(x,\mathcal{B})=-\lambda(x,\mathcal{B}^c)$, where $\mathcal{B}^c=[0,+\infty)-\mathcal{B}$.

The two cases can be well understood by means of the results for a continuous-time Markov chain on a discrete state space.

We now provide conditions on the function $\lambda(x,\mathcal{B})$ such that $\lambda(x,\mathcal{B})$ for $x\geqslant 0$, $\mathcal{B}\in\sigma([0,+\infty))$ forms the kernel of a continuous-time Markov chain on continuous state space. These conditions resemble that for the infinitesimal generator of a continuous-time Markov chain on a discrete state space.

Definition 5.1 *For the continuous-time Markov chain on continuous state space* $[0,+\infty)$, $\lambda(x,\mathcal{B})$ *on* $[0,+\infty)\times\sigma([0,+\infty))$ *is called the kernel of the Markov chain if*

(1) *for each fixed* $x\geqslant 0,\lambda(x,\cdot)$ *is a signed measure on* $\sigma([0,+\infty))$*; and for each fixed* $\mathcal{B}\in\sigma\left([0,+\infty)\right),\lambda\left(\cdot,\mathcal{B}\right)$ *is a real-value measurable function on* $x\in[0,+\infty)$;

(2) $\lambda(x,\mathcal{B})\leqslant 0$ *for* $x\in\mathcal{B}$; *while* $0\leqslant\lambda(x,\mathcal{B})<+\infty$ *for* $x\notin\mathcal{B}$; *and*

(3) *when* $\mathcal{B}=\bigcup_{i=1}^{\infty}\mathcal{A}_i$ *and* $\mathcal{A}_i\cap\mathcal{A}_j=\varnothing$ *for all* $i\neq j$, *where* $\mathcal{A}_i\in\sigma([0,+\infty))$ *for* $i\geqslant 1$ *and* $\varnothing$ *is an empty set, we have*

$$\lambda(x,\mathcal{B}) = \sum_{i=1}^{\infty} \lambda(x,\mathcal{A}_i).$$

Based on Definition 5.1, we easily obtain the following useful properties. The proof is obvious and is omitted here.

Proposition 5.2 (1) $\lambda(x,[0,x)) \geqslant 0$ *and* $\lambda(x,(x,+\infty)) \geqslant 0$.

(2) *For the set sequence* $\{\mathcal{B}_n\}$ *on* $\sigma([0,+\infty))$ *with* $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ *for* $n \geqslant 1$, $0 \leqslant \lambda(x,\mathcal{B}_1) \leqslant \lambda(x,\mathcal{B}_2) \leqslant \ldots \leqslant \lambda(x,[0,x)) + \lambda(x,(x,+\infty))$ *if* $x \notin \mathcal{B}_\infty$ *where* $\mathcal{B}_\infty = \lim_{n\to\infty} \mathcal{B}_n$; *while* $\lambda(x,\{x\}) \leqslant \lambda(x,\mathcal{B}_1) \leqslant \lambda(x,\mathcal{B}_2) \leqslant \ldots \leqslant \lambda(x,[0,+\infty)) \leqslant 0$ *if* $x \in \mathcal{B}_1$.

For the continuous-time Markov chain on continuous state space $[0,+\infty)$, the irreducibility and the associated conditions may be similarly provided as that in Tweedie [27] (p. 370). Intuitively, if the Markov chain with the kernel $\lambda(x,\mathcal{B})$ is irreducible, then there exist at least two sets $\mathcal{B}_1 \in \sigma([0,x))$ and $\mathcal{B}_2 \in \sigma((x,+\infty))$ such that $\lambda(x,\mathcal{B}_1) > 0$ and $\lambda(x,\mathcal{B}_2) > 0$.

The negative number $\lambda(x,\{x\})$ is crucial for the classification of the continuous-time Markov chains on continuous state space. For example,

(1) if $\lambda(x,\{x\}) = -[\lambda(x,[0,x)) + \lambda(x,(x,+\infty))]$ for all $x \geqslant 0$, then the Markov chain is said to be conservative;

(2) if $\lambda(x,\{x\}) > -\infty$ for all $x \geqslant 0$ (resp. $\lambda(x,\{x\}) = -\infty$ for some $x \geqslant 0$), then the Markov chain is said to be stable (resp. instantaneous).

Henceforth we always assume that the Markov chain is conservative, irreducible, stable and regular, see, e.g. Anderson [1].

If $\mathcal{B}_F \in \sigma([0,x))$, $\mathcal{B}_B \in \sigma((x,+\infty))$ and $x \in \mathcal{B}$, then $\mathcal{B} = \mathcal{B}_F \bigcup \{x\} \bigcup \mathcal{B}_B \in \sigma([0,+\infty))$. We assume that there exists an almost everywhere nonnegative function $\lambda(x,y)$ for x, $y \geqslant 0$ such that

$$\lambda(x,\mathcal{B}_F) = \int_{\mathcal{B}_F} \lambda(x,y)\mathrm{d}y$$

and

$$\lambda(x,\mathcal{B}_B) = \int_{\mathcal{B}_B} \lambda(x,y)\mathrm{d}y.$$

Then, we write $\lambda(x,\mathcal{B})$ as

$$\lambda(x,\mathcal{B}) = \int_{\mathcal{B}_F} \lambda(x,y)\mathrm{d}y + \lambda(x,\{x\}) + \int_{\mathcal{B}_B} \lambda(x,y)\mathrm{d}y. \tag{5.28}$$

We call $\lambda(x,y)$ the generalized density function of the kernel $\lambda(x,\mathcal{B})$ on $[0,+\infty) \times \sigma([0,+\infty))$.

To understand the generalized density function $\lambda(x,y)$, we now provide a useful expression for $\lambda(x,y)$. Let

$$\Lambda(x) = -\left[\lambda(x,[0,x)) + \lambda(x,(x,+\infty))\right].$$

Then

$$\lambda(x,\{x\}) = \lambda(x,x) = \Lambda(x)$$

and

$$\lambda(x,y) = \lambda(x,y)(1-\delta_{xy}) + \Lambda(x)\delta_{xy},$$

where δ_{xy} is equal to one or zero according to $x = y$ or $x \neq y$, respectively.

Remark 5.1 (1) *For a continuous-time Markov chain on a discrete state space* $\Omega = \{0,1,2,...\}$ *whose infinitesimal generator is given by* $Q = (q_{i,j})_{i,j=0,1,2,...}$, *the integral expression Eq. (5.28) has the sum form:* $\int_{\mathcal{B}} q_{i,y}\mathrm{d}y = \sum_{j\in\mathcal{B}} q_{i,j}$.

(2) *We now provide a concrete example to construct the generalized density function* $\lambda(x,y)$, *which illustrates that the continuous-time Markov chain on continuous state space is a convenient mathematical tool for studying many real systems. Let*

$$\lambda(x,y) = \begin{cases} \dfrac{1}{(y-x+1)^3}, & y \geqslant x, \\ \dfrac{y^4}{(x+1)^3}, & y < x. \end{cases}$$

Then

$$\Lambda(x) = -\int_0^x \frac{y^4}{(x+1)^3}\mathrm{d}y - \int_x^{+\infty} \frac{1}{(y-x+1)^3}\mathrm{d}y = -\frac{1}{2} - \frac{x^5}{5(x+1)^3}.$$

Thus, for $x, y \geqslant 0$ *we have*

$$\lambda(x,y) = \lambda(x,y)(1-\delta_{xy}) + \Lambda(x)\delta_{xy}.$$

For the continuous-time Markov chain $\{X(t), t \geqslant 0\}$ on continuous state space $[0,+\infty)$ whose kernel is given by $\lambda(x,\mathcal{B})$ on $[0,+\infty)\times\sigma([0,+\infty))$, we define its transition probability as

$$P(t;x,\mathcal{B}) = P\{X(t)\in\mathcal{B} \mid X(0) = x\}, \quad t > 0.$$

We assume that the initial transition probability $P(0;x,\mathcal{B}) = 1_{\mathcal{B}}(x)$, where $1_{\mathcal{B}}(x)$ stands for the indicator function, which is equal to one or zero according to $x \in \mathcal{B}$ or $x \notin \mathcal{B}$, respectively. The transition probability $P(t;x,\mathcal{B})$ can be regarded as a solution to the Kolmogorov differential equation. The Kolmogorov's backward and forward equations are respectively given by

$$\frac{\partial}{\partial t}P(t;x,\mathcal{B})=\int_0^{+\infty}\lambda(x,y)P(t;\mathrm{d}y,\mathcal{B}) \tag{5.29}$$

and

$$\frac{\partial}{\partial t}P(t;x,\mathcal{B})=\int_0^{+\infty}P(t;x,\mathrm{d}y)\lambda(y,\mathcal{B}). \tag{5.30}$$

We assume that the continuous-time Markov chain with the kernel $\lambda(x,\mathcal{B})$ on $[0,+\infty)\times\sigma([0,+\infty))$ is Harris recurrent and ergodic (e.g., see Athreya and Ney [3]). Let $\Pi(\mathcal{B})$ for $\mathcal{B}\in\sigma([0,+\infty))$ be the stationary probability distribution of the Markov chain. We also assume that there exists a generalized density function $\pi(x)\geqslant 0$ for $x\geqslant 0$ such that $\Pi(\mathcal{B})=\int_{\mathcal{B}}\pi(x)\mathrm{d}x$. It is worthwhile to note that such a nonnegative function $\pi(x)\in L_2([0,+\infty))$ can be explicitly expressed below if the kernel $\lambda(x,y)\in L_2([0,+\infty)^2)$.

Note that $\int_0^{+\infty}\pi(x)\lambda(x,\mathcal{B})\mathrm{d}x=0$ for each $\mathcal{B}\in\sigma([0,+\infty))$, it is clear that $\pi(x)$ for $x\geqslant 0$ is a non-zero solution to the integral equation

$$\int_0^{+\infty}\pi(x)\lambda(x,y)\mathrm{d}x=0,\quad \text{for each } y\geqslant 0, \tag{5.31}$$

with the normalization condition $\int_0^{+\infty}\pi(x)\mathrm{d}x=1$.

Using the orthogonal basis $\{\psi_k(x),k\geqslant 0\}$, for $\lambda(x,y)\in L_2([0,+\infty)^2)$ we write

$$\lambda(x,y)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_{k,l}\psi_k(x)\psi_l(y), \tag{5.32}$$

where these coefficients $a_{k,l}$ for k, $l\geqslant 0$ are uniquely determined by the bivariate function $\lambda(x,y)$ and the orthogonal basis $\{\psi_k(x),k\geqslant 0\}$. Therefore, for each $\mathcal{B}\in\sigma([0,+\infty))$,

$$\lambda(x,\mathcal{B})=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_{k,l}\psi_k(x)\int_{\mathcal{B}}\psi_l(y)\mathrm{d}y.$$

On the other hand, for $\pi(x)\in L_2([0,+\infty))$ we have

$$\pi(x)=\sum_{i=0}^{\infty}b_i\psi_i(x), \tag{5.33}$$

where these coefficients b_i for $i\geqslant 0$ are uniquely determined by the bivariate function $\pi(x)$ and the orthogonal basis $\{\psi_k(x),k\geqslant 0\}$. It is clear that

$$\Pi(\mathcal{B})=\int_{\mathcal{B}}\pi(x)\mathrm{d}x=\sum_{i=0}^{\infty}b_i\int_{\mathcal{B}}\psi_i(x)\mathrm{d}x. \tag{5.34}$$

Based on the expression Eq. (5.34), the stationary probability distribution $\Pi(\mathcal{B})$ can be given once all the unknown coefficients b_i for $i \geqslant 0$ are determined.

It follows from Eq. (5.31), Eq. (5.32) and Eq. (5.33) that for each $y \geqslant 0$,

$$\int_0^{+\infty} \sum_{i=0}^{\infty} b_i \psi_i(x) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} \psi_k(x) \psi_l(y) \mathrm{d}x = 0,$$

which leads to the unknown row vector $(b_0, b_1, b_2, ...)$ is a non-zero solution to the system of linear equations

$$(b_0, b_1, b_2, \ldots) \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} = 0 \tag{5.35}$$

with the normalization condition $\sum_{i=0}^{\infty} b_i \int_0^{+\infty} \psi_i(x) \mathrm{d}x = 1$.

5.5 The QBD Processes

In this section, we deal with continuous-time QBD processes with continuous phase variable, and derive the UL-type *RG*-factorization. Based on this, the stationary probability distribution of the QBD process is shown to be an operator-multiplicative solution by means of the UL-type *RG*-factorization.

We describe a more general continuous-time Markov chain with a discrete level and continuous phase. The kernel matrix of the Markov chain is denoted as the function matrix $B(x, \mathcal{B})$ for $x \geqslant 0$, $\mathcal{B} \in \sigma([0, +\infty))$ with the (i, j)th entry $b_{i,j}(x, \mathcal{B})$ for i, $j \geqslant 0$. Then it is easy to see from Definition 5.1 that

(1) $b_{i,i}(x, \mathcal{B})$ is the kernel of a Markov chain for each $i \geqslant 0$,

(2) $b_{i,j}(x, \mathcal{B}) \geqslant 0$ for each $i \neq j$,

(3) $\sum_{j \geqslant 0} b_{i,j}(x, \mathcal{B}) \leqslant 0$ for each $i \geqslant 0$, and

(4) $B(x, \mathcal{B}) = \int_{\mathcal{B}} B(x, y) \mathrm{d}y$.

The continuous-time Markov chains with a discrete level and continuous phase have many important examples, including *GI*/*G*/1 type, *GI*/*M*/1 type, *M*/*G*/1 type and QBD processes. For simplification of discussion, here we mainly deal with the QBD processes.

Let $\{(L_t, P_t), t \geqslant 0\}$ be a continuous-time QBD process on a semi-continuous state space $\{0, 1, 2, ...\} \times [0, +\infty)$. The first coordinate $L_t \in \{0, 1, 2, ...\}$ is called level variable and the second coordinate $P_t \in [0, +\infty)$, phase variable. To write the kernel of the QBD process, we need to introduce an element $A_j^{(k)}(x,\mathcal{B})$ for $x \geqslant 0$, $\mathcal{B} \in \sigma([0,+\infty))$, and for either $k=0$, $i=1$, 2 or $k \geqslant 1$, $j=0$, 1, 2. It is obvious that $A_j^{(k)}(x,\mathcal{B})$ is the transition rate from an initial state (k,x) to the state set $(k+1-j,\mathcal{B})$ for either $k=0$, $j=0$, 1 or $k \geqslant 1$, $j=0$, 1, 2. Henceforth we assume that the continuous-time QBD process with continuous phase variable is conservative, stable and regular. According to Definition 5.1, we further show additional conditions on these elements as follows:

(1) $-\infty < A_1^{(k)}(x,\{x\}) < 0$ for $k \geqslant 0$ and

$$A_j^{(k)}(x,\mathcal{B})\begin{cases} \leqslant 0, & \text{if } j=1 \text{ and } x \in \mathcal{B}, \\ \geqslant 0, & \text{otherwise.} \end{cases}$$

(2) For each pair (k,j) for either $k=0$, $j=1$, 0 or $k \geqslant 1$, $j=0$, 1, 2, there exists a generalized density function $A_j^{(k)}(x,y)$ such that

$$A_j^{(k)}(x,\mathcal{B}) = \int_{\mathcal{B}} A_j^{(k)}(x,y)\mathrm{d}y.$$

(3)

$$\int_0^{+\infty}\left[A_1^{(0)}(x,y) + A_0^{(0)}(x,y)\right]\mathrm{d}y = 0$$

and

$$\int_0^{+\infty}\left[A_2^{(l)}(x,y) + A_1^{(l)}(x,y) + A_0^{(l)}(x,y)\right]\mathrm{d}y = 0, \quad l \geqslant 1.$$

(4) There exist two non-empty sets $\mathcal{B}_2$, $\mathcal{B}_0 \in \sigma([0,+\infty))$ such that $A_2^{(l)}(x,\mathcal{B}_2) > 0$ for all $l \geqslant 1$ and $A_0^{(k)}(x,\mathcal{B}_0) > 0$ for all $k \geqslant 0$.

Based on these elements, we write the kernel matrix of the QBD process as $Q(x,\mathcal{B}) = \int_{\mathcal{B}} Q(x,y)\mathrm{d}y$ with elementwise integrals for $\mathcal{B} \in \sigma([0,+\infty))$, where the matrix $Q(x,y)$ of generalized density functions for $x,\ y \geqslant 0$ is given by

$$Q(x,y) = \begin{pmatrix} A_1^{(0)}(x,y) & A_0^{(0)}(x,y) & & & \\ A_2^{(1)}(x,y) & A_1^{(1)}(x,y) & A_0^{(1)}(x,y) & & \\ & A_2^{(2)}(x,y) & A_1^{(2)}(x,y) & A_0^{(2)}(x,y) & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{5.36}$$

Now, we always use the matrix of generalized density functions to study the continuous-time QBD processes with continuous phase variable. To use the censoring technique, we need to define the matrix-integral inverse.

Definition 5.2 (1) *Let $B(x,\mathcal{B})$ for $x \geqslant 0$, $\mathcal{B} \in \sigma([0,+\infty))$ be the kernel matrix of a continuous-time Markov chain with a discrete level and continuous phase and $B(x,y)$ for $x,\ y \geqslant 0$, the associated matrix of generalized density functions.*

(2) *For the bivariate function matrix* $A(x,y)$ *for* x, $y \geqslant 0$, *if there exists a bivariate function matrix* $A(x,y)$ *such that for each* x, $y \geqslant 0$,

$$\int_0^{+\infty} A(x,z)B(z,y)\mathrm{d}z = I(x,y)$$

and

$$\int_0^{+\infty} B(x,z)A(z,y)\mathrm{d}z = I(x,y),$$

then $A(x,y)$ *is called the matrix-integral inverse of* $B(x,y)$, *we set* $A(x,y) = B^{\langle - \rangle}(x,y)$.

(3) *Let* Ω *be the set of all negative matrix-integral inverses of* $B(x,y)$. *If* $B^{\langle - \rangle}_{\max}(x,y) \in \Omega$ *and* $B^{\langle - \rangle}(x,y) \leqslant B^{\langle - \rangle}_{\max}(x,y) \leqslant 0$ *for all* $B^{\langle - \rangle}(x,y) \in \Omega$, *then* $B^{\langle - \rangle}_{\max}(x,y)$ *is called the maximal negative matrix-integral inverse of* $B(x,y)$.

5.5.1 The UL-Type *RG*-Factorization

To derive the UL-type *RG*-factorization, from Eq. (5.36) we write

$$_kQ(x,y) = \begin{pmatrix} A_1^{(k)}(x,y) & A_0^{(k)}(x,y) & & & \\ A_2^{(k+1)}(x,y) & A_1^{(k+1)}(x,y) & A_0^{(k+1)}(x,y) & & \\ & A_2^{(k+2)}(x,y) & A_1^{(k+2)}(x,y) & A_0^{(k+2)}(x,y) & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

If the continuous-time QBD process with continuous phase variable given in Eq. (5.36) is irreducible, then there must exist the maximal nonpositive matrix-integral inverse of $_kQ(x,y)$ for $k \geqslant 1$. Let $N_k(x,y)$ be the (1,1)st entry of $-_kQ^{\langle - \rangle}_{\max}(x,y)$ for $k \geqslant 1$. We define the *R*- and *G*-measures as follows:

$$R_k(x,y) = \int_0^{+\infty} A_0^{(k)}(x,z) N_{k+1}(z,y)\mathrm{d}z, \quad k \geqslant 0, \tag{5.37}$$

and

$$G_k(x,y) = \int_0^{+\infty} N_k(x,z) A_2^{(k)}(z,y)\mathrm{d}z, \quad k \geqslant 1. \tag{5.38}$$

We write the *U*-measure as

$$U_l(x,y) = A_1^{(l)}(x,y) + \int_0^{+\infty} R_l(x,z) A_2^{(l+1)}(z,y)\mathrm{d}z \tag{5.39}$$

or

$$U_l(x,y) = A_1^{(l)}(x,y) + \int_0^{+\infty} A_0^{(l)}(x,z)\, G_{l+1}(z,y)\mathrm{d}z, \quad l \geqslant 0. \tag{5.40}$$

It is obvious that for $k \geqslant 1$,

$$N_k(x,y) = -U_{k_{\max}}^{\langle -\rangle}(x,y), \tag{5.41}$$

which leads to

$$\int_0^{+\infty} U_k(x,z)N_k(z,y)\mathrm{d}z = \int_0^{+\infty} N_k(x,z)\,U_k(z,y)\mathrm{d}z = -I(x,y). \tag{5.42}$$

Remark 5.2 $U_0(x,\mathcal{B})$ *is the kernel of the censored Markov chain of the QBD process to level* 0. *Based on a censoring property, we need to identify the following two cases:*

(1) *If the QBD process is positive recurrent* (*resp. null recurrent*), *then the censored Markov chain to level 0 with the kernel* $U_0(x,y)$ *is also positive recurrent* (*resp. null recurrent*), *and* $U_0(x,[0,+\infty)) = \int_0^{+\infty} U_0(x,y)\mathrm{d}y = 0$ *for all* $x \geqslant 0$.

(2) *If the QBD process is transient, then the censored Markov chain to level* 0 *with the kernel* $U_0(x,y)$ *is also transient, and* $U_0(x,[0,+\infty)) = \int_0^{+\infty} U_0(x,y)\mathrm{d}y < 0$ *for some* $x \geqslant 0$.

The following theorem provides the minimal nonnegative solutions to the kernel equations for the *R*-, *U*- and *G*-measures.

Theorem 5.4 (1) *The kernel sequence* $\{R_k(x,y), k \geqslant 0\}$ *is the minimal nonnegative solution to the system of integral equations*

$$\begin{aligned} A_0^{(k)}(x,y) &+ \int_0^{+\infty} R_k(x,z)A_1^{(k+1)}(z,y)\mathrm{d}z \\ &+ \int_0^{+\infty}\int_0^{+\infty} R_k(x,z)R_{k+1}(z,u)A_2^{(k+2)}(u,y)\mathrm{d}u\mathrm{d}z = 0. \end{aligned} \tag{5.43}$$

(2) *The kernel sequence* $\{G_k(x,y), k \geqslant 1\}$ *is the minimal nonnegative solution to the system of integral equations*

$$\begin{aligned} A_2^{(k)}(x,y) &+ \int_0^{+\infty} A_1^{(k)}(x,z)G_k(z,y)\mathrm{d}z \\ &+ \int_0^{+\infty}\int_0^{+\infty} A_0^{(k)}(x,z)\,G_{k+1}(z,u)G_k(u,y)\mathrm{d}u\mathrm{d}z = 0. \end{aligned} \tag{5.44}$$

Proof We only prove Eq. (5.43), while the proof of Eq. (5.44) is similar.

It follows from Eq. (5.37) that

$$\int_0^{+\infty} R_k(x,z)U_{k+1}(z,y)\mathrm{d}z = \int_0^{+\infty}\int_0^{+\infty} A_0^{(k)}(x,u)N_{k+1}(u,z)U_{k+1}(z,y)\mathrm{d}u\mathrm{d}z. \tag{5.45}$$

Substituting Eq. (5.41) into the right-hand side of Eq. (5.42) leads to

$$\int_0^{+\infty}\int_0^{+\infty} A_0^{(k)}(x,u)N_{k+1}(u,z)U_{k+1}(z,y)\mathrm{d}u\mathrm{d}z = -A_0^{(k)}(x,y). \tag{5.46}$$

Substituting Eq. (5.39) into the left-hand side of Eq. (5.45) leads to

$$\int_0^{+\infty} R_k(x,z)U_{k+1}(z,y)\mathrm{d}z = \int_0^{+\infty} R_k(x,z)A_1^{(k+1)}(z,y)\mathrm{d}z$$
$$+ \int_0^{+\infty}\int_0^{+\infty} R_k(x,z)R_{k+1}(z,u)A_2^{(k+2)}(u,y)\mathrm{d}u\mathrm{d}z. \quad (5.47)$$

Thus, Eq. (5.45) and Eq. (5.47), together with Eq. (5.46), illustrate that the kernel sequence $\{R_k(x,y), k \geqslant 0\}$ is a nonnegative solution to the system of integral functions Eq. (5.43). Furthermore, using a similar discussion to Chapter 1 in Neuts [23] yields that $\{R_k(x,y), k \geqslant 1\}$ is the minimal nonnegative solution of Eq. (5.43). This completes the proof.

For the function $R(x,y)$ for $x, y \geqslant 0$, the positive number r is called the spectral radius of $R(x,y)$ if there exist two non-zero nonnegative functions $u(x,y)$ and $v(x,y)$ such that

$$ru(x,y) = \int_0^{+\infty} u(x,z)R(z,y)\mathrm{d}z$$

and

$$rv(x,y) = \int_0^{+\infty} R(x,z)v(z,y)\mathrm{d}z.$$

It is interesting to study the spectral radius of $R(x,y)$, the key issues of which include (1) spectral radius existence, (2) spectral radius uniqueness and (3) spectral radius structure. If the QBD process is Harris recurrent and ergodic, and $R(x,y)$ has the spectral radius r, then $0 < r < 1$, and r is an eigenvalue of the function $R(x,y)$, if any.

For the continuous-time QBD processes with continuous phase variable, the UL-type RG-factorization is given by

$$Q(x,y) = \int_0^{+\infty}\int_0^{+\infty} \left[I(x,z) - R_U(x,z)\right]U_D(z,u)\left[I(u,y) - G_L(u,y)\right]\mathrm{d}u\mathrm{d}z, \quad (5.48)$$

where

$$R_U(x,y) = \begin{pmatrix} 0 & R_0(x,y) & & & \\ & 0 & R_1(x,y) & & \\ & & 0 & R_2(x,y) & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix},$$

$$U_D(x,y) = \mathrm{diag}(U_0(x,y), U_1(x,y), U_2(x,y), \ldots)$$

and

$$G_L(x,y)=\begin{pmatrix} 0 & & & & \\ G_1(x,y) & 0 & & & \\ & G_2(x,y) & 0 & & \\ & & G_3(x,y) & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Proof Computing the matrix product $\left[I(x,z)-R_U(x,z)\right]U_D(z,u)[I(u,y)-G_L(u,y)]$ yields that the (k,k)th entry, the $(k-1,k)$th entry and the $(k,k+1)$th entry of this product are

$$I(x,z)U_k(z,u)I(u,y)+R_k(x,z)U_{k+1}(z,u)G_{k+1}(u,y),$$
$$-I(x,z)U_k(z,u)G_k(u,y)$$

and

$$-R_{k-1}(x,z)U_k(z,u)I(u,y),$$

respectively.

For the (k,k)th entry of this product, since

$$\int_0^{+\infty}\int_0^{+\infty} I(x,z)U_k(z,u)I(u,y)\mathrm{d}u\mathrm{d}z=U_k(x,y)$$

and

$$\int_0^{+\infty}\int_0^{+\infty} R_k(x,z)U_{k+1}(z,u)G_{k+1}(u,y)\mathrm{d}u\mathrm{d}z=-\int_0^{+\infty} A_0^{(k)}(x,z)G_{k+1}(z,y)\mathrm{d}z,$$

it follows from Eq. (5.40) that

$$\int_0^{+\infty}\int_0^{+\infty}[I(x,z)U_k(z,u)I(u,y)+R_k(x,z)U_{k+1}(z,u)$$
$$\cdot G_{k+1}(u,y)]\mathrm{d}u\mathrm{d}z=A_1^{(k)}(x,y).$$

Similarly, we can obtain

$$-\int_0^{+\infty}\int_0^{+\infty} I(x,z)U_k(z,u)G_k(u,y)\mathrm{d}u\mathrm{d}z=A_2^{(k)}(x,y)$$

and

$$-\int_0^{+\infty}\int_0^{+\infty} R_{k-1}(x,z)U_k(z,u)I(u,y)\mathrm{d}u\mathrm{d}z=A_0^{(k)}(x,y).$$

This completes the proof.

For the continuous-time QBD process with continuous phase variable, the following theorem uses the *RG*-factorization to show that the generalized density function of the stationary probability distribution is an operator-multiplicative solution.

Theorem 5.5 *If the continuous-time QBD process with continuous phase variable is Harris recurrent and ergodic, then the generalized density function* $\pi(y)=(\pi_0(y),\pi_1(y),\pi_2(y),...)$ *of its stationary probability distribution is operator-multiplicative, given by*

$$\pi_k(y)=\int_0^{+\infty}\pi_{k-1}(z)R_{k-1}(z,y)\mathrm{d}z,\quad k\geqslant 1, \tag{5.49}$$

where $\pi_0(y)=\kappa v_0(y)$, $v_0(y)$ *is the generalized density function of the stationary probability distribution of the censored chain to level 0 with the kernel* $U_0(x,y)$ *which is Harris recurrent and ergodic, and* κ *is a positive constant such that* $\sum_{l=0}^{\infty}\int_0^{+\infty}\pi_l(x)\mathrm{d}x=1$.

Proof Note that

$$\int_0^{+\infty}\pi(z)Q(z,y)\mathrm{d}z=0, \tag{5.50}$$

substituting Eq. (5.48) into Eq. (5.50) leads to

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\pi(z)\left[I(z,x)-R_U(z,x)\right]U_D(x,u)\cdot\left[I(u,y)-G_L(u,y)\right]\mathrm{d}u\mathrm{d}x\mathrm{d}z=0. \tag{5.51}$$

We now solve the Eq. (5.51) by two steps. In the first step, we set

$$w(y)=\int_0^{+\infty}\pi(z)\left[I(z,y)-R_U(z,y)\right]\mathrm{d}z, \tag{5.52}$$

which is partitioned as $(w_0(y),w_1(y),w_2(y),...)$ according to the levels, then Eq. (5.52) becomes

$$w_0(y)=\int_0^{+\infty}\pi_0(z)I(z,y)\mathrm{d}z=\pi_0(y) \tag{5.53}$$

and for $k\geqslant 1$,

$$\begin{aligned}w_k(y)&=\int_0^{+\infty}\left[-\pi_{k-1}(z)R_{k-1}(z,y)+\pi_k(z)I(z,y)\right]\mathrm{d}z\\&=\pi_k(y)-\int_0^{+\infty}\pi_{k-1}(z)R_{k-1}(z,y)\mathrm{d}z.\end{aligned} \tag{5.54}$$

In the second step, we determine the sequence $\{w_k(y)\}$ in terms of another equation

$$\int_0^{+\infty}\int_0^{+\infty}w(x)U_D(x,u)\left[I(u,y)-G_L(u,y)\right]\mathrm{d}u\mathrm{d}x=0,$$

which is equivalent to

$$\int_0^{+\infty}\int_0^{+\infty}\left[w_l(x)U_l(x,u)I(u,y)-w_{l+1}(x)U_{l+1}(x,u)G_{l+1}(u,y)\right]\mathrm{d}u\mathrm{d}x=0 \tag{5.55}$$

for all $l \geqslant 0$. Since the QBD process is Harris recurrent and ergodic, so is the censored chain to level 0 with the kernel $U_0(x,y)$. Let $v_0(y)$ be the generalized density function of the stationary probability distribution for the censored chain to level 0 with the kernel $U_0(x,y)$. Then it is clear that $(\kappa v_0(y), 0, 0, ...)$ is a non-zero nonnegative solution to the Eq. (5.55), where κ is a positive constant. Therefore,

$$\begin{cases} w_0(y)=\kappa v_0(y), \\ w_k(y)=0, \quad \text{for } k \geqslant 1. \end{cases} \tag{5.56}$$

Substituting Eq. (5.56) into Eq. (5.54) leads to

$$\pi_k(y)=\int_0^{+\infty}\pi_{k-1}(z)R_{k-1}(z,y)\mathrm{d}z, \quad \text{for } k \geqslant 1.$$

Using the normalization condition yields that the positive constant κ is determined by $\sum_{l=0}^{\infty}\int_0^{+\infty}\pi_l(x)\mathrm{d}x=1$. This completes the proof.

Remark 5.3 *For the continuous-time level-independent QBD process with continuous phase variable whose matrix $Q(x,y)$ of generalized density functions is given in Eq. (5.68), by means of the uniformization technique we can transform this continuous-time QBD process to a discrete-time level-independent QBD process with continuous phase variable whose matrix of generalized density functions is given by*

$$P(x,y)=\Delta(x,y)+\Omega(x,y)Q(x,y),$$

where

$$\Delta(x,y)=\operatorname{diag}(\delta_{xy},\delta_{xy},\delta_{xy},...)$$

and

$$\Omega(x,y)=\operatorname{diag}(\Lambda^{-1}(x),\Theta^{-1}(x),\Theta^{-1}(x),...).$$

Based on this, we can compute the stationary probability distribution of the continuous-time QBD process with continuous phase variable in terms of that algorithm proposed for the corresponding discrete-time QBD process.

5.5.2 The LU-Type *RG*-Factorization

We now provide a computable framework for solving the following integral equation

$$\int_0^{+\infty}\varphi(x)Q(x,y)\mathrm{d}x=f(y), \tag{5.57}$$

given that $f(y)\neq 0$ for some $y\geqslant 0$ and $Q(x,\mathcal{B})$ is the kernel matrix of the continuous-time QBD process with continuous phase variable. The solution is a key for deriving expressions for some transient performance measures in stochastic models, for example, the probability distribution at time t, the distribution of a first passage time and the distribution of a sojourn time.

It is easy to obtain that $\varphi(x)=\int_0^{+\infty} f(y)Q_{\max}^{\langle -\rangle}(y,x)\mathrm{d}y$. Thus, it is necessary to provide an expression for the maximal negative matrix-integral inverse of $Q(x,y)$. For simplicity, we introduce the notation

$$f(x,y)\circledast g(x,y)=\int_0^{+\infty} f(x,z)g(z,y)\mathrm{d}z.$$

We write

$$Y_k^{(l)}(x,y)=G_l(x,y)\circledast G_{l-1}(x,y)\circledast\ldots\circledast G_{l-k+1}(x,y),\quad l\geqslant k\geqslant 1,$$

and

$$Z_k^{(l)}(x,y)=R_{l+1}(x,y)\circledast R_{l+2}(x,y)\circledast\ldots\circledast R_{l+k}(x,y),\quad k\geqslant 1, l\geqslant 0.$$

Theorem 5.6 *If the continuous-time QBD process with continuous phase variable given in Eq.* (5.36) *is transient, then*

$$Q_{\max}^{\langle -\rangle}(x,y)=\left[I(x,y)-G_L(x,y)\right]^{\langle -\rangle}\circledast\left[U_D(x,y)\right]_{\max}^{\langle -\rangle}\circledast\left[I(x,y)-R_U(x,y)\right]^{\langle -\rangle},$$

where

$$\left[I(x,y)-G_L(x,y)\right]^{\langle -\rangle}=\begin{pmatrix} I(x,y) & & & \\ Y_1^{(1)}(x,y) & I(x,y) & & \\ Y_2^{(2)}(x,y) & Y_1^{(2)}(x,y) & I(x,y) & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\left[U_D(z,u)\right]_{\max}^{\langle -\rangle}=\mathrm{diag}\left(\left[U_0(x,y)\right]_{\max}^{\langle -\rangle},\left[U_1(x,y)\right]_{\max}^{\langle -\rangle},\left[U_2(x,y)\right]_{\max}^{\langle -\rangle},\ldots\right)$$

and

$$[I(x,y)-R_U(x,y)]^{\langle -\rangle}=\begin{pmatrix} I(x,y) & Z_1^{(0)}(x,y) & Z_2^{(0)}(x,y) & \ldots \\ & I(x,y) & Z_1^{(1)}(x,y) & \ldots \\ & & I(x,y) & \ldots \\ & & & \ddots \end{pmatrix}.$$

Proof We only need to check that

$$Q_{\max}^{\langle -\rangle}(x,y)\circledast Q(x,y)=I(x,y)$$

and

$$Q(x,y) \circledast Q_{\max}^{\langle - \rangle}(x,y) = I(x,y).$$

Note that the matrices $I(x,y) - G_L(x,y)$ and $I(x,y) - R_U(x,y)$ of generalized density functions are lower-triangular and Eupper-triangular, respectively, thus their inverses are both unique. Some integral calculations lead to the stated result.

To solve the integral equation Eq. (5.57), it is necessary to derive another useful factorization: The LU-type RG-factorization for the continuous-time QBD process with continuous phase variable.

We write

$$\overline{U}_0(x,y) = A_1^{(0)}(x,y)$$

and

$$\overline{U}_k(x,y) = A_1^{(k)}(x,y) + A_2^{(k)}(x,y) \circledast \left[-\overline{U}_{k-1}(x,y)\right]^{\langle - \rangle} \circledast A_0^{(k-1)}(x,y), \quad k \geqslant 1. \quad (5.58)$$

It is easy to check that the generalized density function $\overline{U}_k(x,y)$ is invertible for $k \geqslant 0$.

Let the sequence $\{\overline{R}_k(x,y)\}$ of generalized density functions be the unique nonnegative solution to the system of integral equations

$$\overline{R}_{k+1}(x,y) \circledast \overline{R}_k(x,y) \circledast A_0^{(k-1)}(x,y) + \overline{R}_{k+1}(x,y) \circledast A_1^{(k)}(x,y) \\ + A_2^{(k+1)}(x,y) = 0, \quad k \geqslant 2, \quad (5.59)$$

with the boundary condition

$$\overline{R}_1(x,y) = A_2^{(1)}(x,y) \circledast \left[-\overline{U}_0(x,y)\right]^{\langle - \rangle}. \quad (5.60)$$

Similarly, let the matrix sequence $\{\overline{G}_k(x,y)\}$ be the unique nonnegative solution to the system of integral equations

$$A_0^{(k)}(x,y) + A_1^{(k)}(x,y) \circledast \overline{G}_k(x,y) \\ + A_2^{(k)}(x,y) \circledast \overline{G}_{k-1}(x,y) \circledast \overline{G}_k(x,y) = 0, \quad k \geqslant 1, \quad (5.61)$$

with the boundary condition

$$\overline{G}_0(x,y) = \left[-\overline{U}_0(x,y)\right]^{\langle - \rangle} \circledast A_0^{(0)}(x,y). \quad (5.62)$$

The following theorem provides the LU-type RG-factorization for the continuous-time QBD process with continuous phase variable. The proof is obvious and is omitted here.

Theorem 5.7 *For the continuous-time QBD process with continuous phase variable given in Eq.* (5.36)*, the LU-type RG-factorization is given by*

$$Q(x,y) = \left[I(x,y) - \overline{R}_L(x,y)\right] \circledast \overline{U}_D(x,y) \circledast \left[I(x,y) - \overline{G}_U(x,y)\right], \quad (5.63)$$

where

$$\overline{U}_D(x,y) = \text{diag}(\overline{U}_0(x,y), \overline{U}_1(x,y), \overline{U}_2(x,y), \overline{U}_3(x,y), \dots),$$

$$\overline{R}_L(x,y) = \begin{pmatrix} 0 & & & & \\ \overline{R}_1(x,y) & 0 & & & \\ & \overline{R}_2(x,y) & 0 & & \\ & & \overline{R}_3(x,y) & 0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

and

$$\overline{G}_U = \begin{pmatrix} 0 & \overline{G}_0(x,y) & & & \\ & 0 & \overline{G}_1(x,y) & & \\ & & 0 & \overline{G}_2(x,y) & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

We write

$$\overline{X}_k^{(l)}(x,y) = \overline{R}_l(x,y) \circledast \overline{R}_{l-1}(x,y) \circledast \dots \circledast \overline{R}_{l-k+1}(x,y), \quad l \geqslant k \geqslant 1, \tag{5.64}$$

and

$$\overline{Y}_k^{(l)}(x,y) = \overline{G}_l(x,y) \circledast \overline{G}_{l+1}(x,y) \circledast \dots \circledast \overline{G}_{l+k-1}(x,y), \quad k \geqslant 1, l \geqslant 0. \tag{5.65}$$

The following Theorem provides expressions for each entry in the maximal nonpositive inverse $Q_{\max}^{\langle - \rangle}(x,y)$. We set $Q_{\max}^{-1}(x,y) = (q_{m,n}(x,y))_{m,n \geqslant 0}$ partitioned according to the levels.

Theorem 5.8 *For the continuous-time QBD process with continuous phase variable given in Eq.* (5.36)*, we have*

$$q_{m,n}(x,y) = \begin{cases} \overline{U}_m^{\langle - \rangle}(x,y) \circledast \overline{X}_{m-n}^{(m)}(x,y) + \sum_{i=1}^{\infty} \overline{Y}_i^{(m)}(x,y) \circledast \overline{U}_{i+m}^{\langle - \rangle}(x,y) \\ \quad \circledast \overline{X}_{i+m-n}^{(i+m)}(x,y), \qquad \text{if } 0 \leqslant n \leqslant m-1, \\ \overline{U}_m^{\langle - \rangle}(x,y) + \sum_{i=1}^{\infty} \overline{Y}_i^{(m)}(x,y) \circledast \overline{U}_{i+m}^{\langle - \rangle}(x,y) \circledast \overline{X}_i^{(i+m)}(x,y), \\ \qquad \text{if } n = m, \\ \overline{Y}_{n-m}^{(m)}(x,y) \circledast \overline{U}_n^{\langle - \rangle}(x,y) + \sum_{i=n-m+1}^{\infty} \overline{Y}_i^{(m)}(x,y) \circledast \overline{U}_{i+m}^{\langle - \rangle}(x,y) \\ \quad \circledast \overline{X}_{i-(n-m)}^{(i+m)}(x,y), \qquad \text{if } n \geqslant m+1. \end{cases}$$

5.6 Structured Matrix Expressions

In this section, if the matrix of generalized density functions of the continuous-time QBD process with continuous phase variable is in $L_2([0,+\infty)^2)$, then the integral equations given in the above section can be converted into the associated matrix equations. Therefore, some useful matrix-structured formulae, which resemble that in Section 1.3, are derived again. This leads to an algebraically algorithmic framework for calculating some performance measures of the continuous-time QBD processes with continuous phase variable.

Since $Q(x,y)\in L_2([0,+\infty)^2)$, we obtain that for either $k=0, i=0,1$ or $k\geqslant 1$, $i=0,1,2$,

$$A_i^{(k)}(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{i,m,n}^{(k)}\psi_m(x)\psi_n(y)=\Psi(x)\mathbf{A}_i^{(k)}\Psi(y)^{\mathrm{T}},$$

where $\{\psi_i(x)\}$ is an orthogonal basis in $L_2([0,+\infty))$,

$$\Psi(x)=(\psi_0(x),\psi_1(x),\psi_2(x),\ldots)$$

and

$$\mathbf{A}_i^{(k)}=\begin{pmatrix} a_{i,00}^{(k)} & a_{i,01}^{(k)} & a_{i,02}^{(k)} & \cdots \\ a_{i,10}^{(k)} & a_{i,11}^{(k)} & a_{i,12}^{(k)} & \cdots \\ a_{i,20}^{(k)} & a_{i,21}^{(k)} & a_{i,22}^{(k)} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

We write

$$D(x)=\mathrm{diag}\,(\Psi(x),\Psi(x),\Psi(x),\ldots)$$

and

$${}_kQ=\begin{pmatrix} \mathbf{A}_1^{(k)} & \mathbf{A}_0^{(k)} & & & \\ \mathbf{A}_2^{(k+1)} & \mathbf{A}_1^{(k+1)} & \mathbf{A}_0^{(k+1)} & & \\ & \mathbf{A}_2^{(k+2)} & \mathbf{A}_1^{(k+2)} & \mathbf{A}_0^{(k+2)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Note that ${}_0Q(x,y)=Q(x,y)$ and ${}_0Q=Q$. It is easy to check that for $k\geqslant 0$,

$${}_kQ(x,y)=D(x)({}_kQ)D(y)^{\mathrm{T}}.$$

Proposition 5.3 (1) *The matrix-identity kernel is given by*

$$I(x,y)=D(x)D(y)^{\mathrm{T}}.$$

(2) *For* $k \geqslant 1$, *the maximal negative matrix-integral inverse of the matrix* ${}_kQ(x,y)$ *is given by*

$$ {}_kQ_{\max}^{\langle -\rangle}(x,y) = D(x)\left[{}_kQ^{\langle -\rangle}\right]D(y)^{\mathrm{T}}, \tag{5.66}$$

where ${}_kQ^{\langle -\rangle}$ *is an inverse of the matrix* ${}_kQ$.

Proof We only prove (1), while (2) can be proved similarly.

We assume that the matrix-identity kernel $I(x,y) = D(x)CD(y)^{\mathrm{T}}$, where the matrix C is unknown. Let $B(x,y)$ be an arbitrary matrix of generalized density functions in $L_2([0,+\infty)^2)$. Then

$$B(x,y) = D(x)BD(y)^{\mathrm{T}}.$$

Notice that

$$\int_0^{+\infty} I(x,z)B(z,y)\mathrm{d}z = B(x,y)$$

and

$$\int_0^{+\infty} B(x,z)I(z,y)\mathrm{d}z = B(x,y),$$

we obtain

$$D(x)(I-C)BD(y)^{\mathrm{T}} = 0$$

and

$$D(x)B(I-C)D(y)^{\mathrm{T}} = 0.$$

Since $\{\psi_i(x)\}$ is an orthogonal basis in $L_2([0,+\infty))$ and the matrix B is arbitrary, we obtain $C = I$. This completes the proof.

The following proposition provides an algebraically computational framework for calculating the kernel sequences $\{R_k(x,y), k \geqslant 0\}$, $\{G_k(x,y), k \geqslant 1\}$ and $\{U_l(x,y), l \geqslant 0\}$. The proof is obvious and is omitted here.

Proposition 5.4 *Let* N_k *be the* $(1,1)st$ *block of the matrix* $-{}_kQ^{\langle -\rangle}$ *corresponding to the block-structure of the matrix* ${}_kQ$, *given in Eq.* (5.66).

(1) *For* $k \geqslant 0$

$$R_k(x,y) = \Psi(x)R_k\Psi(y)^{\mathrm{T}},$$

where $R_k = \mathbf{A}_0^{(k)}N_{k+1}$ *satisfies the system of matrix equations*

$$\mathbf{A}_0^{(k)} + R_k\mathbf{A}_1^{(k+1)} + R_kR_{k+1}\mathbf{A}_2^{(k+2)} = 0. \tag{5.67}$$

(2) *For* $k \geqslant 1$

$$G_k(x,y) = \Psi(x)G_k\Psi(y)^{\mathrm{T}},$$

where $G_k = N_k \mathbf{A}_2^{(k)}$ *satisfies the system of matrix equations*

$$\mathbf{A}_2^{(k)} + \mathbf{A}_1^{(k)} G_k + \mathbf{A}_0^{(k)} G_{k+1} G_k = 0.$$

(3) *For* $l \geqslant 0$,

$$U_l(x, y) = \Psi(x) U_l \Psi(y)^{\mathrm{T}},$$

where

$$U_l = \mathbf{A}_1^{(l)} + R_l \mathbf{A}_2^{(l+1)} = \mathbf{A}_1^{(l)} + \mathbf{A}_0^{(l)} G_{l+1}.$$

Furthermore,

$$N_k = -U_k^{\langle -\rangle}, \quad k \geqslant 1,$$

while U_0 *is singular if the QBD process is recurrent, otherwise it is invertible.*

Remark 5.4 *The sequence* $\{R_k\}$ *is a non-zero solution to the system of matrix Eq.* (5.67). *However, a non-zero solution to the system of matrix Eq.* (5.67) *may not be the sequence* $\{R_k\}$, *because the system of matrix Eq. (5.67) possibly has different non-zero solutions, e.g., see Gail, Hantler and Taylor [13]. From the viewpoint of solutions, it is necessary but more difficult to provide conditions for identifying whether a non-zero solution to the system of matrix Eq.* (5.67) *is the sequence* $\{R_k\}$. *At the same time, the sequence* $\{G_k\}$ *also has the same characteristics.*

The following theorem provides a matrix-structured form for the *RG*-factorization, which will be useful in constructing algebraically algorithmic solutions for the continuous-time QBD processes with continuous phase variable.

Theorem 5.9

$$Q = (I - R_U) U_D (I - G_L),$$

where

$$U_D = \mathrm{diag}(U_0, U_1, U_2, \ldots),$$

$$R_U = \begin{pmatrix} 0 & R_0 & & & \\ & 0 & R_1 & & \\ & & 0 & R_2 & \\ & & & \ddots & \ddots \end{pmatrix}$$

and

$$G_L = \begin{pmatrix} 0 & & & \\ G_1 & 0 & & \\ & G_2 & 0 & \\ & & \ddots & \ddots \end{pmatrix}.$$

Proof Note that

$$Q(x,y)=D(x)QD(y)^{\mathrm{T}},$$

$$U_D(x,y)=D(x)U_D D(y)^{\mathrm{T}},$$

$$R_U(x,y)=D(x)R_U D(y)^{\mathrm{T}},$$

$$G_L(x,y)=D(x)G_L D(y)^{\mathrm{T}},$$

it follows from Eq. (5.48) that

$$D(x)[Q-(I-R_U)U_D(I-G_L)]D(y)^{\mathrm{T}}=0.$$

Therefore, we obtain

$$Q=(I-R_U)U_D(I-G_L).$$

This completes the proof.

The following theorem provides an algebraically computational framework for calculating the generalized density function $\pi(x)$ of stationary probability distribution.

Theorem 5.10 *If the continuous-time QBD process with continuous phase variable is Harris recurrent and ergodic, then*

$$\pi_l(y)=S_l\Psi(y)^{\mathrm{T}},\quad l\geqslant 0,$$

where the row vector S_l *is iteratively determined by*

$$S_l=S_{l-1}R_{l-1},\quad l\geqslant 1,$$

and S_0 *is a non-zero solution to the equation* $S_0U_0=0$ *satisfying* $S_0\Psi(y)^{\mathrm{T}}>0$ *for some* $y\geqslant 0$.

Proof It follows from Eq. (5.49) that

$$(S_l-S_{l-1}R_{l-1})\Psi(y)^{\mathrm{T}}=0.$$

Since $\{\psi_i(x)\}$ is an orthogonal basis in $L_2([0,+\infty))$, we obtain $S_l=S_{l-1}R_{l-1}$. On the other hand, $\int_0^{+\infty}\pi_0(x)U_0(x,y)\mathrm{d}x=0$ and $\sum_{k=0}^{\infty}\int_0^{+\infty}\pi_k(x)\mathrm{d}x=1$ imply $S_0U_0=0$ and $S_0\neq 0$. Note that $\pi_0(y)>0$ for some $y\geqslant 0$, hence $S_0\Psi(y)^{\mathrm{T}}>0$ for some $y\geqslant 0$. This completes the proof.

We now discuss an important special case: A continuous-time level-independent QBD process with continuous phase variable whose matrix of generalized density functions is given by

$$Q(x,y)=\begin{pmatrix} B_1(x,y) & B_0(x,y) & & & \\ B_2(x,y) & A_1(x,y) & A_0(x,y) & & \\ & A_2(x,y) & A_1(x,y) & A_0(x,y) & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{5.68}$$

In this case, we have

$$R_k(x,y)=R(x,y), \quad k\geqslant 1,$$

and the generalized density function of the stationary probability is given by

$$\begin{cases} \pi_0(x) &= \kappa v_0(x) \\ \pi_1(x) &= \int_0^{+\infty}\pi_0(z)R_0(z,x)\mathrm{d}z \\ \pi_k(x) &= \int_0^{+\infty}\int_0^{+\infty}\cdots\int_0^{+\infty}\pi_1(z_1)R(z_1,z_2) \\ & \quad \cdot R(z_2,z_3)\ldots R(z_{k-1},x)\mathrm{d}z_1\mathrm{d}z_2\ldots\mathrm{d}z_{k-1}, \quad k\geqslant 2. \end{cases}$$

At the same time, we have

$$\begin{cases} S_1 = S_0R_0, \\ S_k = S_1R^{k-1}, \quad \text{for } k\geqslant 2. \end{cases}$$

Now, we provide an algorithm framework for computing the stationary distribution of the continuous-time level-independent QBD process with continuous phase variable whose matrix of generalized density functions is given in Eq. (5.68). We assume that the QBD process is Harris recurrent and ergodic, and each entry of its matrix of generalized density functions is in $L([0,+\infty)^2)$.

To compute the stationary distribution of the QBD process, we describe an orthogonal algorithm as follows:

Step 1 Orthogonal basis and orthogonal expansion

Let the function sequence $\{\psi_n(x), n\geqslant 0\}$ be an orthogonal basis in $L_2([0,+\infty))$. Then for an arbitrary function $f(x,y)\in L_2([0,+\infty)^2)$, we have

$$f(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n}\psi_m(x)\psi_n(y),$$

where

$$f_{m,n}=\int_0^{+\infty}\int_0^{+\infty} f(x,y)\psi_m(x)\psi_n(y)\mathrm{d}x\mathrm{d}y, \quad m,n\geqslant 0.$$

Step 2 Orthogonal expansions for kernel elements

Using the orthogonal basis $\{\psi_n(x), n\geqslant 0\}$ given in Step 1, we provide the orthogonal expansion for each entry of the matrix $Q(x,y)$ given in Eq. (5.68).

First, we give the orthogonal expansions for the off-diagonal entries $A_0(x,y)$, $A_2(x,y)$, $B_0(x,y)$, $B_2(x,y)$ as follows:

$$A_i(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}^{(i)}\psi_m(x)\psi_n(y),\quad i=0,2,$$

and

$$B_i(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}b_{m,n}^{(i)}\psi_m(x)\psi_n(y),\quad i=0,2.$$

We take the matrix A_i with the (m,n)th entry $a_{m,n}^{(i)}$ and the matrix B_i with the (m,n)th entry $b_{m,n}^{(i)}$ for $i=0,2$.

Then, we give the orthogonal expansions for the diagonal entries $A_1(x,y)$ and $B_1(x,y)$, respectively. To that end, we write

$$\Lambda(x)=-\left[B_0(x,[0,+\infty))+B_1(x,[0,x)\cup(x,+\infty))\right]$$

and

$$\Theta(x)=-\left[A_0(x,[0,+\infty))+A_1(x,[0,x)\cup(x,+\infty))+A_2(x,[0,+\infty))\right].$$

Note that

$$\Theta(x)=-\left[A_0(x,[0,+\infty))+A_1(x,[0,x)\cup(x,+\infty))+B_2(x,[0,+\infty))\right]$$

due to $B_2(x,[0,+\infty))=A_2(x,[0,+\infty))$. It is easy to see that for any $x,y\geqslant 0$,

$$A_1(x,y)=A_1(x,y)[1-\delta_{xy}]+\Theta(x)\delta_{xy}$$

and

$$B_1(x,y)=B_1(x,y)[1-\delta_{xy}]+\Lambda(x)\delta_{xy}.$$

Let

$$A_1(x,y)[1-\delta_{xy}]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}^{(1,1)}\psi_m(x)\psi_n(y)$$

and

$$\Theta(x)\delta_{xy}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}^{(1,2)}\psi_m(x)\psi_n(y).$$

Then

$$A_1(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}^{(1)}\psi_m(x)\psi_n(y),$$

where

$$a_{m,n}^{(1)} = a_{m,n}^{(1,1)} + a_{m,n}^{(1,2)}, \quad m,n \geqslant 0.$$

Therefore, we take the matrix A_1 with the (m,n)th entry $a_{m,n}^{(1)}$.

Similarly, let

$$B_1(x,y)[1-\delta_{xy}] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{m,n}^{(1,1)} \psi_m(x)\psi_n(y)$$

and

$$\Lambda(x)\delta_{xy} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{m,n}^{(1,2)} \psi_m(x)\psi_n(y).$$

Then

$$B_1(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{m,n}^{(1)} \psi_m(x)\,\psi_n(y),$$

where

$$b_{m,n}^{(1)} = b_{m,n}^{(1,1)} + b_{m,n}^{(1,2)}, \quad m,n \geqslant 0.$$

We take the matrix B_1 with the (m,n)th entry $b_{m,n}^{(1)}$.

Step 3 Computation of the stationary probability distribution

Let R be a non-zero solution to the matrix equation $A_0 + RA_1 + R^2 A_2 = 0$ with the setting that $R = \lim_{N\to\infty} R(N)$, where $R(0)=0$ and $R(N) = -A_0 A_1^{-1} - R(N-1)^2 A_2 A_1^{-1}$ for $N \geqslant 1$. If S_0 is the non-zero solution to the matrix equation $S_0 U_0 = 0$, where $U_0 = B_1 + B_0[-(A_1 + RA_2)]^{-1} B_2$, then the stationary probability distribution is given by

$$\pi_l(y) = S_0 R_0 R^l \Psi(y)^{\mathrm{T}}, \quad l \geqslant 0, y \geqslant 0,$$

where $\Psi(y) = (\psi_0(y), \psi_1(y), \psi_3(y), \ldots)$.

Step 4 A truncated approximation

When the sizes of the matrices A_1 and / or B_1 are infinite, it is necessary to truncate the matrices in order to compute the matrix R and / or the vector S_0 approximately. After choosing a larger positive integer N, for $i = 0, 1, 2$ we take

$$A_i = \begin{pmatrix} a_{0,0}^{(i)} & a_{0,1}^{(i)} & \cdots & a_{0,N}^{(i)} \\ a_{1,0}^{(i)} & a_{1,1}^{(i)} & \cdots & a_{1,N}^{(i)} \\ \vdots & \vdots & & \vdots \\ a_{N,0}^{(i)} & a_{N,1}^{(i)} & \cdots & a_{N,N}^{(i)} \end{pmatrix}$$

and

$$B_i = \begin{pmatrix} b_{0,0}^{(i)} & b_{0,1}^{(i)} & \cdots & b_{0,N}^{(i)} \\ b_{1,0}^{(i)} & b_{1,1}^{(i)} & \cdots & b_{1,N}^{(i)} \\ \vdots & \vdots & & \vdots \\ b_{N,0}^{(i)} & b_{N,1}^{(i)} & \cdots & b_{N,N}^{(i)} \end{pmatrix}$$

For these new matrices A_i and B_i for $i = 0, 1, 2$, after going to step 3 again, we can give an approximate probability distribution below

$$\tilde{\pi}_l(y) = S_0 R_0 R^l \Psi(y)^{\mathrm{T}}, \quad l \geqslant 0, y \geqslant 0,$$

where $\Psi(y) = (\psi_0(y), \psi_1(y), ..., \psi_N(y))$, S_0 is a row vector of dimension $N+1$, R_0 and R are two matrices of order $N+1$.

Now, we provide a structure for the numbers $a_{m,n}^{(1,2)}$ and $b_{m,n}^{(1,2)}$ for $m, n \geqslant 0$. To that end, we need to note a crucial result for the δ-function δ_{xy} for $x, y \geqslant 0$ as follows:

$$\int_0^{+\infty} \Theta(x)\delta_{xy}\mathrm{d}y = \Theta(x).$$

Let

$$\Theta(x) = \sum_{m=0}^{\infty} a_m \psi_m(x).$$

Then

$$\begin{aligned} a_{m,n}^{(1,2)} &= \int_0^{+\infty}\int_0^{+\infty} \Theta(x)\delta_{xy}\psi_m(x)\psi_n(y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^{+\infty} \Theta(x)\psi_m(x)\psi_n(x)\mathrm{d}x \\ &= \begin{cases} a_m, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases} \end{aligned}$$

Similarly, we have

$$b_{m,n}^{(1,2)} = \begin{cases} b_m, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

where

$$\Lambda(x) = \sum_{m=0}^{\infty} b_m \psi_m(x).$$

The following theorem constructs a basic relationship among these numbers $a_{m,n}^{(i)}$ and $b_{m,n}^{(i)}$ for $i=0,1,2$ and $m,n \geqslant 0$. This relationship is similar to the ordinary result that each row sum is zero in the infinitesimal generator of a continuous-time QBD process with countable phases.

Theorem 5.11 *Let* $c_n = \int_0^{+\infty} \psi_n(y)\mathrm{d}y$. *For each given* $m \geqslant 0$, *we have*

(1)

$$a_m = -\left[\sum_{n=0}^{\infty} c_n \left[a_{m,n}^{(1,1)} + \sum_{i=0,2} a_{m,n}^{(i)}\right]\right].$$

(2)

$$b_m = -\left[\sum_{n=0}^{\infty} c_n \left[b_{m,n}^{(1,1)} + b_{m,n}^{(0)}\right]\right].$$

Proof We only prove (1), while the proof of (2) is similar.

Note the QBD process is Harris recurrent and ergodic, we yield

$$\int_0^{+\infty} \left[A_0(x,y) + A_1(x,y) + A_2(x,y)\right]\mathrm{d}y = 0.$$

Using the orthogonal expansions given in Step 2, we obtain

$$\int_0^{+\infty} A_0(x,y)\mathrm{d}y = \sum_{m=0}^{\infty} \psi_m(x) \sum_{n=0}^{\infty} c_n a_{m,n}^{(0)},$$

$$\int_0^{+\infty} A_2(x,y)\mathrm{d}y = \sum_{m=0}^{\infty} \psi_m(x) \sum_{n=0}^{\infty} c_n a_{m,n}^{(2)},$$

and the relation $A_1(x,y) = A_1(x,y)[1-\delta_{xy}] + \Theta(x)\delta_{xy}$ leads to

$$\int_0^{+\infty} A_1(x,y)[1-\delta_{xy}]\mathrm{d}y = \sum_{m=0}^{\infty} \psi_m(x) \sum_{n=0}^{\infty} c_n a_{m,n}^{(1,1)}$$

and

$$\int_0^{+\infty} \Theta(x)\delta_{xy}\mathrm{d}y = \sum_{m=0}^{\infty} \psi_m(x) a_m.$$

Thus, we obtain

$$\sum_{m=0}^{\infty} \psi_m(x) \left\{ a_m + \left[\sum_{n=0}^{\infty} c_n \left[a_{m,n}^{(1,1)} + \sum_{i=0,2} a_{m,n}^{(i)}\right]\right]\right\} = 0.$$

Since $\{\psi_m(x), m \geqslant 0\}$ is an orthogonal basis in $L_2([0.+\infty))$, we obtain that for each given $m \geqslant 0$,

$$a_m = -\left[\sum_{n=0}^{\infty} c_n \left[a_{m,n}^{(1,1)} + \sum_{i=0,2} a_{m,n}^{(i)} \right]\right].$$

This completes the proof.

We now extend the mean drift condition of the continuous-time level-independent QBD process with countable phases to the corresponding QBD process $Q(x,y)$ for $x, y \geqslant 0$ with continuous phase. To do this, for $x, y \geqslant 0$ we write

$$A(x,y) = A_0(x,y) + A_1(x,y) + A_2(x,y)$$

and

$$\begin{aligned} U_0(x,y) &= B_1(x,y) + \int_0^{+\infty} R_0(x,z) B_2(z,y) \mathrm{d}z \\ &= B_1(x,y) + \int_0^{+\infty} B_0(x,z) G_1(z,y) \mathrm{d}z. \end{aligned}$$

We assume that the two Markov chains $A(x,y)$ and $U_0(x,y)$ on continuous state space are all Harris recurrent and ergodic. Let $\omega(x)$ be the stationary probability distribution of the Markov chain $A(x,y)$. Then $\int_0^{+\infty} \omega(x) A(x,y) \mathrm{d}x = 0$ for each $y \geqslant 0$ and $\int_0^{+\infty} \omega(x) \mathrm{d}x = 1$.

Using a similar analysis on the mean drift condition of the continuous-time level-independent QBD process with countable phases, it is easy to see that the QBD process $Q(x,y)$ is Harris recurrent and ergodic if for all $y \geqslant 0$,

$$\int_0^{+\infty} \omega(x) A_0(x,y) \mathrm{d}x < \int_0^{+\infty} \omega(x) A_2(x,y) \mathrm{d}x.$$

When using an orthogonal base to provide orthogonal expansions for some entries of the matrix of the generalized density functions given in Eq. (5.68), it is easy to see that the signs of the orthogonal coefficients always alternate. For this, the subtractive cancellation may be catastrophic in order to compute the matrix R in Step 3 according to the basic iterative procedure: $R = \lim_{N \to \infty} R(N)$, where $R(0) = 0$ and $R(N) = -A_0 A_1^{-1} - R(N-1)^2 A_2 A_1^{-1}$ for $N \geqslant 1$. In general, the sign alternation and the subtractive cancellation may result in the fact that the basic iterative computation is not numerically stable unless the size of the orthogonal expansion is small or otherwise is well structured.

We now provide an example to indicate how the orthogonal algorithm works, which demonstrates that the algorithm is full of promise for numerical computation of the QBD process with continuous phase. We take the entries of the matrix $Q(x,y)$ given in Eq. (5.68) as follows:

$$B_1(x,y) \equiv -1,\ B_0(x,y) \equiv 1, \quad \text{for all } x, y \geqslant 0,$$

$$A_0(x,y) = xy\exp\left\{-\frac{1}{2}(x+y)\right\},$$

$$A_1(x,y) = \frac{1}{3}x(y+2)\exp\left\{-\frac{1}{2}(x+y)\right\}[1-\delta_{xy}]$$

$$+\left[-9\frac{2}{3}+6(1-x)\right]\exp\left\{-\frac{1}{2}x\right\}\delta_{xy}$$

and

$$A_2(x,y) = B_2(x,y) = \frac{1}{4}(x+1)(y+1)\exp\left\{-\frac{1}{2}(x+y)\right\}.$$

It is easy to check that

$$A_0(x,y) = \exp\left\{-\frac{1}{2}x\right\}(1,1-x)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1-y \end{pmatrix}\exp\left\{-\frac{1}{2}y\right\},$$

$$A_2(x,y) = \exp\left\{-\frac{1}{2}x\right\}(1,1-x)\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 \\ 1-y \end{pmatrix}\exp\left\{-\frac{1}{2}y\right\}$$

and

$$A_1(x,y) = \exp\left\{-\frac{1}{2}x\right\}(1,1-x)\begin{pmatrix} 1 & -\frac{1}{3} \\ -1 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 1-y \end{pmatrix}\exp\left\{-\frac{1}{2}y\right\}$$

$$+\exp\left\{-\frac{1}{2}x\right\}(1,1-x)\begin{pmatrix} -9\frac{2}{3} & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1-y \end{pmatrix}\exp\left\{-\frac{1}{2}y\right\}$$

$$=\exp\left\{-\frac{1}{2}x\right\}(1,1-x)\begin{pmatrix} -8\frac{2}{3} & -\frac{1}{3} \\ -1 & 6\frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 1-y \end{pmatrix}\exp\left\{-\frac{1}{2}y\right\}.$$

Note that $\left\{\exp\left\{-\frac{1}{2}x\right\}, (1-x)\exp\left\{-\frac{1}{2}x\right\}\right\}$ are orthogonal bases in $L_2([0,+\infty))$, According to (1) in Remark 10, we obtain

$$A_0 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -8\frac{2}{3} & -\frac{1}{3} \\ -1 & 6\frac{1}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} -8\frac{2}{3} & -\frac{1}{3} \\ -1 & 6\frac{1}{3} \end{pmatrix}.$$

Let R be a non-zero solution to the matrix equation $A_0 + RA_1 + R^2 A_2 = 0$, and this solution is iteratively obtained by using step 3 of the orthogonal algorithm. Then

$$\pi_l(y) = \kappa(r_1, r_2) R^l \begin{pmatrix} 1 \\ 1-y \end{pmatrix} \exp\left\{-\frac{1}{2} y\right\},$$

where $(r_1, r_2) = R_0$ and κ is constant such that $\sum_{l=0}^{\infty} \int_0^{+\infty} \pi_l(y)\mathrm{d}y = 1$.

5.7 A *CMAP*/*CPH*/1 Queue

In this section, we discuss a $CMAP/CPH/1$ queue in terms of the continuous-time QBD process with continuous phase variable.

5.7.1 The CPH Distribution

Let $\lambda(x, \mathcal{B})$ on $[0, +\infty) \times \sigma([0, +\infty))$ and $\lambda(x, y)$ for $x, y \geqslant 0$ be the kernel and the generalized density function of the continuous-time Markov chain on continuous state space $[0, +\infty)$, respectively. We assume that

$$\lambda(x, [0, +\infty)) = \int_0^x \lambda(x, y)\mathrm{d}y + \lambda(x, x) + \int_x^{+\infty} \lambda(x, y)\mathrm{d}y \leqslant 0$$

and there exists an interval $[a, b) \subset [0, +\infty)$ such that

$$\lambda(x, [0, +\infty)) < 0 \text{ for all } x \in [a, b). \tag{5.69}$$

Let $\lambda^{(0)}(x) = -\lambda(x, [0, +\infty))$. Then it is easy to see that the absorbing rate of the Markov chain $\lambda(x, \mathcal{B})$ from state x into the absorbing state is $\lambda^{(0)}(x)$ for $x \geqslant 0$. Using a similar discussion on the PH distribution with an irreducible expression (α, T) of size m (e.g., see Neuts [23] for more details), we write the kernel expression of the CPH distribution as $(\alpha(x), \lambda(x, y))$, where $\alpha(x) \geqslant 0$ for $x \geqslant 0$ and $\int_0^{+\infty} \alpha(x)\mathrm{d}x = 1$. A simple computation leads to explicit expression of the CPH distribution below:

$$F(t)=1-\int_0^{+\infty}\alpha(x)\exp\{\lambda(x,x)t\}\left[\int_0^{+\infty}\exp\{\lambda(x,y)t\}\mathrm{d}y\right]\mathrm{d}x. \tag{5.70}$$

Note that

$$0\leqslant\exp\{\lambda(x,x)t\}\left[\int_0^{+\infty}\exp\{\lambda(x,y)t\}\mathrm{d}y\right]\leqslant 1 \quad \text{for all } x\in[0,+\infty).$$

Remark 5.5 *To understand Eq.* (5.70)*, we analyze the expression for the ordinary PH distribution with irreducible representation* $\{\beta,S\}$ *of size m, where* $\beta=(\beta_1,\beta_2,\ldots,\beta_m)$ *and* $S=(s_{i,j})_{1\leqslant i,j\leqslant m}$. *It is easy to check that*

$$\begin{aligned}F(t)&=1-\beta\exp\{St\}e\\&=1-\sum_{i=1}^{m}\beta_i\exp\{s_{i,i}t\}\sum_{j\neq i}^{m}\left(\exp\{(S-\Lambda)\}\right)_{i,j},\end{aligned}$$

where $\Lambda=\mathrm{diag}(s_{1,1},s_{2,2},\ldots,s_{m,m})$.

It is well known from Neuts [23] that the PH distribution with an irreducible expression (α,T) of size m for $m<\infty$ must be light-tailed. However, the CPH distribution with the kernel expression $(\alpha(x),\lambda(x,y))$ is either light-tailed or heavy-tailed. To construct the heavy-tailed case, we let

$$\alpha(x)=\begin{cases}1, & \text{for } x\in[0,1],\\ 0, & \text{otherwise,}\end{cases}$$

$\lambda(x,y)=0$ for all $x\neq y$ and $\lambda(x,x)<0$ for all $x\in[0,+\infty)$. Then

$$F(t)=1-\int_0^1\exp\{\lambda(x,x)t\}\mathrm{d}x.$$

We take that $\lambda(x,x)=-\ln(a(x))$, where $a(x)\in(1,+\infty)$. It is clear that

$$F(t)=1-\int_0^1\frac{\mathrm{d}x}{[a(x)]^t}.$$

When $a(x)=x^{-1}$, we have $F(t)=1-1/(1+t)$; when $a(x)=x^{-t}$, we have $F(t)=1-1/(1+t^2)$. Obviously, $F(t)$ is heavy-tailed in each of the two cases.

5.7.2 The CMAP

Let $C(x,\mathcal{B})$ and $C(x,\mathcal{B})+D(x,\mathcal{B})$ on $[0,+\infty)\times\sigma([0,+\infty))$ be the kernels of the two continuous-time Markov chains on continuous state space $[0,+\infty)$, and $C(x,y)$ and $C(x,y)+D(x,y)$ the generalized density functions of the two Markov chains.

We further assume that

(1)

$$C(x,[0,+\infty)) \leqslant 0$$

and there exists an interval $[a,b)$ such that for each $x \in [a,b)$

$$C(x,[0,+\infty)) < 0.$$

(2) The continuous-time Markov chain $C(x,\mathcal{B})+D(x,\mathcal{B})$ on $[0,+\infty) \times \sigma([0,+\infty))$ is Harris recurrent and ergodic. Specifically, for $x \in [0,+\infty)$,

$$C(x,[0,+\infty)) + D(x,[0,+\infty)) = 0.$$

We write

$$Q(x,\mathcal{B}) = \begin{pmatrix} C(x,\mathcal{B}) & D(x,\mathcal{B}) & & & \\ & C(x,\mathcal{B}) & D(x,\mathcal{B}) & & \\ & & C(x,\mathcal{B}) & D(x,\mathcal{B}) & \\ & & & \ddots & \ddots \end{pmatrix}$$

for $x \in [0,+\infty)$ and $\mathcal{B} \in \sigma([0,+\infty))$. The Markov process with kernel $Q(x,\mathcal{B})$ for $x \in [0,+\infty)$ and $\mathcal{B} \in \sigma([0,+\infty))$ is called a Markovian arrival process on continuous state space, denoted by CMAP.

Let $N(t)$ and $J(t)$ be the arrival number and the continuous phase of the CMAP at time t. We write

$$P_{x,\mathcal{B}}(k,t) = P\{N(t)=k, J(t) \in \mathcal{B} \mid N(0)=0, J(t)=x\}.$$

Then the probability sequence $\{P_{x,\mathcal{B}}(k,t), k \geqslant 0\}$ for $t \geqslant 0, x \in [0,+\infty)$ and $\mathcal{B} \in \sigma([0,+\infty))$ satisfies the forward Chapman-Kolmogorov differential equations

$$\frac{\partial}{\partial t} P_{x,\mathcal{B}}(0,t) = \int_0^{+\infty} P_{x,y}(0,t) C(y,\mathcal{B}) \mathrm{d}y$$

and for $k \geqslant 1$

$$\frac{\partial}{\partial t} P_{x,\mathcal{B}}(k,t) = \int_0^{+\infty} P_{x,y}(k,t) C(y,\mathcal{B}) \mathrm{d}y + \int_0^{+\infty} P_{x,y}(k-1,t) D(y,\mathcal{B}) \mathrm{d}y;$$

or the backward Chapman-Kolmogorov differential equations

$$\frac{\partial}{\partial t} P_{x,\mathcal{B}}(0,t) = \int_0^{+\infty} C(x,y) P_{y,\mathcal{B}}(0,t) \mathrm{d}y$$

and for $k \geqslant 1$

$$\frac{\partial}{\partial t} P_{x,\mathcal{B}}(k,t) = \int_0^{+\infty} C(x,y) P_{y,\mathcal{B}}(k,t)\mathrm{d}y + \int_0^{+\infty} D(x,y) P_{y,\mathcal{B}}(k-1,t)\mathrm{d}y.$$

Let

$$P^*_{x,\mathcal{B}}(z,t) = \sum_{k=0}^{\infty} z^k P_{x,\mathcal{B}}(k,t).$$

Then

$$\frac{\partial}{\partial t} P^*_{x,\mathcal{B}}(z,t) = \int_0^{+\infty} P^*_{x,y}(z,t)[C(y,\mathcal{B}) + zD(y,\mathcal{B})]\mathrm{d}y$$

or

$$\frac{\partial}{\partial t} P^*_{x,\mathcal{B}}(z,t) = \int_0^{+\infty} [C(x,y) + zD(x,y)] P^*_{y,\mathcal{B}}(z,t)\mathrm{d}y.$$

We assume that the continuous-time Markov chain with kernel $C(x,\mathcal{B}) + D(x,\mathcal{B})$ on $[0,+\infty)\times\sigma([0,+\infty))$ is Harris recurrent and ergodic. Let $\theta(B)$ be the stationary probability of this Markov chain with kernel $C(x,\mathcal{B}) + D(x,\mathcal{B})$. Then the stationary arrival rate of the CMAP is given by

$$\lambda = \int_0^{+\infty}\int_0^{+\infty} \theta(x) D(x,y)\mathrm{d}x\mathrm{d}y.$$

5.7.3 The *CMAP*/*CPH*/1 Queue

Now, we use the CMAP and the CPH distribution to study a single server queue, where the customer arrivals form a CMAP with the kernel expression $(C(x,y), D(x,y))$, and the service times are i.i.d. with a CPH distribution with the kernel expression $(\beta(x), \mu(x,y))$. $\mu^{(0)}(x) = \int_0^{+\infty} \mu(x,y)\mathrm{d}y$. We denote such a queue as the *CMAP*/*CPH*/1 queue. Let $N(t)$, $I(t)$ and $J(t)$ be the number of customers in this system at time *t*, the phases of the arrival and service processes at time *t*, respectively. $\{N(t), I(t), J(t), t \geqslant 0\}$ is a continuous-time QBD process with continuous phase variable whose matrix of generalized density functions is given by

$$Q^*(x,y) = \begin{pmatrix} C(x,y) & D(x,y) & & & \\ \mu^{(0)}(x)\beta(y) & C(x,y)+\mu(x,y) & D(x,y) & & \\ & \mu^{(0)}(x)\beta(y) & C(x,y)+\mu(x,y) & D(x,y) & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let $\theta(x)$ be the generalized density function of the stationary probability of the continuous-time Markov chain with continuous state space $[0,+\infty)$ whose generalized density function is $\Psi(x,y)=C(x,y)+D(x,y)+\mu(x,y)+\mu^{(0)}(x)\beta(y)$. Then it is easy to check that this system is stable if and only if

$$\int_0^{+\infty}\int_0^{+\infty}\theta(x)D(x,y)\mathrm{d}x\mathrm{d}y < \int_0^{+\infty}\int_0^{+\infty}\theta(x)\mu^{(0)}(x)\beta(y)\mathrm{d}x\mathrm{d}y.$$

We denote by $R(x,y)$ the minimal nonnegative solution to the integral equation

$$D(x,y)+\int_0^{+\infty}R(x,z)\left[C(z,y)+\mu(z,y)\right]\mathrm{d}z \\ +\int_0^{+\infty}\int_0^{+\infty}R(x,z)R(z,u)\mu^{(0)}(u)\beta(y)\mathrm{d}u\mathrm{d}z=0.$$

Let $\phi(x)$ be the generalized density function of the stationary probability of the censored Markov chain whose generalized density function is given by

$$U_0(x,y)=C(x,y)+\int_0^{+\infty}R(x,z)\mu^{(0)}(z)\beta(y)\mathrm{d}z.$$

We write the generalized density function $\pi(y)=(\pi_0(y),\pi_1(y),\pi_2(y),\ldots)$ of stationary probability distribution of the QBD process with the matrix $Q^*(x,y)$ of generalized density functions. Then

$$\begin{cases}\pi_0(y)=\kappa\phi(y)\\ \pi_k(y)=\int_0^{+\infty}\pi_{k-1}(z)R(z,y)\mathrm{d}z, \quad \text{for } k\geqslant 1,\end{cases}$$

where $\phi(y)$ is the stationary probability of the censored chain with kernel $U_0(x,y)$, and κ is a positive constant such that $\sum_{k=0}^{\infty}\int_0^{+\infty}\pi_k(y)\mathrm{d}y=1$.

5.8 Piecewise Deterministic Markov Processes

In this section, we first study a piecewise deterministic Markov process (PDMP). Then we apply the PDMP to analyze the $GI/G/c$ queue, and provide the stationary and transient probability distributions of the queue length.

5.8.1 Semi-Dynamic Systems

A deterministic system is described as follows. Given an initial state x_0, the state x_t of a deterministic system at time t is completely determined by the initial state

x_0 and time t. We denote by $x_t = \phi(t, x_0)$ the state change of this system. Obviously, x_t is a function of x_0 and t. It can be seen that the function ϕ satisfies the following two conditions:

(1) $\phi(0, x) = x$;

(2) $\phi(t, \phi(s, x)) = \phi(t+s, x)$.

In this case, the function $x_t = \phi(t, x_0)$ defines a dynamic system.

Let E denote the state space of the dynamic system, and ε the σ-algebra generated by the Borel subsets of E. Now, we define a local semi-dynamic system, which is useful in the study of the PDMPs later.

Definition 5.3 *Let (E, ε) be a Polish space. A local semi-dynamic system or a local semi-flow on E is a measurable mapping $\phi : D \to E$, where $D \subset E \times \mathbb{R}_+$, such that for each $x \in E$*

$$D(x) = \{t \in \mathbb{R}_+ : (t, x) \in D\} = [0, c(x))$$

and ϕ satisfies the following two conditions:

(1) $\phi(0, x) = x$ *;*

(2) *for all $x \in E$ and all $s \in D(x)$,*

$$t \in D(\phi(s, x)) \text{ if and only if } s + t \in D(x),$$

and in this case

$$\phi(t, \phi(s, x)) = \phi(t+s, x).$$

Condition (2) in Definition 5.3 can be interpreted as the future of this system can be precisely predicted as long as the present state of this system is known, and is independent of its history. This is the Markov property for the local semi-dynamic system, which indicates that a Markov process is a random generalization of the local semi-dynamic system.

Now, we introduce some basic concepts for the local semi-dynamic system. The function $t \to \phi(t, x)$ is called a motion of the local semi-dynamic system, starting from state x. For each $x \in E$, the subset $E(x) = \{\phi(t, x) : t \in D(x)\}$ is called a trajectory of the local semi-dynamic system, starting from state x. A state $x \in E$ is called an equilibrium point if $\phi(t, x) = x$ for all $t \in D(x)$. A trajectory $E(x)$ is said to be periodic if it is the trajectory of a periodic motion, that is, there exists the minimal positive number $T \in D(x)$ such that $\phi(t+T, x) = \phi(t, x)$ for all $t \in D(x)$, where state x is also called to be periodic.

Theorem 5.12 *If state $x \in E$ is periodic, then $D(x) = [0, +\infty)$.*

Proof If $T > 0$ is the period of state x, then $\phi(T, x) = \phi(0, x) = x$. Since

$$[0, c(x)) = D(x) = D(\phi(T, x)) = [0, T + c(x)),$$

we obtain

$$c(x) = T + c(x).$$

Note that $T > 0$ and $c(x) > 0$, it is clear that $c(x) = +\infty$. This completes the proof.

Definition 5.4 *State $x \in E$ is called a joint point of ϕ if there exist two states x_1 and x_2 with*

$$x_1 \notin E(x_2), \quad x_2 \notin E(x_1), \quad E(x_1) \bigcap E(x_2) \neq \varnothing$$

such that

$$x = \phi(t^*(x_1, x_2), x_1),$$

where

$$t^*(x_1, x_2) = \inf\{t \in D(x_1) : \phi(t, x_1) \in E(x_2)\}.$$

We call ϕ to be time reversible if for a fixed $t \in \mathbb{R}_+$, the map $\phi(t, \cdot)$: $\{x \in E : t < c(x)\} \to E$ is reversible. Let $y = \phi^-(t, x)$. Then $x = \phi(t, y)$. Obviously, a time reversible semi-dynamic system has no joint point.

A semi-dynamic system is said to be smoothing if $\phi(t, x)$ is differentiable for time t. In this case, we have

$$\begin{aligned}\frac{\mathrm{d}x_t}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{\phi(t + \Delta t, x) - \phi(t, x)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\phi(\Delta t, \phi(t, x)) - \phi(t, x)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\phi(\Delta t, x_t) - x_t}{\Delta t}.\end{aligned}$$

Thus, we can write

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = v(x_t),$$

where

$$v(x_t) = \lim_{\Delta t \to 0} \frac{\phi(\Delta t, x_t) - x_t}{\Delta t}.$$

Clearly, the smoothing semi-dynamic system can be expressed by a stationary ordinary differential equation.

5.8.2 The ϕ-Memoryless Distribution Family

Let $\{F(x, \cdot) : x \in E\}$ be a probability distribution family satisfying the following two conditions:

(1) for a fixed $x \in E$, $F(x, \cdot)$ is a probability distribution on $\mathcal{B}(\mathbb{R}_+)$, and

(2) for a fixed $A \in \mathcal{B}(\mathbb{R}_+)$, $F(\cdot, A)$ is ε-measurable.

Definition 5.5 *The probability distribution family* $\{F(x,\cdot): x \in E\}$ *is said to be memoryless if*

$$F(x, s+t) = F(x,s) F(\phi(s,x), t), \tag{5.71}$$

and for each $x \in E$, *the support of* $F(x,\cdot)$: *supp*$\{F(x,\cdot)\} \subset D(x)$, *where* $F(x,t) = F(x,[t,+\infty))$.

Theorem 5.13 *Suppose* $\{F(x,\cdot): x \in E\}$ *is a memoryless probability distribution family. If the function* $F(x,t)$ *has right derivative at* $t = s \in$ *supp*$\{F(x,\cdot)\}$, *then* $F(\phi(s,x), t)$ *has right derivative at* $t = 0$, *and*

$$\frac{\partial^+ F(\phi(s,x), t)}{\partial t}\Big|_{t=0} = [F(x,s)]^{-1} \frac{\partial^+ F(x,t)}{\partial t}\Big|_{t=s}.$$

Proof It follows from Eq. (5.71) that

$$\begin{aligned}
\frac{\partial^+ F(x,t)}{\partial t}\Big|_{t=s} &= \lim_{\Delta t \to 0^+} \frac{F(x, s+\Delta t) - F(x,s)}{\Delta t} \\
&= \lim_{\Delta t \to 0^+} \frac{F(x,s)F(\phi(s,x),\Delta t) - F(x,s)}{\Delta t} \\
&= F(x,s) \lim_{\Delta t \to 0^+} \frac{F(\phi(s,x),\Delta t) - 1}{\Delta t} \\
&= F(x,s) \frac{\partial^+ F(\phi(s,x),t)}{\partial t}\Big|_{t=0},
\end{aligned}$$

since

$$\lim_{t \to 0^+} F(\phi(s,x),t) = \lim_{t \to 0^+} \frac{F(x,s+t)}{F(x,s)} = 1.$$

This completes the proof.

For each $x \in E$, we write

$$\lambda(x) = \begin{cases} -\dfrac{\partial^+ F(x,t)}{\partial t}\Big|_{t=0}, & \text{if } \dfrac{\partial^+ F(x,t)}{\partial t}\Big|_{t=0} \text{ exists}, \\ 0, & \text{otherwise}. \end{cases}$$

The measurability of $F(x,t)$ with respet to $x \in E$ indicates that $\lambda(x)$ is ε-measurable.

Theorem 5.14 *Suppose* $\{F(x,\cdot): x \in E\}$ *is a memoryless probability distribution family. If* $F(x,t)$ *is absolutely continuous in* $[0, c_F(x))$ *for a fixed* $x \in E$ *and there exist at most countable points* $t \in [0, c_F(x))$ *such that* $\phi(t,x)$ *is a* ϕ*-joint point, where* $c_F(x) = \inf\{t : F(x,t) = 0\}$, *then*

$$F(x,t)=\begin{cases}\exp\left\{-\int_0^t \lambda(\phi(u,x))\mathrm{d}u\right\}, & t\in[0,c_F(x))\\ 0, & t\in[c_F(x),+\infty).\end{cases}$$

Proof It follows from Theorem 5.13 that

$$\frac{\frac{\partial F(x,t)}{\partial t}}{F(x,t)}=-\lambda(\phi(t,x)),\quad \text{a.e. on } [0,c_F(x)).$$

Hence we obtain

$$\frac{\partial}{\partial t}\ln F(x,t)=-\lambda(\phi(t,x)),\quad \text{a.e. on } [0,c_F(x)),$$

which leads to

$$F(x,t)=\kappa\cdot\exp\left\{-\int_0^t \lambda(\phi(u,x))\mathrm{d}u\right\},\quad \text{a.e. on } [0,c_F(x)).$$

Note that $F(x,0)=1$, it is clear that $\kappa=1$ and we obtain the desired result. This completes the proof.

Corollary 5.1 *Let E_e be the set of all equilibrium points of a semi-dynamic system ϕ. Then there exists a nonnegative function λ defined on E_e such that for each $x\in E_e$,*

$$F(x,t)=\exp\{-\lambda(x)t\}.$$

Proof If $x\in E_e$, then $\phi(u,x)=x$ for $0\leqslant u\leqslant t$, and x is a periodic point which leads to $c_F(x)=+\infty$. Thus we obtain

$$\begin{aligned}F(x,t)&=\exp\left\{-\int_0^t \lambda(\phi(u,x))\mathrm{d}u\right\}\\&=\exp\left\{-\lambda(x)t\right\},\quad \text{a.e. on } [0,+\infty).\end{aligned}$$

This completes the proof.

Definition 5.6

Let

$$J(E)=\{\phi(t,x):F(x,t^-)-F(x,t)>0,x\in E,t\in\mathbb{R}_+\}.$$

If $x\in J(E)$, state x is called an F-jump point.

For any $x\in E$ and $t\in[0,c_F(x))$, we write

$$a(x,t)=a(\phi(t,x))=\frac{F(x,t^-)-F(x,t)}{F(x,t^-)}$$

with convention $\frac{0}{0}=0$.

Theorem 5.15 (1) *If* $y=\phi(t,x)\notin J(E)$, *then* $a(x,t)=a(y)=0$.

(2) *If* $y=\phi(t,x)\in J(E)$ *but it is not a* ϕ*-joint point, then* $a(x,t)=a(y)$ *which is independent of the pair* (t,x).

Proof (1) is clear. We only need to prove (2).

If $y=\phi(t,x)\in J(E)$ but it is not a ϕ-joint point, then for any two states x_1, $x_2\in E$ with $y=\phi(t_1,x_1)$ and $y=\phi(t_2,x_2)$, there exists a u for $0<u<\min\{t_1,t_2\}$ and a state z such that

$$y=\phi(u,z),\quad z=\phi(t_1-u,x_1),\quad z=\phi(t_2-u,x_2).$$

Hence

$$\begin{aligned}a(x_1,t_1)&=\frac{F(x_1,t_1^-)-F(x_1,t_1)}{F(x_1,t_1^-)}\\&=\frac{F(x_1,t_1-u)[F(z,u^-)-F(z,u)]}{F(x_1,t_1-u)F(z,u^-)}\\&=\frac{F(z,u^-)-F(z,u)}{F(z,u^-)}.\end{aligned}$$

Similarly, we have

$$a(x_2,t_2)=\frac{F(z,u^-)-F(z,u)}{F(z,u^-)}.$$

Thus, we obtain

$$a(x_1,t_1)=a(x_2,t_2)=a(y)$$

which is independent of the pair (t,x). This completes the proof.

The function $a(y)$ has the following probabilistic setting

$$a(y)=\sup\{F(x,t^-)-F(x,t):y=\phi(t,x)\}.$$

Clearly, $a(y)\leqslant 1$. Specifically, a necessary and sufficient condition under which $a(y)=1$ is that $y=\phi(t,x)\in J(E)$ and $y=\phi(c_F(x),x)$ for some $x\in E$.

Theorem 5.16 *Let* $F(x,t)$ *be discrete type for* $x\in E$ *and its jump times* $\{t_n\}$ *can be ordered as* $t_1<t_2<\ldots$. *If for each n,* $x_n=\phi(t_n,x)$ *is not a* ϕ*-joint point, then*

$$p_n=a(x_n)\prod_{m=1}^{n-1}[1-a(x_m)],$$

where $p_n=F(x,t_n^-)-F(x,t_n)$.

Proof Note that $a(x_n) = p_n / F(x, t_n^-)$ and $F(x, t_n^-) = \sum_{m=n}^{\infty} p_m$, we obtain

$$a(x_n) = \frac{p_n}{\sum_{m=n}^{\infty} p_m},$$

hence we have

$$\begin{aligned} p_n &= \frac{a(x_n)}{a(x_{n-1})}[1 - a(x_{n-1})]p_{n-1} \\ &= a(x_n)\prod_{m=1}^{n-1}[1 - a(x_m)]. \end{aligned}$$

This completes the proof.

5.8.3 Time Shift ϕ-Invariant Transition Kernel

Let $Q(x,t,B)$ for $x \in E$, $t \in \mathbb{R}_+$ and $B \in E$ satisfy

(1) for a fixed pair $(x,t), Q(x,t,\cdot)$ is a probability measure on ε,

(2) for a fixed B, $Q(\cdot\,,\cdot\,,B)$ is $\varepsilon \times \mathcal{B}(\mathbb{R}_+)$-measurable.

Definition 5.7 *The transition kernel $Q(x,t,B)$ for $x \in E$, $t \in \mathbb{R}_+$ and $B \in \varepsilon$ is called time shift ϕ-invariant if for any $x \in E$, $B \in \varepsilon$, $s,t \in \mathbb{R}_+$ with $s+t \in$,* $[0, c(x))$,

$$Q(x, s+t, B) = Q(\phi(s,x), t, B).$$

We introduce a ϕ-exit boundary as follows:

$$\partial^+ E = \{\phi(c(x), x) : x \in E\},$$

which extends the concept of the ϕ-joint point.

Theorem 5.17 *Let $Q(x,t,B)$ for $x \in E$, $t \in \mathbb{R}_+$ and $B \in \varepsilon$ be the time shift ϕ-invariant transition kernel. The following two statements are true.*

(1) *There exists a transition kernel $q : (E \cup \partial^+ E) \times \varepsilon \to [0,1]$ such that if $x \in E \cup \partial^+ E$ is neither a ϕ-joint point nor an equilibrium point, then for any $x_0 \in E$ and $t \in [0, c(x))$ with $x = \phi(t, x_0)$, we have*

$$Q(x_0, t, B) = q(x, B),$$

which is independent of the pair (x_0, t).

(2) *There exists a transition kernel $q_e : E_e \times \varepsilon \to [0,1]$ such that if $x \in E_e$ is an*

equilibrium point, then

$$Q(x, t, B) = q_e(x, B),$$

which is independent of time t.

Proof (1) If $x \in E \bigcup \partial^+ E$ is neither a ϕ-joint point nor an equilibrium point, then for any $x_1, x_2 \in E, t_1 \in [0, c(x_1)), t_2 \in [0, c(x_2))$ satisfying $x = \phi(t_1, x_1) = \phi(t_2, x_2)$, there exists a t_0 for $0 < t_0 < t_1 \Lambda t_2$ such that

$$x_0 = \phi(t_1 - t_0, x_1) = \phi(t_2 - t_0, x_2).$$

Thus we obtain

$$\begin{aligned} Q(x_1, t_1, B) &= Q(\phi(t_1 - t_0, x_1), t_0, B) \\ &= Q(x_0, t_0, B) \\ &= Q(\phi(t_2 - t_0, x_2), t_0, B) \\ &= Q(x_2, t_2, B), \end{aligned}$$

which shows that $Q(x_0, t, B)$ is independent of the pair (x_0, t), we write it as $q(x, B)$.

(2) If $x \in E_e$ is an equilibrium point, then $\phi(s, x) = x$. Thus we obtain

$$Q(x, s + t, B) = Q(\phi(s, x), t, B) = Q(x, t, B) = Q(x, 0, B)$$

which is denoted as $q_e(x, B)$.

This completes the proof.

5.8.4 Piecewise Deterministic Markov Processes

PDMPs are a generalization from Markov jump processes with respect to three main features. (1) The state space is not constrained to a countable set anymore, while it may be allowed to be continuous. (2) Between two jump points, the process is not restricted to remain constant, but may change deterministically. (3) The possibility of movement among jump points gives rise to a new kind of jumps which occur immediately upon reaching a certain state. The new kind of jump is called an intrinsic jump, since it is induced exclusively by the state of the system, while the other kind of jump, as induced by Markovian arrivals, is called an extrinsic jump.

The PDMP $\mathcal{X} = \{X_t, t \in \mathbb{R}_+\}$ is a continuouzs-time Markov process on a Polish state space *E*, and can be determined by the following four characteristic representations:

(1) A flow $\phi : E \times \mathbb{R}_+ \rightarrow E$ is a local semi-dynamic system on the state space *E*.

(2) A closed set $\varDelta$ contains the states that induce the intrinsic jumps.

(3) A function $\lambda : E \to \mathbb{R}_+$ satisfies

$$\sup_{x \in E}\{\lambda(x)\} \leqslant c < +\infty.$$

The function $\lambda(x)$ indicates the intensity of an extrinsic jump occurring if the Markov process X is at state x. At the same time,

$$F(x,t) = \exp\left\{-\int_0^t \lambda(\phi(u,x))\mathrm{d}u\right\},$$

which leads to the conclusion that $\{F(x,\cdot), x \in E\}$ is a ϕ-memoryless distribution family.

(4) $Q(x,t,\cdot)$ for $x \in E$ and $t \in \mathbb{R}_+$ is the time shift ϕ-invariant transition kernel. Specifically, the transition measure $q : E \times \varepsilon^0 \to [0,1]$ with $\varepsilon^0 = \varepsilon \bigcap (E \setminus \varDelta)$, describes the behavior upon (extrinsic and intrinsic) jumps.

We first define for all $x \in E^0 = E \setminus \varDelta$ the deterministic variable

$$t_* = \inf\{t \in \mathbb{R}_+ : \phi(x,t) \in \varDelta\}$$

as the time until the set $\varDelta$ is reached from a state $x \in E$. Then we define the random vaiable $T(x)$ as the first (intrinsic and extrinsic) jump time after starting in state x. This is distributed as

$$P\{T(x) > t\} = \begin{cases} \exp\left\{-\int_0^t \lambda(\phi(x,u))\mathrm{d}u\right\}, & \text{if } t < t_*, \\ 0, & \text{if } t \geqslant t_*, \end{cases}$$

for all $t \in \mathbb{R}_+$. For simplicity of description on the distribution $P\{T(x) > t\}$, we assume that the Markov process has only finitely many jumps in any finite interval.

The PDMP χ evolves in the following way: Starting in any state $x \in E \setminus \varDelta$, it changes deterministically according to the flow ϕ until it enters $\varDelta$, inducing an intrinsic jump, or an extrinsic jump. Upon a jump, the state of χ changes immediately according to the transition measure Q, leading to a state $y \in E \setminus \varDelta$. Then the PDMP starts a new cycle, behaving as described until the next jump.

5.8.5 The Stationary Distribution

We consider a PDMP $\chi = \{X_t, t \in \mathbb{R}_+\}$ with characteristic triple (ϕ, F, q). For $x \in E$ and $A \in \varepsilon$, we write

$$p(x,A) = K(x,A) + L(x,A),$$

where

$$K(x, A) = \int_0^{c(x)} \mathrm{e}^{-s} q(\phi(s, x), A) F(x, \mathrm{d}s)$$

and

$$L(x, A) = \int_0^{c(x)} \mathrm{e}^{-s} F(x, s) I_A(\phi(s, x)) \mathrm{d}s.$$

We denote by U the resolvent kernel associated with the PDMP χ, that is,

$$U(x, A) = E_x\left[\int_0^{+\infty} I_A(X_t) \mathrm{e}^{-t} \mathrm{d}t\right].$$

Lemma 5.6 *If the PDMP χ is regular, then the resolvent kernel U is the minimal nonnegative solution to*

$$KU(x, A) + L(x, A) = U(x, A),$$

where

$$KU(x, A) = \int_0^{c(x)} \mathrm{e}^{-s} F(x, \mathrm{d}s) \int_E q(\phi(s, x), \mathrm{d}y) U(y, A).$$

Proof If the PDMP χ is regular, then the random time sequence $\{\tau_n\}$ satisfies $\lim_{n \to \infty} \tau_n = +\infty$. Let σ be an exponential distributed random variable with rate 1 which is independent of the PDMP χ. We write

$$U_n(x, A) = P_x\{X_{\tau_n \Lambda \sigma} \in A\}, \quad U(x, A) = P_x\{X_\sigma \in A\}.$$

It is seen that

$$\lim_{n \to \infty} U_n(x, A) = \lim_{n \to \infty} P_x\{X_{\tau_n \Lambda \sigma} \in A\} = P_x\{X_\sigma \in A\} = U(x, A)$$

and

$$\begin{aligned}
U_{n+1}(x, A) &= P_x\{X_{\tau_{n+1} \Lambda \sigma} \in A\} \\
&= P_x\{X_{\tau_{n+1} \Lambda \sigma} \in A, \sigma < \tau_1\} + P_x\{X_{\tau_{n+1} \Lambda \sigma} \in A, \sigma \geqslant \tau_1\} \\
&= L(x, A) + E_x[P_x\{X_{\tau_{n+1} \Lambda \sigma} \in A \mid X_{\tau_1}\} I_{[\sigma \geqslant \tau_1]}] \\
&= L(x, A) + E_x[U_n(X_{\tau_1}, A) I_{[\sigma \geqslant \tau_1]}] \\
&= L(x, A) + \int_0^{c(x)} \mathrm{e}^{-s} F(x, \mathrm{d}s) \int_E q(\phi(s, x), \mathrm{d}y)\, U_n(y, A) \\
&= L(x, A) + KU_n(x, A).
\end{aligned}$$

We take

$$\begin{aligned}
U_0(x, A) &= 0, \\
U_1(x, A) &= L(x, A), \\
&\vdots \\
U_{n+1}(x, A) &= L(x, A) + KU_n(x, A), \quad n \geqslant 2.
\end{aligned}$$

It is clear that for any $x \in E$ and $A \in \varepsilon$, the kernel sequence $\{U_n(x, A)\}$ is monotonely increasing for $n \geqslant 0$. This indicates that the resolvent kernel $U = \lim_{n\to\infty} U_n$ is the minimal nonnegative solution to

$$KU(x, A) + L(x, A) = U(x, A).$$

This completes the proof.

For $x \in E$ and $A \in \varepsilon$, we write

$$R(x, A) = \frac{I_A(x)}{L(x, A)}$$

and

$$S(x, A) = L(x, A) I_A(x).$$

Lemma 5.7 *Let α be a σ-finite measure on (E, ε). Then the following two statements are equivalent.*

(1) $\alpha = cvR$*, where v is a probability measure and c is a positive constant.*

(2) $\alpha S(E) < +\infty$.

Proof (1) $\Rightarrow$ (2).

$$\begin{aligned}
\alpha S(E) &= \int_E \alpha(\mathrm{d}x) S(x, E) \\
&= c \int_E v(\mathrm{d}y) \int_E R(y, \mathrm{d}x) S(x, E) \\
&= c \int_E v(\mathrm{d}y) \int_E \frac{I_{\mathrm{d}x}(y)}{L(y, E)} L(x, E) \\
&= c < +\infty.
\end{aligned}$$

(2) $\Rightarrow$ (1).

We define the probability measure as

$$v = \frac{\alpha S}{\alpha S(E)}.$$

Hence we obtain that $\alpha = cvR$, where $c = \alpha S(E)$.

This completes the proof.

For $x \in E$ and $A \in \varepsilon$, we define

$$J(x, A) = \sum_{n=0}^{\infty} \int_A K^n(x, \mathrm{d}y) L(y, E).$$

Obviously, $J(x, E) = U(x, E) = 1$ if the PDMP χ is regular, hence J is a stochastic kernel.

The following theorem provides useful relations between the existence of a stationary distribution of the PDMP χ and the existence of a σ-finite invariant measure of the transition kernel p.

Theorem 5.18 *The following three statements are equivalent.*

(1) *There exists a stationary distribution* π *of the PDMP* χ.

(2) *There exists a probability measure* v *such that the positive* σ*-finite measure* $\mu = vR$ *is invariant for the transition kernel* p.

(3) *There exists a positive* σ*-finite measure* μ, *which is invariant for the transition kernel* p, *such that* $\mu S(E) < +\infty$.

Proof (1) $\Rightarrow$ (2).

Let π be a stationary distribution of the PDMP χ. Then $\pi U = \pi$. We define $v = \pi J$, which is clearly a probability measure, since $\pi J(E) = \pi U(E) = \pi(E) = 1$. Let $\mu = vR$. Then $\mu = \sum_{n=0}^{\infty} \pi K^n$. Note that $\mu L = \pi$ and $\mu K + \pi = \mu$, we obtain

$$\mu p = \mu(K + L) = \mu K + \pi = \mu.$$

(2) $\Rightarrow$ (1).

If there exists a probability measure v such that the positive σ-finite measure $\mu = vR$ is invariant for the transition kernel p, and $\pi = \mu L$, then for each $A \in \varepsilon$,

$$\pi(A) = \int_E \frac{L(y, A)}{L(y, E)} v(\mathrm{d}y)$$

is a probability measure. Since μ is a positive σ-finite measure, there exists a partition $\{E_i\}$ of E such that $\mu(E_i) < +\infty$. Note that $L = p - K$ and $\mu P = \mu$, we obtain that for each $A \in \varepsilon$,

$$\begin{aligned}
\pi U(A) &= \sum_i \int_{E_i} U(x, A) \mu L(\mathrm{d}x) \\
&= \sum_i \int_{E_i} U(x, A) \mu (I - K)(\mathrm{d}x) \\
&= \sum_i \int_{E_i} L(x, A) \mu(\mathrm{d}x) \\
&= \pi(A).
\end{aligned}$$

Therefore, π is the stationary distribution of the PDMP χ.

(2) $\Leftrightarrow$ (3) can be easily proved by means of Lemma 5.7.

This completes the proof.

Corollary 5.2 (1) *If* π *is a stationary distribution of the PDMP* χ, *then the positive* σ*-finite measure* πJR *is invariant for the transition kernel* p, *and* $\pi JRL = \pi$.

(2) *If* v *is a probability measure such that* vR *is invariant for the transition kernel* p, *then* v *is a stationary distribution* π *of the PDMP* χ, *and* $vRLJ = v$.

In what follows we provide an approach to express the stationary distribution of the PDMP $\mathcal{X}$.

Let Z_0 denote the initial state of $\mathcal{X}$, and Z_n the state of $\mathcal{X}$ after the nth jump. Then $\mathcal{Z} = \{Z_n : n \geqslant 0\}$ is called the Markov chain associated with $\mathcal{X}$. If $\mathcal{Z}$ has a stationary distribution π satisfying

$$\int_E \int_0^{t_*(x)} \mathrm{e}^{-\Lambda(x,t)} \mathrm{d}t\mathrm{d}\pi(x) < +\infty,$$

where $\Lambda(x,t) = \int_0^t \lambda(\phi(x,u))\mathrm{d}u$, then a stationary distribution of $\mathcal{X}$ can be constructed as follows: We define the set $M = \{(x,t) \in E \times \mathbb{R}_+ : t < t_*(x)\}$, and denote by $\mathcal{M}$ the Borel σ-algebra on M. For any set $A \in \varepsilon$ and measurable functions $t_1, t_2 : E \to \mathbb{R}_+$ with $t_1(x) < t_2(x) < t_*(x)$ for all $x \in E$, we write

$$B_A^{t_1,t_2} = \{(x,t) \in M : t_1(x) < t < t_2(x), x \in A\}$$

and

$$v_\pi\left(B_A^{t_1,t_2}\right) = \frac{\int_A \int_{t_1(x)}^{t_2(x)} \mathrm{e}^{-\Lambda(x,t)} \mathrm{d}t\mathrm{d}\pi(x)}{\int_E \int_0^{t_*(x)} \mathrm{e}^{-\Lambda(x,t)} \mathrm{d}t\mathrm{d}\pi(x)}.$$

Obviously, v_π can be uniquely extended to a measure on $\mathcal{M}$. Using the measurable restriction of the flow function $\phi : M \to E$ to the set M, the measure $v_\pi \phi^{-1}$ is the stationary distribution of $\mathcal{X}$.

5.8.6 The *GI*/*G*/*k* Queue

We consider a $GI/G/k$ queue. Arrivals occur independently with i.i.d. interarrival time distributed by A, and only single arrivals are allowed. In order to avoid multiple events (for example, arrivals and departures) occurring at the same time instant, we assume that A has a Lebesgue density a. There are k independent and identical servers in the queueing system, and each customer has identical service time distribution B in each server. The service discipline is FCFS and the capacity of the waiting room is infinite.

5.8.6.1 The State Change is Induced by a Service Event

Let $Q(t)$ and $J_i(t)$ be the queue length and the remaining service time of the ith server at time t for $1 \leqslant i \leqslant k$ and $t \geqslant 0$. Then the Markov process

$$\mathcal{I} = \left\{(Q(t), J_1(t), J_2(t), \ldots, J_k(t)) : t \geqslant 0\right\}$$

has the state space $E = \mathbb{N}_+ \times (\mathbb{R}_+)^k$. Specifically, if the *i*th server is idle, then $J_i(t) = x_i = 0$ for $1 \leqslant i \leqslant k$. For $y \in E$, we express $y = (n, x)$ with $x = (x_1, x_2, \dots, x_k)$.

We now describe the Markov process $\mathcal{I}$ as a PDMP as follows:

(1) The flow function A flow ϕ on E is defined by

$$\phi_t(n,t) = (n, (x_1 - t)^+, (x_2 - t)^+, \dots, (x_k - t)^+),$$

where $(x_i - t)^+ = \max\{0, x_i - t\}$. Obviously, this flow represents the proceeding service time.

(2) The first passage time We define

$$t_*(x) = \begin{cases} \min\{x_i : x_i > 0, 1 \leqslant i \leqslant k\}, & \text{if } x \neq 0, \\ +\infty, & \text{if } x = 0. \end{cases}$$

(3) The transition measure The transition measure Q_S describes the state changes of the system in the case of a server becoming idle. Let $x \in (\mathbb{R}_+)^k$ and

$$A = A_1 \times A_2 \times \dots \times A_k \in \mathcal{B}^k,$$

where $\mathcal{B}^k$ is the σ-algebra of the Borel sets on $(\mathbb{R}_+)^k$. Then

$$Q_S((n,x), \{m\} \times A) = \begin{cases} \delta_{m,n-1} \prod\limits_{\substack{j \neq i \\ x_i = 0}} 1_{A_j}(x_j) B(A_j), & \text{for } n \geqslant 1, \\ \delta_{m,n} 1_A(x), & \text{for } n = 0. \end{cases}$$

(4) The transition probability kernel For the case $n \geqslant 1$, only one server can become idle at a time. Since the queue has the Lebesgue-dominated single arrival input and the servers work independently, the probability that two servers finish their work (or an arrival and a service) at the same time instant is zero. Let $P(t; (n,x), \{m\} \times A)$ be the probability that at time t after the last arrival, the PDMP is in state set $\{m\} \times A$ under the condition that it was in state (n, x) immediately after the last arrival. Further, let $P^{(r)}(t; (n,x), \{m\} \times A)$ denote the same probability, but restricted to the set of paths with r service completions until time t. Then the transition probability kernel and hence the transient distribution of the interarrival process is given iteratively by

$$P(t; (n,x), \{m\} \times A) = \sum_{r=0}^{\infty} P^{(r)}(t; (n,x), \{m\} \times A),$$

where

$$P^{(0)}(t;(n,x),\{m\}\times A)=\begin{cases}\delta_{m,n}1_A(x-te), & \text{for } n\geqslant 1, t<t_*(x),\\ \delta_{m,0}1_A((x-te)^+), & \text{for } n=0,\\ 0, & \text{otherwise;}\end{cases}$$

and

$$P^{(i+1)}(t;(n,x),\{m\}\times A)=\begin{cases}\int_E P^{(i)}(t-t_*(x);(l,y),\{m\}\times A) & \\ \qquad \cdot Q_S((n,x),\mathrm{d}(l,y)), & t>t_*(x),\\ 0, & \text{otherwise.}\end{cases}$$

5.8.6.2 The State Change is Induced by an Arrival Event

We define another transition measure Q_A, which describes the state changes of the queueing process induced by an arrival event, as follows:

$$Q_A((n,x),\{m\}\times A)=\begin{cases}\delta_{m,n+1}1_A(x), & \text{if } x>0,\\ \delta_{m,n}\prod_{\substack{j\neq i\\ x_i=0}}1_{A_j}(x_j)B(A_j), & \text{there exists } x_i=0.\end{cases}$$

Then the transient distribution of the queueing process is given iteratively by

$$P(t;(n,x),\{m\}\times A)=\sum_{r=0}^{\infty}P^{(r)}(t;(n,x),\{m\}\times A),$$

where

$$P^{(0)}(t;(n,x),\{m\}\times A)=P^I(t;(n,x),\{m\}\times A)$$

which is an initial probability, and

$$\begin{aligned}P^{(i+1)}(t;(n,x),\{m\}\times A)=&\int_0^t\int_{(h,z)}P^{(i)}(t-u;(h,z),\{m\}\times A)\\ &\cdot\int_{(l,y)}P^I(u;(n,x),\mathrm{d}(l,y))Q_A((l,y),\mathrm{d}(h,z))a(u)\mathrm{d}u\end{aligned}$$

5.8.6.3 An Embedded Markov Chain of *GI*/*M*/1 Type

Let $\{t_l: l\geqslant 0\}$ denote the time instants of successive arrivals. Then $\{t_l: l\geqslant 0\}$ is a series of stopping times with respect to the canonical filtration of the queueing process $\{Q(t): t\geqslant 0\}$. We define $X_l=Q(t_l^-)$ as the system state immediately before the lth arrival. Then $X=\{X_l: l\geqslant 0\}$ is the embedded Markov chain immediately before arrival instants whose transition probabilities are defined by

$$P^X((n,x),\{m\}\times A)=P\{X_{l+1}\in\{m\}\times A\mid X_l=(n,x)\}.$$

We have

$$P^X((n,x),\{m\}\times A)=\int_E\int_0^{+\infty}Q_A((n,x),\mathrm{d}(l,y))P^I(t;(l,y),\{m\}\times A)a(t)\mathrm{d}t.$$

Let

$$\begin{aligned}A_i(x,A)&=P^X((n,x),\{n-i+1\}\times A)\\&=\int_0^{+\infty}P^I(t;(n+1,x),\{n-i+1\}\times A)a(t)\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}B_i(x,A)&=P^X((i,x),\{0\}\times A)\\&=\begin{cases}\displaystyle\int_0^{+\infty}P^I(t;(n+1,x),\{0\}\times A)a(t)\mathrm{d}t, & x>0,\\ \displaystyle\int_0^{+\infty}\int_0^{+\infty}P^I(t;(0,y),\{0\}\times A)\mathrm{d}B(y_i)a(t)\mathrm{d}t, & x_l=0,\end{cases}\end{aligned}$$

for all $n\geqslant i\geqslant 0$, $x\in(\mathbb{R}_+)^k$ and $A\in\mathcal{B}^k$. In this case, the transition probability kernel is given by

$$P^X=\begin{pmatrix}B_0 & A_0 & & & \\ B_1 & A_1 & A_0 & & \\ B_2 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

It is clear that

$$B_n(x,(\mathbb{R}_+)^k)+\sum_{i=n+1}^{\infty}A_i(x,(\mathbb{R}_+)^k)=1$$

for all $n\geqslant 0$ and $x\in(\mathbb{R}_+)^k$.

For the sequence $\{A_n(x,A),n\geqslant 0\}$ of probability kernels, let $\hat{A}(x,A)=\sum_{n=0}^{\infty}A_n(x,A)$. We assume that the Markov chain with transition probability kernel $\hat{A}(x,A)$ is Harris recurrent and ergodic, and let $\theta(A)$ denote the stationary probability of the the Markov chain with transition probability kernel $\hat{A}(x,A)$. Note that this kernel $\hat{A}(x,A)$ is equal to the transition probability kernel of the remaining service times immediately before arrival instants if there is always at least one waiting customer, the stationary distribution of the kernel $\hat{A}(x,A)$ equals the k-fold convolution of the respective stationary distribution $\gamma(A)$ for one server. Hence we obtain that $\theta([0,x))=\gamma^{k*}([0,x))$, where $\gamma([0,x))=\mu\int_0^x[1-B(v)]\mathrm{d}v$

with $\mu = 1/\int_0^{+\infty}[1-B(v)]\mathrm{d}v$. We define $\beta(x) = \sum_{n=1}^{\infty} nA_n\left(x,(\mathbb{R}_+)^k\right)$ for $x \in (\mathbb{R}_+)^k$. Then using the mean drift condition, we obtain that $\int_{(\mathbb{R}_+)^k} \beta(x)\mathrm{d}\theta(x) > 1$.

Since

$$\mathrm{d}\theta(x) = \prod_{i=1}^{k} \mu[1-B(x_i)]\mathrm{d}x_i,$$

we obtain

$$\begin{aligned}\int_{(\mathbb{R}_+)^k} \beta\left(x\right)\mathrm{d}\theta\left(x\right) &= \int_0^{+\infty}\int_{(\mathbb{R}_+)^k}\sum_{n=1}^{\infty} nP^I(t;(n,x),\{1\}\times(\mathbb{R}_+)^k) \\ &\quad \cdot\prod_{i=1}^{k}\mu[1-B(x_i)]\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_k a(t)\mathrm{d}t \\ &= k\mu\int_0^{+\infty} ta(t)\mathrm{d}t = \frac{k\mu}{\lambda},\end{aligned}$$

where $\lambda = 1/\int_0^{+\infty} ta(t)\mathrm{d}t$. Therefore, this queueing system is stable if and only if $\lambda < k\mu$.

Let $R(x,y)$ be the minimal nonnegative solution to the nonlinear kernel equation

$$R(x,y) = \sum_{n=0}^{\infty}\int_0^{+\infty} R^n(x,z)A_n(z,y)\mathrm{d}z.$$

We write

$$\varPsi_0(x,y) = \sum_{n=1}^{\infty}\int_0^{+\infty} R^{n-1}(x,z)B_{n-1}(z,y)\mathrm{d}z.$$

It is easy to check that $\varPsi_0(x,y)$ is the transition probability kernel of the censored chain to level 0. If $\lambda < k\mu$, then the stationary distribution of the embedded Markov chain with kernel P^X is given by

$$\pi_0(x) = \kappa w_0(x)$$

and

$$\pi_n(x) = \kappa\int_0^{+\infty} w_0(z)R^n(z,x)\mathrm{d}z, \quad n \geqslant 1,$$

where $w_0(x)$ is the stationary probability of the Markov chain with transition probability kernel $\varPsi_0(x,y)$ and

$$\kappa = \frac{1}{1+\sum_{n=1}^{\infty}\int_0^{+\infty}\int_0^{+\infty} w_0(x)R^n(z,x)\mathrm{d}z\mathrm{d}x}.$$

Now, we compute the stationary distribution $p_n(A)$ of the queueing process. To do this, we write

$$A^c(t) = 1 - \int_0^t a(u)\mathrm{d}u,$$

$$\begin{aligned} K(t;(n,x),\{m\}\times A) &= P\{Q(t)\in\{m\}\times A, H > t \mid Q(0)=(n,x)\} \\ &= A^c(t)P^I(t;(n,x),\{m\}\times A), \end{aligned}$$

where H denotes the first inter-arrival time. Therefore, we obtain

$$p_m(A) = \lambda\sum_{n=m}^{\infty}\int_{(\mathbb{R}_+)^k}\int_0^{+\infty}\pi_n(x)K(t;(n,x),\{m\}\times A)\mathrm{d}x\mathrm{d}t$$

for $m \geqslant 0$ and $A\in B^k$.

5.9 Notes in the Literature

Early results on discrete-time Markov chains on continuous state space were well documented in Finch [12], Athreya and Ney [3,4], Meyn and Tweedie [22], Hernández-Lerma and Lasserre [15] and Prieto-Rumeau and Hernández-Lerma [25]. As an important example, Tweedie [27] considered a discrete-time Markov chain of $GI/M/1$ type with continuous phase variable. He showed that the stationary probability distribution is operator-geometric. Sengupta [26] used the operator-geometric solution to study the stationary buffer occupancy distribution in a data communication model. Nielsen and Ramaswami [24] studied orthonormal expansion for a discrete-time level-independent QBD process with continuous phase variable under appropriate and regular conditions, which lead to a computational framework in order to implement Tweedie's operator-geometric solution. Breuer [5] described the infinitesimal generator for a Markov jump process on continuous state space, and provided the Kolmogorov's forward and backward differential equations.

Available results for the continuous-time Markov chains on continuous state space are few. From the standard theory of Markov chains (e.g. Anderson [1] and Kemeny, Snell and Knapp [17]), it is well known that the study of the continuous-time Markov chains is different from that of discrete-time Markov chains, although both of them can be related by using the uniformization technique. However, it is possible that the conditions for the uniformization technique cannot be satisfied by some practical continuous-time Markov chains, e.g., see Artalejo and Gómez-Corral [2] for some simple retrial queues. Li and Lin [18] provides a new theoretical framework for studying the continuous-time QBD process with continuous phase variable. Finally, this chapter introduces the PDMPs which enables us to deal with more general queueing systems such as the $GI/G/c$ queue. For the PDMPs, reader may refer to Davis [9], Gugerli [14], Costa [8], Dempster [10], Liu [20,21], Dufour and Costa [11] and Hou and Liu [16]. At the same time, Breuer [6,7] used the PDMPs to deal with some general queueing systems.

This chapter is written by means of Tweedie [27], Nielsen and Ramaswami [24], Li and Lin [18], Davis [9], Costa [8], Hou and Liu [16] and Breuer [6, 7].

Problems

5.1 Compute the stationary distribution of the queue length for the $G/SM/1$ queue, where the interarrival time distribution is denoted by $F(x)$, and the service times form a semi-Markov process with the transition probability matrix $G(x)$ of size k.

5.2 Compute the stationary distribution of the queue length for the $SM/G/1$ queue, where the arrival process is a semi-Markov process with the transition probability matrix $F(x)$ of size k, and the service time distribution is denoted by $G(x)$.

5.3 Consider the $SM/CPH/1$ queue by means of the Markov chain of $GI/M/1$ type on a continuous state space.

5.4 Discuss the $CMAP/G/1$ queue by means of the Markov chain of $M/G/1$ type on a continuous state space.

5.5 Study the $CMAP/CPH/1$ queue by means of the continuous-time QBD process.

5.6 For a fluid queue driven by the $GI/G/1$ queue, please apply the continuous-time Markov chain on continuous state space to analyze this fluid model.

5.7 Construct a more general fluid queue driven by a continuous-time Markov chain on continuous state space.

5.8 For the $CMAP/CPH/1$ queue, apply the orthogonal algorithm to compute the means of the stationary queue length and the stationary waiting time.

5.9 Define a batch Markovian process with continuous phase variable, and then discuss its useful properties.

5.10 Use the PDMP to study the $M/G/c$ retrial queue, and derive its stationary distributions for the queue length and the waiting time.

5.11 Use the PDMP to study the $G/G/1$ queue with server vacations whose time distribution is general, and derive its stationary distributions for the queue length and the waiting time.

5.12 Use the PDMP to study the $G/G/1$ queue with negative customers whose interarrival time distribution is general, and derive its stationary distributions for the queue length and the waiting time.

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6 Block-Structured Markov Renewal Processes

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Abstract In this chapter, we provide the UL- and LU-types of *RG*-factorizations for the transition probability mass matrix of any irreducible Markov renewal process in terms of the censoring technique. Specifically, we deal with Markov renewal processes of *GI*/*G*/1 type, including the *RG*-factorization, the *RG*-factorization for the repeated blocks, the spectral analysis and the first passage time.

Keywords Markov renewal process, Markov renewal processes of *GI*/*G*/1 type, *RG*-factorization, spectral analysis, the first passage time.

This chapter provides the UL- and LU-types of *RG*-factorizations for the transition probability mass matrix of an irreducible Markov renewal process in terms of the Wiener-Hopf equations. Specifically, Markov renewal processes of *GI*/*G*/1 type are dealt with for such as, the *RG*-factorization for the repeated blocks, the spectral analysis and the first passage time. Note that Markov renewal process is a useful mathematical tool in the study of non-Markovian stochastic models.

This chapter is organized as follows. Section 6.1 discusses the censoring Markov renewal processes for block-structured Markov renewal processes. Sections 6.2 and 6.3 derive the UL- and LU-types *RG*-factorizations for the transition probability mass matrix based on the Wiener-Hopf equations, respectively. Section 6.4 deals with block-structured Markov renewal processes with finitely-many levels. Section 6.5 studies Markov renewal processes of *GI*/*G*/1 type. Section 6.6 considers spectral properties for the *R*- and *G*-measures. Section 6.7 analyzes the first passage times with effective algorithms, and also provides conditions for the state classification of Markov renewal processes of *GI*/*G*/1 type in terms of the *R*-, *U*- and *G*-measures. Finally, Section 6.8 gives some notes to the references on Markov renewal processes.

6.1 The Censoring Markov Renewal Processes

In this section, the censoring technique is applied to deal with an irreducible block-structured Markov renewal process. Based on the censored processes, conditions on the state classification of the Markov renewal process are provided.

We consider a Markov renewal process $\{(X_n, T_n), n \geqslant 0\}$ on the state space $\Omega \times [0, +\infty)$ with $\Omega = \{(k, j) : k \geqslant 0, 1 \leqslant j \leqslant m_k\}$, where X_n is the state of the process at the nth renewal epoch and T_n is the total renewal time up to the nth renewal, or $T_n = \sum_{i=0}^{n} \tau_i$ with $\tau_0 = 0$ and τ_n being the inter-renewal interval time between the $(n-1)$st and the nth renewal epochs for $n \geqslant 1$. The transition probability mass matrix of the Markov renewal process $\{(X_n, T_n), n \geqslant 0\}$ is given by

$$P(x) = \begin{pmatrix} P_{0,0}(x) & P_{0,1}(x) & P_{0,2}(x) & \cdots \\ P_{1,0}(x) & P_{1,1}(x) & P_{1,2}(x) & \cdots \\ P_{2,0}(x) & P_{2,1}(x) & P_{2,2}(x) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \tag{6.1}$$

where $P_{i,j}(x)$ is a matrix of size $m_i \times m_j$ whose (r, r')th entry is

$$(P_{ij}(x))_{r,r'} = P\{X_{n+1} = (j, r'), T_{n+1} \leqslant x + T_n \mid X_n = (i, r), T_n\}.$$

The matrix $P(+\infty)$, defined as $\lim_{x \to +\infty} P(x)$ entry-wisely, is referred to as the embedded Markov chain of the Markov renewal process. Throughout this chapter, we assume that the Markov renewal process $P(x)$ is irreducible and $P(x)e \lneqq e$ for all $x \geqslant 0$, Where e is a column vector of ones with suitable size. Because of the block-partitioned structure of $P(x)$, the Markov renewal process $\{(X_n, T_n), n \geqslant 0\}$ is referred to as a block-structured Markov renewal process. Many application problems can be naturally modeled as a block-structured Markov renewal process.

We first define a censored process for a Markov renewal process whose transition probability mass matrix consists of scalar entries. We then treat a block-structured Markov renewal process as a special case.

Definition 6.1 *Suppose that* $\{(X_n, T_n), n \geqslant 0\}$ *is an irreducible Markov renewal process on the state space* $\Omega \times [0, +\infty)$, *where* $X_n \in \Omega = \{0, 1, 2, ...\}$ *and* $T_n \in [0, +\infty)$. *Let E be a non-empty subset of* Ω. *If the successive visits of* X_n *to the subset E take place at the* n_k*th step of state transition, then the inter-visit time* τ_k^E *between the* $(k-1)$*st and the* k*th visits to* E *is given by* $\tau_k^E = \tau_{n_{k-1}+1} + \tau_{n_{k-1}+2} + \ldots + \tau_{n_k}$ *for* $k \geqslant 1$. *Let* $X_k^E = X_{n_k}$ *and* $T_k^E = \sum_{i=1}^{k} \tau_i^E$ *for* $k \geqslant 1$. *Then the* *sequence* $\left\{\left(X_k^E, T_k^E\right), k \geqslant 1\right\}$ *is called the censored process with censoring set E.*

Throughout this chapter, we denote by $(B)_{r,r'}$ the (r,r')th entry of the matrix B and by $B*C(x)$ (or $B(x)*C(x)$) the convolution of two matrix functions $B(x)$ and $C(x)$, i.e., $B*C(x)=\int_0^x B(x-u)\mathrm{d}C(u)$. We then recursively define $B^{n^*}(x)=B*B^{(n-1)^*}(x)$ for $n \geqslant 1$ with $B^{0^*}(x)=I$ where I is the identity matrix.

For convenience of description, we write $P^{[\leqslant n]}(x)$ for the censored transition probability mass matrix $P^E(x)$ if the censored set $E=L_{\leqslant n}$, in particular, $P^{[\leqslant +\infty]}(x)=P(x)$ and $P^{[0]}(x)=P^{[\leqslant 0]}(x)$. On the other hand, we write $P^{[\geqslant n]}(x)$ for the censored transition probability mass matrix with the censored set $E=L_{\geq n}$, specifically, $P^{[\geqslant 0]}(x)=P(x)$.

Let $E^c=\Omega-E$. According to the subsets E and E^c, the transition probability mass matrix $P(x)$ is partitioned as

$$P(x)=\begin{array}{c c} & \begin{array}{cc} E & E^c \end{array} \\ \begin{array}{c} E \\ E^c \end{array} & \begin{pmatrix} T(x) & U(x) \\ V(x) & Q(x) \end{pmatrix} \end{array}. \tag{6.2}$$

Lemma 6.1 *If* $P(x)$ *is irreducible, then each element of* $\widehat{Q}(x)=\sum_{n=0}^{\infty} Q^{n^*}(x)$ *is finite for* $x \geqslant 0$.

Proof If $P(x)$ is irreducible, then $P(+\infty)$ is irreducible, since $0 \leqslant P(x) \leqslant P(+\infty)$. It is obvious that $Q(+\infty)$ is strictly substochastic due to $V(+\infty) \gneqq 0$. Hence, using Lemma 2.1 we have

$$\widehat{Q}(+\infty)=\sum_{n=0}^{\infty} Q^n(+\infty)=[I-Q(+\infty)]^{-1}<+\infty,$$

where $[I-Q(+\infty)]^{-1}$ is the minimal nonnegative inverse of $I-Q(+\infty)$. Since each element of $\widehat{Q}(+\infty)$ is finite and $0 \leqslant \widehat{Q}(x) \leqslant \widehat{Q}(+\infty)$ for $x>0$, each element of $\widehat{Q}(x)$ is finite. This completes the proof.

The matrix $\widehat{Q}(x)$ is referred to as the fundamental matrix of $Q(x)$. In the following, we show that the censored process $\{(X_k^E, T_k^E), k \geqslant 1\}$ is a Markov renewal process again.

Theorem 6.1 *The censored process* $\{(X_k^E, T_k^E), k \geqslant 1\}$ *is a Markov renewal process whose transition probability mass matrix is given by*

$$P^E(x)=T(x)+U*\widehat{Q}*V(x). \tag{6.3}$$

Proof To show that $\{(X_k^E, T_k^E), k \geqslant 1\}$ is a Markov renewal process, we need to show that the T_{n+1}^E is independent of X_0^E, X_1^E, …, X_{n-1}^E, T_0^E, T_1^E, …, T_n^E, given the state of X_n^E. This is clear from the fact that $\{(X_n, T_n), n \geqslant 0\}$ is a Markov renewal process with the strong Markov property. The (i,j)th entry of the transition probability mass matrix of the Markov renewal process $\{(X_k^E, T_k^E), k \geqslant 1\}$ is

$$\begin{aligned}(P^E(x))_{i,j} &= P\left\{X_{n+1}^E = j, T_{n+1}^E \leqslant x + T_n^E \mid X_n^E = i, T_n^E\right\}\\ &= P\left\{X_1^E = j, T_1^E \leqslant x \mid X_0^E = i, T_0^E = 0\right\}.\end{aligned}$$

To explicitly express $P^E(x)$ in terms of the original transition probability mass matrix, we consider the following two possible cases:

Case Ⅰ $n_1 = 1$. In this case, $i, j \in E$, $X_1^E = X_1$, $T_1^E = \tau_1$ and

$$P\{X_1^E = j, T_1^E \leqslant x \mid X_0^E = i, T_0^E = 0\} = (T(x))_{i,j}. \tag{6.4}$$

Case Ⅱ $n_1 = k$ for $k \geqslant 2$. In this case, $i, j \in E$, $X_1^E = X_k$, $T_1^E = \sum_{l=1}^{k} \tau_l$ and

$$\begin{aligned}&P\{X_1^E = j, T_1^E \leqslant x \mid X_0^E = i, T_0^E = 0\} = P\{X_k = j, X_j \notin E\\ &\text{for } j = 1, 2, \ldots, k-1, \sum_{l=1}^{k} \tau_l \leqslant x \mid X_0 = i, \tau_0 = 0\}\\ &\quad = (U * Q^{(k-2)*} * V(x))_{i,j}.\end{aligned} \tag{6.5}$$

It follows from Eq. (6.4) and Eq. (6.5) that

$$\begin{aligned}P\{X_1^E = j, T_1^E \leqslant x \mid X_0^E = i, T_0^E = 0\} &= (T(x))_{i,j} + \sum_{k=2}^{\infty} (U * Q^{(k-2)*} * V(x))_{i,j}\\ &= (T(x))_{i,j} + (U * \widehat{Q} * V(x))_{i,j}.\end{aligned}$$

This completes the proof.

Remark 6.1 *The censored process* $\left\{\left(X_k^{E^c}, T_k^{E^c}\right), k \geqslant 1\right\}$ *is a Markov renewal process whose transition probability mass matrix is given by*

$$P^{E^c}(x) = Q(x) + V * \widehat{T} * U(x).$$

As seen in Chapter 2, the UL- and LU-types of RG-factorizations are obtained by means of the two different censored processes $P^E(x)$ *and* $P^{E^c}(x)$, *respectively.*

Based on the censored renewal processes above, a probabilistic interpretation for each component in the expression Eq. (6.3) for $P^E(x)$ is available and useful. For the Markov renewal process $P(x)$, let $T_{E^c, E^c}(i, j)$ be total renewal time until the process visits state $j \in E^c$ for the last time before entering E, given that the process starts in state $i \in E^c$. Formally, assume that at the kth transition the process visits state $j \in E^c$ for the last time before entering E, given that the process starts in state $i \in E^c$. Then $T_{E^c, E^c}(i,j) = \sum_{l=1}^{k} \tau_l$. Similarly, let $T_{E,E^c}(i,j)$ be the total renewal time until the process visits state $j \in E^c$ before returning to E, given that the process starts in state $i \in E$; $T_{E^c, E}(i,j)$ the total renewal time until the process enters E and upon entering E the first state visited is $j \in E$, given that the process started at state $i \in E^c$; and $T_{E,E}(i,j)$ the total renewal time until the process enters E and upon returning to E the first state visited is $j \in E$, given that the process started at state $i \in E$.

(1) $(\widehat{Q}(x))_{i,j}$ is the expected number of visits to state $j \in E^c$ before entering E and $T_{E^c,E^c}(i,j) \leqslant x$, given that the process starts in state $i \in E^c$.

(2) $(U * \widehat{Q}(x))_{i,j}$ is the expected number of visits to state $j \in E^c$ before returning to E and $T_{E,E^c}(i,j) \leqslant x$, given that the process starts in state $i \in E$.

(3) $(\widehat{Q} * V(x))_{i,j}$ is the probability that the process enters E and upon entering E the first state visited is $j \in E$ and $T_{E^c,E}(i,j) \leqslant x$, given that the process starts in state $i \in E^c$.

(4) $(U * \widehat{Q} * V(x))_{i,j}$ is the probability that upon returning to E the first state visited is $j \in E$ and $T_{E,E}(i,j) \leqslant x$, given that the process starts in state $i \in E$.

Define the double transformation of n and x for the censored Markov renewal process as

$$\widetilde{P^E}^*(z,s) = \left(\widetilde{P^E}^*_{i,j}(z,s)\right)_{i,j \in E},$$

where

$$\widetilde{P^E}^*_{i,j}(z,s) = \sum_{n=1}^{\infty} z^n \int_0^{+\infty} \mathrm{e}^{-sx} \mathrm{d}P\{X_n = j, T_n \leqslant x \mid X_0 = i, T_0 = 0\}.$$

The single transformations $\widetilde{T}(s)$, $\widetilde{U}(s)$, $\widetilde{V}(s)$ and $\widetilde{Q}(s)$ are defined conventionally, for example, $\widetilde{T}(s) = \int_0^{+\infty} \mathrm{e}^{-sx} \mathrm{d}T(x)$.

The following corollary provides a useful result for studying the two-dimensional random vector (X_1^E, T_1^E), the proof of which is obvious from Eq. (6.4) and Eq. (6.5). (X_1^E, T_1^E) is important for the study of the Markov renewal process $\{(X_n, T_n), n \geqslant 1\}$. It is worthwhile to notice that an important example is analyzed in Section 2.4 of Neuts [17].

Corollary 6.1

$$\widetilde{P^E}^*(z,s) = z\widetilde{T}(s) + z^2\widetilde{U}(s)[I - z\widetilde{Q}(s)]^{-1}\widetilde{V}(s),$$

where

$$[I - z\widetilde{Q}(s)]^{-1} = \sum_{n=0}^{\infty} z^n [\widetilde{Q}(s)]^n.$$

Based on Definition 6.1, we have the following two useful properties.

Property 6.1 For $E_1 \subset E_2$, $P^{E_1}(x) = (P^{E_2})^{E_1}(x)$.

Property 6.2 $P(x)$ is irreducible if and only if $P^E(x)$ is irreducible for all the subsets E of Ω.

Now, we consider the state classification for an irreducible Markov renewal process. Çinlar [6] shows that $P(x)$ is recurrent or transient if and only if $P(+\infty)$ is recurrent or transient, respectively. Therefore, we illustrate the following useful relations.

Proposition 6.1 (1) $P(x)$ *is recurrent if and only if* $P^E(x)$ *is recurrent for every subset* $E \subset \Omega$.

(2) $P(x)$ *is transient if and only if* $P^E(x)$ *is transient for every subset* $E \subset \Omega$.

Proposition 6.2 *If* $P(x)$ *is irreducible, then*

(1) $P(x)$ *is recurrent if and only if* $P^E(x)$ *is recurrent for some subset* $E \subset \Omega$*; and*

(2) $P(x)$ *is transient if and only if* $P^E(x)$ *is transient for some subset* $E \subset \Omega$.

The following proposition provides a sufficient condition under which a Markov renewal process $P(x)$ is positive recurrent. The proof is easy according to the fact that $\sum_{j=0}^{\infty} \int_0^{+\infty} x \mathrm{d}P_{i,j}(x)e$ is the mean total sojourn time of the Markov renewal process $P(x)$ in state i. It is worthwhile to note that $P(x)$ may not be positive recurrent when $P(+\infty)$ is positive recurrent. This is a further result of Proposition 6.2 from the recurrent to the positive recurrent.

Proposition 6.3 *The Markov renewal process* $P(x)$ *is positive recurrent if*

(1) $P(+\infty)$ *is positive recurrent and*

(2) $\sum_{j=0}^{\infty} \int_0^{+\infty} x \mathrm{d}P_{i,j}(x)e$ *is finite for all* $i \geqslant 0$.

Remark 6.2 (1) *Remark b in Section 3.2 of Neuts [17] (p. 140) illustrates that condition* (2) *in Proposition 6.3 is strong. For example, for a Markov renewal process of* $M/G/1$ *type, the sufficient condition only requires that* $\sum_{j=0}^{\infty} \int_0^{+\infty} x \mathrm{d}P_{i,j}(x)e$ *for* $i = 0$ *and* $\int_0^{+\infty} x \mathrm{d}P_{1,0}(x)$ *are finite. Therefore, condition* (2) *in Proposition 6.3 can further be weakened.*

6.2 The UL-Type *RG*-Factorization

In this section, we define the UL-type *R*-, *U*- and *G*-measures for the Markov renewal process, and derive the UL-type *RG*-factorization for the transition probability mass matrix.

For $0 \leqslant i < j$, $R_{i,j}(k,x)$ is an $m_i \times m_j$ matrix whose (r,r')th entry $(R_{i,j}(x))_{r,r'}$ is the probability that starting in state (i,r) at time 0, the Markov renewal process makes its kth transition in the renewal time interval $[0,x]$ for a visit into state (j,r') without visiting any states in $L_{\leqslant(j-1)}$ during intermediate steps; or

$$(R_{i,j}(k,x))_{r,r'} = P\{X_k = (j,r'), X_l \notin L_{\leqslant(j-1)} \text{ for } l = 1, 2, \ldots, k-1, \\ T_k \leqslant x \mid X_0 = (i,r)\}. \tag{6.6}$$

Let $R_{i,j}(x) = \sum_{k=1}^{\infty} R_{i,j}(k,x)$. Then the (r,r')th entry of $R_{i,j}(x)$ is the expected number of visits to state (j,r') made in the renewal time interval $[0,x]$ without visiting any states in $L_{\leqslant(j-1)}$ during intermediate steps, given that the process starts in state (i,r) at time 0.

For $0 \leqslant j < i$, $G_{i,j}(k,x)$ is an $m_i \times m_j$ matrix whose (r,r')th entry $(G_{i,j}(x))_{r,r'}$ is the probability that starting in state (i,r) at time 0, the Markov renewal process makes its kth transition in the renewal time interval $[0,x]$ for a visit into state (j,r') without visiting any states in $L_{\leqslant(i-1)}$ during intermediate steps; or

$$(G_{i,j}(k,x))_{r,r'} = P\{X_k = (j,r'), X_l \notin L_{\leqslant(i-1)} \text{ for } l = 1, 2, \ldots, k-1, \\ T_k \leqslant x \mid X_0 = (i,r)\}. \tag{6.7}$$

Let $G_{i,j}(x) = \sum_{k=1}^{\infty} G_{i,j}(k,x)$. Then the (r,r')th entry of $G_{i,j}(x)$ is the probability that starting in state (i,r) at time 0, the Markov renewal process makes its first visit into $L_{\leqslant(i-1)}$ in the renewal time interval $[0,x]$ and upon entering $L_{\leqslant(i-1)}$ it visits state (j,r').

The two matrix sequences $\{R_{i,j}(x)\}$ and $\{G_{i,j}(x)\}$ are called the UL-type *R*- and *G*-measures of the Markov renewal process $P(x)$, respectively.

We partition the transition probability mass matrix $P(x)$ according to the three subsets $L_{\leqslant(n-1)}$, L_n and $L_{\geqslant(n+1)}$ as

$$P(x) = \begin{pmatrix} T(x) & U_0(x) & U_1(x) \\ V_0(x) & T_0(x) & U_2(x) \\ V_1(x) & V_2(x) & Q_0(x) \end{pmatrix}. \tag{6.8}$$

Let

$$Q(x)=\begin{pmatrix} T_0(x) & U_2(x) \\ V_2(x) & Q_0(x) \end{pmatrix} \tag{6.9}$$

and

$$Q^{n*}(x)=\begin{pmatrix} D_{11}(n,x) & D_{12}(n,x) \\ D_{21}(n,x) & D_{22}(n,x) \end{pmatrix}, \quad n \geqslant 0. \tag{6.10}$$

Partition $\widehat{Q}(x)=\sum_{n=0}^{\infty} Q^{n*}(x)$ accordingly as

$$\widehat{Q}(x)=\begin{pmatrix} H_{11}(x) & H_{12}(x) \\ H_{21}(x) & H_{22}(x) \end{pmatrix}. \tag{6.11}$$

We write

$$R_{<n}(x)=(R_{0,n}^{\mathrm{T}}(x), R_{1,n}^{\mathrm{T}}(x), R_{2,n}^{\mathrm{T}}(x), \ldots, R_{n-1,n}^{\mathrm{T}}(x))^{\mathrm{T}} \tag{6.12}$$

and

$$G_{<n}(x)=(G_{n,0}(x), G_{n,1}(x), G_{n,2}(x), \ldots, G_{n,n-1}(x)). \tag{6.13}$$

The following lemma provides expressions for the matrices $R_{<n}(x)$ and $G_{<n}(x)$.

Lemma 6.2 *For $x>0$ and $n \geqslant 1$,*

$$R_{<n}(x)=U_0 * H_{11}(x)+U_1 * H_{21}(x) \tag{6.14}$$

and

$$G_{<n}(x)=H_{11} * V_0(x)+H_{12} * V_1(x). \tag{6.15}$$

Proof We only prove Eq. (6.14) while Eq. (6.15) can be proved similarly.

For $0 \leqslant i \leqslant n-1$, we consider two possible cases for $(R_{i,n}(k,x))_{r,r'}$ as follows:

Case Ⅰ $k=1$. In this case,

$$\begin{aligned}(R_{i,n}(k,x))_{r,r'} &= P\{X_1=(n,r'), T_1 \leqslant x \mid X_0=(i,r), T_0=0\} \\ &= (U_0(x))_{r,r'}^{\langle i \rangle}.\end{aligned} \tag{6.16}$$

Case Ⅱ $k \geqslant 2$. In this case,

$$\begin{aligned}(R_{i,n}(k,x))_{r,r'} = P\{&X_k=(n,r'), X_l \notin L_{\leqslant(j-1)} \text{ for } l=1,2,\ldots,k-1, \\ &T_k \leqslant x \mid X_0=(i,r), T_0=0\} \\ = (U_0 &* D_{11}(k-1,x)+U_1 * D_{21}(k-1,x))_{r,r'}^{\langle i \rangle}.\end{aligned} \tag{6.17}$$

Noting that $D_{11}(0,x)=I$ and $D_{21}(0,x)=0$, it follows from Eq. (6.16) and Eq. (6.17) that

$$\begin{aligned}(R_{i,n}(x))_{r,r'} &= \sum_{k=1}^{\infty}(R_{i,n}(k,x))_{r,r'}\\ &= (U_0(x))_{r,r'}^{\langle i\rangle} + \sum_{k=2}^{\infty}(U_0 * D_{11}(k-1,x) + U_1 * D_{21}(k-1,x))_{r,r'}^{\langle i\rangle}\\ &= \left(\sum_{k=0}^{\infty}(U_0 * D_{11}(k,x) + U_1 * D_{21}(k,x))\right)_{r,r'}^{\langle i\rangle}\\ &= (U_0 * H_{11}(x) + U_1 * H_{21}(x))_{r,r'}^{\langle i\rangle}.\end{aligned}$$

This completes the proof.

For the Markov renewal process $P(x)$, let $Q_n(x)$ be the southeast corner of $P(x)$ beginning from level n, i.e., $Q_n(x)=(P_{i,j}(x))_{i,j\geqslant n}$. Let $\widehat{Q}_n(x)=\sum_{k=0}^{\infty}Q_n^{k*}(x)$, and $\widehat{Q}_n^{(k,\cdot)}(x)$ and $\widehat{Q}_n^{(\cdot,l)}(x)$ be the kth block-row and the lth block-column of $\widehat{Q}_n(x)$, respectively. The following corollary easily follows from Lemma 6.2.

Corollary 6.2 *For* $0\leqslant i<j$,

$$R_{i,j}(x)=(P_{i,j}(x), P_{i,j+1}(x), P_{i,j+2}(x), \ldots) * \widehat{Q}_j^{(\cdot,1)}(x) \tag{6.18}$$

and for $0\leqslant j<i$,

$$G_{i,j}(x)=\widehat{Q}_i^{(1,\cdot)}(x) * \left(P_{i,j}(x)^{\mathrm{T}}, P_{i+1,j}(x)^{\mathrm{T}}, P_{i+2,j}(x)^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}. \tag{6.19}$$

It follows from Eq. (6.9) that

$$\widetilde{Q}(s)=\begin{pmatrix}\widetilde{T}_0(s) & \widetilde{U}_2(s)\\ \widetilde{V}_2(s) & \widetilde{Q}_0(s)\end{pmatrix}$$

and from Eq. (6.10) that

$$\widetilde{\widetilde{Q}}(s)=\begin{pmatrix}\widetilde{H}_{11}(s) & \widetilde{H}_{12}(s)\\ \widetilde{H}_{21}(s) & \widetilde{H}_{22}(s)\end{pmatrix}.$$

From either Lemma 6.2 or Corollary 6.2 it is clear that the determination of the R- and G-measures relies on the entries of the fundamental matrix $\widehat{Q}_k(x)$. The following lemma provides a formula for expressing the transformation of the fundamental matrix.

Lemma 6.3 *For* $\mathrm{Re}(s)\geqslant 0$,

$$\widetilde{H}_{11}(s)=\left[I-\widetilde{T}_0(s)-\widetilde{U}_2(s)\widetilde{\widehat{Q}}_0(s)\widetilde{V}_2(s)\right]^{-1},$$

$$\widetilde{H}_{12}(s)=\left[I-\widetilde{T}_0(s)-\widetilde{U}_2(s)\widetilde{\widehat{Q}}_0(s)\widetilde{V}_2(s)\right]^{-1}\widetilde{U}_2(s)\widetilde{\widehat{Q}}_0(s),$$

$$\widetilde{H}_{21}(s)=\widetilde{\widehat{Q}}_0(s)\widetilde{V}_2(s)\left[I-\widetilde{T}_0(s)-\widetilde{U}_2(s)\widetilde{\widehat{Q}}_0(s)\widetilde{V}_2(s)\right]^{-1},$$

$$\widetilde{H}_{22}(s)=\widetilde{\widehat{Q}}_0(s)+\widetilde{\widehat{Q}}_0(s)\widetilde{V}_2(s)\left[I-\widetilde{T}_0(s)-\widetilde{U}_2(s)\widetilde{\widehat{Q}}_0(s)\widetilde{V}_2(s)\right]^{-1}\widetilde{U}_2(s)\widetilde{\widehat{Q}}_0(s).$$

Symmetrically,

$$\widetilde{H}_{11}(s)=\widetilde{\widehat{T}}_0(s)+\widetilde{\widehat{T}}_0(s)\widetilde{U}_2(s)\left[I-\widetilde{Q}_0(s)-\widetilde{V}_2(s)\widetilde{\widehat{T}}_0(s)\widetilde{U}_2(s)\right]^{-1}\widetilde{V}_2(s)\widetilde{\widehat{T}}_0(s),$$

$$\widetilde{H}_{12}(s)=\widetilde{\widehat{T}}_0(s)\widetilde{U}_2(s)\left[I-\widetilde{Q}_0(s)-\widetilde{V}_2(s)\widetilde{\widehat{T}}_0(s)\widetilde{U}_2(s)\right]^{-1},$$

$$\widetilde{H}_{21}(s)=\left[I-\widetilde{Q}_0(s)-\widetilde{V}_2(s)\widetilde{\widehat{T}}_0(s)\widetilde{U}_2(s)\right]^{-1}\widetilde{V}_2(s)\widetilde{\widehat{T}}_0(s),$$

$$\widetilde{H}_{22}(s)=I-\widetilde{Q}_0(s)-\widetilde{V}_2(s)\widetilde{\widehat{T}}_0(s)\widetilde{U}_2(s).$$

Theorem 6.2 *For* $x\geqslant 0$ *and* $n\geqslant 1$,

$$R_{<n}(x)=\left[U_0(x)+U_1*\widehat{Q}_0*V_2(x)\right]*\sum_{l=0}^{\infty}\sum_{k=0}^{l}\binom{l}{k}T_0^{k*}(x)*\left[U_2*\widehat{Q}_0*V_2(x)\right]^{(l-k)*}$$

and

$$G_{<n}(x)=\sum_{l=0}^{\infty}\sum_{k=0}^{l}\binom{l}{k}T_0^{k*}(x)*\left[U_2*\widehat{Q}_0*V_2(x)\right]^{(l-k)*}*\left[V_0(x)+U_2*\widehat{Q}_0*V_1(x)\right].$$

Proof It follows from Lemmas 6.2 and 6.3 that

$$\widetilde{R}_{<n}(s)=\left[\widetilde{U}_0(s)+\widetilde{U}_1(s)\widetilde{\widehat{Q}}_0(s)\widetilde{V}_2(s)\right]\left[I-\widetilde{T}_0(s)-\widetilde{U}_2(s)\widetilde{\widehat{Q}}_0(s)\widetilde{V}_2(s)\right]^{-1}$$

and

$$\widetilde{Q}_{<n}(s)=\left[I-\widetilde{T}_0(s)-\widetilde{U}_2(s)\widetilde{\widehat{Q}}_0(s)\widetilde{V}_2(s)\right]^{-1}\left[\widetilde{V}_0(s)+\widetilde{U}_2(s)\widetilde{\widehat{Q}}_0(s)\widetilde{V}_1(s)\right].$$

The inverse transform for the above two equations immediately leads to the desired result.

The following theorem provides a censoring invariance for the R- and G-measures of Markov renewal processes. We denote by $R_{i,j}^{[\leqslant n]}(x)$ and $G_{i,j}^{[\leqslant n]}(x)$ the R- and G-measures of the censored Markov renewal process $P^{[\leqslant n]}(x)$.

Theorem 6.3 (1) *For* $0 \leqslant i < j \leqslant n$, $R_{i,j}^{[\leqslant n]}(x) = R_{i,j}(x)$.

(2) *For* $0 \leqslant j < i \leqslant n$, $G_{i,j}^{[\leqslant n]}(x) = G_{i,j}(x)$.

Proof We only prove (1) while (2) can be proved similarly.

First, we assume that $n = j$ and $P(x)$ is partitioned according to the three subsets $L_{<n}$, L_n and $L_{>n}$ as in Eq. (6.8). It follows from Theorem 6.1 that

$$P^{[\leqslant n]}(x) = \begin{pmatrix} T(x) & U_0(x) \\ V_0(x) & T_0(x) \end{pmatrix} + \begin{pmatrix} U_1(x) \\ U_2(x) \end{pmatrix} * \widehat{Q}_0(x) * (V_1(x), V_2(x))$$
$$= \begin{pmatrix} T(x) + U_1 * \widehat{Q}_0 * V_1(x) & U_0(x) + U_1 * \widehat{Q}_0 * V_2(x) \\ V_0(x) + U_2 * \widehat{Q}_0 * V_1(x) & T_0(x) + U_2 * \widehat{Q}_0 * V_2(x) \end{pmatrix}. \tag{6.20}$$

Hence, simple calculations lead to

$$R_{<n}^{[\leqslant n]}(x) = \left[U_0(x) + U_1 * \widehat{Q}_0 * V_2(x)\right] * \sum_{l=0}^{\infty} \left[T_0(x) + U_2 * \widehat{Q}_0 * V_2(x)\right]^{l*}$$
$$= \left[U_0(x) + U_1 * \widehat{Q}_0 * V_2(x)\right] * \sum_{l=0}^{\infty} \sum_{k=0}^{l} \binom{l}{k} T_0^{k*}(x)$$
$$* \left[U_2 * \widehat{Q}_0 * V_2(x)\right]^{(l-k)*}. \tag{6.21}$$

Therefore, $R_{<n}^{[\leqslant n]}(x) = R_{<n}(x)$ according to Theorem 6.2.

If $n > j$, we first censor the matrix $P(x)$ in the set $L_{\leqslant j}$, $R_{i,j}^{[\leqslant j]}(x) = R_{i,j}(x)$ based on the fact just proved. Next, we censor the matrix $P(x)$ in the set $L_{\leqslant n}$. Since according to Property 1 the censored matrix $P^{[\leqslant j]}$ can be obtained by the censored matrix $P^{[\leqslant n]}$, $R_{i,j}^{[\leqslant n]}(x) = R_{i,j}^{[\leqslant j]}(x)$ based on the fact just proved, hence, $R_{i,j}^{[\leqslant n]}(x) = R_{i,j}(x)$ for $j < n$. This completes the proof.

Let

$$P^{[\leqslant n]}(x) = \begin{pmatrix} \phi_{0,0}^{(n)}(x) & \phi_{0,1}^{(n)}(x) & \dots & \phi_{0,n}^{(n)}(x) \\ \phi_{1,0}^{(n)}(x) & \phi_{1,1}^{(n)}(x) & \dots & \phi_{1,n}^{(n)}(x) \\ \vdots & \vdots & & \vdots \\ \phi_{n,0}^{(n)}(x) & \phi_{n,1}^{(n)}(x) & \dots & \phi_{n,n}^{(n)}(x) \end{pmatrix}, \quad n \geqslant 0,$$

be block-partitioned according to levels.

The equations in the following lemma provide a relationship among the entries of censored Markov renewal processes, which are essentially the Wiener-Hopf equations for the Markov renewal process.

Lemma 6.4 *For* $n \geqslant 0$, $0 \leqslant i,\ j \leqslant n$,

$$\phi_{i,j}^{(n)}(x) = P_{i,j}(x) + \sum_{k=n+1}^{\infty} \phi_{i,k}^{(k)}(x) * \sum_{l=0}^{\infty} \left[\phi_{k,k}^{(k)}(x)\right]^{l*} * \phi_{k,j}^{(k)}(x).$$

Proof Consider the censored matrix $P^{[\leqslant n]}(x)$ based on $P^{[\leqslant (n+1)]}(x)$. It follows from Theorem 6.1 that

$$\begin{aligned} P^{[\leqslant n]}(x) = &\begin{pmatrix} \phi_{0,0}^{(n+1)}(x) & \phi_{0,1}^{(n+1)}(x) & \dots & \phi_{0,n}^{(n+1)}(x) \\ \phi_{1,0}^{(n+1)}(x) & \phi_{1,1}^{(n+1)}(x) & \dots & \phi_{1,n}^{(n+1)}(x) \\ \vdots & \vdots & & \vdots \\ \phi_{n,0}^{(n+1)}(x) & \phi_{n,1}^{(n+1)}(x) & \dots & \phi_{n,n}^{(n+1)}(x) \end{pmatrix} \\ &+ \begin{pmatrix} \phi_{0,n+1}^{(n+1)}(x) \\ \phi_{1,n+1}^{(n+1)}(x) \\ \vdots \\ \phi_{n,n+1}^{(n+1)}(x) \end{pmatrix} * \sum_{l=0}^{\infty} \left[\phi_{n+1,n+1}^{(n+1)}(x)\right]^{l*} \\ &* \left(\phi_{n+1,0}^{(n+1)}(x), \phi_{n+1,1}^{(n+1)}(x), \dots, \phi_{n+1,n}^{(n+1)}(x)\right). \end{aligned}$$

Therefore, from repeatedly using Theorem 6.1 we obtain

$$\begin{aligned} \phi_{i,j}^{(n)}(x) &= \phi_{i,j}^{(n+1)}(x) + \phi_{i,n+1}^{(n+1)}(x) * \sum_{l=0}^{\infty} \left[\phi_{n+1,n+1}^{(n+1)}(x)\right]^{l*} * \phi_{n+1,j}^{(n+1)}(x) \\ &= \phi_{i,j}^{(n+2)}(x) + \phi_{i,n+2}^{(n+2)}(x) * \sum_{l=0}^{\infty} \left[\phi_{n+2,n+2}^{(n+2)}(x)\right]^{l*} * \phi_{n+2,j}^{(n+2)}(x) \\ &\quad + \phi_{i,n+1}^{(n+1)}(x) * \sum_{l=0}^{\infty} \left[\phi_{n+1,n+1}^{(n+1)}(x)\right]^{l*} * \phi_{n+1,j}^{(n+1)}(x) \\ &= \dots = P_{i,j}(x) + \sum_{k=n+1}^{\infty} \phi_{i,k}^{(k)}(x) * \sum_{l=0}^{\infty} \left[\phi_{k,k}^{(k)}(x)\right]^{l*} * \phi_{k,j}^{(k)}(x), \end{aligned}$$

where $P_{i,j}(x) = \phi_{i,j}^{(\infty)}(x)$. This completes the proof.

The following lemma provides expressions for the *R*- and *G*-measures.

Lemma 6.5 (1) *For* $0 \leqslant i < j$,

$$R_{i,j}(x) = \phi_{i,j}^{(j)}(x) * \sum_{l=0}^{\infty} \left[\phi_{j,j}^{(j)}(x)\right]^{l*}.$$

(2) *For* $0 \leqslant j < i$,

$$G_{i,j}(x) = \sum_{l=0}^{\infty} \left[\phi_{i,i}^{(i)}(x)\right]^{l*} * \phi_{i,j}^{(j)}(x).$$

Proof Applying Corollary 6.2 to the censored process $P^{[\leqslant j]}(x)$ gives that

$$R_{i,j}^{[\leqslant j]}(x)=\phi_{i,j}^{(j)}(x)*\sum_{l=0}^{\infty}\left[\phi_{j,j}^{(j)}(x)\right]^{l*},\quad 0\leqslant i<j,$$

and

$$G_{i,j}^{[\leqslant i]}(x)=\sum_{l=0}^{\infty}\left[\phi_{i,i}^{(i)}(x)\right]^{l*}*\phi_{i,j}^{(j)}(x),\quad 0\leqslant j<i.$$

The rest of the proof follows from the censoring invariance for the R- and G-measures proved in Theorem 6.3.

Let

$$\Psi_n(x)=\phi_{n,n}^{(n)}(x),\quad n\geqslant 0.$$

The following theorem provides an equivalent form, to the equations in Lemma 6.4, of the Wiener-Hopf equations stated in terms of the R- and G-measures.

Theorem 6.4 (1) *For* $0\leqslant i<j$,

$$R_{i,j}(x)*[I-\Psi_j(x)]=P_{i,j}(x)+\sum_{k=j+1}^{\infty}R_{i,k}(x)*[I-\Psi_k(x)]*G_{k,j}(x).$$

(2) *For* $0\leqslant j<i$,

$$[I-\Psi_i(x)]*G_{i,j}(x)=P_{i,j}(x)+\sum_{k=i+1}^{\infty}R_{i,k}(x)*[I-\Psi_k(x)]*G_{k,j}(x).$$

(3) *For* $n\geqslant 0$,

$$\Psi_n(x)=P_{n,n}(x)+\sum_{k=n+1}^{\infty}R_{n,k}(x)*[I-\Psi_k(x)]*G_{k,n}(x).$$

Proof We only prove (1) while (2) and (3) can be proved similarly.

It follows from (1) in Lemma 6.5 that

$$R_{i,j}(x)*[I-\Psi_j(x)]=\phi_{i,j}^{(j)}(x).$$

Using Lemma 6.4 and Theorem 6.5 leads to

$$\begin{aligned}\phi_{i,j}^{(j)}(x)&=P_{i,j}(x)+\sum_{k=j+1}^{\infty}\phi_{i,k}^{(k)}(x)*\sum_{l=0}^{\infty}\left[\phi_{k,k}^{(k)}(x)\right]^{l*}*\phi_{k,j}^{(k)}(x)\\&=P_{i,j}(x)+\sum_{k=j+1}^{\infty}R_{i,k}^{[\leqslant k]}(x)*\left[I-\Psi_k(x)\right]*G_{k,j}^{[\leqslant k]}(x)\\&=P_{i,j}(x)+\sum_{k=j+1}^{\infty}R_{i,k}(x)*\left[I-\Psi_k(x)\right]*G_{k,j}(x).\end{aligned}$$

This completes the proof.

Based on the Wiener-Hopf equations, the following theorem gives the UL-type RG-factorization for the transition probability mass matrix.

Theorem 6.5 *For the Markov renewal process* $P(x)$ *given in Eq.* (6.1),

$$I - P(x) = [I - R_U(x)] * [I - \Psi_D(x)] * [I - G_L(x)], \quad x \geqslant 0, \tag{6.22}$$

or

$$I - \widetilde{P}(s) = \left[I - \widetilde{R}_U(s)\right]\left[I - \widetilde{\Psi}_D(s)\right]\left[I - \widetilde{G}_L(s)\right], \quad \mathrm{Re}(s) \geqslant 0,$$

where

$$R_U(x) = \begin{pmatrix} 0 & R_{0,1}(x) & R_{0,2}(x) & R_{0,3}(x) & \cdots \\ & 0 & R_{1,2}(x) & R_{1,3}(x) & \cdots \\ & & 0 & R_{2,3}(x) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$\Psi_D(x) = \mathrm{diag}(\Psi_0(x), \Psi_1(x), \Psi_2(x), \Psi_3(x), \ldots)$$

and

$$G_L(x) = \begin{pmatrix} 0 & & & & \\ G_{1,0}(x) & 0 & & & \\ G_{2,0}(x) & G_{2,1}(x) & 0 & & \\ G_{3,0}(x) & G_{3,1}(x) & G_{3,2}(x) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof We only prove Eq. (6.22) for the entries in the first block-row and first lock-column. The rest can be proved similarly.

The entry $(0, 0)$ on the right-hand side is

$$I - \Psi_0(x) + \sum_{k=1}^{\infty} R_{0,k}(x) * [I - \Psi_k(x)] * G_{k,0}(x),$$

which is equal to $I - P_{0,0}(x)$ according to (3) of Theorem 6.4.

The entry $(0, l)$ with $l \geqslant 1$ on the right-hand side is

$$-R_{0,l}(x) * [I - \Psi_l(x)] + \sum_{k=l+1}^{\infty} R_{0,k}(x) * [I - \Psi_k(x)] * G_{k,l}(x),$$

which is equal to $-P_{0,l}(x)$ according to (1) of Theorem 6.4.

Finally, to see that the entry $(l,0)$ with $l \geqslant 1$ on the right-hand side is equal to the corresponding entry on the left-hand side, it follows from (2) of Theorem 6.4 that

$$-[I-\varPsi_l(x)]*G_{l,0}(x)+\sum_{k=l+1}^{\infty}R_{l,k}(x)*\left[I-\varPsi_k(x)\right]*G_{k,0}(x)=-P_{l,0}(x).$$

This completes the proof.

In what follows we list the main results for two important examples: Level-dependent Markov renewal processes of $M/G/1$ type, and level-dependent Markov renewal processes of $GI/M/1$ type.

6.2.1 Level-Dependent Markov Renewal Processes of *M*/*G*/1 Type

Let

$$Q_k(x)=\begin{pmatrix} P_{k,k}(x) & P_{k,k+1}(x) & P_{k,k+2}(x) & \cdots \\ P_{k+1,k}(x) & P_{k+1,k+1}(x) & P_{k+1,k+2}(x) & \cdots \\ & P_{k+2,k+1}(x) & P_{k+2,k+2}(x) & \cdots \\ & & \ddots & \ddots \end{pmatrix}, \quad k\geqslant 1.$$

We denote by $(\widehat{Q}_{1,1}^{(k)}(x)^{\mathrm{T}}, \widehat{Q}_{2,1}^{(k)}(x)^{\mathrm{T}}, \cdots)^{\mathrm{T}}$ the first block-column of the matrix $\widehat{Q}_k(x) = \sum_{l=0}^{\infty}\left[Q_k(x)\right]^{l*}$. Thus, the *R*- and *G*-measures are defined as

$$R_{i,j}(x)=\sum_{l=0}^{\infty}P_{i,j+l}(x)*\widehat{Q}_{l+1,1}^{(j)}(x), \quad 0\leqslant i<j, \tag{6.23}$$

$$G_k(x)\overset{\text{def}}{=}G_{k,k-1}(x)=\widehat{Q}_{1,1}^{(k)}(x)*P_{k,k-1}(x), \quad k\geqslant 1, \tag{6.24}$$

$G_{i,j}(x)=0$ for $0\leqslant j\leqslant i-2$, and

$$\varPsi_k(x)=P_{k,k}(x)+\sum_{i=1}^{\infty}P_{k,k+i}(x)*G_{k+i}(x)*G_{k+i-1}(x)\ldots*G_{k+1}(x), \quad k\geqslant 0, \tag{6.25}$$

and

$$[I-\varPsi_k(x)]*\widehat{Q}_{1,1}^{(k)}(x)=\widehat{Q}_{1,1}^{(k)}(x)*[I-\varPsi_k(x)]=I.$$

Therefore, we can obtain

(1) For $0\leqslant i<j$,

$$R_{i,j}(x)=[P_{i,j}(x)+\sum_{l=1}^{\infty}P_{i,j+l}(x)*G_{j+l}(x)*G_{j+l-1}(x)$$
$$*\ldots*G_{j+1}(x)]*\widehat{\varPsi}_j(x).$$

(2) The matrix sequence $\{G_i(x)\}$ is the minimal nonnegative solution to the system of matrix equations

$$G_i(x) = P_{i,i-1}(x) + \sum_{l=0}^{\infty} P_{i,i+l}(x) * G_{i+l}(x) * G_{i+l-1}(x) * \ldots * G_i(x), \quad i \geqslant 1.$$

(3) In the RG-factorization, we have

$$G_L(x) = \begin{pmatrix} 0 & & & & \\ G_1(x) & 0 & & & \\ & G_2(x) & 0 & & \\ & & G_3(x) & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

6.2.2 Level-Dependent Markov Renewal Processes of *GI*/*M*/1 Type

We write

$$W_k(x) = \begin{pmatrix} P_{k,k}(x) & P_{k,k+1}(x) & & \\ P_{k+1,k}(x) & P_{k+1,k+1}(x) & P_{k+1,k+2}(x) & \\ P_{k+2,k}(x) & P_{k+2,k+1}(x) & P_{k+2,k+2}(x) & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad k \geqslant 1.$$

We denote by $\left(\widehat{W}_{1,1}^{(k)}(x), \widehat{W}_{1,2}^{(k)}(x), \ldots\right)$ the first block-row of the matrix $\widehat{W}_k(x) = \sum_{l=0}^{\infty} \left[W_k(x)\right]^{l*}$. Based on Corollary 2.2, we have

$$R_k(x) \stackrel{\text{def}}{=} R_{k,k+1}(x) = P_{k,k+1} * \widehat{W}_{1,1}^{(k+1)}(x), \quad k \geqslant 0,$$

$R_{i,j}(x) = 0$ for $j \geqslant i+2$,

$$G_{i,j}(x) = \sum_{l=1}^{\infty} \widehat{W}_{1,l}^{(i)} * P_{i+l,j}(x), \quad 0 \leqslant j < i,$$

and

$$\Psi_k(x) = P_{k,k}(x) + \sum_{i=1}^{\infty} R_k * R_{k+1} * \ldots * R_{k+i} * P_{k+i,k}(x), \quad k \geqslant 0.$$

The matrix sequence $\{R_k(x)\}$ is the minimal nonnegative solution to the system of matrix equations

$$R_k(x) = P_{k,k+1}(x) + \sum_{i=1}^{\infty} R_k * R_{k+1} * \ldots * R_{k+i-1} * P_{k+i,k+1}(x), \quad k \geqslant 0.$$

In the *RG*-factorization, we have

$$R_U(x) = \begin{pmatrix} 0 & R_0(x) & & & \\ & 0 & R_1(x) & & \\ & & 0 & R_2(x) & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

6.2.3 Markov Renewal Equations

We use the *R*-, *U*- and *G*-measures to express the block-structured Markov renewal matrix. Note that the Markov renewal equation plays an important role in the study of Markov renewal processes.

Consider a block-structured Markov renewal equation

$$U(t) = H(t) + \int_0^t P(\mathrm{d}x)U(t-x), \quad t \geqslant 0, \tag{6.26}$$

where $P(x)$ is a block-structured transition probability mass matrix and $H(x)$ is a given matrix. We partition $U(x)$ and $H(x)$ according to the levels, and denote their block-entries by $U_{i,j}(x)$ and $H_{i,j}(x)$ for i, $j \geqslant 0$, respectively.

To solve the Eq. (6.26), we need to compute the Markov renewal matrix

$$M(t) = \sum_{n=0}^{\infty} [P(t)]^{n*}. \tag{6.27}$$

It is obvious that

$$\widetilde{M}(s) = \sum_{n=0}^{\infty} [\widetilde{P}(s)]^{n*} = [I - \widetilde{P}(s)]^{-1}, \tag{6.28}$$

which is the minimal nonnegative inverse of $I - \widetilde{P}(s)$ for $s \geqslant 0$. It follows from Eq. (6.26) and Eq. (6.28) that

$$\widetilde{U}(s) = [I - \widetilde{P}(s)]^{-1}\widetilde{H}(s) = \widetilde{M}(s)\widetilde{H}(s). \tag{6.29}$$

It is clear from Eq. (6.29) that the computation of the inverse of the matrix $I - \widetilde{P}(s)$ is crucial for expressing $\widetilde{U}(s)$. For $\mathrm{Re}(s) \geqslant 0$, using the UL-type *RG*-factorization we obtain

$$\widetilde{M}(s) = [I - \widetilde{P}(s)]^{-1} = [I - \widetilde{G_L}(s)]^{-1}[I - \widetilde{\Psi_D}(s)]^{-1}[I - \widetilde{R_U}(s)]^{-1}$$

or

$$M(x)=\sum_{n=0}^{\infty}\left[G_L(x)\right]^{n^*}*\sum_{n=0}^{\infty}\left[\Psi_D(x)\right]^{n^*}*\sum_{n=0}^{\infty}\left[R_U(x)\right]^{n^*}.$$

6.3 The LU-Type *RG*-Factorization

In this section, we define the LU-type *R*-, *U*- and *G*-measures by means of another censored process, and derive the LU-type *RG*-factorization for the transition probability mass matrix.

Let

$$P^{[\geqslant n]}(x)=Q(x)+V*\hat{T}*U(x), \tag{6.30}$$

where

$$\hat{T}(x)=\sum_{n=0}^{\infty}T^{n^*}(x).$$

The block-entry expression of the matrix $P^{[\geqslant n]}(x)$ is given by

$$P^{[\geqslant n]}(x)=\begin{pmatrix} \eta_{n,n}^{(n)}(x) & \eta_{n,n+1}^{(n)}(x) & \eta_{n,n+2}^{(n)}(x) & \cdots \\ \eta_{n+1,n}^{(n)}(x) & \eta_{n+1,n+1}^{(n)}(x) & \eta_{n+1,n+2}^{(n)}(x) & \cdots \\ \eta_{n+2,n}^{(n)}(x) & \eta_{n+2,n+1}^{(n)}(x) & \eta_{n+2,n+2}^{(n)}(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Lemma 6.6 *For i,* $j\geqslant n+1$*, we have*

$$\eta_{i,j}^{(n+1)}(x)=P_{i,j}(x)+\sum_{k=0}^{n}\eta_{i,k}^{(k)}*\hat{\eta}_{k,k}^{(k)}*\eta_{k,j}^{(k)}(x). \tag{6.31}$$

Proof Since

$$P^{[\geqslant(n+1)]}(x)=\begin{pmatrix} \eta_{n+1,n+1}^{(n)}(x) & \eta_{n+1,n+2}^{(n)}(x) & \eta_{n+1,n+3}^{(n)}(x) & \cdots \\ \eta_{n+2,n+1}^{(n)}(x) & \eta_{n+2,n+2}^{(n)}(x) & \eta_{n+2,n+3}^{(n)}(x) & \cdots \\ \eta_{n+3,n+1}^{(n)}(x) & \eta_{n+3,n+2}^{(n)}(x) & \eta_{n+3,n+3}^{(n)}(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$+\begin{pmatrix} \eta_{n+1,n}^{(n)} \\ \eta_{n+2,n}^{(n)} \\ \eta_{n+3,n}^{(n)} \\ \vdots \end{pmatrix}*\hat{\eta}_{n,n}^{(n)}*\left(\eta_{n,n+1}^{(n)}(x) \quad \eta_{n,n+2}^{(n)}(x) \quad \eta_{n,n+3}^{(n)}(x) \quad \cdots\right),$$

we obtain

$$\begin{aligned}\eta_{i,j}^{(n+1)}(x) &= \eta_{i,j}^{(n)}(x) + \eta_{i,n}^{(n)} * \widehat{\eta}_{n,n}^{(n)} * \eta_{n,j}^{(n)}(x) \\ &= \eta_{i,j}^{(n-1)}(x) + \eta_{i,n-1}^{(n-1)} * \widehat{\eta}_{n-1,n-1}^{(n-1)} * \eta_{n-1,j}^{(n-1)}(x) \\ &\quad + \eta_{i,n}^{(n)} * \widehat{\eta}_{n,n}^{(n)} * \eta_{n,j}^{(n)}(x) \\ &\quad \cdots \\ &= \eta_{i,j}^{(0)}(x) + \sum_{k=0}^{n} \eta_{i,k}^{(k)} * \widehat{\eta}_{k,k}^{(k)} * \eta_{k,j}^{(k)}(x).\end{aligned}$$

Note that $\eta_{i,j}^{(0)}(x) = P_{i,j}(x)$ for all i, $j \geqslant 0$. This completes the proof.

Using the censoring invariance, we define the U-measure as

$$\Phi_n(x) = \eta_{n,n}^{(n)}(x), \quad n \geqslant 0, \tag{6.32}$$

the R-measure as

$$\overline{R}_{i,j}(x) = \eta_{i,j}^{(j)} * \widehat{\eta}_{j,j}^{(j)}(x), \quad 0 \leqslant j < i,$$

and the G-measure as

$$\overline{G}_{i,j}(x) = \widehat{\eta}_{i,i}^{(i)} * \eta_{i,j}^{(i)}(x), \quad 0 \leqslant i < j.$$

It is obvious that

$$\overline{R}_{i,j}(x) = \eta_{i,j}^{(j)} * \widehat{\Phi}_j(x), \quad 0 \leqslant j < i, \tag{6.33}$$

and

$$\overline{G}_{i,j}(x) = \widehat{\Phi}_i * \eta_{i,j}^{(i)}(x), \quad 0 \leqslant i < j. \tag{6.34}$$

The following theorem provides the important Wiener-Hopf equations, which are satisfied by the R-, U- and G-measures.

Theorem 6.6 *The R-, U- and G-measures defined above satisfy the following Wiener-Hopf equations,*

$$\overline{R}_{i,j} * (I - \Phi_i(x)) = P_{i,j}(x) + \sum_{k=0}^{j-1} \overline{R}_{i,k} * (I - \Phi_k) * \overline{G}_{k,j}(x), \quad 0 \leqslant j < i, \tag{6.35}$$

$$(I - \Phi_i) * \overline{G}_{i,j}(x) = P_{i,j}(x) + \sum_{k=0}^{i-1} \overline{R}_{i,k} * (I - \Phi_k) * \overline{G}_{k,j}(x), \quad 0 \leqslant i < j, \tag{6.36}$$

and

$$\varPhi_n(x) = P_{n,n}(x) + \sum_{k=0}^{n-1} \overline{R}_{n,k} * (I - U_k) * \overline{G}_{k,n}(x), \quad n \geqslant 0. \tag{6.37}$$

Proof We only prove Eq. (6.35), while Eq. (6.36) and Eq. (6.37) can be proved similarly.

It follows from Eq. (6.33) that

$$\overline{R}_{i,j} * (I - \varPhi_j(x)) = \eta_{i,j}^{(j)}(x). \tag{6.38}$$

By Lemma 6.6, we have

$$\eta_{i,j}^{(j)}(x) = P_{i,j}(x) + \sum_{k=0}^{j-1} \eta_{i,k}^{(k)} * \hat{\eta}_{k,k}^{(k)} * \eta_{k,j}^{(k)}(x). \tag{6.39}$$

From Eq. (6.33), Eq. (6.34) and Eq. (6.39) we obtain

$$\eta_{i,j}^{(j)}(x) = P_{i,j}(x) + \sum_{k=0}^{j-1} \overline{R}_{i,k} * (I - \varPhi_k) * \overline{G}_{k,j}(x),$$

which, together with Eq. (6.38), leads to the stated result.

By the Wiener-Hopf Eq. (6.35), Eq. (6.36) and Eq. (6.37), the following theorem constructs an LU-type RG-factorization.

Theorem 6.7 *The Markov renewal process $P(x)$ defined in Eq. (6.1) can be factorized as follows,*

$$I - P(x) = (I - \overline{R}_L(x)) * (I - \varPhi_D(x)) * (I - \overline{G}_U(x)), \tag{6.40}$$

where

$$\overline{R}_L(x) = \begin{pmatrix} 0 & & & & \\ \overline{R}_{1,0}(x) & 0 & & & \\ \overline{R}_{2,0}(x) & \overline{R}_{2,1}(x) & 0 & & \\ \overline{R}_{3,0}(x) & \overline{R}_{3,1}(x) & \overline{R}_{3,2}(x) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\varPhi_D(x) = \mathrm{diag}(\varPhi_0(x), \varPhi_1(x), \varPhi_2(x), \varPhi_3(x), \ldots)$$

and

$$\overline{G}_U(x) = \begin{pmatrix} 0 & \overline{G}_{0,1}(x) & \overline{G}_{0,2}(x) & \overline{G}_{0,3}(x) & \cdots \\ & 0 & \overline{G}_{1,2}(x) & \overline{G}_{1,3}(x) & \cdots \\ & & 0 & \overline{G}_{2,3}(x) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Proof We prove Eq. (6.40) for the block-entries of the first two block-rows. The rest can be proved similarly. For the first block-row, the entry $(0,0)$ is

$$I-\varPhi_0(x)=I-\eta_{0,0}^{(0)}(x)=I-P_{0,0}(x),$$

and the entry $(0,l)$ for $l\geqslant 1$ is, from Eq. (6.34)

$$\begin{aligned}-(I-\varPhi_0)*\overline{G}_{0,l}(x)&=-\left[I-\eta_{0,0}^{(0)}\right]*\hat{\eta}_{0,0}^{(0)}*\eta_{0,l}^{(0)}(x)\\&=-\eta_{0,l}^{(0)}(x)=-P_{0,l}(x).\end{aligned}$$

For the second block-row, the entry $(1,0)$ is

$$\begin{aligned}-\overline{R}_{1,0}*(I-\varPhi_0(x))&=-\eta_{1,0}^{(0)}*\hat{\eta}_{0,0}^{(0)}*\left[I-\eta_{0,0}^{(0)}(x)\right]\\&=-\eta_{1,0}^{(0)}(x)=-P_{1,0}(x)\end{aligned}$$

by Eq. (6.33). By Lemma 6.6, the entry $(1,1)$ is

$$\begin{aligned}&\overline{R}_{1,0}*(I-\varPhi_0)*\overline{G}_{0,1}(x)+I-\varPhi_1(x)\\&\quad=\eta_{1,0}^{(0)}*\hat{\eta}_{0,0}^{(0)}*\left[I-\eta_{0,0}^{(0)}\right]*\hat{\eta}_{0,0}^{(0)}*\eta_{0,1}^{(0)}(x)+I-\eta_{1,1}^{(1)}(x)\\&\quad=\eta_{1,0}^{(0)}*\hat{\eta}_{0,0}^{(0)}*\eta_{0,1}^{(0)}(x)+I-\eta_{1,1}^{(1)}(x)\\&\quad=I-P_{1,1}(x),\end{aligned}$$

and the entry $(1,k)$ for $k\geqslant 2$ is

$$\begin{aligned}&\overline{R}_{1,0}*(I-\varPhi_0)*\overline{G}_{0,k}(x)-(I-\varPhi_1)*\overline{G}_{1,k}(x)\\&\quad=\eta_{1,0}^{(0)}*\hat{\eta}_{0,0}^{(0)}*\left[I-\eta_{0,0}^{(0)}\right]*\hat{\eta}_{0,0}^{(0)}*\eta_{0,k}^{(0)}(x)-\left[I-\eta_{1,1}^{(1)}\right]*\hat{\eta}_{1,1}^{(1)}*\eta_{1,k}^{(1)}(x)\\&\quad=\eta_{1,0}^{(0)}*\hat{\eta}_{0,0}^{(0)}*\eta_{0,k}^{(0)}(x)-\eta_{1,k}^{(1)}(x)\\&\quad=P_{1,k}(x).\end{aligned}$$

This completes the proof.

6.4 Finite Levels

In this section, as an important example we considers an irreducible Markov renewal process with finite levels, and derive the UL- and LU-types of *RG*-factorizations.

We consider an irreducible block-structured Markov renewal process with finite levels whose transition probability mass matrix is given by

$$P(x)=\begin{pmatrix} P_{0,0}(x) & P_{0,1}(x) & \dots & P_{0,M}(x) \\ P_{1,0}(x) & P_{1,1}(x) & \dots & P_{1,M}(x) \\ \vdots & \vdots & & \vdots \\ P_{M,0}(x) & P_{M,1}(x) & \dots & P_{M,M}(x) \end{pmatrix},$$

where $P_{i,i}(x)$ is a matrix of size $m_i \times m_i$ for all $0 \leqslant i \leqslant M$, and the sizes of the other blocks are determined accordingly.

6.4.1 The UL-Type *RG*-Factorization

For $0 \leqslant i, j \leqslant k$ and $0 \leqslant k \leqslant M$, it is clear from Section 6.2 that

$$P_{i,j}^{[\leqslant k]}(x)=P_{i,j}(x)+\sum_{n=k+1}^{M} P_{i,n}^{[\leqslant n]} * \widehat{P}_{n,n}^{[\leqslant n]} * P_{n,j}^{[\leqslant n]}(x).$$

Note that $P_{i,j}^{[\leqslant M]}(x)=P_{i,j}(x)$ and $P_{i,j}^{[\leqslant 0]}(x)=P_{i,j}^{[0]}(x)$.

Let

$$\Psi_n(x)=P_{n,n}^{[\leqslant n]}(x), \quad 0 \leqslant n \leqslant M,$$
$$R_{i,j}(x)=P_{i,j}^{[\leqslant j]} * \widehat{\Psi}_j(x), \quad 0 \leqslant j < i \leqslant M,$$

and

$$G_{i,j}(x)=\widehat{\Psi}_i * P_{i,j}^{[\leqslant i]}(x), \quad 0 \leqslant i < j \leqslant M.$$

Then the UL-type *RG*-factorization is given by

$$I-P(x)=(I-R_U)*(I-\Psi_D)*(I-G_L(x)),$$

where

$$R_U(x)=\begin{pmatrix} 0 & R_{0,1}(x) & R_{0,2}(x) & R_{0,3}(x) & \dots & R_{0,M-1}(x) & R_{0,M}(x) \\ & 0 & R_{1,2}(x) & R_{1,3}(x) & \dots & R_{1,M-1}(x) & R_{1,M}(x) \\ & & 0 & R_{2,3}(x) & \dots & R_{2,M-1}(x) & R_{2,M}(x) \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & R_{M-2,M-1}(x) & R_{M-2,M}(x) \\ & & & & & 0 & R_{M-1,M}(x) \\ & & & & & & 0 \end{pmatrix},$$

$$\Psi_D(x)=\operatorname{diag}(\Psi_0(x),\Psi_1(x),\Psi_2(x),\Psi_3(x),\dots,\Psi_{M-1}(x),\Psi_M(x))$$

and

$$G_L(x) = \begin{pmatrix} 0 & & & & & & \\ G_{1,0}(x) & 0 & & & & & \\ G_{2,0}(x) & G_{2,1}(x) & 0 & & & & \\ G_{3,0}(x) & G_{3,1}(x) & G_{3,2}(x) & 0 & & & \\ G_{4,0}(x) & G_{4,1}(x) & G_{4,2}(x) & G_{4,3}(x) & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ G_{M,0}(x) & G_{M,1}(x) & G_{M,2}(x) & G_{M,3}(x) & \dots & G_{M,M-1}(x) & 0 \end{pmatrix}.$$

6.4.2 The LU-Type *RG*-Factorization

For $k \leqslant i, j \leqslant M$ and $0 \leqslant k \leqslant M$, it is clear from Section 6.3 that

$$P_{i,j}^{[\geqslant k+1]}(x) = P_{i,j}(x) + \sum_{n=0}^{k} P_{i,n}^{[\geqslant n]} * \widehat{P}_{n,n}^{[\geqslant n]} * P_{n,j}^{[\geqslant n]}(x).$$

Note that $P_{i,j}^{[\geqslant M]}(x) = P_{i,j}^{[M]}(x)$ and $P_{i,j}^{[\geqslant 0]}(x) = P_{i,j}(x)$.

Let

$$\varPhi_n(x) = P_{n,n}^{[\geqslant n]}(x), \quad 0 \leqslant n \leqslant M,$$

$$\overline{R}_{i,j}(x) = P_{i,j}^{[\geqslant j]} * \widehat{\varPhi}_j(x), \quad 0 \leqslant i < j \leqslant M,$$

and

$$\overline{G}_{i,j}(x) = \widehat{\varPhi}_i * P_{i,j}^{[\geqslant i]}(x), \quad 0 \leqslant j < i \leqslant M.$$

Then the LU-type *RG*-factorization is given by

$$I - P(x) = (I - \overline{R}_L) * (I - \varPhi_D) * (I - \overline{G}_U(x)),$$

where

$$\overline{R}_L(x) = \begin{pmatrix} 0 & & & & & & \\ \overline{R}_{1,0}(x) & 0 & & & & & \\ \overline{R}_{2,0}(x) & \overline{R}_{2,1}(x) & 0 & & & & \\ \overline{R}_{3,0}(x) & \overline{R}_{3,1}(x) & \overline{R}_{3,2}(x) & 0 & & & \\ \overline{R}_{4,0}(x) & \overline{R}_{4,1}(x) & \overline{R}_{4,2}(x) & \overline{R}_{4,3}(x) & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ \overline{R}_{M,0}(x) & \overline{R}_{M,1}(x) & \overline{R}_{M,2}(x) & \overline{R}_{M,3}(x) & \dots & \overline{R}_{M,M-1}(x) & 0 \end{pmatrix},$$

$$\varPhi_D(x) = \operatorname{diag}(\varPhi_0(x), \varPhi_1(x), \varPhi_2(x), \varPhi_3(x), \dots, \varPhi_{M-1}(x), \varPhi_M(x))$$

and

$$\overline{G}_U(x)=\begin{pmatrix} 0 & \overline{G}_{0,1}(x) & \overline{G}_{0,2}(x) & \overline{G}_{0,3}(x) & \dots & \overline{G}_{0,M-1}(x) & \overline{G}_{0,M}(x) \\ & 0 & \overline{G}_{1,2}(x) & \overline{G}_{1,3}(x) & \dots & \overline{G}_{1,M-1}(x) & \overline{G}_{1,M}(x) \\ & & 0 & \overline{G}_{2,3}(x) & \dots & \overline{G}_{2,M-1}(x) & \overline{G}_{2,M}(x) \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & \overline{G}_{M-2,M-1}(x) & \overline{G}_{M-2,M}(x) \\ & & & & & 0 & \overline{G}_{M-1,M}(x) \\ & & & & & & 0 \end{pmatrix}.$$

6.5 Markov Renewal Processes of *GI*/*G*/1 Type

In this section, we simplify the UL-type expressions for the R-, U- and G-measures and the RG-factorization for Markov renewal processes of $GI/G/1$ type. Furthermore, we derive the RG-factorization for the repeated blocks and four Wiener-Hopf inequalities for the boundary blocks.

Consider a Markov renewal process of $GI/G/1$ type whose transition probability mass matrix is given by

$$P(x)=\begin{pmatrix} D_0(x) & D_1(x) & D_2(x) & D_3(x) & \dots \\ D_{-1}(x) & A_0(x) & A_1(x) & A_2(x) & \dots \\ D_{-2}(x) & A_{-1}(x) & A_0(x) & A_1(x) & \dots \\ D_{-3}(x) & A_{-2}(x) & A_{-1}(x) & A_0(x) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{6.41}$$

where the sizes of the matrices $D_0(x)$, $D_i(x)$, $D_{-i}(x)$ for $i \geqslant 1$ and $A_j(x)$ for $-\infty < j < \infty$ are $m_0 \times m_0$, $m_0 \times m$, $m \times m_0$ and $m \times m$, respectively.

Comparing Eq. (6.41) with Eq. (6.1), it is easy to yield $P_{0,j}(x)=D_j(x)$ for $j \geqslant 0$, $P_{i,0}(x)=D_{-i}(x)$ for $i \geqslant 1$ and $P_{i,j}(x)=A_{j-i}(x)$ for i, $j \geqslant 1$. It is clear that $Q_n(x)=Q(x)$ for all $n \geqslant 1$. Thus, we write $\widehat{Q}^{(\cdot,1)}(x)=\widehat{Q}_j^{(\cdot,1)}(x)$ and $\widehat{Q}^{(1,\cdot)}(x)=\widehat{Q}_i^{(1,\cdot)}(x)$ for all i, $j \geqslant 1$. Therefore,

$$R_{0,j}(x)=(D_j(x), D_{j+1}(x), D_{j+2}(x), \dots) * \widehat{Q}^{(\cdot,1)}(x), \quad j \geqslant 1,$$

$$R_{i,j}(x)=(A_{j-i}(x), A_{j-i+1}(x), A_{j-i+2}(x), \dots) * \widehat{Q}^{(\cdot,1)}(x), \quad 1 \leqslant i < j.$$

It is obvious that the matrices $R_{i,j}(x)$ for $1 \leqslant i < j$ only depend on the difference $j-i$. We write $R_{i,j}(x)$ as $R_{j-i}(x)$ for all $1 \leqslant i < j$. Therefore, for $k \geqslant 1$

$$R_k(x)=(A_k(x), A_{k+1}(x), A_{k+2}(x), \dots) * \widehat{Q}^{(\cdot,1)}(x).$$

Similarly, for $i \geqslant 1$,

$$G_{i,0}(x) = \widehat{Q}^{(1,\cdot)}(x) * \left(D_{-i}^{\mathrm{T}}(x), D_{-(i+1)}^{\mathrm{T}}(x), D_{-(i+2)}^{\mathrm{T}}(x), \ldots \right)^{\mathrm{T}}$$

and for $1 \leqslant j < i$,

$$G_{i,j}(x) = \widehat{Q}^{(1,\cdot)}(x) * \left(A_{-(i-j)}^{\mathrm{T}}(x), A_{-(i-j+1)}^{\mathrm{T}}(x), A_{-(i-j+2)}^{\mathrm{T}}(x), \ldots \right)^{\mathrm{T}}.$$

The matrices $G_{i,j}(x)$ for $1 \leqslant j < i$ only depend on the difference $i-j$. We write $G_{i,j}(x)$ as $G_{i-j}(x)$ for all $1 \leqslant j < i$. Therefore, for $k \geqslant 1$,

$$G_k(x) = \widehat{Q}^{(1,\cdot)}(x) * \left(A_{-k}^{\mathrm{T}}(x), A_{-(k+1)}^{\mathrm{T}}(x), A_{-(k+2)}^{\mathrm{T}}(x), \ldots \right)^{\mathrm{T}}.$$

For the Markov renewal process of $GI/G/1$ type, the following lemma is a consequence of the repeating property of the transition probability mass matrix, which also leads to the censoring invariance.

Lemma 6.7 *For* $n \geqslant 1$, i, $j = 1, 2, 3, \ldots, n$,

$$P_{n-i,n-j}^{[\leqslant n]}(x) = P_{n+1-i,n+1-j}^{[\leqslant (n+1)]}(x) = P_{n+2-i,n+2-j}^{[\leqslant (n+2)]}(x) = \cdots.$$

Proof For $n \geqslant 1$, i, $j = 1, 2, 3, \ldots, n$, it is easy to see that

$$\begin{aligned} P_{n-i,n-j}^{[\leqslant n]}(x) = A_{i-j}(x) + \left(A_{i+1}(x), A_{i+2}(x), \ldots \right) * \widehat{Q}(x) \\ * \left(A_{-(j+1)}^{\mathrm{T}}(x), A_{-(j+2)}^{\mathrm{T}}(x), \ldots \right)^{\mathrm{T}}, \end{aligned}$$

which is independent of $n \geqslant 1$. Thus

$$P_{n-i,n-j}^{[\leqslant n]}(x) = P_{n+1-i,n+1-j}^{[\leqslant (n+1)]}(x) = P_{n+2-i,n+2-j}^{[\leqslant (n+2)]}(x) = \cdots.$$

This completes the proof.

Based on Lemma 6.7, we can define for $1 \leqslant i, \ j \leqslant n$,

$$\begin{aligned} \Phi_0(x) &= P_{n,n}^{[\leqslant n]}(x), \\ \Phi_i(x) &= P_{n-i,n}^{[\leqslant n]}(x), \\ \Phi_{-j}(x) &= P_{n,n-j}^{[\leqslant n]}(x). \end{aligned}$$

The following theorem explicitly expresses the R- and G-measures in terms of the matrix sequence $\{\Phi_i(x): \text{for} -\infty < i < +\infty\}$.

Theorem 6.8 (1) *For* $i \geqslant 0$,

$$R_i(x) = \Phi_i(x) * \sum_{l=0}^{\infty} \Phi_0^{l*}(x). \tag{6.42}$$

(2) *For* $j > 1$,

$$G_j(x) = \sum_{l=0}^{\infty} \Phi_0^{l*}(x) * \Phi_{-j}(x). \tag{6.43}$$

Proof We only prove Eq. (6.42) while Eq. (6.43) can be proved similarly. It follows from Lemma 6.7 and Eq. (6.42) that

$$\begin{aligned} R_i(x) = R_i^{[\leqslant n]}(x) = R_{n-i,n}^{[\leqslant n]}(x) = P_{n-i,n}^{[\leqslant n]}(x) * \sum_{l=0}^{\infty} \left[P_{n,n}^{[\leqslant n]}(x) \right]^{l*} \\ = \Phi_i(x) * \sum_{l=0}^{\infty} \Phi_0^{l*}(x). \end{aligned}$$

This completes the proof.

The following theorem provides Wiener-Hopf equations for the repeating matrix sequence and also for the boundary matrix sequence.

Theorem 6.9 (1) *For* $i \geqslant 1$,

$$R_i(x) * [I - \Phi_0(x)] = A_i(x) + \sum_{k=1}^{\infty} R_{i+k}(x) * [I - \Phi_0(x)] * G_k(x), \tag{6.44}$$

for $j \geqslant 1$,

$$[I - \Phi_0(x)] * G_j(x) = A_{-j}(x) + \sum_{k=1}^{\infty} R_k(x) * [I - \Phi_0(x)] * G_{j+k}(x) \tag{6.45}$$

and

$$\Phi_0(x) = A_0(x) + \sum_{k=1}^{\infty} R_k(x) * [I - \Phi_0(x)] * G_k(x). \tag{6.46}$$

(2) *For* $i \geqslant 1$,

$$R_{0,i}(x) * [I - \Phi_0(x)] = D_i(x) + \sum_{k=1}^{\infty} R_{0,i+k}(x) * [I - \Phi_0(x)] * G_k(x), \tag{6.47}$$

for $j \geqslant 1$,

$$[I - \Phi_0(x)] * G_{j,0}(x) = D_{-j}(x) + \sum_{k=1}^{\infty} R_k(x) * [I - \Phi_0(x)] * G_{j+k,0}(x) \tag{6.48}$$

and

$$\Psi_0(x) = D_0(x) + \sum_{k=1}^{\infty} R_{0,k}(x) * [I - \Phi_0(x)] * G_{k,0}(x). \tag{6.49}$$

Proof We only prove Eq. (6.44) while Eq. (6.45) to Eq. (6.49) can be proved similarly.

When n is big enough, it follows from Theorem 6.8 that

$$R_i(x) * [I - \Phi_0(x)] = \Phi_i(x) * \sum_{l=0}^{\infty} \Phi_0^{l*}(x) * [I - \Phi_0(x)] = \Phi_i(x) = P_{n-i,n}^{[\leqslant n]}(x),$$

and

$$\begin{aligned} P_{n-i,n}^{[\leqslant n]}(x) &= P_{n-i,n}^{[\leqslant (n+1)]}(x) + R_{i+1}^{[\leqslant (n+1)]}(x) * P_{n+1,n}^{[\leqslant (n+1)]}(x) \\ &= P_{n-i,n}^{[\leqslant (n+1)]}(x) + R_{i+1}^{[\leqslant (n+1)]}(x) * \left[I - P_{n+1,n+1}^{[\leqslant (n+1)]}(x) \right] * G_1^{[\leqslant (n+1)]}(x) \\ &= P_{n-i,n}^{[\leqslant (n+1)]}(x) + R_{i+1}(x) * [I - \Phi_0(x)] * G_1(x) \\ &= \ldots = P_{n-i,n}^{[\leqslant (n+N)]}(x) + \sum_{k=1}^{N} R_{i+k}(x) * [I - \Phi_0(x)] * G_k(x) \\ &= A_i(x) + \sum_{k=1}^{\infty} R_{i+k}(x) * [I - \Phi_0(x)] * G_k(x). \end{aligned}$$

This completes the proof.

For an irreducible Markov renewal process of $GI/G/1$ type, the UL-type RG-factorization given in Eq. (6.22) can be simplified as

$$I - P(x) = [I - R_U(x)] * [I - \Phi_D(x)] * [I - G_L(x)], \tag{6.50}$$

where

$$R_U(x) = \begin{pmatrix} 0 & R_{0,1}(x) & R_{0,2}(x) & R_{0,3}(x) & \cdots \\ & 0 & R_1(x) & R_2(x) & \cdots \\ & & 0 & R_1(x) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$\Phi_D(x) = \operatorname{diag}\left(\Psi_0(x), \Phi_0(x), \Phi_0(x), \Phi_0(x), \ldots\right)$$

and

$$G_L(x) = \begin{pmatrix} 0 & & & & \\ G_{1,0}(x) & 0 & & & \\ G_{2,0}(x) & G_1(x) & 0 & & \\ G_{3,0}(x) & G_2(x) & G_1(x) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is worthwhile to note that the LU-type RG-factorization given in Eq. (6.40) can not be further simplified. Thus we will not discuss the LU-type RG-factorization here.

For the Markov renewal process of $GI/G/1$ type, we write

$$\widetilde{A}_i(s) = \int_0^{+\infty} \mathrm{e}^{-sx}\mathrm{d}A_i(x), \ \widetilde{\varPhi}_0(s) = \int_0^{+\infty} \mathrm{e}^{-sx}\mathrm{d}\varPhi_0(x),$$

$$\widetilde{R}_i(s) = \int_0^{+\infty} \mathrm{e}^{-sx}\mathrm{d}R_i(x), \ \widetilde{G}_i(s) = \int_0^{+\infty} \mathrm{e}^{-sx}\mathrm{d}G_i(x),$$

$$\widetilde{A}^*(z,s) = \sum_{i=-\infty}^{\infty} z^i \widetilde{A}_i(s),$$

$$\widetilde{R}^*(z,s) = \sum_{i=1}^{\infty} z^i \widetilde{R}_i(s), \ \widetilde{G}^*(z,s) = \sum_{j=1}^{\infty} z^{-j} \widetilde{G}_j(s).$$

The following theorem provides the RG-factorization for the repeated blocks based on the double transformations. Note that this RG-factorization is very useful for transient performance analysis such as the first passage time and the sojourn time.

Theorem 6.10

$$I - \widetilde{A}^*(z,s) = [I - \widetilde{R}^*(z,s)][I - \widetilde{\varPhi}_0(s)][I - \widetilde{G}^*(z,s)]. \tag{6.51}$$

Proof It follows from Eq. (6.44), Eq. (6.45) and Eq. (6.46) that

$$\widetilde{R}_i(s)[I - \widetilde{\varPhi}_0(s)] = \widetilde{A}_i(s) + \sum_{k=1}^{\infty} \widetilde{R}_{i+k}(s)[I - \widetilde{\varPhi}_0(s)]\widetilde{G}_k(s), \quad i \geqslant 1,$$

$$[I - \widetilde{\varPhi}_0(s)]\widetilde{G}_j(s) = \widetilde{A}_{-j}(s) + \sum_{k=1}^{\infty} \widetilde{R}_k(s)[I - \widetilde{\varPhi}_0(s)]\widetilde{G}_{j+k}(s), \quad j \geqslant 1,$$

$$\widetilde{\varPhi}_0(s) = \widetilde{A}_0(s) + \sum_{k=1}^{\infty} \widetilde{R}_k(s)[I - \widetilde{\varPhi}_0(s)]\widetilde{G}_k(x).$$

Therefore, we obtain

$$\begin{aligned}\widetilde{R}^*(z,s)[I - \widetilde{\varPhi}_0(s)] + [I - \widetilde{\varPhi}_0(s)]\widetilde{G}^*(z,s) + \widetilde{\varPhi}_0(s) \\ = I_1 + I_2 + \sum_{k=1}^{\infty} z^k \widetilde{R}_k(s)[I - \widetilde{\varPhi}_0(s)]z^{-k}\widetilde{G}_k(x),\end{aligned}$$

where

$$\begin{aligned}I_1 &= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} z^{i+k}\widetilde{R}_{i+k}(s)[I - \widetilde{\varPhi}_0(s)]z^{-k}\widetilde{G}_k(x) \\ &= \widetilde{R}^*(z,s)[I - \widetilde{\varPhi}_0(s)][I - \widetilde{G}^*(z,s)] \\ &\quad - \sum_{k=1}^{\infty}\sum_{i=1}^{k} z^i \widetilde{R}_i(s)[I - \widetilde{\varPhi}_0(s)]z^{-k}\widetilde{G}_k(x)\end{aligned}$$

and

$$I_2 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} z^k \widetilde{R}_k(s)\left[I-\widetilde{\varPhi}_0(s)\right] z^{-(j+k)} \widetilde{G}_{j+k}(x)$$
$$= \widetilde{R}^*(z,s)\left[I-\widetilde{\varPhi}_0(s)\right]\left[I-\widetilde{G}^*(z,s)\right]$$
$$-\sum_{k=1}^{\infty}\sum_{j=k}^{\infty} z^j \widetilde{R}_j(s)\left[I-\widetilde{\varPhi}_0(s)\right] z^{-k} \widetilde{G}_k(x).$$

Since

$$I_1 + I_2 + \sum_{k=1}^{\infty} z^k \widetilde{R}_k(s)[I-\widetilde{\varPhi}_0(s)] z^{-k} \widetilde{G}_k(x) = \widetilde{A}^*(z,s)$$
$$+ \widetilde{R}^*(z,s)[I-\widetilde{\varPhi}_0(s)]\widetilde{G}^*(z,s),$$

we get

$$\widetilde{R}^*(z,s)[I-\widetilde{\varPhi}_0(s)] + [I-\widetilde{\varPhi}_0(s)]\widetilde{G}^*(z,s) + \widetilde{\varPhi}_0(s) = \widetilde{A}^*(z,s)$$
$$+ \widetilde{R}^*(z,s)[I-\widetilde{\varPhi}_0(s)]\widetilde{G}^*(z,s),$$

which is equivalent to Eq. (6.51). This completes the proof.

Let

$$\widetilde{D}_i(s) = \int_0^{+\infty} \mathrm{e}^{-sx} \mathrm{d}D_i(x),$$

$$\widetilde{D}_+^*(z,s) = \sum_{i=1}^{\infty} z^i \widetilde{D}_i(s), \quad \widetilde{D}_-^*(z,s) = \sum_{i=1}^{\infty} z^{-i} \widetilde{D}_{-i}(s),$$

$$\widetilde{R}_{0,i}(s) = \int_0^{+\infty} \mathrm{e}^{-sx} \mathrm{d}R_{0,i}(x), \quad \widetilde{G}_{i,0}(s) = \int_0^{+\infty} \mathrm{e}^{-sx} \mathrm{d}G_{i,0}(x),$$

and

$$\widetilde{R}_0^*(z,s) = \sum_{i=1}^{\infty} z^i \widetilde{R}_{0,i}(s), \quad \widetilde{G}_0^*(z,s) = \sum_{j=1}^{\infty} z^{-j} \widetilde{G}_{j,0}(s). \tag{6.52}$$

Theorem 6.11 *For* $z > 0$ *and* $s \geqslant 0$,

$$\widetilde{R}_0^*(z,s) \geqslant \widetilde{D}_+^*(z,s)[I-\widetilde{\varPhi}_0(s)]^{-1}, \tag{6.53}$$

$$\widetilde{R}_0^*(z,s)[I-\widetilde{\varPhi}_0(s)][I-\widetilde{G}^*(z,s)] \leqslant \widetilde{D}_+^*(z,s), \tag{6.54}$$

$$\widetilde{G}_0^*(z,s) \geqslant [I-\widetilde{\varPhi}_0(s)]^{-1}\widetilde{D}_-^*(z,s) \tag{6.55}$$

and

$$[I-\widetilde{R}^*(z,s)][I-\widetilde{\varPhi}_0(s)]\widetilde{G}_0^*(z,s) \leqslant \widetilde{D}_-^*(z,s). \tag{6.56}$$

Proof We only prove Eq. (6.53) and Eq. (6.54), while Eq. (6.55) and Eq. (6.56) can be proved similarly.

It follows from Eq. (6.47) that

$$\widetilde{R}_{0,i}(s)[I-\widetilde{\Phi}_0(s)] = \widetilde{D}_i(s) + \sum_{k=1}^{\infty} \widetilde{R}_{0,i+k}(s)[I-\widetilde{\Phi}_0(s)]\widetilde{G}_k(s), \tag{6.57}$$

and from Eq. (6.43) that

$$\widetilde{G}_k(s) = [I-\widetilde{\Phi}_0(s)]^{-1}\widetilde{\Phi}_{-k}(s).$$

It is obvious that for $s \geqslant 0$,

$$\sum_{k=1}^{\infty} \widetilde{R}_{0,i+k}(s)[I-\widetilde{\Phi}_0(s)]\widetilde{G}_k(s) = \sum_{k=1}^{\infty} \widetilde{R}_{0,i+k}(s)\widetilde{\Phi}_{-k}(s) \geqslant 0.$$

Hence it follows from Eq. (6.57) that

$$\widetilde{R}_{0,i}(s)[I-\widetilde{\Phi}_0(s)] \geqslant \widetilde{D}_i(s)$$

and for $z > 0$,

$$\widetilde{R}_0^*(z,s)\left[I-\widetilde{\Phi}_0(s)\right] \geqslant \widetilde{D}_+^*(z,s). \tag{6.58}$$

Since the Markov renewal process is irreducible, the spectral radius

$$sp(\widetilde{\Phi}_0(s)) \leqslant sp(\widetilde{\Phi}_0(0)) = sp(\Phi_0(+\infty)) < 1$$

for all $s > 0$. Furthermore, the matrix $I-\widetilde{\Phi}_0(s)$ is invertible and $[I-\widetilde{\Phi}_0(s)]^{-1} \geqslant 0$. Therefore, it follows from Eq. (6.19) that

$$\widetilde{R}_0^*(z,s) \geqslant \widetilde{D}_+^*(z,s)[I-\widetilde{\Phi}_0(s)]^{-1}.$$

It follows from Eq. (6.58) that

$$\widetilde{R}_0^*(z,s)[I-\widetilde{\Phi}_0(s)] \leqslant \widetilde{D}_+^*(z,s) + \widetilde{R}_0^*(z,s)[I-\widetilde{\Phi}_0(s)]\widetilde{G}^*(z,s),$$

simple computations lead to

$$\widetilde{R}_0^*(z,s)[I-\widetilde{\Phi}_0(s)][I-\widetilde{G}^*(z,s)] \leqslant \widetilde{D}_+^*(z,s).$$

This completes the proof.

6.6 Spectral Analysis

In this section, we provide spectral properties for the R- and G-measures of a Markov renewal process of $GI/G/1$ type. These spectral properties are important in the study of stochastic models. For simplicity, we assume that the matrix $A = \widetilde{A}^*(1,0)$ is irreducible and stochastic.

To discuss the equations $\det(I - \widetilde{R}^*(z,s)) = 0$ and $\det(I - \widetilde{G}^*(z,s)) = 0$, we first need to provide the relations among the radii of convergence for some matrix functions.

For $s \geqslant 0$, we denote by $\phi_R(s)$, $\phi_G(s)$, $\phi_{R_0}(s)$, $\phi_{G_0}(s)$, $\phi_{A+}(s)$, $\phi_{A-}(s)$, $\phi_{D+}(s)$ and $\phi_{D-}(s)$ the convergence radii of the matrices $\widetilde{R}^*(z,s)$, $\widetilde{G}^*(z,s)$, $\widetilde{R}_0^*(z,s)$, $\widetilde{G}_0^*(z,s)$, $\widetilde{A}_+^*(z,s)$, $\widetilde{A}_-^*(z,s)$, $\widetilde{D}_+^*(z,s)$ and $\widetilde{D}_-^*(z,s)$, respectively, where $\widetilde{A}_+^*(z,s) = \sum_{k=1}^{\infty} z^k \widetilde{A}_k(s)$ and $\widetilde{A}_-^*(z,s) = \sum_{k=1}^{\infty} z^{-k} \widetilde{A}_{-k}(s)$.

Theorem 6.12 *For* $s \geqslant 0$,

(1)

$$\phi_{A+}(s) = \phi_R(s) \geqslant 1, \quad 0 \leqslant \phi_{A-}(s) = \phi_G(s) \leqslant 1; \tag{6.59}$$

(2)

$$\phi_{D+}(s) = \phi_{R_0}(s), \quad \phi_{D-}(s) = \phi_{G_0}(s). \tag{6.60}$$

Proof We first prove (1). It follows from Eq. (6.51) that

$$\begin{aligned}\widetilde{A}^*(z,s) = {} & \widetilde{R}^*(z,s)[I - \widetilde{\Phi}_0(s)][I - \widetilde{G}^*(z,s)] + \widetilde{\Phi}_0(s) \\ & + [I - \widetilde{\Phi}_0(s)]\widetilde{G}^*(z,s),\end{aligned} \tag{6.61}$$

and

$$\begin{aligned}\widetilde{A}^*(z,s) = {} & [I - \widetilde{R}^*(z,s)][I - \widetilde{\Phi}_0(s)]\widetilde{G}^*(z,s) + \widetilde{\Phi}_0(s) \\ & + \widetilde{R}^*(z,s)[I - \widetilde{\Phi}_0(s)].\end{aligned} \tag{6.62}$$

Noting that $\phi_R(s)$ is increasing in $s \geqslant 0$ and $\phi_G(s)$ is decreasing in $s \geqslant 0$, it follows from Corollary 3.10 that $\phi_R(s) \geqslant \phi_R(0) \geqslant 1$ and $0 \leqslant \phi_G(s) \leqslant \phi_G(0) \leqslant 1$. It follows from Eq. (6.52) that $\widetilde{R}^*(z,s)$ is analytic in $|z| < \phi_R(s)$ and $\widetilde{G}^*(z,s)$ is analytic in $|z| > \phi_G(s)$. Also, noting that $I - \widetilde{\Phi}_0(s)$ is invertible, it follows from Eq. (6.61) that $\phi_{A+}(s) = \phi_R(s) \geqslant 1$ and from Eq. (6.62) that $0 \leqslant \phi_{A-}(s) = \phi_G(s) \leqslant 1$.

Next we prove (2). We only prove the first one while the second one can be proved similarly.

For $s \geqslant 0$, it is easy to see that either $\phi_{D+}(s) > 1$ or $\phi_{D+}(s) = 1$. We consider the following two possible cases:

Case Ⅰ $\phi_{D+}(s) > 1$. In this case, since $\widetilde{A}^*(z,s)$ is irreducible, we obtain that for $s > 0$ and $1 < z < \phi_{D+}(s)$,

$$sp(\widetilde{G}^*(z,s)) < sp(\widetilde{G}^*(1,0)) = sp(G) \leqslant 1,$$

thus it follows from Eq. (6.54) that

$$\widetilde{R}_0^*(z,s) \leqslant \widetilde{D}_+^*(z,s)[I - \widetilde{G}^*(z,s)]^{-1}(I - \Phi_0(s))^{-1},$$

it is clear that $\phi_{R_0}(s) \geqslant \phi_{D+}(s)$. On the other hand, it follows from Eq. (6.53) that $\phi_{R_0}(s) \leqslant \phi_{D+}(s)$. Hence $\phi_{R_0}(s) = \phi_{D+}(s)$.

Case Ⅱ $\phi_{D+}(s) = 1$. In this case, it follows from Eq. (6.53) that $\phi_{R_0}(s) \leqslant \phi_{D+}(s) = 1$. Since $\widetilde{R}_0^*(1,s)$ is always finite and has the probabilistic interpretation in Theorem 6.11 $\phi_{R_0}(s) \geqslant 1$. Hence, $\phi_{R_0}(s) = \phi_{D+}(s)$. This completes the proof.

For $z > 0$ and $s > 0$, let $\chi(z,s)$, $r(z,s)$ and $g(z,s)$ be the maximal eigenvalues of $\widetilde{A}^*(z,s)$, $\widetilde{R}^*(z,s)$ and $\widetilde{G}^*(z,s)$, respectively.

The following Lemma provides the useful relations among the minimal positive solutions for the R-measure and among the maximal positive solutions for the G-measure.

Lemma 6.8 (1) *For* $s \geqslant 0$, *the minimal positive solution of the equation* $\det(I - \widetilde{R}^*(z,s)) = 0$ *has the following relations*

$$\min\{z(s) \geqslant 1 : \det(I - \widetilde{R}^*(z,s)) = 0\} = \min\{z(s) \geqslant 1 : 1 - r(z,s) = 0\},$$
$$\min\{z(s) \geqslant 1 : \det(I - \widetilde{A}^*(z,s)) = 0\} = \min\{z(s) \geqslant 1 : 1 - \chi(z,s) = 0\},$$
$$\min\{z(s) \geqslant 1 : 1 - r(z,s) = 0\} = \min\{z(s) \geqslant 1 : 1 - \chi(z,s) = 0\}.$$

(2) *For* $s \geqslant 0$, *the maximal positive solution of the equation* $\det(I - \widetilde{G}^*(z,s)) = 0$ *has the following relations*

$$\max\{z(s) \leqslant 1 : \det(I - \widetilde{G}^*(z,s)) = 0\} = \max\{z(s) \leqslant 1 : 1 - g(z,s) = 0\},$$
$$\max\{z(s) \leqslant 1 : \det(I - \widetilde{A}^*(z,s)) = 0\} = \max\{z(s) \leqslant 1 : 1 - \chi(z,s) = 0\},$$
$$\max\{z(s) \leqslant 1 : 1 - g(z,s) = 0\} = \max\{z(s) \leqslant 1 : 1 - \chi(z,s) = 0\}.$$

We now determine the distribution of the roots of $\det(I - \widetilde{R}^*(z,s)) = 0$ and $\det(I - \widetilde{G}^*(z,s)) = 0$.

Let

$$\eta = \min\{|z|, 1 \leqslant |z| \leqslant \phi_{A+}, \det(I - R^*(z)) = 0\}$$

and

$$\zeta = \max\{|z|, \phi_{A-} \leqslant |z| \leqslant 1, \det(I - G^*(z)) = 0\}.$$

Note that η is the minimal positive solution of the equation $\det(I - R^*(z)) = 0$, and ξ is the maximal positive solution of the equation $\det(I - G^*(z)) = 0$.

Theorem 6.13 *Suppose that* η *and* ξ *are given in* (1) *and* (2) *of Theorem 3.22, respectively.*

(1) *If* $P(x)$ *is positive recurrent, then for* $s \geqslant 0$,

$$\{z(s):\det\{I-\widetilde{R}^*(z,s)\}=0\}\subset\{z:|z|>\eta\} \tag{6.63}$$

and

$$\{z(s):\det\{I-\widetilde{G}^*(z,s)\}=0\}\subset\{z:|z|\leqslant\xi\}.$$

(2) *If* $P(x)$ *is null recurrent, then for* $s\geqslant 0$,

$$\{z(s):\det\{I-\widetilde{R}^*(z,s)\}=0 \text{ or } \det\{I-\widetilde{G}^*(z,s)\}=0\}\subset\{z:\xi\leqslant|z|\leqslant\eta\}.$$

(3) *If* $P(x)$ *is transient, then for* $s\geqslant 0$,

$$\{z(s):\det\{I-\widetilde{R}^*(z,s)\}=0\}\subset\{z:|z|\geqslant\eta\}$$

and

$$\{z(s):\det\{I-\widetilde{G}^*(z,s)\}=0\}\subset\{z:|z|<\xi\}.$$

Proof We only prove Eq. (6.63) while the other four can be proved similarly. It follows from Lemma 6.8 that

$$\min\{z(s)\geqslant 1:\det(I-\widetilde{R}^*(z,s))=0\}=\min\{z(s)\geqslant 1:$$
$$\det\{I-\widetilde{A}^*(z,s)\}=0\}=\min\{z(s)\geqslant 1:1-\chi(z,s)=0\}.$$

Since $\widetilde{A}^*(z,s)$ is irreducible for $z\geqslant 0$ and $s\geqslant 0$, $\chi(z,s)$ is strictly increasing for $z\geqslant 0$ and strictly decreasing for $s\geqslant 0$, hence $z_0(s)=\min\{z(s)\geqslant 1:1-\chi(z,s)=0\}$ is increasing for $s\geqslant 0$. A similar analysis to that used for proving Theorem 3.22 leads to the fact that if $P(+\infty)$ is positive recurrent, then for any solution $z=z(s)$ to equation $\det\{I-\widetilde{R}^*(z,s)\}=0$,

$$|z(s)|\geqslant z_0(s)>z_0(0)\geqslant\eta.$$

This completes the proof.

The following theorem provides the positive roots for the equations $\det(I-\widetilde{R}^*(z,s))=0$ and $\det(I-\widetilde{G}^*(z,s))=0$.

Theorem 6.14 (1) *If* $P(x)$ *is positive recurrent, then for each* $s>0$, *there must exist a unique* $z_0(s)$ *with* $1<\eta<z_0(s)$ *such that* $\det(I-\widetilde{R}^*(z_0(s),s))=0$.

(2) *If* $P(x)$ *is transient, then for each* $s>0$, *there must exist a unique* $z_0(s)$ *for* $0<z_0(s)<\xi<1$ *such that* $\det(I-\widetilde{G}^*(z_0(s),s))=0$.

Proof We only prove 1) while 2) can be proved similarly.

We write $f(z,s)=1-\chi(z,s)$. Since $\widetilde{A}^*(z,s)$ is nonnegative and irreducible for $z>0$ and $s\geqslant 0$, it is obvious that $\chi(z,s)$ is strictly increasing for $z>0$ and strictly decreasing for $s\geqslant 0$. Noting that $\chi(\eta,0)=1$, we obtain that for any given $s>0$,

$$f(\eta,s)=1-\chi(\eta,s)>1-\chi(\eta,0)=0. \tag{6.64}$$

On the other hand, since for any given $s>0$, $\lim_{z \nearrow +\infty} \widetilde{A}^*(z,s)=+\infty$, so that $\chi(+\infty,s)=+\infty$, we get

$$f(+\infty,s)=1-\chi(+\infty,s)<0, \tag{6.65}$$

Noting that for an arbitrarily given $s>0$, $f(z,s)$ is continuous and strictly increasing for $z \in [\eta,+\infty)$, it follows from Eq. (6.64) and Eq. (6.65) that there always exists a unique positive solution $z_0(s) \in (\eta,+\infty)$ such that

$$f(z_0(s),s)=1-\chi(z_0(s),s)=0.$$

Therefore, for $z>0$ and $s \geqslant 0$ it follows from Theorem 3.22 and Eq. (6.64) that

$$\begin{aligned} z_0(s) &= \min\{z(s)>\eta>1:1-\chi(z,s)=0\} \\ &= \min\{z(s)>\eta>1:\det(I-\widetilde{R}^*(z,s))=0\}. \end{aligned}$$

This completes the proof.

6.7 The First Passage Times

In this section, we first provide an algorithmic framework for computing the first passage time of any irreducible Markov renewal process, and then consider the first passage times for a Markov renewal process of $GI/G/1$ type.

6.7.1 An Algorithmic Framework

Consider a Markov renewal process $\{(X_n,T_n),n\geqslant 0\}$ whose transition probability mass matrix is given in Eq. (6.1). Let $E=\{0\}$ and $E^c=\{1,2,3,4,\ldots\}$. According to the subsets E and E^c, the transition probability mass matrix $P(x)$ is partitioned as

$$P(x)= \begin{array}{c} \\ E \\ E^c \end{array} \!\!\begin{array}{c} \begin{array}{cc} E & \quad E^c \end{array} \\ \begin{pmatrix} T(x) & U(x) \\ V(x) & Q(x) \end{pmatrix} \end{array}. \tag{6.66}$$

Let (α_0,α) be the initial probability vector of the Markov renewal process $\{(X_n,T_n),n\geqslant 0\}$, where $\alpha=(\alpha_1,\alpha_2,\alpha_3,\ldots)$ and $\alpha_0+\alpha e=1$. We denote by $f_k(x)$ the joint probability that the Markov renewal process $\{(X_n,T_n),n\geqslant 0\}$ first visits state 0 at the kth step no later than time $x\geqslant 0$. It is easy to check that

$$f_0(x)=\alpha_0\delta(x) \tag{6.67}$$

and

$$f_k(x)=\alpha Q^{(k-1)*}(x)*V(x), \quad k\geqslant 1. \tag{6.68}$$

Taking the Laplace-Stieltjes transforms for Eq. (6.67) and Eq. (6.68), we obtain

$$\widetilde{f}_0^*(s) = \alpha_0$$

and

$$\widetilde{f}_k^*(s) = \alpha[\widetilde{Q}^*(s)]^{k-1}\widetilde{V}^*(s), \quad k \geqslant 1.$$

We write

$$\widetilde{F}^*(s,z) = \sum_{k=0}^{\infty} z^k \widetilde{f}_k^*(s).$$

It is clear that

$$\widetilde{F}^*(s,z) = \alpha_0 + z\alpha[I - z\widetilde{Q}^*(s)]^{-1}\widetilde{V}^*(s).$$

When $s > 0$ and $0 < z \leqslant 1$, it is clear that $z\widetilde{Q}^*(s)$ is the transition probability matrix of a Markov chain. In this case, we can obtain the UL-type *RG*-factorization

$$I - z\widetilde{Q}^*(s) = [I - R_U(s,z)][I - U_D(s,z)][I - G_L(s,z)],$$

which leads to that

$$\widetilde{F}^*(s,z) = \alpha_0 + z\alpha[I - G_L(s,z)]^{-1}[I - U_D(s,z)]^{-1}[I - R_U(s,z)]^{-1}\widetilde{V}^*(s);$$

and the LU-type *RG*-factorization

$$I - z\widetilde{Q}^*(s) = [I - \overline{R}_L(s,z)][I - \overline{U}_D(s,z)][I - \overline{G}_U(s,z)],$$

which yields that

$$\widetilde{F}^*(s,z) = \alpha_0 + z\alpha[I - \overline{G}_U(s,z)]^{-1}[I - \overline{U}_D(s,z)]^{-1}[I - \overline{R}_L(s,z)]^{-1}\widetilde{V}^*(s).$$

6.7.2 Markov Renewal Processes of *GI*/*G*/1 Type

Let $\widetilde{A}(s) = \widetilde{A}^*(1,s)$, $\widetilde{R}(s) = \widetilde{R}^*(1,s)$ and $\widetilde{G}(s) = \widetilde{G}^*(1,s)$. Then it follows from Eq. (6.51) that

$$I - \widetilde{A}(s) = [I - \widetilde{R}(s)][I - \widetilde{\varPhi}_0(s)][I - \widetilde{G}(s)]. \tag{6.69}$$

Hence, we obtain

$$I - A(x) = [I - R(x)] * [I - \varPhi_0(x)] * [I - G(x)]. \tag{6.70}$$

Notice that $G(x) = \sum_{k=1}^{\infty} G_k(x)$ is the distribution matrix of the first passage times of the Markov renewal process of $GI/G/1$ type from $L_{\geqslant i}$ to $L_{\leqslant(i-1)}$ for $i \geqslant 2$.

Specifically, for a Markov renewal process of $M/G/1$ type $G(x)=G_1(x)$ is called the matrix distribution of the fundamental period according to Chapter 2 and Chapter 3 in Neuts [17]. It is clear that if the Markov chain $P(+\infty)$ is recurrent, then $G(+\infty)$ is stochastic; otherwise $G(+\infty)$ is strictly substochastic.

For a matrix $\widetilde{M}(s)$ we write $\mu_k(M)=(-1)^k \frac{d^k}{ds^k}\widetilde{M}(s)|_{s=0}$. In the following, we show how to compute conditional moments of the matrix distribution $G(x)$, defined as $\mu_k(G)$, based on Eq. (6.69). It follows from Eq. (6.69) that the first conditional moment of $G(x)$ satisfies

$$\begin{aligned}\mu_1(G)=&\ \mu_1(R)(I-\Phi_0)(I-G)+(I-R)\mu_1(\Phi_0)(I-G)\\&+(I-R)(I-\Phi_0)\mu_1(G),\end{aligned}$$

and the second conditional moment of $G(x)$ satisfies

$$\begin{aligned}\mu_2(G)=&\ \mu_2(R)(I-\Phi_0)(I-G)+(I-R)\mu_2(\Phi_0)(I-G)\\&+(I-R)(I-\Phi_0)\mu_2(G)-2\mu_1(R)\mu_1(\Phi_0)(I-G)\\&-2(I-R)\mu_1(\Phi_0)\mu_1(G)-2\mu_1(R)(I-\Phi_0)\mu_1(G).\end{aligned}$$

Specifically, if the Markov renewal process is recurrent, then the first conditional moment of $G(x)$ is simplified as

$$\mu_1(G)e=(I-\Phi_0)^{-1}(I-R)^{-1}\mu_1(A)e,$$

and the second conditional moment of $G(x)$ is simplified as

$$\begin{aligned}\mu_2(G)e=&\ (I-\Phi_0)^{-1}(I-R)^{-1}\{\mu_2(A)+2(I-R)\mu_1(\Phi_0)\mu_1(G)\\&+2\mu_1(R)(I-\Phi_0)\mu_1(G)\}e\end{aligned}$$

since $(I-G)e=0$.

Now, we consider the matrix distribution $G_0(x)=\sum_{k=1}^{\infty}G_{k,0}(x)$, which often represents the matrix distribution of the busy period. To explicitly express $G_0(x)$, we need to compute $G_{k,0}(x)$ for all $k \geqslant 1$. It follows from Eq. (6.42) that

$$[I-\widetilde{\Phi}_0(s)]\widetilde{G}_{j,0}(s)=\widetilde{D}_{-j}(s)+\sum_{k=1}^{\infty}\widetilde{\Phi}_k(s)\widetilde{G}_{k+j,0}(s), \quad j \geqslant 1,$$

hence,

$$\begin{pmatrix} I & -B_1(s) & -B_2(s) & -B_3(s) & \cdots \\ & I & -B_1(s) & -B_2(s) & \cdots \\ & & I & -B_1(s) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix}\begin{pmatrix} \widetilde{G}_{1,0}(s) \\ \widetilde{G}_{2,0}(s) \\ \widetilde{G}_{3,0}(s) \\ \widetilde{G}_{4,0}(s) \\ \vdots \end{pmatrix}=\begin{pmatrix} F_1(s) \\ F_2(s) \\ F_3(s) \\ F_4(s) \\ \vdots \end{pmatrix}, \qquad (6.71)$$

where $B_i(s)=[I-\widetilde{\Phi}_0(s)]^{-1}\widetilde{\Phi}_i(s)$ and $F_i(s)=[I-\widetilde{\Phi}_0(s)]^{-1}\widetilde{D}_{-i}(s)$ for $i\geqslant 1$.

Lemma 6.9 *Let*

$$\Lambda(s)=\begin{pmatrix} I & -B_1(s) & -B_2(s) & -B_3(s) & \cdots \\ & I & -B_1(s) & -B_2(s) & \cdots \\ & & I & -B_1(s) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Then for $\mathrm{Re}(s)\geqslant 0$*, there always exists a unique inverse matrix* $\Lambda^{-1}(s)$ *such that* $\Lambda^{-1}(s)\Lambda(s)=\Lambda(s)\Lambda^{-1}(s)=I$*, and*

$$\Lambda^{-1}(s)=\begin{pmatrix} I & X_1(s) & X_2(s) & X_3(s) & \cdots \\ & I & X_1(s) & X_2(s) & \cdots \\ & & I & X_1(s) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix}, \tag{6.72}$$

where

$$X_l(s)=\sum_{i=1}^{\infty}\sum_{\substack{n_1+n_2+\ldots n_i=l\\ n_j\geqslant 1, 1\leqslant j\leqslant i}} B_{n_1}(s)B_{n_2}(s)\ldots B_{n_i}(s), \quad l\geqslant 1.$$

Proof Noting that $\Lambda(s)\Lambda^{-1}(s)=I$, we obtain

$$X_k(s)-B_k(s)-\sum_{i=1}^{k-1}B_i(s)X_{k-i}(s)=0, \quad k\geqslant 1. \tag{6.73}$$

Let $X^*(z,s)=\sum_{k=1}^{\infty}z^kX_k(s)$ and $B^*(z,s)=\sum_{k=1}^{\infty}z^kB_k(s)$. Then it follows from Eq. (6.73) that

$$X^*(z,s)-B^*(z,s)-B^*(z,s)X^*(z,s)=0.$$

We obtain

$$\begin{aligned} X^*(z,s)&=[I-B^*(z,s)]^{-1}B^*(z,s)=\sum_{i=1}^{\infty}[B^*(z,s)]^i \\ &=\sum_{l=1}^{\infty}z^l\sum_{i=1}^{\infty}\sum_{\substack{n_1+n_2+\ldots n_i=l\\ n_j\geqslant 1, 1\leqslant j\leqslant i}} B_{n_1}(s)B_{n_2}(s)\ldots B_{n_i}(s). \end{aligned}$$

Therefore,

$$X_l(s)=\sum_{i=1}^{\infty}\sum_{\substack{n_1+n_2+\ldots n_i=l\\ n_j\geqslant 1,\,1\leqslant j\leqslant i}} B_{n_1}(s)B_{n_2}(s)\ldots B_{n_i}(s),\quad l\geqslant 1.$$

This completes the proof.

Theorem 6.15 *For* $\operatorname{Re}(s)\geqslant 0$*,*

$$\begin{aligned}\widetilde{G}_0(s)=&\left[I-\widetilde{\varPhi}_0(s)\right]^{-1}\sum_{k=1}^{\infty}\widetilde{D}_{-k}(s)+\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{i=1}^{\infty}\sum_{\substack{n_1+n_2+\ldots n_i=l\\ n_j\geqslant 1,\,1\leqslant j\leqslant i}}[I-\widetilde{\varPhi}_0(s)]^{-1}\\ &\cdot\widetilde{\varPhi}_{n_1}(s)[I-\widetilde{\varPhi}_0(s)]^{-1}\widetilde{\varPhi}_{n_2}(s)\ldots[I-\widetilde{\varPhi}_0(s)]^{-1}\widetilde{\varPhi}_{n_i}(s)\\ &\cdot[I-\widetilde{\varPhi}_0(s)]^{-1}\widetilde{D}_{-(k+l)}(s).\end{aligned}$$

Proof It follows from Eq. (6.71) that

$$\begin{aligned}\widetilde{G}_{k,0}(s)=&F_k(s)+\sum_{l=1}^{\infty}X_l(s)F_{k+l}(s)\\ =&[I-\widetilde{\varPhi}_0(s)]^{-1}\widetilde{D}_{-k}(s)+\sum_{l=1}^{\infty}\sum_{i=1}^{\infty}\sum_{\substack{n_1+n_2+\ldots n_i=l\\ n_j\geqslant 1,\,1\leqslant j\leqslant i}}[I-\widetilde{\varPhi}_0(s)]^{-1}\\ &\cdot\widetilde{\varPhi}_{n_1}(s)[I-\widetilde{\varPhi}_0(s)]^{-1}\widetilde{\varPhi}_{n_2}(s)\ldots[I-\widetilde{\varPhi}_0(s)]^{-1}\widetilde{\varPhi}_{n_i}(s)\\ &\cdot[I-\widetilde{\varPhi}_0(s)]^{-1}\widetilde{D}_{-(k+l)}(s).\end{aligned}\tag{6.74}$$

Noting that $\widetilde{G}_0(s)=\sum_{k=1}^{\infty}\widetilde{G}_{k,0}(s)$, simple computations yield the proof.

Now, we provide conditions on the state classification for the Markov renewal processes of $GI/G/1$ type.

Based on the result in Theorem 6.15, simple computations can lead to the following corollary.

Corollary 6.3 (1) *If* $P(x)$ *is recurrent, then* $G_0(+\infty)e=e$*.*

(2) *If* $P(x)$ *is transient, then* $G_0(+\infty)e\lneq e$*.*

Remark 6.3 *For a Markov renewal process of* $M/G/1$ *type, since* $\widetilde{D}_{-k}(s)=0$*,* $k\geqslant 2$*, it is clear that* $\widetilde{G}_0(s)=\left[I-\widetilde{\varPhi}_0(s)\right]^{-1}\widetilde{D}_{-1}(s)$*, which is the same as (2.4.3) in Neuts [17] (p. 107). Theorem 6.15 extended Lemma 2.4.1 of Neuts [17] to a Markov renewal process of* $GI/G/1$ *type.*

In what follows we express the transformation of the matrix $\varPsi_0(x)=P^{[0]}(x)$. It follows from Theorem 6.1 that

$$\widetilde{\varPsi}_0(s)=\widetilde{D}_0(s)+\widetilde{U}(s)[I-\widetilde{Q}(s)]^{-1}\widetilde{V}(s).\tag{6.75}$$

Let $\kappa = \frac{\partial}{\partial z}\widetilde{P^{[0]}}^{*}(z,s)e\,|_{(z,s)=(1,0)}$. Then

$$\kappa = D_0 e + (D_1, D_2, D_3, \ldots)(2I - 3Q)(I - Q)^{-2}(D_{-1}^{\mathrm{T}}, D_{-2}^{\mathrm{T}}, D_{-3}^{\mathrm{T}}, \ldots)^{\mathrm{T}} e.$$

Therefore, if the Markov chain is positive recurrent, it follows from Corollary 3.5 in Asmussen [2] that

$$\pi_0 = (x_0 \kappa)^{-1} x_0, \tag{6.76}$$

where x_0 is the stationary probability vector of $\Psi_0(+\infty)$.

Remark 6.4 *If the Markov renewal process is of $M/G/1$ type, Eq.* (6.75) *is the same as (3.2.1) in Neuts [17] (p. 135) and Eq.* (6.76) *is the same as (3.2.6) in Neuts [17] (p. 137).*

It is obvious that the state classification for the Markov chain $P(+\infty)$ is crucial for classifying the Markov renewal process $P(x)$ as recurrent or transient. The following corollary, a consequence of Proposition 6.2, provides a sufficient condition for a recurrent Markov renewal process of $GI/G/1$ type to be positive recurrent.

Corollary 6.4 *A Markov renewal process $P(x)$ of $GI/G/1$ type is positive recurrent if*

(1) $P(+\infty)$ *is positive recurrent and*

(2) $\int_0^{+\infty} x D_{-k}(x)\mathrm{d}x$ *for all* $k \geqslant 0$, $\sum_{k=1}^{\infty}\int_0^{+\infty} x D_k(x)\mathrm{d}x$ *and* $\sum_{k=-\infty}^{\infty}\int_0^{+\infty} x A_k(x)\mathrm{d}x$ *are all finite.*

6.8 Notes in the Literature

The literature on Markov renewal processes is extensive. References, which are closely related to this chapter, include Pyke [19 – 21], Pyke and Schaufele [22 – 23], Teugels [28], Hunter [11], Çinlar [5, 7, 8], Todorovic and Gani [29], Malinovskiĭ [15], Lam [12], Johnson, Liu and Narayana [10], Rossetti and Clark [26], Vesilo [30], Ball and Milne [4]. For a comprehensive discussion on Markov renewal processes, readers may refer to a survey article by Çinlar [6].

The study of block-structured Markov renewal processes provides a useful modelling tool. Readers may refer to Neuts [17] for a study of Markov renewal processes of $M/G/1$ type. Other references on block-structured Markov renewal processes include Neuts [16, 18], Sengupta [27], Asmussen and Ramaswami [3], Ramaswami [24 – 25], Zhao, Li and Alfa [31] and Hsu, Yuan and Li [9] among others. Li and Zhao [13 – 14] systemically studied the UL-types RG-factorization for block-structured Markov renewal processes, while the LU-types RG-factorization is given in this chapter for the first time.

In this chapter, we mainly refer to Li and Zhao [13 − 14], Çinlar [5 − 8] and Neuts [17].

Problems

6.1 Prove Proposition 6.2

6.2 Prove Proposition 6.3

6.3 Provide a concrete example to indicate that Condition (2) of Proposition 6.3 can be further weakened.

6.4 Please provide an example to indicate that the embedded Markov chain $P(+\infty)$ is positive recurrent, but the Markov renewal process $P(x)$ is not positive recurrent.

6.5 Prove Corollary 6.1.

6.6 Prove the censoring invariance for the block-structured Markov renewal process as follows:

(1) for $0 \leqslant i < j \leqslant n$, $R_{i,j}^{[\leqslant n]}(x) = R_{i,j}(x)$; and

(2) for $0 \leqslant j < i \leqslant n$, $G_{i,j}^{[\leqslant n]}(x) = G_{i,j}(x)$.

6.7 Prove Corollary 6.2.

6.8 If $P(x)$ is the transition probability mass matrix of an irreducible QBD renewal process with infinitely-many levels, please compute the Markov renewal matrix $M(x)$.

6.9 If the transition probability mass matrix $P(x)$ is of $GI/G/1$ type, please compute the Markov renewal matrix $M(x)$.

6.10 Consider a $MAP/G/1$ with a repairable server, where the life time and the repair time of the server are exponential and general, respectively. Compute the reliability function by means of the UL-type RG-factorization.

6.11 For the Markov renewal process of $GI/M/1$ type, please express $G_{i,j}(x)$ and $\Psi_j(x)$ by using the matrix $R(x)$.

6.12 In an irreducible Markov renewal process of $GI/G/1$ type, please analyze the asymptotic behavior of the four matrix functions $G(x), G_0(x), R(x)$ and $R_0(x)$ with respect to each of the following three cases:

(1) The matrix function $A(x) = \sum_{k=-\infty}^{\infty} A_k(x)$ is heavy-tailed, and the matrix function $D(x) = \sum_{k=-\infty}^{\infty} D_k(x)$ is light-tailed.

(2) The matrix function $A(x)$ is light-tailed, and the matrix function $D(x)$ is heavy-tailed.

(3) The matrix functions $A(x)$ and $D(x)$ are both heavy-tailed.

6.13 In an irreducible Markov renewal process of $GI/G/1$ type, please analyze the asymptotic behavior of the four matrix functions $G(x), G_0(x), R(x)$ and $R_0(x)$ with respect to each of the following three cases:

(1) The matrix sequence $\{A_k(+\infty)\}$ is heavy-tailed, and the matrix sequence $\{D_k(+\infty)\}$ is light-tailed.

(2) The matrix sequence $\{A_k(+\infty)\}$ is light-tailed, and the matrix sequence $\{D_k(+\infty)\}$ is heavy-tailed.

(3) The matrix sequences $\{A_k(+\infty)\}$ and $\{D_k(+\infty)\}$ are both heavy-tailed.

6.14 In a $MAP/G/1$ queue, if the service time distribution is heavy-tailed, please analyze the asymptotic behavior of the four matrix functions $G_k(x)$ for $k \geqslant 1$, $G(x), R_l(x)$ for $l \geqslant 1$ and $R(x)$, where $G(x) = \sum_{k=1}^{\infty} G_k(x)$ and $R(x) = \sum_{k=1}^{\infty} R_k(x)$.

6.15 In a stable $BMAP/G/1$ queue, please analyze the asymptotic behavior of the stationary queue length, waiting time and busy period with respect to each of the following three cases:

(1) The service time distribution is light-tailed and the BMAP matrix descriptor $\{C_k\}$ is heavy-tailed.

(2) The service time distribution is heavy-tailed and the BMAP matrix descriptor $\{C_k\}$ is light-tailed.

(3) The service time distribution is heavy-tailed and the BMAP matrix descriptor $\{C_k\}$ is heavy-tailed.

6.16 In a $MAP/G/1/N$ queue, when $\rho < 1$ (or $\rho > 1$, or $\rho = 1$), please analyze the asymptotic behavior $\lim_{N \to \infty} q_N$ of the stationary queue length distribution $\{q_k, 0 \leqslant k \leqslant N\}$ with respect to each of the following two cases:

(1) The service time distribution is light-tailed.

(2) The service time distribution is heavy-tailed.

6.17 In a $BMAP/G/1$ queue, compute the transient distributions of the queue length and waiting time.

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7 Examples of Practical Applications

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Abstract In this chapter, we apply the *RG*-factorizations to deal with practical stochastic models, and indicate concrete procedures of performance computation under a unified algorithmic framework. The processor-sharing queue is directly constructed as a block-structured Markov chain, and the fluid queue can be constructed as a block-structured Markov chain by means of the Laplace transform; while the negative-customer queue and the retrial queue need to combine the supplementary variable method and the *RG*-factorizations such that the boundary conditions are simplified as a block-structured Markov chain or a block-structured Markov renewal process. We provide performance analysis of the practical stochastic models.

Keywords practical stochastic models, *RG*-factorization, supplementary variable method, processor-sharing queue, fluid queue, queue with negative customers, retrial queue.

In this chapter, we apply the *RG*-factorizations to deal with practical stochastic models, and indicate concrete procedures both for constructing a block-structured Markov chain and for applying the *RG*-factorizations to provide performance computation under a unified algorithmic framework. The processor-sharing queue is directly constructed as a block-structured Markov chain. The fluid queue can be constructed as a block-structured Markov chain by means of the Laplace transform or the Laplace-Steltjes transform. The negative-customer queue and the retrial queue need to combine the supplementary variable method and the *RG*-factorizations such that the boundary conditions are simplified as a block-structured Markov chain or a block-structured Markov renewal process.

This chapter is organized as follows. Section 7.1 investigates a $BMAP/M/1$ processor-sharing queue with generalized processor-sharing discipline. Section 7.2 discusses an infinite-capacity fluid queue driven by a QBD process. Section 7.3 deals with a $MAP/G/1$ queue with negative customers. Section 7.4 analyzes a

$BMAP/G/1$ retrial queue with a repairable server. Finally, Section 7.5 summarizes the references related to the results of this chapter.

7.1 Processor-Sharing Queues

In this section, we study a $BMAP/M/1$ processor-sharing queue with generalized processor-sharing discipline. Applying the RG-factorizations, we provide the stable condition of the system, and express the stationary distribution of queue length and the Laplace transform of the sojourn time.

The processor-sharing queueing model is described as follows:

The arrival process: The arrivals to the queue are modelled by a BMAP with irreducible matrix descriptor $\{D_k, k \geqslant 0\}$ of size m. We assume that $\sum_{k=1}^{\infty} kD_k$ is finite, and $D = \sum_{k=0}^{\infty} D_k$ is the infinitesimal generator of an irreducible Markov chain with $De = 0$,Where e is a column vector of ones with suitable size. Let σ be the stationary probability vector of D. Then $\lambda = \sigma \sum_{k=1}^{\infty} kD_k e$ is the stationary arrival rate.

The service times: The service times $\{\chi_n, n \geqslant 1\}$ of the customers are assume to be i.i.d. exponential random variables with the service rate μ.

The generalized processor-sharing discipline: When there are n jobs in the system, the attained service of each job increases with rate $f(n)$ relative to being along in the system. Here, $f(n)$ is a positive function such that $0 < C_1 \leqslant nf(n) \leqslant C_2$ for all $n \geqslant 1$, where C_1 and C_2 are two positive constants.

The independence: We assume that all the random variables defined above are mutually independent.

Remark 7.1 (1) *If* $f(n) = 1/n$, *then* $C_1 = C_2 = 1$. *In this case, the generalized processor-sharing becomes the ordinary processor-sharing. In fact, it is easy to see that the generalized processor-sharing describes a wider class of the service disciplines.*

(2) *The condition that* $0 < C_1 < +\infty$ *is necessary for guaranteeing the stability of the generalized processor-sharing queue.*

(3) *If* $C_2 = +\infty$, *then there are two useful cases in practice: (a)* $nf(n) < +\infty$ *for all* $n \geqslant 1$ *but* $\lim_{n \to \infty} nf(n) = +\infty$; *and (b) there exists a finite integer* N *such that* $Nf(N) = +\infty$. *For the two cases, the approach given in this section still works well.*

Let $q(t)$ and $I(t)$ be the number of the jobs in the system and the phase number of the BMAP input at time t, respectively. Then $\{q(t), I(t); t \geqslant 0\}$ is a continuous-time level-dependent Markov chain of $M/G/1$ type whose infinitesimal generator is given by

$$Q=\begin{pmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ f(1)\mu I & D_0-f(1)\mu I & D_1 & D_2 & \cdots \\ & 2f(2)\mu I & D_0-2f(2)\mu I & D_1 & \cdots \\ & & 3f(3)\mu I & D_0-3f(3)\mu I & \cdots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{7.1}$$

Let $C=\inf_{n\geqslant N}\{nf(n)\}$, where N is a large positive integer. Then $0<C<+\infty$ according to the condition $0<C_1\leqslant nf(n)\leqslant C_2$ for all $n\geqslant 1$. Based on the main drift of the Markov chain Q, it is obvious that the generalized processor-sharing queue is stable if $C\mu>\lambda$, since $nf(n)\geqslant C\mu>\lambda$ for all $n\geqslant N$.

For the Markov chain of $M/G/1$ type, let $\{G^{(k)},k\geqslant 1\}$ for the G-measure be the minimal nonnegative solution to the system of matrix equations

$$kf(k)\mu I+[D_0-kf(k)\mu I]G^{(k)}+\sum_{i=1}^{\infty}D_iG^{(k+i)}G^{(k+i-1)}\cdots G^{(k)}=0,\quad k\geqslant 1.$$

Then for $k\geqslant 0$ and $l\geqslant 1$, the R-measure is given by

$$R_l^{(k)}=[D_l+D_{l+1}G^{(k+2)}+D_{l+2}G^{(k+3)}G^{(k+2)}+\cdots]\left(-U_{k+1}^{-1}\right),$$

and for $k\geqslant 0$, the U-measure is given by

$$U_k=D_0-kf(k)\mu I+\sum_{i=1}^{\infty}D_iG^{(k+i)}G^{(k+i-1)}\cdots G^{(k+1)}.$$

Thus, the RG-factorization is given by

$$Q=(I-R_U)U_D(I-G_L),$$

where

$$(I-R_U)=\begin{pmatrix} I & -R_1^{(0)} & -R_2^{(0)} & -R_3^{(0)} & \cdots \\ & I & -R_1^{(1)} & -R_2^{(1)} & \cdots \\ & & I & -R_1^{(2)} & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$U_D=\text{diag}\,(U_0,U_1,U_2,\ldots),$$

and

$$(I - G_L) = \begin{pmatrix} I & & & & \\ -G^{(1)} & I & & & \\ & -G^{(2)} & I & & \\ & & -G^{(3)} & I & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Using the RG-factorization, the stationary probability vector $(\pi_0, \pi_1, \pi_2, \ldots)$ of the Markov chain Q is given by

$$\begin{cases} \pi_0 = \tau x_0, \\ \pi_k = \sum_{i=0}^{k-1} \pi_i R_{k-i}^{(i)}, \quad k \geqslant 1, \end{cases}$$

where x_0 is the stationary probability vector of the transition rate matrix U_0 and the scalar τ is uniquely determined by $\sum_{k=0}^{\infty} \pi_k e = 1$.

Let $q_k = \lim_{t \to +\infty} P\{q(t) = k\}$ for $k \geqslant 0$. Then

$$\begin{aligned} q_0 &= \tau, \\ q_k &= \sum_{i=0}^{k-1} \pi_i R_{k-i}^{(i)} e, \quad k \geqslant 1. \end{aligned}$$

Thus, the mean of the stationary queue length is given by

$$E[q] = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} k \pi_i R_{k-i}^{(i)} e.$$

Now, we derive the Laplace transform for the complementary distribution of the sojourn time by means of the RG-factorization, and also obtain the mean of the sojourn time.

Let W_n denote the sojourn time experienced by a customer when there are $n-1$ customers in the system at his arrival time, and I_n the phase number of the BMAP input at the arrival time of the nth customer. We write

$$W_n(x,i) = P\{W_n \leqslant x, I_n = i\}$$

and

$$W_n(x) = (W_n(x,1), W_n(x,2), \ldots, W_n(x,m)).$$

Using a standard probabilistic analysis, we show that the vector sequence $\{W_k(x)\}$ satisfies the following system of differential-difference equations

$$\frac{\mathrm{d}}{\mathrm{d}x}W_1(x) = W_1(x)[D_0 - \mu f(1)I] + \sum_{l=1}^{\infty} W_{1+l}(x)D_l \tag{7.2}$$

and for $k \geqslant 2$,

$$\frac{\mathrm{d}}{\mathrm{d}x}W_k(x) = \mu k f(k+1)W_{k-1}(x) + W_k(x)[D_0 - \mu k f(k)I] + \sum_{l=1}^{\infty} W_{k+l}(x)D_l, \tag{7.3}$$

with the initial conditions

$$W_n(0) = \alpha \text{ and } W_k(0) = 0 \text{ for all } k \neq n, \tag{7.4}$$

where α is a probability vector with $\alpha e = 1$.

Let

$$W(x) = (W_1(x), W_2(x), W_3(x), \dots)$$

and

$$\mathcal{A} = \begin{pmatrix} D_0 - f(1)\mu I & D_1 & D_2 & \cdots \\ f(2)\mu I & D_0 - 2f(2)\mu I & D_1 & \cdots \\ & 2f(3)\mu I & D_0 - 3f(3)\mu I & \cdots \\ & & \ddots & \ddots \end{pmatrix}. \tag{7.5}$$

Then the system of equations Eq. (7.2) to Eq. (7.5) is rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}W(x) = W(x)\mathcal{A} \tag{7.6}$$

with an initial condition

$$W(0) = \left(\underbrace{0, 0, \dots, 0,}_{n-1 \text{ vectors } 0 \text{ of size } m} \quad \alpha, 0, 0, \dots \right).$$

Thus, we obtain

$$W(x) = W(0)\exp\{Ax\}.$$

We denote by $w^*(s)$ and $w_n^*(s)$ the Laplace transform of the row vectors $W(x)$ and $W_n(x)$, respectively. For example, $w^*(s) = \int_0^{+\infty} \mathrm{e}^{-sx} W(x)\mathrm{d}x$. It is easy to check that

$$\int_0^{+\infty} \mathrm{e}^{-sx} \mathrm{d}W(x) = -W(0) + sw^*(s).$$

Thus, it follows from Eq. (7.6) that

$$w^*(s)(\mathcal{A}-sI)=-W(0). \tag{7.7}$$

Let $(\mathcal{A}-sI)^{-1}_{\max}$ denote the maximal non-positive inverse of the matrix $\mathcal{A}-sI$ of infinite size with the setting that $(\mathcal{A}-sI)^{-1}\leqslant(\mathcal{A}-sI)^{-1}_{\max}\leqslant 0$ for an arbitrary inverse $(\mathcal{A}-sI)^{-1}$ of $\mathcal{A}-sI$. Then

$$w^*(s)=-W(0)(\mathcal{A}-sI)^{-1}_{\max}.$$

Let $\{G^{(k)}(s), k\geqslant 1\}$ for the G-measure be the minimal nonnegative solution to the system of matrix equations

$$\begin{aligned}kf(k)\mu I+\{D_0-\left[s+kf(k)\mu\right]I\}G^{(k)}(s)+D_1G^{(k+1)}(s)G^{(k)}(s)\\ +D_2G^{(k+2)}(s)G^{(k+1)}(s)G^{(k)}(s)+\ldots=0,\quad k\geqslant 1.\end{aligned}$$

For $k\geqslant 0$, the U-measure is given by

$$U_k(s)=D_0-[s+kf(k)\mu]I+\sum_{i=1}^{\infty}D_iG^{(k+i)}(s)G^{(k+i-1)}(s)\ldots G^{(k+1)}(s),$$

and for $k\geqslant 0$ and $l\geqslant 1$ the R-measure is given by

$$R_l^{(k)}(s)=[D_l+D_{l+1}G^{(k+2)}(s)+D_{l+2}G^{(k+3)}(s)G^{(k+2)}(s)+\ldots][-U_{k+1}(s)]^{-1}.$$

Therefore, we obtain

$$\mathcal{A}-sI=[I-R_U(s)]U_D(s)[I-G_L(s)], \tag{7.8}$$

where

$$I-R_U(s)=\begin{pmatrix} I & -R_1^{(0)}(s) & -R_2^{(0)}(s) & -R_3^{(0)}(s) & \cdots \\ & I & -R_1^{(1)}(s) & -R_2^{(1)}(s) & \cdots \\ & & I & -R_1^{(2)}(s) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$U_D(s)=\mathrm{diag}(U_0(s),U_1(s),U_2(s),U_3(s),\ldots),$$

$$I-G_L(s)=\begin{pmatrix} I & & & & \\ -G^{(1)}(s) & I & & & \\ & -G^{(2)}(s) & I & & \\ & & -G^{(3)}(s) & I & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Let

$$X_1^{(l)}(s) = R_1^{(l)}(s), \quad l \geqslant 0,$$

$$X_{k+1}^{(l)}(s) = R_1^{(l)}(s)X_k^{(l+1)}(s) + R_2^{(l)}(s)X_{k-1}^{(l+2)}(s) + \ldots + R_k^{(l)}(s)X_1^{(l+k)}(s),$$
$$l \geqslant 0, k \geqslant 1,$$

and

$$Y_k^{(l)}(s) = G^{(l)}(s)G^{(l-1)}(s)\ldots G^{(l-k+1)}(s), \quad l \geqslant 1, k \geqslant 1.$$

If $\mathrm{Re}(s) > 0,$ then $I - R_U(s), U_D(s)$ and $I - G_L(s)$ are invertible,

$$[I - R_U(s)]^{-1} = \begin{pmatrix} I & X_1^{(0)}(s) & X_2^{(0)}(s) & X_3^{(0)}(s) & \cdots \\ & I & X_1^{(1)}(s) & X_2^{(1)}(s) & \cdots \\ & & I & X_1^{(2)}(s) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$U_D(s)^{-1} = \mathrm{diag}(U_0(s)^{-1}, U_1(s)^{-1}, U_2(s)^{-1}, U_3(s)^{-1}, \ldots)$$

and

$$[I - G_L(s)]^{-1} = \begin{pmatrix} I & & & & \\ Y_1^{(1)}(s) & I & & & \\ Y_2^{(2)}(s) & Y_1^{(2)}(s) & I & & \\ Y_3^{(3)}(s) & Y_2^{(3)}(s) & Y_1^{(3)}(s) & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $w_n^*(s)$ be the Laplace transform of the sojourn time distribution $W_n(x)$. It follows from Eq. (7.7) and Eq. (7.8) that

$$w^*(s) = (w_1^*(s), w_2^*(s), w_3^*(s), \ldots)$$
$$= \left(\underbrace{0, 0, \ldots, 0,}_{n-1 \text{ vectors } 0 \text{ of size } m} \quad \alpha, 0, 0, \ldots \right) [I - G_L(s)]^{-1}[-U_D(s)]^{-1}[I - R_U(s)]^{-1}.$$

Thus, for $\mathrm{Re}(s) > 0$ we obtain

$$w_n^*(s) = \alpha[-U_{n-1}(s)]^{-1} + \alpha \sum_{i=1}^{n-1} Y_i^{(n-1)}(s)[-U_{n-1-i}(s)]^{-1} X_i^{(n-1-i)}(s).$$

It follows from Eq. (7.7) that

$$E[W_n]=\alpha[-U_{n-1}(0)]^{-1}e$$
$$+\alpha\sum_{i=1}^{n-1}Y_i^{(n-1)}(0)[-U_{n-1-i}(0)]^{-1}X_i^{(n-1-i)}(0)e. \tag{7.9}$$

7.2 Fluid Queues

In this section, we study an infinite-capacity fluid queue driven by a level-dependent QBD process, and derive the stationary probability distribution of the buffer content.

Consider an infinite capacity buffer in which the fluid input rate and the fluid output rate are influenced by a stochastic environment. The fluid model is described as a continuous-time QBD process $\{Z(t),t\geqslant 0\}$ whose infinitesimal generator is given by

$$Q=\begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & & \\ & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{7.10}$$

where the size of the matrix $A_1^{(k)}$ is $m_k\times m_k$ for $k\geqslant 0$, the sizes of the other block-entries are determined accordingly, and all empty entries are zero. The QBD process $\{Z(t),t\geqslant 0\}$ is assumed to be irreducible and positive recurrent. Let $\pi=(\pi_0,\pi_1,\pi_2,\ldots)$ be the stationary probability vector of the QBD process $\{Z(t),t\geqslant 0\}$ partitioned according to the levels. Whenever the environment $Z(t)$ stays in state (k,i), the net input rate of fluid (i.e., the input rate - the output rate) is $\mu_{k,i}$ for $k\geqslant 0$, $1\leqslant i\leqslant m_k$. We assume that

$$(\mu_{0,1},\mu_{0,2},\ldots,\mu_{0,m_0})\lneqq 0,$$
$$(\mu_{k,1},\mu_{k,2},\ldots,\mu_{k,m_k})\geqslant 0,\quad k\geqslant 1,$$

and there exists at least a $k_0\geqslant 1$ such that $(\mu_{k_0,1},\mu_{k_0,2},\ldots,\mu_{k_0,m_{k_0}})\gneqq 0.$

Let $X(t)$ be the buffer content at time t. Then it can not be negative. We write

$$\varLambda_k=\operatorname{diag}(\mu_{k,1},\mu_{k,2},\ldots,\mu_{k,m_k}),\quad k\geqslant 0.$$

Let

$$d=\sum_{k=0}^{\infty}\pi_k\varLambda_k e.$$

Since the change of the process $X(t)$ depends only on its rate, which in turn changes according to the Markov chain $\{Z(t), t \geqslant 0\}$, it is clear that $\{(X(t), Z(t)), t \geqslant 0\}$ is a Markov process. The state space of the Markov process $\{(X(t), Z(t)), t \geqslant 0\}$ is given by

$$\Omega = \{(x, k, j) : x \geqslant 0, k = 0, 1, 2, \ldots, j = 1, 2, \ldots, m_k\}.$$

If the stochastic environment $\{Z(t), t \geqslant 0\}$ is ergodic, the quantity d is called the mean drift of the process $\{X(t), t \geqslant 0\}$. When the buffer is infinite, the bivariate Markov process $\{(X(t), Z(t)), t \geqslant 0\}$ is ergodic under the mean drift $d < 0$.

We define

$$F(t, x, (k, j); y, (l, i)) = P\{X(t) < x, Z(t) = (k, j) \mid X(0) = y, Z(0) = (l, i)\}$$

for $x, y \geqslant 0, k, l = 0, 1, 2, \ldots, 1 \leqslant i \leqslant m_l$ and $1 \leqslant j \leqslant m_k$. Note that $F(t, x, (k, j); y, (l, i))$ is the joint conditional probability distribution of the process $\{(X(t), Z(t)), t \geqslant 0\}$ at time t. When the process $\{(X(t), Z(t)), t \geqslant 0\}$ is ergodic, we write

$$F_{k,j}(x) = \lim_{t \to \infty} F(t, x, (k, j); y, (l, i)),$$

which is irrelevant to the initial state $X(0) = y$ and $Z(0) = (l, i)$ according to a standard result in the theory of Markov processes.

Let $\Lambda = \operatorname{diag}(\Lambda_0, \Lambda_1, \Lambda_2, \ldots)$, $\mathcal{F}(x) = (\mathcal{F}_0(x), \mathcal{F}_1(x), \mathcal{F}_2(x), \ldots)$, where

$$\mathcal{F}_k(x) = (F_{k,1}(x), F_{k,2}(x), \ldots, F_{k,m_k}(x)), \quad k \geqslant 0.$$

Using a standard probabilistic analysis, the vector function $\mathcal{F}(x)$ for $x \geqslant 0$ can be shown to satisfy the following system of differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathcal{F}(x) \Lambda = \mathcal{F}(x) Q \tag{7.11}$$

with the boundary conditions

$$\mathcal{F}_0(0) = \pi_0 \text{ and } \mathcal{F}_k(0) = 0 \quad \text{for} \quad k \geqslant 1. \tag{7.12}$$

Now, we provide an approach for solving the system of differential Eq. (7.11) and Eq. (7.12). The approach is based on the *RG*-factorization of the matrix $Q - s\Lambda$ for an arbitrary $s \geqslant 0$.

We write the Laplace transform of the vector function $\mathcal{F}(x)$ as $\mathcal{F}^*(s)$, that is, $\mathcal{F}^*(s) = \int_0^{+\infty} \mathrm{e}^{-sx} \mathcal{F}(x) \mathrm{d}x$. Note that $0 < \mathcal{F}^*(s) \leqslant 1$ for $s \geqslant 0$. Then

$$\int_0^{+\infty} \mathrm{e}^{-sx} \mathrm{d}\mathcal{F}(x) = -F(0) + s\mathcal{F}^*(s).$$

It follows from Eq. (7.11) and Eq. (7.12) that

$$\mathcal{F}^*(s)[Q - s\Lambda] = -\mathcal{F}(0)\Lambda = -(\pi_0\Lambda_0, 0, 0, 0, \ldots). \tag{7.13}$$

To solve the Eq. (7.13), we first define the *R*- and *G*-measures of the matrix $Q - s\Lambda$ for an arbitrary $s \geqslant 0$, and then provide the *RG*-factorization.

Let $\{R_k(s)\}$ and $\{G_k(s)\}$ for $s \geqslant 0$ be the minimal nonnegative solutions to the systems of matrix equations

$$A_0^{(l)} + R_l(s)\left[A_1^{(l+1)} - s\Lambda_{l+1}\right] + R_l(s)R_{l+1}(s)A_2^{(l+2)} = 0, \quad l \geqslant 0,$$

and

$$A_0^{(k)}G_{k+1}(s)G_k(s) + [A_1^{(k)} - s\Lambda_k]G_k(s) + A_2^{(k)} = 0, \quad k \geqslant 1,$$

respectively. Thus, we obtain

$$U_l(s) = A_1^{(l)} - s\Lambda_l + R_l(s)A_2^{(l+1)}, \quad l \geqslant 0,$$

or

$$U_l(s) = A_1^{(l)} - s\Lambda_l + A_0^{(l)}G_{l+1}(s), \quad l \geqslant 0.$$

Clearly, the matrix $U_k(s)$ is invertible for $k \geqslant 1$ and the matrix $U_0(s)$ may not be invertible for some $s \geqslant 0$ due to $\Lambda_0 \lneqq 0$.

The *RG*-factorization of the matrix $Q - s\Lambda$ for $s \geqslant 0$ is given by

$$Q - s\Lambda = [I - R_U(s)]U_D(s)[I - G_L(s)], \tag{7.14}$$

where

$$U_D(s) = \mathrm{diag}(U_0(s), U_1(s), U_2(s), U_3(s), \ldots),$$

$$R_U(s) = \begin{pmatrix} 0 & R_0(s) & & & \\ & 0 & R_1(s) & & \\ & & 0 & R_2(s) & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}$$

and

$$G_L(s) = \begin{pmatrix} 0 & & & & \\ G_1(s) & 0 & & & \\ & G_2(s) & 0 & & \\ & & G_3(s) & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Let

$$X_k^{(l)}(s) = R_l(s)R_{l+1}(s)\ldots R_{l+k-1}(s), \quad l \geqslant 0, k \geqslant 1, \tag{7.15}$$

and

$$Y_k^{(l)}(s) = G_l(s)G_{l-1}(s)\ldots G_{l-k+1}(s), \quad l \geqslant k \geqslant 1. \tag{7.16}$$

Then for $s \geqslant 0$,

$$[I - R_U(s)]^{-1} = \begin{pmatrix} I & X_1^{(0)}(s) & X_2^{(0)}(s) & X_3^{(0)}(s) & \ldots \\ & I & X_1^{(1)}(s) & X_2^{(1)}(s) & \ldots \\ & & I & X_1^{(2)}(s) & \ldots \\ & & & I & \ldots \\ & & & & \ddots \end{pmatrix} \tag{7.17}$$

and

$$[I - G_L(s)]^{-1} = \begin{pmatrix} I & & & & \\ Y_1^{(1)}(s) & I & & & \\ Y_2^{(2)}(s) & Y_1^{(2)}(s) & I & & \\ Y_3^{(3)}(s) & Y_2^{(3)}(s) & Y_1^{(3)}(s) & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7.18}$$

To solve the Eq. (7.13), we need the following assumption:

The matrix $U_0(s)$ is invertible for $s \in (a,b)$, where $0 \leqslant a < b \leqslant +\infty$.

The following theorem provides the unique solution to the Eq. (7.13).

Theorem 7.1 *If the matrix $U_0(s)$ is invertible for $s \in (a,b)$, then*

$$\mathcal{F}_0^*(s) = \pi_0 \Lambda_0 [-U_0(s)^{-1}]$$

and

$$\mathcal{F}_k^*(s) = \pi_0 \Lambda_0 [-U_0(s)^{-1}] R_0(s) R_1(s) \ldots R_{k-1}(s), \quad k \geqslant 1.$$

Proof It follows from Eq. (7.13) and Eq. (7.14) that

$$\begin{aligned} \mathcal{F}^*(s) &= -(\pi_0 \Lambda_0, 0, 0, 0, \ldots)(Q - s\Lambda)^{-1} \\ &= -(\pi_0 \Lambda_0, 0, 0, 0, \ldots)[I - G_L(s)]^{-1} U_D(s)^{-1} [I - R_U(s)]^{-1}. \end{aligned}$$

Thus, we obtain

$$\mathcal{F}_0^*(s) = \pi_0 \Lambda_0 [-U_0(s)^{-1}]$$

and

$$\begin{aligned}\mathcal{F}_k^*(s) &= -\pi_0 \varLambda_0 U_0(s)^{-1} X_k^{(0)}(s)\\ &= \pi_0 \varLambda_0 [-U_0(s)^{-1}] R_0(s) R_1(s) \ldots R_{k-1}(s), \quad k \geqslant 1.\end{aligned}$$

This completes the proof.

We define

$$G(t,x;y) = P\{X(t) \leqslant x \mid X(0) = y\},$$

which is the conditional probability distribution of the process $\{X(t), t \geqslant 0\}$ at time t. When the fluid model is ergodic, we write $\mathcal{G}(x) = \lim_{t \to +\infty} G(t,x;y)$ for all $y \geqslant 0$.

The following corollary provides the Laplace transform of the stationary probability distribution $\mathcal{G}(x)$ of the buffer content.

Corollary 7.1 *If the matrix $U_0(s)$ is invertible for $s \in (a,b)$, then*

$$\mathcal{G}^*(s) = \pi_0 \varLambda_0 [-U_0(s)^{-1}] \left[I + \sum_{l=0}^{\infty} R_0(s) R_1(s) \ldots R_l(s) e \right].$$

In particular, if the QBD process is level-independent, that is, $A_2^{(k)} = A_2$ for all $k \geqslant 2$, and $A_1^{(k)} = A_1$ and $A_0^{(k)} = A_0$ for all $k \geqslant 1$, then

$$\mathcal{G}^*(s) = \pi_0 \varLambda_0 [-U_0(s)^{-1}] \{I + R_0(s)[I - R(s)]^{-1}\} e.$$

Proof Note that $\mathcal{G}^*(s) = \sum_{k=0}^{\infty} \mathcal{F}_k^*(s) e$, we obtain

$$\mathcal{G}^*(s) = \mathcal{F}_0^*(s) e + \mathcal{F}_1^*(s) e + \mathcal{F}_2^*(s) e + \ldots.$$

Simple computation leads to the stated result.

Now, we express the Laplace-Stieltjes transforms for both the conditional distribution and the conditional mean of a first passage time.

We define a first passage time as

$$T = \inf\{t > 0 : X(t) = 0\}.$$

It is clear that the first passage time T is finite a.s. if the mean drift $d < 0$.

We write the conditional distribution of the first passage time T as

$$B_{k,i}(t,x) = P\{T \leqslant t \mid Z(0) = (k,i), X(0) = x\}.$$

Let

$$B_k(t,x) = (B_{k,1}(t,x), B_{k,2}(t,x), \ldots, B_{k,m_k}(t,x))$$

and

$$\mathcal{B}(t,x) = (B_0(t,x), B_1(t,x), B_2(t,x), \ldots).$$

Then the vector function $\mathcal{B}(t,x)$ satisfies the following system of differential equation

$$\frac{\partial}{\partial t}\mathcal{B}(t,x) + \frac{\partial}{\partial x}\mathcal{B}(t,x)\Lambda = \mathcal{B}(t,x)Q, \quad x > 0, \tag{7.19}$$

with the initial conditions

$$\mathcal{B}(t,0) = \frac{\pi_0}{\pi_0 e} \stackrel{\text{def}}{=} \theta, \quad \text{if } t \geqslant 0,$$
$$\mathcal{B}(0,x) = 0, \quad \text{if } x > 0,$$

and

$$B_k(0,0) = 0, \quad \text{if } k \geqslant 1.$$

Let

$$\mathcal{B}^*(t,\eta) = \int_0^{+\infty} \mathrm{e}^{-\eta x}\mathcal{B}(t,x)\mathrm{d}x, \quad \eta \geqslant 0.$$

Then

$$\int_0^{+\infty} \mathrm{e}^{-\eta x}\mathcal{B}(t,x)\mathrm{d}x = \frac{1}{\eta}[\mathcal{B}(t,0) + \mathcal{B}^*(t,\eta)].$$

It follows from Eq. (7.19) that

$$\frac{\partial}{\partial t}\left\{\frac{1}{\eta}[\mathcal{B}(t,0) + \mathcal{B}^*(t,\eta)]\right\} + \mathcal{B}^*(t,\eta)\Lambda = \frac{1}{\eta}[\mathcal{B}(t,0) + \mathcal{B}^*(t,\eta)]Q.$$

Note that $\mathcal{B}(t,0) = (\theta, 0, 0, 0, \ldots)$, we obtain

$$\frac{\partial}{\partial t}\mathcal{B}^*(t,\eta) = \mathcal{B}^*(t,\eta)(Q - \eta\Lambda) + (\theta, 0, 0, 0, \ldots)Q. \tag{7.20}$$

Let

$$\widetilde{\mathcal{B}^*}(\xi,\eta) = \int_0^{+\infty} \mathrm{e}^{-\xi t}\mathcal{B}^*(t,\eta)\mathrm{d}t, \quad \xi \geqslant 0.$$

Then

$$\int_0^{+\infty} \mathrm{e}^{-\xi t}\mathcal{B}^*(t,\eta)\mathrm{d}t = \frac{1}{\xi}\left[\mathcal{B}^*(0,\eta) + \widetilde{\mathcal{B}^*}(\xi,\eta)\right].$$

It follows from Eq. (7.20) that

$$\widetilde{\mathcal{B}^*}(\xi,\eta)=\frac{1}{\xi}\left[\mathcal{B}^*(0,\eta)+\widetilde{\mathcal{B}^*}(\xi,\eta)\right](Q-\eta\Lambda)+\frac{1}{\xi}(\theta,0,0,0,\ldots)Q.$$

According to the initial conditions we obtain $\mathcal{B}^*(0,\eta)=0.$ Therefore,

$$\widetilde{\mathcal{B}^*}(\xi,\eta)(Q-\eta\Lambda-\xi I)=-(\theta A_1^{(0)},\theta A_0^{(0)},0,0,\ldots). \tag{7.21}$$

We now study the conditional distribution of the first passage time T. Let $H(t)=P\{T\leqslant t\}$ and $\widetilde{H}(\xi)=\int_0^{+\infty}\mathrm{e}^{-\xi t}\mathrm{d}H(t)$ for $\xi\geqslant 0$.

Theorem 7.2 *Suppose that*

$$P\{Z(0)=(k,i),X(0)=x\}=\begin{cases}\phi_i(x), & \text{if } k=0,\\ 0, & \text{if } k\geqslant 1.\end{cases}$$

Let $\Phi(x)=(\phi_1(x),\phi_2(x),\ldots,\phi_{m_0}(x))$. *Then*

$$\widetilde{H}(\xi)=\int_0^{+\infty}\Phi(x)\widetilde{\mathcal{B}}_0(\xi,x)\mathrm{d}x,$$

where $\widetilde{\mathcal{B}}_0(\xi,x)$ *is determined by*

$$\widetilde{\mathcal{B}^*}_0(\xi,\eta)=-\theta[I+(\eta\Lambda_0+\xi I)U_0(\xi,\eta)^{-1}].$$

Proof Applying the *RG*-factorization, it follows from Eq. (7.21) that

$$\widetilde{\mathcal{B}^*}_0(\xi,\eta)=-\theta[I+(\eta\Lambda_0+\xi I)U_0(\xi,\eta)^{-1}].$$

Let the initial probability $\Phi_k(x)$ of level k be equal to $\Phi(x)$ or 0 according as $k=0$ or $k\geqslant 1$, respectively. By the formula of total probability we obtain

$$\widetilde{H}(\xi)=\sum_{k=0}^{\infty}\int_0^{+\infty}\Phi_k(x)\widetilde{\mathcal{B}}_k(\xi,x)\mathrm{d}x=\int_0^{+\infty}\Phi(x)\widetilde{\mathcal{B}}_0(\xi,x)\mathrm{d}x.$$

This completes the proof.

We write the conditional mean of the first passage time T as

$$H_{k,i}(x)=E[T\mid Z(0)=(k,i),X(0)=x].$$

Let

$$H_k(x)=(H_{k,1}(x),H_{k,2}(x),\ldots,H_{k,m_k}(x)).$$

and

$$\mathcal{H}(x)=(H_0(x),H_1(x),H_2(x),\ldots).$$

Then the vector function $\mathcal{H}(x)$ satisfies the following system of differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{H}(x)\Lambda = \mathcal{H}(x)Q + \mathrm{e}^{\mathrm{T}}, \quad x > 0, \tag{7.22}$$

with the boundary conditions

$$\mathcal{H}_0(0) = 0 \text{ and } \mathcal{H}_k(x) = \frac{\pi_k}{1-\pi_0 e} \overset{\text{def}}{=} \theta_k, \quad k \geqslant 1. \tag{7.23}$$

Let

$$\mathcal{H}^*(s) = \int_0^{+\infty} \mathrm{e}^{-sx}\mathrm{d}\mathcal{H}(x), \quad s \geqslant 0.$$

Then

$$\int_0^{+\infty} \mathrm{e}^{-sx}\mathcal{H}(x)\mathrm{d}x = \frac{1}{s}[\mathcal{H}(0) + \mathcal{H}^*(s)].$$

It follows from Eq. (7.22) and Eq. (7.23) that

$$\mathcal{H}^*(s)(Q - s\Lambda) = -[(0,\theta_1,\theta_2,\ldots) + s\mathrm{e}^{\mathrm{T}}].$$

Therefore, we obtain

$$\begin{aligned}\mathcal{H}^*(s) &= -[(0,\theta_1,\theta_2,\ldots) + s\mathrm{e}^{\mathrm{T}}](Q - s\Lambda)^{-1} \\ &= -[(0,\theta_1,\theta_2,\ldots) + s\mathrm{e}^{\mathrm{T}}][I - G_L(s)]^{-1}U_D(s)^{-1}[I - R_U(s)]^{-1}.\end{aligned}$$

We can further express $\mathcal{H}^*(s)$ by means of the R-, U- and G-measures of the matrix $Q - s\Lambda$ without any difficulty.

7.3 A Queue with Negative Customers

In this section, we apply the *RG*-factorizations to analyze a queue with negative customers, and obtain the distributions of the stationary queue length and the busy period.

Consider a single-server FCFS queue with two types of independent arrivals, positive and negative. Positive arrivals join the queue with the intention of being served and then leaving the system. At a negative arrival epoch, the system is affected if and only if customers are present. The arrival of a negative customer removes all the customers in the system.

We assume that the arrivals of both positive and negative customers are MAPs with matrix descriptors (C_1, D_1) and (C_2, D_2), respectively. Also, the infinitesimal generators $C_1 + D_1$ and $C_2 + D_2$ of sizes $m_1 \times m_1$ and $m_2 \times m_2$ are both irreducible. Thus the two Markov chains $C_1 + D_1$ and $C_2 + D_2$ are positive recurrent. Let θ_1 and θ_2 be the stationary probability vectors of $C_1 + D_1$ and $C_2 + D_2$, respectively. Then $\lambda_1 = \theta_1 D_1 e$ and $\lambda_2 = \theta_2 D_2 e$ are the stationary arrival rates of positive and negative customers, respectively. Further, we assume that the first customer to join the queue, when the server is idle, has the service time distribution given by

$$B_0(x) = 1 - \exp\left\{-\int_0^x \mu_0(v)\mathrm{d}v\right\}$$

with mean $1/\mu_0 \in (0, +\infty)$. While the service times of all the other customers are i.i.d. random variables whose distribution function is given by

$$B(x) = 1 - \exp\left\{-\int_0^x \mu(v)\mathrm{d}v\right\}$$

with mean $1/\mu \in (0, +\infty)$. $B_0(x)$ and $B(x)$ may be different and are referred to as distributions of the special and regular service times, respectively. The service process and the arrival processes of positive and negative customers are assumed to be mutually independent.

7.3.1 The Supplementary Variables

We introduce several supplementary variables to construct the differential equations for the queueing model. Further, we provide an approach for solving these equations. The crucial step of solving these equations is the connection of the boundary equations to a Markov chain of $GI/G/1$ type.

Let $N(t)$ be the number of customers in the system at time t, and let $J_1(t)$ and $J_2(t)$ be the phases of the arrivals of positive and negative customers at time t, respectively. We define the states of the server as

$$I(t) = \begin{cases} 0, & \text{if the server is idle,} \\ S, & \text{if the server is working with service time distribution } B_0(x), \\ G, & \text{if the server is working with service time distribution } B(x). \end{cases}$$

For $t > 0$, we define the random variable $S(t)$ as follows: i) If $I(t) = S$, then $S(t)$ represents the elapsed service time received by a customer with the special service time up to time t. ii) If $I(t) = G$, then $S(t)$ represents the elapsed service time received by a customer with the regular service time up to time t. iii) If $I(t) = 0$, then $S(t)$ represents the elapsed time since the last service completion during a busy period up to time t. Obviously, $\{I(t), N(t), J_1(t), J_2(t), S(t) : t \geqslant 0\}$ is a Markov process. Note that $I(t) = 0$ is equivalent to $N(t) = 0$, the state space of the process is expressed as

$$\begin{aligned}\Omega = &\{(0, j_1, j_2, x) : 1 \leqslant j_1 \leqslant m_1, 1 \leqslant j_2 \leqslant m_2, x \geqslant 0\} \\ &\bigcup \{(S, k, j_1, j_2, x) : k \geqslant 1, 1 \leqslant j_1 \leqslant m_1, 1 \leqslant j_2 \leqslant m_2, x \geqslant 0\} \\ &\bigcup \{(G, k, j_1, j_2, x) : k \geqslant 1, 1 \leqslant j_1 \leqslant m_1, 1 \leqslant j_2 \leqslant m_2, x \geqslant 0\}.\end{aligned}$$

We write

$$p_{0,i,j}(t,x)\mathrm{d}x = P\{I(t)=0, J_1(t)=i, J_2(t)=j, x \leqslant S(t) < x+\mathrm{d}x\},$$
$$p_{Sk,i,j}(t,x)\mathrm{d}x = P\{I(t)=S, N(t)=k, J_1(t)=i, J_2(t)=j, x \leqslant S(t) < x+\mathrm{d}x\},$$
$$p_{Gk,i,j}(t,x)\mathrm{d}x = P\{I(t)=G, N(t)=k, J_1(t)=i, J_2(t)=j, x \leqslant S(t) < x+\mathrm{d}x\},$$

$$\begin{aligned}p_{0,i,j}(x) &= \lim_{t\to+\infty} p_{0,i,j}(t,x), \\ p_{Sk,i,j}(x) &= \lim_{t\to+\infty} p_{Sk,i,j}(t,x), \\ p_{Gk,i,j}(x) &= \lim_{t\to+\infty} p_{Gk,i,j}(t,x);\end{aligned}$$

$$\begin{aligned}P_0(x) &= (p_{0,1,1}(x), \ldots, p_{0,1,m_2}(x), \ldots, p_{0,m_1,1}(x), \ldots, p_{0,m_1,m_2}(x)), \\ P_{Sk}(x) &= (p_{Sk,1,1}(x), \ldots, p_{Sk,1,m_2}(x), \ldots, p_{Sk,m_1,1}(x), \ldots, p_{Sk,m_1,m_2}(x)), \\ P_{Gk}(x) &= (p_{Gk,1,1}(x), \ldots, p_{Gk,1,m_2}(x), \ldots, p_{Gk,m_1,1}(x), \ldots, p_{Gk,m_1,m_2}(x)).\end{aligned}$$

It is easy to see that $P_0(x)$, $P_{Sk}(x)$ and $P_{Gk}(x)$ for $k \geqslant 1$ are row vectors of size $m_1 m_2$.

Consider the number $N(t)$ of customers in the system at time t, the stability conditions of the system can be easily discussed in the same way as that in Jain and Sigman [87]. The arrival of a negative customer removes all the customers in the system. Therefore, the arrival epochs of negative customers with the irreducible MAP descriptor (C_2, D_2) form positive recurrent regenerative times of the system. Clearly, $\{N(t), t \geqslant 0\}$ is a positive recurrent regenerative process with a unique stationary distribution. Therefore, the queueing system is stable.

If this system is stable, then the system of stationary differential equations of the joint probability density $\{P_0(x), P_{Sk}(x), P_{Gk}(x), k \geqslant 1\}$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x} P_0(x) = P_0(x)[(C_1 \oplus C_2) + (I \otimes D_2)], \tag{7.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} P_{S1}(x) = P_{S1}(x)[C_1 \oplus C_2 - \mu_0(x) I], \tag{7.25}$$

for $k \geqslant 2$,

$$\frac{\mathrm{d}}{\mathrm{d}x} P_{Sk}(x) = P_{Sk}(x)[C_1 \oplus C_2 - \mu_0(x) I] + P_{Sk-1}(x)(D_1 \otimes I), \tag{7.26}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{G1}(x) = P_{G1}(x)[C_1 \oplus C_2 - \mu(x)I], \tag{7.27}$$

for $k \geqslant 2$,

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{Gk}(x) = P_{Gk}(x)[C_1 \oplus C_2 - \mu(x)I] + P_{Gk-1}(x)(D_1 \otimes I). \tag{7.28}$$

The joint probability density function $\{P_0(x), P_{Sk}(x), P_{Gk}(x), k \geqslant 1\}$ should satisfy the boundary conditions

$$\begin{aligned} P_0(0) = & \int_0^{+\infty} \mu_0(x)P_{S1}(x)\mathrm{d}x + \int_0^{+\infty} \mu(x)P_{G1}(x)\mathrm{d}x \\ & + \sum_{k=1}^{\infty}\left[\int_0^{+\infty} P_{Sk}(x)\mathrm{d}x + \int_0^{+\infty} P_{Gk}(x)\mathrm{d}x\right](I \otimes D_2), \end{aligned} \tag{7.29}$$

$$P_{S1}(0) = \int_0^{+\infty} P_0(x)\mathrm{d}x(D_1 \otimes I), \tag{7.30}$$

$$P_{Sk}(0) = 0, \quad k \geqslant 2, \tag{7.31}$$

$$P_{Gk}(0) = \int_0^{+\infty} \mu_0(x)P_{Sk+1}(x)\mathrm{d}x + \int_0^{+\infty} \mu(x)P_{Gk+1}(x)\mathrm{d}x \tag{7.32}$$

for $k \geqslant 1$, and the normalization condition

$$\left\{\int_0^{+\infty} P_0(x)\mathrm{d}x + \sum_{k=1}^{\infty}\left[\int_0^{+\infty} P_{Sk}(x)\mathrm{d}x + \int_0^{+\infty} P_{Gk}(x)\mathrm{d}x\right]\right\}e = 1. \tag{7.33}$$

7.3.2 A Markov Chain of *GI/G/*1 Type

We provide an approach to solve the equations Eq. (7.24) to Eq. (7.33), which can be described as a Markov chain of $GI/G/1$ type.

It follows from Eq. (7.24) that

$$P_0(x) = P_0(0)\exp\{[C_1 \oplus (C_2 + D_2)]x\}. \tag{7.34}$$

To solve the equations Eq. (7.25) to Eq. (7.33), we define

$$Q_S^*(z,x) = \sum_{k=1}^{\infty} z^k P_{Sk}(x), \quad Q_G^*(z,x) = \sum_{k=1}^{\infty} z^k P_{Gk}(x).$$

It follows from Eq. (7.25) and Eq. (7.26) that

$$\frac{\partial}{\partial x}Q_S^*(z,x) = Q_S^*(z,x)[(C_1 + zD_1) \oplus C_2 - \mu_0(x)I],$$

which leads to

$$Q_S^*(z,x) = Q_S^*(z,0)[\exp\{(C_1 + zD_1)x\} \otimes \exp\{C_2 x\}]\overline{B}_0(x). \tag{7.35}$$

Similarly, it follows from Eq. (7.26) and Eq. (7.27) that

$$Q_G^*(z,x) = Q_G^*(z,0)[\exp\{(C_1 + zD_1)x\} \otimes \exp\{C_2 x\}]\overline{B}(x). \tag{7.36}$$

To obtain expressions of the vectors $P_{Sk}(x)$ and $P_{Gk}(x)$ for $k \geqslant 1$, we need to define the conditional probabilities of the MAP with matrix descriptor (C_1, D_1) as

$$P_{i,j}(n,t) = P\{K(t) = n, J_1(t) = j \mid K(0) = 0, J_1(0) = i\},$$

where $K(t)$ denotes the number of arrivals of the MAP during $[0,t)$. Let $P(n,t) = (P_{i,j}(n,t))_{m_1 \times m_1}$ and $P^*(z,t) = \sum_{n=0}^{\infty} z^n P(n,t)$. Then it follows from Chapter 5 of Neuts [24] that

$$P^*(z,t) = \exp\{(C_1 + zD_1)t\}. \tag{7.37}$$

Substituting Eq. (7.37) into Eq. (7.35) and Eq. (7.36), we have

$$P_{Sk}(x) = \sum_{j=1}^{k} P_{Sj}(0)[P(k-j,x) \otimes \exp\{C_2 x\}]\overline{B}_0(x) \tag{7.38}$$

and

$$P_{Gk}(x) = \sum_{j=1}^{k} P_{Gj}(0)[P(k-j,x) \otimes \exp\{C_2 x\}]\overline{B}(x). \tag{7.39}$$

Clearly, all the probability vectors $P_{Sj}(0) = 0$ for $j \geqslant 2$ according to Eq. (7.31) and $P_{S1}(0)$ can be determined from Eq. (7.34) and Eq. (7.30) as

$$P_{S1}(0) = P_0(0)\{-[C_1 \oplus (C_2 + D_2)]^{-1}\}(D_1 \otimes I). \tag{7.40}$$

Therefore

$$P_{Sk}(x) = P_0(0)L[P(k-1,x) \otimes \exp\{C_2 x\}]\overline{B}_0(x), \tag{7.41}$$

where

$$L = -[C_1 \oplus (C_2 + D_2)]^{-1}(D_1 \otimes I),$$

which is a stochastic matrix.

The equations Eq. (7.40) and Eq. (7.41) provide a solution for the system of differential equations Eq. (7.24) to Eq. (7.28). Further, the boundary Eq. (7.29) to

Eq. (7.33) we used to determine the vectors $P_0(0)$ and $P_{Gk}(0)$ for $k \geqslant 1$, which are complicated. To that end, we define

$$T^{(S)} = \int_0^{+\infty} \exp\{(C_1 + D_1)x\} \otimes \exp\{C_2 x\} \overline{B}_0(x)\mathrm{d}x,$$

$$A_k^{(S)} = \int_0^{+\infty} P(k,x) \otimes \exp\{C_2 x\}\mathrm{d}B_0(x),$$

$$A_k^{(G)} = \int_0^{+\infty} P(k,x) \otimes \exp\{C_2 x\}\mathrm{d}B(x),$$

$$B_k^{(G)} = \int_0^{+\infty} P(k,x) \otimes \exp\{C_2 x\}\overline{B}(x)\mathrm{d}x,$$

$$H_0 = L[A_0^{(S)} + T^{(S)}(I \otimes D_2)], \quad H_k = LA_k^{(S)}, \quad k \geqslant 1,$$

$$H_{-1} = A_0^{(G)} + H_{-2}, \quad H_{-2} = \sum_{j=0}^{\infty} B_j^{(G)}(I \otimes D_2).$$

Then it follows from Eq. (7.29) to Eq. (7.33) that

$$P_G \varLambda = P_G, \tag{7.42}$$

where

$$P_G = (P_0(0), P_{G1}(0), P_{G2}(0), P_{G3}(0), \ldots)$$

and

$$\varLambda = \begin{pmatrix} H_0 & H_1 & H_2 & H_3 & \cdots \\ H_{-1} & A_1^{(G)} & A_2^{(G)} & A_3^{(G)} & \cdots \\ H_{-2} & A_0^{(G)} & A_1^{(G)} & A_2^{(G)} & \cdots \\ H_{-2} & & A_0^{(G)} & A_1^{(G)} & \cdots \\ H_{-2} & & & A_0^{(G)} & \cdots \\ \vdots & & & & \ddots \end{pmatrix}. \tag{7.43}$$

Theorem 7.3 *The matrix $\varLambda$ is irreducible, stochastic and positive recurrent.*

Proof According to the definition of H_i for $-2 \leqslant i < +\infty$ and $A_k^{(G)}$ for $k \geqslant 0$, it is not difficult to see that $\varLambda$ is irreducible.

To prove that $\varLambda$ is stochastic, we only need to check that $\sum_{k=0}^{\infty} H_k e = e$, $H_{-1}e + \sum_{k=1}^{\infty} A_k^{(G)} e = e$ and $H_{-2}e + \sum_{k=0}^{\infty} A_k^{(G)} e = e$. It is easy to see that $H_{-1}e + \sum_{k=1}^{\infty} A_k^{(G)} e = H_{-2}e + \sum_{k=0}^{\infty} A_k^{(G)} e$. We can obtain

$$\sum_{k=0}^{\infty} H_k e = L\left[T^{(S)}(I \otimes D_2) + \sum_{k=0}^{\infty} A_k^{(S)}\right] e$$
$$= L\int_0^{+\infty} \exp\{(C_1 + D_1)x\} \otimes \exp\{C_2 x\} \mathrm{d}B_0(x) e$$
$$+ L\int_0^{+\infty} \exp\{(C_1 + D_1)x\} \otimes \exp\{C_2 x\} \overline{B}_0(x) \mathrm{d}x (I \otimes D_2) e$$
$$= L\int_0^{+\infty} \exp\{[(C_1 + D_1) \oplus C_2]x\} \mathrm{d}B_0(x) e$$
$$+ L\int_0^{+\infty} \exp\{[(C_1 + D_1) \oplus C_2]x\} \overline{B}_0(x) \mathrm{d}x (I \otimes D_2) e.$$

Note that

$$\int_0^{+\infty} \exp\{[(C_1 + D_1) \oplus C_2]x\} \overline{B}_0(x) \mathrm{d}x (I \otimes D_2) e$$
$$= e - \int_0^{+\infty} \exp\{[(C_1 + D_1) \oplus C_2]x\} \mathrm{d}B_0(x) e$$

and the matrix L is stochastic, we obtain $\sum_{k=0}^{\infty} H_k e = e$. Similarly, we can prove that $H_{-2} e + \sum_{k=0}^{\infty} A_k^{(G)} e = e.$

Since $H_{-2} + \sum_{k=0}^{\infty} A_k^{(G)}$ is stochastic and $H_{-2} e \gneq 0$ the matrix $\sum_{k=0}^{\infty} A_k^{(G)}$ is substochastic. The matrix $\varLambda$ is irreducible and stochastic, and the matrix $\sum_{k=0}^{\infty} A_k^{(G)}$ is substochastic. Thus $\varLambda$ is positive recurrent by means of theorem 3.16. This completes the proof.

Let $(x_0, x_1, x_2, \ldots)$ be the stationary probability vector of the matrix $\varLambda$. Then it follows from Eq. (7.42) that

$$P_G = \alpha(x_0, x_1, x_2, \ldots),$$

where α is determined by Eq. (7.33) as

$$\alpha = \frac{1}{x_0[V + LT^{(S)}]e + \sum_{k=1}^{\infty} x_k T^{(G)} e}$$

with

$$V = -[C_1 \oplus (C_2 + D_2)]^{-1}$$

and

$$T^{(G)} = \int_0^{+\infty} \exp\{(C_1 + D_1)x\} \otimes \exp\{C_2 x\} \overline{B}(x) \mathrm{d}x.$$

We solve the equation $(x_0, x_1, x_2, \ldots) = (x_0, x_1, x_2, \ldots)\varLambda$ for $(x_0, x_1, x_2, \ldots)$ based on the censoring technique and the RG-factorization.

Note that the Markov chain $\varLambda$ is of $GI/G/1$ type, and is analyzed in Chapter 3 in detail. Let $Q = \varLambda^{[\geqslant 1]}$. Then the matrix Q is of $M/G/1$ type. We denote by G the minimal nonnegative solution to the matrix equation $G = \sum_{i=0}^{\infty} A_i^{(G)} G^i$. Thus we have

$$\varPhi_0 = A_1^{(G)} + \sum_{i=1}^{\infty} A_{i+1}^{(G)} G^i, \tag{7.44}$$

$$R_{0,j} = \sum_{i=0}^{\infty} H_{i+j} G^i [I - \varPhi_0]^{-1}, \quad j \geqslant 1, \tag{7.45}$$

and

$$R_j = \sum_{i=1}^{\infty} A_{i+j}^{(G)} G^{i-1} [I - \varPhi_0]^{-1}, \quad j \geqslant 1. \tag{7.46}$$

The following lemma is useful for studying the generating function of the matrix sequence $\{R_k\}$.

Lemma 7.1 *The matrix $I - A_G^*(1)$ is invertible, where $A_G^*(z) = \sum_{k=0}^{\infty} z^k A_k^{(G)}$.*

Proof Let $\lambda_1, \lambda_2, \ldots, \lambda_{m_1}$ be the m_1 eigenvalues of the matrix $C_1 + D_1$ of size $m_1 \times m_1$, and $\mu_1, \mu_2, \ldots, \mu_{m_2}$ the m_2 eigenvalues of the matrix C_2 of size $m_2 \times m_2$. We denote by λ_{i_0} and μ_{j_0} the eigenvalues with largest real parts of the matrices $C_1 + D_1$ and C_2, respectively. Note that (C_1, D_1) and (C_2, D_2) are the irreducible matrix descriptors of the MAPs for the positive and negative customers, respectively, it is clear that the matrices $C_1 + D_1$ and C_2 are infinitesimal generators of two continuous-time Markov chains. Therefore, $\lambda_{i_0} = 0$ due to $(C_1 + D_1)e = 0$, and $\mu_{j_0} < 0$ due to both $C_2 e \lneqq 0$ and the fact that $C_2 + D_2$ is irreducible. At the same time, the real part $\mathrm{Re}(\lambda_i) < 0$ for $i \neq i_0$, since $C_1 + D_1$ is irreducible, and $\mathrm{Re}(\mu_j) \leqslant \mu_{j_0} < 0$ for $j \neq j_0$. Note that the eigenvalues of the matrix $(C_1 + D_1) \oplus C_2$ are $\lambda_i + \mu_j$ for $1 \leqslant i \leqslant m_1$ and $1 \leqslant j \leqslant m_2$ (a standard result of the Kronecker sum, for example, see 2.4 in Graham [12]), the eigenvalue with largest real part of the matrix $(C_1 + D_1) \oplus C_2$ is μ_{j_0} due to the fact

$$\mathrm{Re}(\lambda_i + \mu_j) = \mathrm{Re}(\lambda_i) + \mathrm{Re}(\mu_j) \leqslant \mathrm{Re}(\lambda_{i_0}) + \mathrm{Re}(\mu_{j_0}) = \lambda_{i_0} + \mu_{j_0} = \mu_{j_0} < 0.$$

Since

$$A_G^*(1) = \int_0^{+\infty} \exp\{[(C_1 + D_1) \oplus C_2]x\} \mathrm{d}B(x),$$

the eigenvalues of the matrix $A_G^*(1)$ are $\int_0^{+\infty} \exp\{(\lambda_i + \mu_j)x\}\mathrm{d}B(x)$ for $1 \leqslant i \leqslant m_1$ and $1 \leqslant j \leqslant m_2$. Using $\mathrm{Re}(\lambda_i + \mu_j) < 0$, we obtain

$$\left| \int_0^{+\infty} \exp\{(\lambda_i + \mu_j)x\}\mathrm{d}B(x) \right| \leqslant \int_0^{+\infty} \exp\{[\mathrm{Re}(\lambda_i + \mu_j)]x\}\mathrm{d}B(x) < 1$$

for $1 \leqslant i \leqslant m_1$ and $1 \leqslant j \leqslant m_2$. This implies that all the $m_1 m_2$ eigenvalues of the matrix $I - A_G^*(1)$ are not equal to zero. Therefore, the matrix $I - A_G^*(1)$ is invertible. This completes the proof.

It follows from Theorem 3.5 that

$$zI - A_G^*(z) = [I - R^*(z)](I - \Phi_0)(zI - G). \tag{7.47}$$

Since $A_G^*(1)e = e - H_2 e \lneqq e$, it is clear that $I - A_G^*(1)$ is invertible, which implies that $I - R^*(1)$, $I - \Phi_0$ and $I - G$ in Eq. (7.47) are all invertible.

Let $X^*(z) = \sum_{k=1}^{\infty} z^k x_k$. Then

$$X^*(z) = \frac{1 - x_0 e}{x_0 R_0^*(1)[I - R^*(1)]^{-1} e} x_0 R_0^*(z)[I - R^*(z)]^{-1}, \tag{7.48}$$

or

$$x_k = \frac{1 - x_0 e}{x_0 R_0^*(1)[I - R^*(1)]^{-1} e} x_0 R_{0,k}^* \sum_{n=0}^{\infty} R_k^{n*}, \quad k \geqslant 1. \tag{7.49}$$

The following lemma provides the G-measure and the censored chain $\varPsi_0$, both of which are necessary for determining the crucial vector x_0 given in Eq. (7.49).

Lemma 7.2 *The transition probability matrix $\varPsi_0$ of the censored Markov chain of $\varLambda$ to level* 0 *is given by*

$$\varPsi_0 = H_0 + \left(\sum_{i=0}^{\infty} H_{i+1} G^i \right) G_{1,0} + \left(\sum_{j=2}^{\infty} \sum_{i=0}^{\infty} H_{i+j} G^i \right) G_{2,0}, \tag{7.50}$$

where

$$G_{2,0} = \left[I - \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} A_{k+l}^{(G)} G^{k-1} \right]^{-1} H_{-2},$$

and

$$G_{1,0} = \left[I - \sum_{i=1}^{\infty} A_i^{(G)} G^{i-1} \right]^{-1} \left\{ H_{-1} + \left[\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} A_{i+j}^{(G)} G^{i-1} \right] G_{2,0} \right\}.$$

Proof It follows from Eq. (3.17) that

$$\Psi_0 = H_0 + \sum_{j=1}^{\infty} R_{0,j}(I-\Phi_0)G_{j,0},$$

which leads to

$$\Psi_0 = H_0 + \left(\sum_{i=0}^{\infty} H_{i+1}G^i\right)G_{1,0} + \left(\sum_{j=2}^{\infty}\sum_{i=0}^{\infty} H_{i+j}G^i\right)G_{2,0},$$

by using Eq. (7.45) and $G_{i,0} = G_{2,0}$ for $i \geqslant 2$. It follows from Eq. (3.16) that

$$(I-\Phi_0)G_{k,0} = H_{-k} + \sum_{j=1}^{\infty} R_j(I-\Phi_0)G_{j+k,0}, \quad k \geqslant 1,$$

which leads to

$$(I-\Phi_0)G_{1,0} = H_{-1} + \sum_{j=1}^{\infty} R_j(I-\Phi_0)G_{2,0} \tag{7.51}$$

and

$$(I-\Phi_0)G_{2,0} = H_{-2} + \sum_{j=1}^{\infty} R_j(I-\Phi_0)G_{2,0} \tag{7.52}$$

by using $G_{i,0} = G_{2,0}$ for $i \geqslant 2$. Eq. (7.51) and Eq. (7.52) imply

$$\left[I - \sum_{i=1}^{\infty} A_i^{(G)} G^{i-1}\right] G_{1,0} = H_{-1} + \left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} A_{i+j}^{(G)} G^{i-1}\right] G_{2,0},$$

and

$$\left[I - \sum_{i=1}^{\infty} A_i^{(G)} G^{i-1}\right] G_{2,0} = H_{-2} + \left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} A_{i+j}^{(G)} G^{i-1}\right] G_{2,0},$$

respectively, by applying Eq. (7.44) and Eq. (7.46). Since the matrices $I - \sum_{j=0}^{\infty}\sum_{i=1}^{\infty} A_{i+j}^{(G)} G^{i-1}$ and $I - \sum_{i=1}^{\infty} A_i^{(G)} G^{i-1}$ are invertible, we obtain

$$G_{2,0} = \left[I - \sum_{l=0}^{\infty}\sum_{k=1}^{\infty} A_{k+l}^{(G)} G^{k-1}\right]^{-1} H_{-2}$$

and

$$G_{1,0} = \left[I - \sum_{i=1}^{\infty} A_i^{(G)} G^{i-1}\right]^{-1} \left\{H_{-1} + \left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} A_{i+j}^{(G)} G^{i-1}\right] G_{2,0}\right\}.$$

This completes the proof.

We now summarize the above discussions into the following theorem.

Theorem 7.4 *For the stable system, let* y_0 *be the stationary probability vector of the censored Markov chain* Ψ_0 *given in Eq.* (7.50). *Then,*

$$P_0(x) = P_0(0)\exp\{[C_1 \oplus (C_2 + D_2)]x\},$$

$$P_{Sk}(x) = P_0(0)L[P(k-1,x) \otimes \exp\{C_2 x\}]\overline{B}_0(x), \quad k \geqslant 1,$$

and

$$P_{Gk}(x) = \sum_{j=1}^{k} P_{Gj}(0)[P(k-j,x) \otimes \exp\{C_2 x\}]\overline{B}(x), \quad k \geqslant 1,$$

where

$$P_0(0) = \frac{x_0}{x_0[V + LT^{(S)}]e + X^*(1)T^{(G)}e},$$

and

$$P_{Gk}(0) = \frac{x_0 R_{0,k} * \sum_{n=0}^{\infty} R_k^{n*}}{x_0[V + LT^{(S)}]e + X^*(1)T^{(G)}e} \frac{1 - x_0 e}{x_0 R_0^*(1)[I - R^*(1)]^{-1}}, \quad k \geqslant 1,$$

with

$$x_0 = \frac{y_0}{1 + \pi_0 R_0^*(1)[I - R^*(1)]^{-1} e}.$$

7.3.3 The Stationary Queue Length

Now, we consider the distribution of the stationary queue length. Note that $N(t)$ is the number of customers in the system at time t, we write

$$p_k = \lim_{t \to +\infty} P\{N(t) = k\}, \quad k \geqslant 0,$$
$$p_k^{(S)} = \lim_{t \to +\infty} P\{N(t) = k, I(t) = S\}, \quad k \geqslant 1,$$

and

$$p_k^{(G)} = \lim_{t \to +\infty} P\{N(t) = k, I(t) = G\}, \quad k \geqslant 1.$$
$$p_k = p_k^{(S)} + p_k^{(G)}, \quad k \geqslant 1.$$

Theorem 7.5 *If the system is stable, then*

$$\begin{cases} p_0 = \alpha x_0 V e, \\ p_k = \alpha \left[x_0 L B_{k-1}^{(S)} + \sum_{j=1}^{k} x_j B_{k-j}^{(G)} \right] e, \quad k \geqslant 1, \end{cases}$$

where

$$B_k^{(S)} = \int_0^{+\infty} P(k,x) \otimes \exp\{C_2 x\} \overline{B}_0(x) \mathrm{d}x, \quad k \geqslant 1.$$

Proof It follows from Eq. (7.34) that

$$p_0 = \int_0^{+\infty} P_0(x) \mathrm{d}x e = P_0(0)\{-[C_1 \oplus (C_2 + D_2)]^{-1}\} e = \alpha x_0 V e,$$

and from Eq. (7.39) and Eq. (7.41) that

$$p_k^{(S)} = \int_0^{+\infty} P_{Sk}(x) \mathrm{d}x e = P_0(0) L B_{k-1}^{(S)} e = \alpha x_0 L B_{k-1}^{(S)} e$$

and

$$p_k^{(G)} = \int_0^{+\infty} P_{Gk}(x) \mathrm{d}x e = \sum_{j=1}^{k} P_{Gj}(0) B_{k-j}^{(G)} e = \alpha \sum_{j=1}^{k} x_j B_{k-j}^{(G)} e.$$

This completes the proof.

If the system is stable and $N = \lim_{t \to +\infty} N(t)$, then

$$\begin{aligned} E[z^N] = {} & \alpha x_0 V e + \alpha x_0 L z \int_0^{+\infty} \exp\{[(C_1 + z D_1) \oplus C_2] x\} \overline{B}_0(x) \mathrm{d}x\, e \\ & + \frac{\alpha(1 - x_0 e)}{x_0 R_0^*(1)[I - R^*(1)]^{-1} e} x_0 R_0^*(z) [I - R^*(z)]^{-1} \\ & \cdot \int_0^{+\infty} \exp\{[(C_1 + z D_1) \oplus C_2] x\} \overline{B}(x) \mathrm{d}x\, e. \end{aligned}$$

7.3.4 The Busy Period

We provide an analysis for the busy period of the negative-customer queue. Let us introduce a new absorbing state 0 and modify the Markov process in the following way: Whenever the process visits state $(0, j_1, j_2, x)$, it is absorbed into state 0. We study the absorbing time of the modified process, given that it starts from the state set $\{(S, 1, j_1, j_2, 0) : 1 \leqslant j_1 \leqslant m_1, 1 \leqslant j_2 \leqslant m_2\}$ with probability vector θ_0 of size $m_1 m_2$. A similar analysis to the equations Eq. (7.24) to Eq. (7.28) and Eq. (7.29) to Eq. (7.33) can be used to obtain the following differential equations:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)P_{S1}(t,x)=P_{S1}(t,x)[C_1\oplus C_2-\mu_0(x)I], \tag{7.53}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)P_{Sk}(t,x)=P_{Sk}(t,x)[C_1\oplus C_2-\mu_0(x)I] + P_{Sk-1}(t,x)(D_1\otimes I), \tag{7.54}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)P_{G1}(t,x)=P_{G1}(t,x)[C_1\oplus C_2-\mu(x)I], \tag{7.55}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)P_{Gk}(t,x)=P_{Gk}(t,x)[C_1\oplus C_2-\mu(x)I] + P_{Gk-1}(t,x)(D_1\otimes I), \tag{7.56}$$

with the boundary conditions:

$$P_{S1}(t,0)=\theta_0\delta(t), \quad P_{Sl}(t,0)=0, \quad l\geqslant 2, \tag{7.57}$$

for $k\geqslant 1$

$$P_{Gk}(t,0)=\int_0^{+\infty}\mu_0(x)P_{Sk+1}(t,x)\mathrm{d}x + \int_0^{+\infty}\mu(x)P_{Gk+1}(t,x)\mathrm{d}x, \tag{7.58}$$

and the initial conditions:

$$P_{S1}(0,x)=\theta_0\delta(x), \quad P_{Sl}(0,x)=0, \quad l\geqslant 2, \tag{7.59}$$

and

$$P_{Gk}(0,x)=0, \quad k\geqslant 1, \tag{7.60}$$

where

$$\delta(t)=\begin{cases}0, & \text{if } t=0,\\ 1, & \text{if } t>1.\end{cases}$$

Let

$$\widetilde{P}_{Sk}(s,x)=\int_0^{+\infty}\mathrm{e}^{-st}P_{Sk}(t,x)\mathrm{d}t, \quad \widetilde{Q}_S^{*}(s,z,x)=\sum_{k=1}^{\infty}z^k\widetilde{P}_{Sk}(s,x),$$

$$\widetilde{P}_{Gk}(s,x)=\int_0^{+\infty}\mathrm{e}^{-st}P_{Gk}(t,x)\mathrm{d}t, \quad \widetilde{Q}_G^{*}(s,z,x)=\sum_{k=1}^{\infty}z^k\widetilde{P}_{Gk}(s,x).$$

Then it follows from Eq. (7.53), Eq. (7.54) and Eq. (7.57) that

$$\begin{aligned}\widetilde{Q}_S^*(s,z,x) &= z\theta_0 \exp\{[-sI + (C_1 + zD_1) \oplus C_2]x\}\overline{B}_0(x) \\ &= \theta_0 \sum_{k=1}^{\infty} z^k [P(k-1,x) \otimes \exp\{C_2 x\}]\mathrm{e}^{-sx}\overline{B}_0(x).\end{aligned}$$

Hence,

$$\widetilde{P}_{Sk}(s,x) = \theta_0 [P(k-1,x) \otimes \exp\{C_2 x\}]\mathrm{e}^{-sx}\overline{B}_0(x), \quad k \geqslant 1. \tag{7.61}$$

It follows from Eq. (7.55) and Eq. (7.56) that

$$\begin{aligned}\widetilde{Q}_G^*(s,z,x) &= \widetilde{Q}_G^*(s,z,0)\exp\{[-sI + (C_1 + zD_1) \oplus C_2]x\}\overline{B}(x) \\ &= \sum_{k=1}^{\infty} z^k \sum_{j=1}^{k} \widetilde{P}_{Gj}(s,0)[P(k-j,x) \otimes \exp\{C_2 x\}]\mathrm{e}^{-sx}\overline{B}(x).\end{aligned}$$

We obtain that for $k \geqslant 1$,

$$\widetilde{P}_{Gk}(s,x) = \sum_{j=1}^{k} \widetilde{P}_{Gj}(s,0)[P(k-j,x) \otimes \exp\{C_2 x\}]\mathrm{e}^{-sx}\overline{B}(x). \tag{7.62}$$

Clearly, $\widetilde{P}_{Sk}(s,x)$ is explicitly expressed by the given information, while $\widetilde{P}_{Gk}(s,x)$ is explicitly expressed by both the given information and the vectors $\widetilde{P}_{Gj}(s,0)$ for $1 \leqslant j \leqslant k$, where $\widetilde{P}_{Gj}(s,0)$ can be determined by the boundary condition Eq. (7.58). It follows from Eq. (7.58) that for $k \geqslant 1$,

$$\widetilde{P}_{Gk}(s,0) = \int_0^{+\infty} \mu_0(x)\widetilde{P}_{Sk+1}(s,x)\mathrm{d}x + \int_0^{+\infty} \mu(x)\widetilde{P}_{Gk+1}(s,x)\mathrm{d}x. \tag{7.63}$$

Substituting Eq. (7.61) and Eq. (7.62) into Eq. (7.63) leads to

$$\widetilde{P}_G(s)[I - \widetilde{Q}(s)] = \widetilde{\varGamma}(s), \tag{6.64}$$

where

$$\widetilde{P}_G(s) = (\widetilde{P}_{G1}(s,0), \widetilde{P}_{G2}(s,0), \widetilde{P}_{G3}(s,0), \widetilde{P}_{G4}(s,0), \ldots),$$

$$\widetilde{\varGamma}(s) = (\theta_0 \widetilde{A}_1^{(S)}(s), \theta_0 \widetilde{A}_2^{(S)}(s), \theta_0 \widetilde{A}_3^{(S)}(s), \theta_0 \widetilde{A}_4^{(S)}(s), \ldots)$$

and

$$\widetilde{Q}(s) = \begin{pmatrix} \widetilde{A}_1^{(G)}(s) & \widetilde{A}_2^{(G)}(s) & \widetilde{A}_3^{(G)}(s) & \cdots \\ \widetilde{A}_0^{(G)}(s) & \widetilde{A}_1^{(G)}(s) & \widetilde{A}_2^{(G)}(s) & \cdots \\ & \widetilde{A}_0^{(G)}(s) & \widetilde{A}_1^{(G)}(s) & \cdots \\ & & \widetilde{A}_0^{(G)}(s) & \cdots \\ & & & \ddots \end{pmatrix}$$

with

$$\widetilde{A}_k^{(S)}(s) = \int_0^{+\infty} \left[P(k,x) \otimes \exp\{C_2 x\} \right] \mathrm{e}^{-sx} \mathrm{d}B_0(x), \quad k \geqslant 0,$$

and

$$\widetilde{A}_k^{(G)}(s) = \int_0^{+\infty} \left[P(k,x) \otimes \exp\{C_2 x\} \right] \mathrm{e}^{-sx} \mathrm{d}B(x), \quad k \geqslant 0.$$

For the discrete-time Markov chain $\widetilde{Q}(s)$ of $M/G/1$ type without the boundary, we easily obtain the R-measure $\{\widetilde{R}_k(s)\}$, the U-measure $\widetilde{\Phi}_0(s)$, and the G-measure $\widetilde{G}(s)$. Therefore, the RG-factorization of the matrix $I-\widetilde{Q}(s)$ is given by

$$I - \widetilde{Q}(s) = [I - \widetilde{R}_U(s)][I - \widetilde{\Psi}_D(s)][I - \widetilde{G}_L(s)], \tag{7.65}$$

where

$$\widetilde{R}_U(s) = \begin{pmatrix} 0 & \widetilde{R}_1(s) & \widetilde{R}_2(s) & \widetilde{R}_3(s) & \cdots \\ & 0 & \widetilde{R}_1(s) & \widetilde{R}_2(s) & \cdots \\ & & 0 & \widetilde{R}_1(s) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$\widetilde{\Psi}_D(s) = \mathrm{diag}(\widetilde{\Phi}_0(s), \widetilde{\Phi}_0(s), \widetilde{\Phi}_0(s), \widetilde{\Phi}_0(s), \ldots)$$

and

$$\widetilde{G}_L(s) = \begin{pmatrix} 0 & & & & \\ \widetilde{G}(s) & 0 & & & \\ & \widetilde{G}(s) & 0 & & \\ & & \widetilde{G}(s) & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

If $\mathrm{Re}(s) > 0$, then $I - \widetilde{R}_U(s)$, $I - \widetilde{\Psi}_D(s)$ and $I - \widetilde{G}_L(s)$ are invertible, and their inverses are respectively given by

$$[I - \widetilde{R}_U(s)]^{-1} = \begin{pmatrix} I & X_1(s) & X_2(s) & X_3(s) & \cdots \\ & I & X_1(s) & X_2(s) & \cdots \\ & & I & X_1(s) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix},$$

where

$$X_l(s)=\sum_{i=1}^{\infty}\sum_{\substack{n_1+n_2+\dots+n_i=l\\ n_j\geqslant 1,1\leqslant j\leqslant i}}\widetilde{R}_{n_1}(s)\widetilde{R}_{n_2}(s)\dots\widetilde{R}_{n_i}(s),\quad l\geqslant 1,$$

$$[I-\widetilde{\Psi}_D(s)]^{-1}=\operatorname{diag}([I-\widetilde{\Phi}_0(s)]^{-1},[I-\widetilde{\Phi}_0(s)]^{-1},[I-\widetilde{\Phi}_0(s)]^{-1},\dots)$$

and

$$[I-\widetilde{G}_L(s)]^{-1}=\begin{pmatrix} I &&&& \\ \widetilde{G}(s) & I &&& \\ \widetilde{G}^2(s) & \widetilde{G}(s) & I && \\ \widetilde{G}^3(s) & \widetilde{G}^2(s) & \widetilde{G}(s) & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $\mathfrak{B}$ be the absorbing time of the modified process, given that it starts from the state set $\{(S,1,j_1,j_2,0):1\leqslant j_1\leqslant m_1,1\leqslant j_2\leqslant m_2\}$ with probability vector θ_0, where

$$\theta_0=\frac{\alpha x_0 V(D_1\otimes I)}{\alpha x_0 V(D_1\otimes I)e}.$$

We write

$$\overline{B}(t)=P\{\mathfrak{B}>t\}\quad\text{and}\quad\widetilde{\overline{B}}(s)=\int_0^{+\infty}\mathrm{e}^{-st}\,\overline{B}(t)\mathrm{d}t.$$

Theorem 7.6 *If* $\operatorname{Re}(s)>0$, *then*

$$\begin{aligned}\widetilde{\overline{B}}(s)=&\,\theta_0\int_0^{+\infty}\exp\{[(C_1+D_1)\oplus C_2]x\}\mathrm{e}^{-sx}\,\overline{B}_0(x)\mathrm{d}xe\\ &+\theta_0\left\{\sum_{l=1}^{\infty}\left[\widetilde{A}_l^{(G)}(s)+\sum_{k=l+1}^{\infty}\widetilde{A}_k^{(G)}(s)\widetilde{G}^{k-1}(s)\right]\right\}[I-\widetilde{\Phi}_0(s)]^{-1}\\ &\cdot\left[I-\sum_{k=1}^{\infty}\widetilde{R}_k(s)\right]^{-1}\int_0^{+\infty}\exp\{[(C_1+D_1)\oplus C_2]x\}\mathrm{e}^{-sx}\,\overline{B}(s)\mathrm{d}xe.\end{aligned}$$

Proof By using the supplementary variable method we can obtain

$$\widetilde{\overline{B}}(s)=\int_0^{+\infty}\widetilde{Q}_S^{*}(s,1,x)\mathrm{d}xe+\int_0^{+\infty}\widetilde{Q}_G^{*}(s,1,x)\mathrm{d}xe.$$

Note that

$$\int_0^{+\infty}\widetilde{Q}_S^{*}(s,1,x)\mathrm{d}x=\theta_0\int_0^{+\infty}\exp\{[(C_1+D_1)\oplus C_2]x\}\mathrm{e}^{-sx}\,\overline{B}_0(x)\mathrm{d}x$$

and

$$\int_0^{+\infty} \widetilde{Q}_G^*(s,1,x)\mathrm{d}x = \widetilde{Q}_G^*(s,1,0)\int_0^{+\infty} \exp\{[(C_1+D_1)\oplus C_2]x\}\mathrm{e}^{-sx}\overline{B}(x)\mathrm{d}x,$$

we only need to compute $\widetilde{Q}_G^*(s,1,0)$.

Since $\widetilde{Q}_G^*(s,1,0) = \sum_{k=1}^{\infty} \widetilde{P}_{Gk}(s,0)$, $\widetilde{Q}_G^*(s,1,0) = \widetilde{P}_G(s)(I,I,I,\dots)^{\mathrm{T}}$. It follows from Eq. (7.64) and Eq. (7.65) that

$$\begin{aligned}
\widetilde{Q}_G^*(s,1,0) &= \widetilde{\Gamma}(s)[I-\widetilde{Q}(s)]^{-1}(I,I,I,\dots)^{\mathrm{T}} \\
&= \widetilde{\Gamma}(s)[I-\widetilde{G}_L(s)]^{-1}[I-\widetilde{\Psi}_D(s)]^{-1}[I-\widetilde{R}_U(s)]^{-1}(I,I,I,\dots)^{\mathrm{T}} \\
&= \theta_0 \sum_{l=1}^{\infty}\left[\widetilde{A}_l^{(G)}(s) + \sum_{k=l+1}^{\infty} \widetilde{A}_k^{(G)}(s)\widetilde{G}^{k-1}(s)\right] \\
&\quad \cdot [I-\widetilde{\Phi}_0(s)]^{-1}\left[I-\sum_{k=1}^{\infty}\widetilde{R}_k(s)\right]^{-1}.
\end{aligned}$$

Some simple computations can lead to the desired result.

It is easy to see from Theorem 7.6 that

$$\begin{aligned}
E[\mathfrak{B}] &= \theta_0 \int_0^{+\infty} \exp\{[(C_1+D_1)\oplus C_2]x\}\overline{B}_0(x)\mathrm{d}xe \\
&\quad +\theta_0 \left\{\sum_{l=1}^{\infty}\left[\widetilde{A}_l^{(G)}(0) + \sum_{k=l+1}^{\infty}\widetilde{A}_k^{(G)}(0)\widetilde{G}^{k-1}(0)\right]\right\}[I-\widetilde{\Phi}_0(0)]^{-1} \\
&\quad \cdot\left[I-\sum_{k=1}^{\infty}\widetilde{R}_k(0)\right]^{-1}\int_0^{+\infty}\exp\{[(C_1+D_1)\oplus C_2]x\}\overline{B}(x)\mathrm{d}xe.
\end{aligned}$$

7.4 A Repairable Retrial Queue

In this section, we consider a $BMAP/G/1/1$ retrial queue with a server subject to breakdowns and repairs, obtain the distribution of stationary queue length, the stationary availability and the stationary failure frequency.

The retrial queueing model is described as follows:

The arrival process: The arrivals to the retrial queue are modelled by a BMAP with irreducible matrix descriptor $\{D_k, k \geqslant 0\}$ of size m. We assume that $D = \sum_{k=0}^{\infty} D_k$ is the infinitesimal generator of an irreducible Markov chain with $De = 0$. Let σ be the stationary probability vector of the Markov chain D. Then $\lambda = \sigma \sum_{k=1}^{\infty} kD_k e$ is the stationary arrival rate of the BMAP.

The service times: The service times $\{\chi_n, n \geqslant 1\}$ of the customers are assume to be i.i.d. random variables whose distribution function is given by

$$B(t) = P\{\chi_n \leqslant t\} = 1 - \exp\left\{-\int_0^t \mu(v)\mathrm{d}v\right\}$$

with $E[\chi_n] = 1/\mu < +\infty$.

The life time and the repair time: The life time X of the server is exponential with a mean life time $1/\alpha$, and does not change during the idle period of the server. The repair time Y of the server has the following distribution function

$$V(y) = P\{Y \leqslant y\} = 1 - \exp\left\{-\int_0^y \beta(v)\mathrm{d}v\right\}$$

with $E[Y] = 1/\beta < +\infty$.

The retrial rule: We assume that there is no waiting space in the retrial queue, and the size of the orbit is infinite. If an arrival, either a primary or a retrial customer, finds that there is no customer in the server, then it enters the server immediately and receives service. Otherwise, it enters the orbit and makes a retrial at a later time. Returning customers behave independently of each other, and are persistently keep making retrials until they receive their requested service. Successive inter-retrial times $\{\xi_k, k \geqslant 1\}$ are i.i.d. exponential random variables with mean inter-retrial time $1/\theta$.

The service discipline: If the server is busy at the arrival epoch, then all calls join the orbit, If the server is free, then one of the arriving customers begins his service and the others form sources of repeated calls.

The repair discipline: When the server fails, it enters the state of failure and undergoes repair immediately. The customer who has been partially served has to wait to continue service. As soon as the repair of the server is completed, the server enters the working state immediately and continues to serve the customer. We assume that the repaired server is as good as a new server, and the service time is cumulative.

The independence: We assume that all the random variables defined above are mutually independent.

7.4.1 The Supplementary Variables

Now, we introduce several supplementary variables to make the model Markovian and set up the system of stationary differential equations for the model.

Let $\widetilde{\chi}_n$ be the generalized service time of the nth customer, which is the length of time since the beginning of the service for the nth customer until the completion of the service. Clearly, $\widetilde{\chi}_n$ includes the down time of the server due to server failures during the service period of the nth customer. It is easy to see that the sequence $\widetilde{\chi}_n$ for $n \geqslant 1$ are i.i.d. random variables and $E\left[\widetilde{\chi}_n\right] = \frac{1}{\mu}\left(1 + \frac{\alpha}{\beta}\right)$. It follows from Theorem 3 of Dudin and Klimenok [107] or Liang and Kulkarni [119] that if $\rho = \lambda E\left[\widetilde{\chi}_n\right] = \frac{\lambda}{\mu}\left(1 + \frac{\alpha}{\beta}\right) < 1$, then the queueing system is stable.

For the repairable $BMAP/G/1$ retrial queue defined above, we denote by $N(t)$ the number of sources of repeated calls at time t, and define the states of the server as

$$L(t) = \begin{cases} I, & \text{if the server is idle at time } t, \\ W, & \text{if the server is working at time } t, \\ R, & \text{if the server is under repair at time } t. \end{cases}$$

We introduce three variables $J(t)$, $S(t)$ and $R(t)$ representing the phase of the arrival process, the elapsed service time and the elapsed repair time at time t, respectively. Then $\{(L(t), N(t), J(t), S(t), R(t)) : t \geqslant 0\}$ is a Markov process with state space expressed as

$$\begin{aligned} \Omega = {} & \{(I, k, j) : k \geqslant 0, 1 \leqslant j \leqslant m\} \cup \{(W, k, j, x) : k \geqslant 0, 1 \leqslant j \leqslant m, x \geqslant 0\} \\ & \cup \{(R, k, j, x, y) : k \geqslant 0, 1 \leqslant j \leqslant m, x \geqslant 0, y \geqslant 0\}, \end{aligned}$$

where k, j, x and y denote the number of customers in the orbit, the phase of the arrival process, the amount of the elapsed service time and the elapsed repair time, respectively.

For $k \geqslant 0$, we define

$$P_{(I,k,j)}(t) = P\{L(t) = I, N(t) = k, J(t) = j\},$$
$$P_{(W,k,j)}(t,x)\mathrm{d}x = P\{L(t) = W, N(t) = 1 + k, J(t) = j, x \leqslant S(t) < x + \mathrm{d}x\}$$

and

$$\begin{aligned} P_{(R,k,j)}(t,x,y)\mathrm{d}y = P\{ & L(t) = R, N(t) = 1 + k, J(t) = j, \\ & S(t) = x, y \leqslant R(t) < y + \mathrm{d}y\}. \end{aligned}$$

We write the above probabilities into vector form as

$$P_{I,k}(t) = (P_{(I,k,1)}(t), P_{(I,k,2)}(t), \ldots, P_{(I,k,m)}(t)),$$
$$P_{W,k}(t,x) = (P_{(W,k,1)}(t,x), P_{(W,k,2)}(t,x), \ldots, P_{(W,k,m)}(t,x))$$

and

$$P_{R,k}(t,x,y) = (P_{(R,k,1)}(t,x,y), P_{(R,k,2)}(t,x,y), \ldots, P_{(R,k,m)}(t,x,y)).$$

Since we are interested in the stationary behavior of the system, we define

$$P_{I,k} = \lim_{t \to +\infty} P_{I,k}(t),$$
$$P_{W,k}(x) = \lim_{t \to +\infty} P_{W,k}(t,x),$$
$$P_{R,k}(x,y) = \lim_{t \to +\infty} P_{R,k}(t,x,y).$$

The joint probability density $\{P_{I,k}, P_{W,k}(x), P_{R,k}(x,y), k \geqslant 0\}$ satisfies the following system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}x} P_{W,0}(x) = P_{W,0}(x)\{D_0 - [\alpha + \mu(x)]I\} + \int_0^{+\infty} \beta(y) P_{R,0}(x,y)\mathrm{d}y, \tag{7.66}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} P_{W,k}(x) = P_{W,k}(x)\{D_0 - [\alpha + \mu(x)]I\} + \sum_{i=0}^{k} P_{W,i}(x) D_{k-i}$$
$$+ \int_0^{+\infty} \beta(y) P_{R,k}(x,y)\mathrm{d}y, \quad k \geqslant 1, \tag{7.67}$$

$$\frac{\partial}{\partial y} P_{R,0}(x,y) = P_{R,0}(x,y)[D_0 - \beta(y) I], \tag{7.68}$$

$$\frac{\partial}{\partial y} P_{R,k}(x,y) = P_{(R,k)}(x,y)[D_0 - \beta(y)I] + \sum_{i=0}^{k} P_{R,i}(x,y) D_{k-i}, \quad k \geqslant 1, \tag{7.69}$$

with the boundary conditions

$$P_{I,k}(k\theta I - D_0) = \int_0^{+\infty} \mu(x) P_{W,k}(x)\mathrm{d}x, \quad k \geqslant 0, \tag{7.70}$$

$$P_{W,k}(0) = \sum_{i=0}^{k} P_{I,i} D_{k+1-i} + (k+1)\theta P_{I,k+1}, \quad k \geqslant 0, \tag{7.71}$$

$$P_{R,k}(x,0) = \alpha P_{W,k}(x), \quad k \geqslant 0, \tag{7.72}$$

and the normalization condition

$$\sum_{k=0}^{\infty} \left[P_{I,k} + \int_0^{+\infty} P_{W,k}(x)\mathrm{d}x + \int_0^{+\infty}\int_0^{+\infty} P_{R,k}(x,y)\mathrm{d}x\mathrm{d}y \right] e = 1. \tag{7.73}$$

7.4.2 A Level-Dependent Markov Chain of *M*/*G*/1 Type

Now, we provide an approach for solving the Eq. (7.66) to Eq. (7.73) with two crucial steps: Express $P_{W,k}(x)$ and $P_{R,k}(x,y)$ in terms of boundary probabilistic vectors $P_{W,k}(0)$ by recognizing a new BMAP, and obtain $P_{I,k}$ and $P_{W,k}(0)$ by converting boundary equations into a level-dependent Markov chain of $M/G/1$ type.

Let

$$D^*(z) = \sum_{k=0}^{\infty} z^k D_k, \quad \tilde{v}(D^*(z)) = \int_0^{+\infty} \exp\{D^*(z)y\} \mathrm{d}V(y).$$

The following lemma recognizes a new BMAP, which appears in the process of solving the system of differential equations.

Lemma 7.3 *Let*

$$\Psi^*(z) = \sum_{k=0}^{\infty} z^k \Psi_k = D^*(z) - \alpha[I - \tilde{v}(D^*(z))]. \tag{7.74}$$

Then Ψ_k *for* $k \geqslant 0$ *are coefficient matrices of a BMAP of size m.*

Proof To prove this lemma, we need to show that the following three conditions are satisfied: i) The diagonal entries of Ψ_0 are strictly negative, the off-diagonal entries are nonnegative, and Ψ_0 is invertible. ii) For $k \geqslant 1$, $\Psi_k \geqslant 0$ and $\sum_{k=1}^{\infty} k\Psi_k < +\infty$. iii) $\Psi = \sum_{k=0}^{\infty} \Psi_k$ is irreducible and $\Psi e = 0$.

(1) It follows from Eq. (7.74) that

$$\Psi_0 = D_0 - \alpha[I - \tilde{v}(D_0)]. \tag{7.75}$$

It is clear from Eq. (7.75) that the off-diagonal entries of Ψ_0 are nonnegative due to $\tilde{v}(D_0) \geqslant 0$. Note that the i th diagonal entry of the matrix $\tilde{v}(D_0)$ is the conditional probability that the BMAP returns to state i and no arrival occurs during a repair time, given that the BMAP starts in state i, we obtain that the i th diagonal entry of the matrix $I - \tilde{v}(D_0)$ is nonnegative. Hence, the diagonal entries of Ψ_0 are strictly negative according to the assumption of D_0. It is easy to see from Eq. (7.75) that the off-diagonal entries of Ψ_0 are nonnegative, and $\Psi_0 e \lneqq 0$. Therefore, the real parts of the eigenvalues of Ψ_0 are all negative according to Gerŝgorin Theorem given in Horn and Johnson [112], and so Ψ_0 is invertible.

(2) It is clear that for $k \geqslant 1$, $\frac{\mathrm{d}^k}{\mathrm{d}z^k}[\tilde{v}(D^*(z))]_{|z=0} \geqslant 0$. Since

$$\Psi_k = D_k + \alpha \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}z^k}[\tilde{v}(D^*(z))]_{|z=0},$$

$D_k \geqslant 0$ for $k \geqslant 1$ and $\alpha > 0$, we obtain that for $k \geqslant 1$, $\Psi_k \geqslant 0$. Using $\sum_{k=1}^{\infty} kD_k < +\infty$, we obtain

$$\sum_{k=1}^{\infty} k\Psi_k = \left[I + \alpha \int_0^{+\infty} x \exp\{Dx\} \mathrm{d}V(x) \right] \sum_{k=1}^{\infty} kD_k < +\infty.$$

(3) Note that

$$\sum_{k=0}^{\infty} \Psi_k = D + \alpha \tilde{v}(D) - \alpha I,$$

it is clear that Ψ is irreducible, since D is irreducible and $\tilde{v}(D) \geqslant 0$. Noting that $De = 0$ and $[I - \tilde{v}(D)]e = 0$, it is obvious that $\Psi e = 0$. This completes the proof.

Remark 7.2 *The BMAP with coefficient matrix sequence $\{\Psi_k\}$ may be regarded as a generalized arrival process, which is composed of the sum of two parts: the first is the original BMAP with coefficient matrix sequence $\{D_k\}$ the is an additional BMAP with coefficient matrix sequence*

$$\left\{ -\alpha[I - \tilde{v}(D_0)], \alpha \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}z^k} [\tilde{v}(D^*(z))]_{|z=0}, k = 1, 2, \dots \right\}.$$

The additional BMAP is related to the server subject to breakdowns and repairs.

For the two BMAPs having coefficient matrix sequences $\{D_k\}$ and $\{\Psi_k\}$, let $K^D(t)$ and $K^\Psi(t)$ denote the numbers of arrivals in the time interval $[0,t)$, respectively, and $J^D(t)$ and $J^\Psi(t)$ the phases at time t, respectively. We introduce the conditional probabilities for the two BMAPs by

$$P_{j,j'}^D(n,t) = P\{K^D(t) = n, J^D(t) = j' \mid K^D(0) = 0, J^D(0) = j\}.$$

and

$$P_{j,j'}^\Psi(n,t) = P\{K^\Psi(t) = n, J^\Psi(t) = j' \mid K^\Psi(0) = 0, J^\Psi(0) = j\}.$$

Let $P^D(n,t)$ and $P^\Psi(n,t)$ be the matrices with entries $P_{j,j'}^D(n,t)$ and $P_{j,j'}^\Psi(n,t)$ for $1 \leqslant j,\ j' \leqslant m$, respectively. Then it follows from Neuts [24] or Lucantoni [120] that

$$P_D^*(z,t) = \sum_{n=0}^{\infty} z^n P^D(n,t) = \exp\{D^*(z)t\} \tag{7.76}$$

and

$$P_\Psi^*(z,t) = \sum_{n=0}^{\infty} z^n P^\Psi(n,t) = \exp\{\Psi^*(z)t\}. \tag{7.77}$$

We write

$$\overline{B}(x) = 1 - B(x), \quad \overline{V}(y) = 1 - V(y),$$

$$P_W^*(z,x) = \sum_{k=0}^{\infty} z^k P_{W,k}(x), \quad P_R^*(z,x,y) = \sum_{k=0}^{\infty} z^k P_{R,k}(x,y).$$

Now, we solve the system of matrix equations Eq. (7.66) to Eq. (7.73). It follows from Eq. (7.68) and Eq. (7.69) that

$$\frac{\partial}{\partial y} P_R^*(z,x,y) = P_R^*(z,x,y)[D^*(z) - \beta(y)],$$

hence from Eq. (7.72) we obtain

$$\begin{aligned} P_R^*(z,x,y) &= P_R^*(z,x,0)\exp\{D^*(z)y\}\overline{V}(y) \\ &= \alpha P_W^*(z,x)\exp\{D^*(z)y\}\overline{V}(y). \end{aligned} \tag{7.78}$$

It follows from Eq. (7.66) and Eq. (7.67), together with Eq. (7.78), that

$$\frac{\partial}{\partial x} P_W^*(z,x) = P_W^*(z,x)\{D^*(z) - \alpha[I - \tilde{v}(D^*(z))] - \mu(x)I\}.$$

Hence,

$$\begin{aligned} P_W^*(z,x) &= P_W^*(z,0)\exp\{\{D^*(z) - \alpha[I - \tilde{v}(D^*(z))]\}x\}\overline{B}(x) \\ &= P_W^*(z,0)\exp\{\Psi^*(z)x\}\overline{B}(x), \end{aligned} \tag{7.79}$$

which, together with Eq. (7.77), leads to

$$P_{W,k}(x) = \sum_{i=0}^{k} P_{W,i}(0)P^{\Psi}(k-i,x)\overline{B}(x). \tag{7.80}$$

Similarly, Eq. (7.78), together with Eq. (7.76) and Eq. (7.80), leads to

$$\begin{aligned} P_{R,l}(x,y) &= \alpha\sum_{k=0}^{l} P_{W,k}(x)P^D(l-k,y)\overline{V}(y) \\ &= \alpha\sum_{k=0}^{l}\sum_{i=0}^{k} P_{W,i}(0)P^{\Psi}(k-i,x)P^D(l-k,y)\overline{B}(x)\overline{V}(y). \end{aligned} \tag{7.81}$$

Equations Eq. (7.80) and Eq. (7.81) provide a solution for $P_{W,k}(x)$ and $P_{R,k}(x,y)$ in terms of $P_{W,k}(0)$, $k \geqslant 0$. In order to completely solve the system of differential equations, we still need to determine the vectors $P_{W,k}(0)$ and $P_{I,k}$ for $k \geqslant 0$ from the boundary equations Eq. (7.70) and Eq. (7.71), and the normalization condition Eq. (7.73). We defiue

$$P_{IW} = (P_{I,0}, P_{W,0}(0), P_{I,1}, P_{W,1}(0), P_{I,2}, P_{W,2}(0), P_{I,3}, P_{W,3}(0), \ldots),$$

$$C_k = \int_0^{+\infty} P^{\Psi}(k,x)\mathrm{d}B(x), \quad k \geqslant 0,$$

$$A_0^{(k)} = \begin{pmatrix} 0 & k\theta I \\ 0 & 0 \end{pmatrix}, \quad k \geqslant 1,$$

$$A_1^{(k)} = \begin{pmatrix} -k\theta I + D_0 & D_1 \\ C_0 & -I \end{pmatrix}, \quad k \geqslant 0,$$

$$A_k = \begin{pmatrix} 0 & D_k \\ C_{k-1} & 0 \end{pmatrix}, \quad k \geqslant 2,$$

and

$$Q = \begin{pmatrix} A_1^{(0)} & A_2 & A_3 & A_4 & \cdots \\ A_0^{(1)} & A_1^{(1)} & A_2 & A_3 & \cdots \\ & A_0^{(2)} & A_1^{(2)} & A_2 & \cdots \\ & & A_0^{(3)} & A_1^{(3)} & \cdots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{7.82}$$

According to the above definitions and the expression for $P_{W,k}(x)$ in Eq. (7.80), the boundary equations Eq. (7.70) and Eq. (7.71) can be written as

$$P_{IW} Q = 0. \tag{7.83}$$

In what follows, we show that the matrix Q is the infinitesimal generator of a continuous-time positive recurrent Markov chain. Therefore, the unique stationary probability vector X of Q can be used to determine the vectors $P_{W,k}(0)$ and $P_{I,k}$ for $k \geqslant 0$. It is clear that $P_{IW} = \gamma X$. Let $X = (x_0, x_1, x_2, \ldots)$, where $x_k = (x_{k,1}, x_{k,2})$ for $k \geqslant 0$ and the size of each vector $x_{k,j}$, $k \geqslant 0$ and $j = 1, 2$, is m. The normalization condition Eq. (7.73) and the expressions for $P_{W,k}(x)$ in Eq. (7.80) and $P_{R,k}(x, y)$ in Eq. (7.82) lead to

$$\gamma = \frac{1}{\sum_{k=0}^{\infty} \sum_{i=0}^{k} (x_{i,1} + x_{i,2} F_{k-i}) e + \alpha \sum_{l=0}^{\infty} \sum_{k=0}^{l} \sum_{i=0}^{k} x_{i,2} F_{k-i} H_{l-k}},$$

where

$$F_k = \int_0^{+\infty} P^{\Psi}(k, x) \overline{B}(x) \mathrm{d}x, \quad H_k = \int_0^{+\infty} P^{D}(k, y) \overline{V}(y) \mathrm{d}y.$$

Theorem 7.7 *The matrix Q is the infinitesimal generator of a continuous-time irreducible positive recurrent Markov chain.*

Proof It can be easily verified that the matrix Q can be regarded as the infinitesimal generator of a continuous-time irreducible Markov chain based on the definition of Q. In what follows we only need to prove that it is positive recurrent.

For each $k \geqslant 1$,

$$A_0^{(k)} + A_1^{(k)} + \sum_{l=2}^{\infty} A_l = \begin{pmatrix} D_0 - k\theta I & D_+ + k\theta I \\ C & -I \end{pmatrix},$$

where $D_+ = \sum_{l=1}^{\infty} D_l$ and $C = \sum_{l=0}^{\infty} C_l$. It is clear that the transition rate matrix $A_0^{(k)} + A_1^{(k)} + \sum_{l=2}^{\infty} A_l$ is irreducible and positive recurrent. Let $\left(y_1^{(k)}, y_2^{(k)}\right)$ be the stationary probability vector of $A_0^{(k)} + A_1^{(k)} + \sum_{l=2}^{\infty} A_l$. Then

$$\left(y_1^{(k)}, y_2^{(k)}\right) \begin{pmatrix} D_0 - k\theta I & D_+ + k\theta I \\ C & -I \end{pmatrix} = 0,$$

hence, solving this equation gives

$$y_1^{(k)} \left[\frac{1}{k\theta}(D_0 + D_+ C) + C - I \right] = 0.$$

Noting that the matrix C is stochastic and the matrix $D_0 + D_+$ is an infinitesimal generator, $\frac{1}{k\theta}\left(D_0 + D_+ C\right) + C - I$ is an irreducible infinitesimal generator of size m for each $k \geqslant 1$. Thus, for each $k \geqslant 1$ the Markov chain $\frac{1}{k\theta}(D_0 + D_+ C) + C - I$ is positive recurrent. Let $w^{(k)}$ be the stationary probability vector of $\frac{1}{k\theta}(D_0 + D_+ C) + C - I$. Then

$$y_1^{(k)} = \frac{w^{(k)}}{1 + k\theta + w^{(k)} D_+ e} \quad \text{and} \quad y_2^{(k)} = \frac{w^{(k)}(D_+ + k\theta I)}{1 + k\theta + w^{(k)} D_+ e}.$$

Noting that $k \to \infty$,

$$\frac{1}{k\theta}(D_0 + D_+ C) + C - I \to C - I,$$

it is clear that $w^{(k)} \to w$, as $k \to \infty$, where w is the stationary probability vector of the irreducible infinitesimal generator $C - I$. Thus, as $k \to \infty$,

$$y_1^{(k)} = \frac{w^{(k)}}{1 + k\theta + w^{(k)} D_+ e} \to 0 \quad \text{and} \quad y_2^{(k)} = \frac{w^{(k)}(D_+ + k\theta I)}{1 + k\theta + w^{(k)} D_+ e} \to w.$$

As $k \to \infty$, simple computation leads to

$$\left(y_1^{(k)}, y_2^{(k)}\right) A_0^{(k)} e = k\theta y_1^{(k)} e = \frac{w^{(k)} C \left(I - \dfrac{1}{k\theta} D_0 \right)^{-1} e}{1 + \dfrac{1}{k\theta} w^{(k)} C \left(I - \dfrac{1}{k\theta} D_0 \right)^{-1} e} \to 1 \tag{7.84}$$

and

$$\left(y_1^{(k)}, y_2^{(k)}\right) \sum_{l=2}^{\infty} (l-1) A_l e = y_1^{(k)} \sum_{l=2}^{\infty} (l-1) D_l e + y_2^{(k)} \sum_{l=2}^{\infty} (l-1) C_{l-1} e$$
$$\to w \sum_{l=2}^{\infty} (l-1) C_{l-1} e = \frac{\lambda}{\mu} < 1 \tag{7.85}$$

due to the stable condition $\rho = \dfrac{\lambda}{\mu}\left(1 + \dfrac{\alpha}{\beta}\right) < 1$. It follows from Eq. (7.84) and Eq. (7.85) that

$$\lim_{k \to \infty} \left(y_1^{(k)}, y_2^{(k)}\right) A_0^{(k)} e > \lim_{k \to \infty} \left(y_1^{(k)}, y_2^{(k)}\right) \sum_{l=2}^{\infty} (l-1) A_l e.$$

Thus, there always exists a positive integer N big enough such that for all $k > N$,

$$\left(y_1^{(k)}, y_2^{(k)}\right) A_0^{(k)} e > \left(y_1^{(k)}, y_2^{(k)}\right) \sum_{l=2}^{\infty} (l-1) A_l e. \tag{7.86}$$

It is easy to check that

$$\sum_{l=2}^{\infty} (l-1) A_l e < +\infty. \tag{7.87}$$

Therefore, it is easy to see from Eq. (7.86) and Eq. (7.87) that the continuous-time irreducible Markov chain Q is positive recurrent based on the principle of the mean drift. This completes the proof.

For the level-dependent Markov chain Q of $M/G/1$ type, let $\{G^{(k)}\}$ be the minimal nonnegative solution to the system of matrix equations

$$A_0^{(k)} + A_1^{(k)} G^{(k)} + A_2 G^{(k+1)} G^{(k)} + A_3 G^{(k+2)} G^{(k+1)} G^{(k)} + \ldots = 0, \quad k \geqslant 1.$$

Then for $k \geqslant 0$ and $l \geqslant 1$,

$$R_l^{(k)} = [A_{l+1} + A_{l+2} G^{(k+2)} + A_{l+3} G^{(k+3)} G^{(k+2)} + \ldots](-U_{k+1}^{-1}),$$

and for $k \geqslant 0$,

$$U_k = A_1^{(k)} + A_2 G^{(k+1)} + A_3 G^{(k+2)} G^{(k+1)} + A_4 G^{(k+3)} G^{(k+2)} G^{(k+1)} + \ldots.$$

Therefore,

$$Q=(I-R_U)U_D(I-G_L), \tag{7.88}$$

where

$$(I-R_U)=\begin{pmatrix} I & -R_1^{(0)} & -R_2^{(0)} & -R_3^{(0)} & \cdots \\ & I & -R_1^{(1)} & -R_2^{(1)} & \cdots \\ & & I & -R_1^{(2)} & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$U_D=\mathrm{diag}(U_0,U_1,U_2,\ldots)$$

$$(I-G_L)=\begin{pmatrix} I & & & & \\ -G^{(1)} & I & & & \\ & -G^{(2)} & I & & \\ & & -G^{(3)} & I & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Using the *RG*-factorization, the stationary probability vector of Q is given by

$$\begin{cases} x_0=\tau y_0, \\ x_k=\sum_{i=0}^{k-1} x_i R_{k-i}^{(i)}, \quad k\geqslant 1, \end{cases}$$

where y_0 is the stationary probability vector of the transition rate matrix U_0 and the scalar τ is uniquely determined by $\sum_{k=0}^{\infty} x_k e=1$.

7.4.3 The Stationary Performance Measures

The solution of the system of differential equations for the vectors $P_{I,k}$, $P_{W,k}(x)$ and $P_{R,k}(x,y)$ for $k\geqslant 0$ is summarized in the following theorem.

Theorem 7.8 *If the system is stable, then for* $k\geqslant 0$,

$$\begin{cases} P_{I,k}=\gamma x_{k,1}, \\ P_{W,k}(x)=\gamma\sum_{i=0}^{k} x_{k,2}P^{\Psi}(k-i,x)\overline{B}(x), \\ P_{R,k}(x,y)=\alpha\gamma\sum_{l=0}^{k}\sum_{i=0}^{l} x_{i,2}P^{\Psi}(l-i,x)P^{D}(k-l,y)\overline{B}(x)\overline{V}(y). \end{cases}$$

Remark 7.3 *The method proposed in this section can be used in principle to deal with a retrial queue with a more general total retrial rate, where the total retrial rate is a function* $f(n,\theta), n$ *is the number of customers in the orbit and* θ *is a parameter. For example,* $f(n,\theta)=\sum_{i=0}^{M} a_i(\theta)n^i$ *or* $f(n,\theta)= C_1 \ln n + C_2 \mathrm{e}^{n\theta}$. *The case with linear retrial rate* $f(n,\theta)=n\theta+\gamma$ *was studied in Dudin and Klimenok* [107].

Now, we express the distribution of stationary queue length. If the system is stable, we write

$$p_k = \lim_{t\to+\infty} P\{N(t)=k\}, \quad k \geqslant 0.$$

Note that

$$p_k^{(I)} = \lim_{t\to+\infty} P\left\{L(t)=I, N(t)=k\right\}, \quad k \geqslant 0,$$
$$p_k^{(W)} = \lim_{t\to+\infty} P\{L(t)=W, N(t)=k\}, \quad k \geqslant 1,$$
$$p_k^{(R)} = \lim_{t\to+\infty} P\left\{L(t)=R, N(t)=k\right\}, \quad k \geqslant 1.$$

we have

$$p_0 = p_0^{(I)}, \quad p_k = p_k^{(I)} + p_k^{(W)} + p_k^{(R)}, \quad k \geqslant 1,$$

which leads to

$$p_0^{(I)} = p_{I,0}e, \quad p_k^{(I)} = p_{I,k}e, \quad k \geqslant 1,$$
$$p_k^{(W)} = \int_0^{+\infty} P_{W,k-1}(x)\mathrm{d}xe, \;\; p_k^{(R)} = \int_0^{+\infty}\int_0^{+\infty} P_{R,k-1}(x,y)\mathrm{d}x\mathrm{d}y, \quad k \geqslant 1.$$

Therefore, the stationary queue length distribution is given by

$$p_0 = \gamma x_{0,1}e$$
$$p_k = \gamma\left(x_{k,1}e + \sum_{i=0}^{k-1} x_{i,2}F_{k-1-i} + \alpha\sum_{j=0}^{k-1}\sum_{i=0}^{j} x_{i,2}F_{j-i}H_{k-1-j}\right)e, \quad k \geqslant 1.$$

To obtain the stationary availability and the stationary failure frequency of the server, we need the following lemma.

Lemma 7.4 *If the system is stable, then*

(1) *the probability that the server is idle is* $P_I = 1-\dfrac{\lambda}{\mu}\left(1+\dfrac{\alpha}{\beta}\right)$,

(2) *the probability that the server is working is* $P_W = \dfrac{\lambda}{\mu}$, *and*

(3) *the probability that the server is under repair is* $P_R = \dfrac{\lambda}{\mu}\dfrac{\alpha}{\beta}$.

Proof It follows from Eq. (7.70) and Eq. (7.79) that

$$\theta z \frac{\mathrm{d}}{\mathrm{d}z} P_I^*(z) - P_I^*(z) D_0 = P_W^*(z,0)\tilde{b}(\Psi^*(z)), \tag{7.89}$$

where

$$P_I^*(z) = \sum_{k=0}^{\infty} z^k P_{I,k}, \quad \tilde{b}(\Psi^*(z)) = \int_0^{+\infty} \exp\{\Psi^*(z)x\}\mathrm{d}B(x).$$

It follows from Eq. (7.71) that

$$P_W^*(z,0) = \frac{1}{z} P_I^*(z)[D^*(z) - D_0] + \theta \frac{\mathrm{d}}{\mathrm{d}z} P_I^*(z). \tag{7.90}$$

From Eq. (7.89) and Eq. (7.90) we obtain

$$P_W^*(z,0) = P_I^*(z) D^*(z)[zI - \tilde{b}(\Psi^*(z))]^{-1}.$$

Noting that

$$\int_0^{+\infty} \exp\{\Psi^*(z)x\}\overline{B}(x)\mathrm{d}x = [I - \tilde{b}(\Psi^*(z))][-\Psi^*(z)]^{-1}$$

and

$$\int_0^{+\infty} \exp\{D^*(z)x\}\overline{V}(x)\mathrm{d}x = [I - \tilde{v}(D^*(z))][-D^*(z)]^{-1},$$

using Eq. (7.78), Eq. (7.79) and Eq. (7.90) yields

$$\begin{aligned} P^*(z) &= P_I^*(z) + \int_0^{+\infty} P_W^*(z,x)\mathrm{d}x + \int_0^{+\infty}\int_0^{+\infty} P_R^*(z,x,y)\mathrm{d}x\mathrm{d}y \\ &= P_I^*(z)\{I + D^*(z)[zI - \tilde{b}(\Psi^*(z))]^{-1}[I - \tilde{b}(\Psi^*(z))][-\Psi^*(z)]^{-1} \\ &\quad \cdot \{I + \alpha[I - \tilde{v}(D^*(z))][-D^*(z)]^{-1}\}\}. \end{aligned} \tag{7.91}$$

For $z \geqslant 0$, we denote by $\chi(z)$ and $e(z)$ the eigenvalue with maximal real part of the matrix $D^*(z)$ and the associated right eigenvector with the first entry normalized to one, respectively. It is obvious that $\lim_{z\to 1}\chi(z) = 0$, $\lim_{z\to 1}\chi'(z) = \lambda$ and $\lim_{z\to 1} e(z) = e$. It follows from Eq. (7.91) that

$$P^*(z)e(z) = P_I^*(z)e(z)\frac{z-1}{z - \tilde{b}(\chi(z) - \alpha[1 - \tilde{v}(\chi(z))])}. \tag{7.92}$$

Noting that, after some calculations,

$$\lim_{z\to 1}\frac{z-1}{z - \tilde{b}(\chi(z) - \alpha[1 - \tilde{v}(\chi(z))])} = \frac{1}{1 - \dfrac{\lambda}{\mu}\left(1 + \dfrac{\alpha}{\beta}\right)}$$

and

$$\lim_{z \to 1} P^*(z)e(z) = 1,$$

we obtain

$$P_I = P_I^*(1)e(1) = 1 - \frac{\lambda}{\mu}\left(1 + \frac{\alpha}{\beta}\right).$$

We can similarly obtain that $P_W = \dfrac{\lambda}{\mu}$ and $P_R = \dfrac{\lambda}{\mu}\dfrac{\alpha}{\beta}$. This completes the proof.

This lemma shows that we can determine the scalar function $P^*(z)e(z)$ from Eq. (7.89) to Eq. (7.92). However, we can not explicitly obtain the vector function $P^*(z)$. This is the main reason why it is necessary for us to provide an approach for solving the Eq. (7.66) to Eq. (7.73). Meanwhile, it is also easy to see the basic difficulty of using the standard method (e.g., see Subsection 1.2.2 in Falin and Templeton [109]) to deal with the *BMAP*/*G*/1 retrial queue and more generally, retrial queues of $M/G/1$ type.

Let

$$A(t) = P\{\text{the server is up at time } t\}$$

and define the stationary availability of the server as $A = \lim_{t \to +\infty} A(t)$. We denote by W_f the stationary failure frequency of the server.

Theorem 7.9 *If the system is stable, then*

(1) *the stationary availability of the server is given by*

$$A = 1 - \frac{\lambda}{\mu}\frac{\alpha}{\beta},$$

(2) *the stationary failure frequency of the server is given by*

$$W_f = \alpha\frac{\lambda}{\mu}.$$

Proof Noting that

$$A = \sum_{k=0}^{\infty}\left[P_{I,k} + \int_0^{+\infty} P_{W,k}(x)\mathrm{d}x\right]e = \left[P_I^*(1) + \int_0^{+\infty} P_W^*(1,x)\mathrm{d}x\right]e = P_I + P_W$$

and

$$W_f = \alpha\left[\sum_{k=0}^{\infty}\int_0^{+\infty} P_{W,k}(x)\mathrm{d}x\right]e = \alpha\int_0^{+\infty} P_W^*(1,x)\mathrm{d}xe = \alpha P_W.$$

This completes the proof.

It is easy to see from Eq. (7.78) to Eq. (7.81) that Lemma 7.3 is a key to express the vector generating function $P_W^*(z,x)$, which is necessary to derive the two reliability indexes: The stationary availability and failure frequency, as shown in the proof of Theorem 7.9.

7.5 Notes in the Literature

In this section, we provide a simple introduction to the literature for processor-sharing queues, fluid queues, queues with negative customers, and retrial queues.

7.5.1 The Processor-Sharing Queues

Processor-sharing queues are useful in the study of computer and communication systems. Early work on processor-sharing queues was motivated by the study of multiuser mainframe computer systems, e.g., see Kleinrock [16], Coffman and Kleinrock [6] and Coffman, Muntz and Trotter [7]. Recent interest in processor-sharing queues is due to their applications to communication networks and web servers, for example, modeling congested links with TCP traffic and job schedulers in web servers. During the last few decades considerable attention has been paid to the study of processor-sharing queues, which have been well documented by, for example, books of Kleinrock [17], Cohen [9] and Asmussen [1] and survey papers of Cohen [8], Yashkov [34,35] and Yashkov and Yashkova [36].

For the $M/M/1$ processor-sharing queue, Coffman, Muntz and Trotter [7] and O'Donovan [26] derived the Laplace–Stieltjes transform of the sojourn time distribution. Morrison [22] and Guillemin and Boyer [13] obtained an integral representation for the complementary distribution of the sojourn time by means of the Laplace-Stieltjes transform expression and the spectral theory, respectively. Sengupta and Jagerman [31] studied moments of the sojourn time conditioned on the number of customers found by an arriving customer. Braband [3,4] discussed the waiting time distributions of the closed $M/M/N$ processor-sharing queue. Núñez-Queija [25] studied a Markovian processor-sharing queue with a service rate that varies over time, depending on the number of customers and on the state of a stochastic environment. Masuyama and Takine [21] provided a recursive formula to compute the stationary sojourn time distribution in the $MAP/M/1$ processor-sharing queue. Li, Liu and Lian [20] applied the RG-factorization and a level-dependent Markov chain of $M/G/1$ type to study a $BMAP/M/1$ generalized processor-sharing queue. For the $GI/M/1$ processor-sharing queue, the first two moments and the Laplace-Stieltjes transform of the sojourn time were derived in Ramaswami [27], Cohen [10], and Jagerman and Sengupta [14]. For the $M/G/1$ processor-sharing queue, reader may refer to Schassberger [30], Resing, Hooghiemstra and Keane [29], Yang and Knessl [33], Zwart and Boxma [37], Sericola, Guillemin and Boyer [32], Li and Lin [19], Cheung, van den Berg and Boucherie [5], and Egorova, Zwart and Boxma [11]. For the processor-sharing queues with bulk arrivals, Kleinrock, Muntz and Rodemich [18] first analyzed a processor-sharing queue with bulk arrivals. Rege and Sengupta [28] studied the $M/G/1$ processor-sharing queue with bulk arrivals under a general job size distribution. Bansal [2] obtained expression for the expected response time of a job as a function of its size, where the service times of jobs have a generalized hyperexponential distribution and more generally, a distribution with rational Laplace transforms. Li, Liu and Lian [20] applied the *RG*-factorizations to study a $BMAP/M/1$ generalized processor-sharing queue. Kim and Kim [15] considered concavity of the conditional mean of sojourn time in the $M/G/1$ processor-sharing queue with batch arrivals.

7.5.2 The Fluid Queues

Fluid queues are motivated as modelling, e.g., high-speed communication networks, transportation systems, and manufacturing systems. During the last few decades considerable attention has been paid to the study of fluid queues, which have been well documented, for example, by survey papers of Kulkarni [51] and Ahn and Ramaswami [40].

In most of the studies of the fluid queues, the state space of the external stochastic environment is assumed to be finite. In this case, the stationary probability distribution of the buffer content is the unique solution to a system of differential equations, for example, see (8) in Anick, Mitra and Sondhi [41]. Until now, four methods for solving the system of differential equations have been presented as follows:

(1) A spectral method was proposed in terms of computing the eigenvalues and eigenvectors of the coefficient matrices involved in a fluid model. Readers may refer to Anick, Mitra and Sondhi [41], van Doorn, Jagers and de Wit [65], Mitra [57], Stern and Elwalid [64], Kontovasilis and Mitrou [50], Blaabjerg, Andersson and Andersson [45], Karandikar and Kulkarni [48], Kulkarni [51], Lenin and Parthasarathy [53], Kulkarni and Tzenova [52], Adan, Resing and Kulkarni [39], and Masuyama and Takine [56]. Further, Asmussen [42] and Karandikar and Kulkarni [48] used this spectral method to analyze second-order fluid models with finite states. Rabehasaina and Sericola [60] studied a second-order Markov-modulated fluid queue with linear service rate.

(2) Sericola and Tuffin [63] proposed a stable algorithm for computing the stationary probability distribution of the buffer content. In the stable algorithm, the stationary probability distribution is first assumed to be the product of an exponential function and a power series. Then the determination of the stationary

probability distribution becomes the computation of the coefficients in the product. Barbot, Sericola and Telek [43] and Parthasarathy, Sericola and Vijayashree [59] further applied the stable algorithm to calculate transient distributions of fluid models such as the distribution of the busy period.

(3) Based on the Wiener-Hopf factorization given in Barlow, Rogers and Williams [44], the stationary probability distribution of the buffer content is expressed as a matrix-exponential form by Rogers [62]. Li and Zhao [55] used the *RG*-factorizations to study a block-structured fluid models driven by level-dependent QBD processes with either infinitely-many levels or finitely-many levels.

(4) Ramaswami [61] provided the matrix-analytic method for deriving the stationary probability distribution of the buffer content. He illustrated that the stationary probability distribution is a PH distribution. The matrix-analytic method was further extended by da Silva Soares and Latouche [46], Ahn and Ramaswami [40], and van Lierde, da Silva Soares and Latouche [66].

When the state space of the external stochastic environment is countably infinite, Virtamo and Norros [67] provided a spectral method for a fluid queue driven by an $M/M/1$ queue, where the generalized eigenvalues are explicitly expressed by means of the Chebyshew Polynomials of the second kind. Adan and Resing [38] presented a method of embedded points, for deriving the stationary probability distribution of the buffer content. Parthasarathy, Vijayashree and Lenin [58] proposed a method of continued fraction to derive Laplace transform of the stationary probability distribution. Li, Liu and Shang [54] discussed the heavy-tailed behavior for the stationary probability distribution of the buffer content in a fluid queue driven by an $M/G/1$ queue. Konovalov [49] analyzed the stability issue of a fluid queue driven by a $GI/G/1$ queue. van Doorn and Scheinhardt [65] used the orthogonal Polynomials to express the stationary probability distribution of the buffer content for a fluid queue driven by an irreducible birth-death process. Li and Zhao [55] analyzed a block-structured fluid models driven by level-dependent QBD processes. Guillemin and Sericola [47] provided stationary analysis of a fluid queue driven by some countable state space Markov chain.

7.5.3 The Queues with Negative Customers

During the last decade, considerable attention has been paid to the study of queueing systems with negative arrivals. Since the introduction of the concept of negative customers by Gelenbe [77], research on queueing systems with negative arrivals has been greatly motivated by some practical applications such as computers, neural networks, manufacturing systems and communication networks. For a comprehensive analysis of queueing networks with negative arrivals, readers may refer to Gelenbe and Pujolle [81], Chao, Miyazawa and Pinedo [74], and Serfozo [90].

Gelenbe [77] introduced negative arrivals to queueing networks and established the product form solution for an open queueing network. Subsequent papers have been published on this theme, among which, see Gelenbe and Schassberger [82], Henderson [86], Gelenbe [78,79], Pitel [89], Harrison and Pitel [84], Chao [73], Gelenbe and Pujolle [81], Artalejo and Gómez-Corral [70], Chao, Miyazawa and Pinedo [74], Serfozo [90], Anisimov and Artalejo [68], Zhu and Zhang [92], Dudin and Semenova [76], and Shin [91]. A recent review can be found in Artalejo [69].

Gelenbe, Glynn and Sigman [80] provided necessary and sufficient conditions for stability of single-server queues with negative arrivals. They illustrated that stability conditions may depend upon not only the arrival and service rates, but also the distributions of the interarrival and service times. For the $M/G/1$ queue with negative arrivals, Harrison and Pitel [85] explicitly expressed the stability conditions. Harrison and Pitel [83] derived expressions for the Laplace transform of the sojourn time density in the $M/M/1$ queue with Poisson arrivals of negative customers. For the stationary workload for $M/G/1$ queues with negative arrivals, Boucherie and Boxma [72] studied a generalization in which a negative arrival removes a random amount of positive customers. Jain and Sigman [87] analyzed the case where a negative arrival removes all the customers in the system. They derived a Pollaczek-Khintchine formula. Bayer and Boxma [71] provided Wiener-Hopf analysis for an $M/G/1$ queue in which positive customers are removed just after a service completion time. Harrison and Pitel [85] studied the distributions of stationary queue length for the $M/G/1$ queue with negative arrivals. The generating functions are expressed in terms of the Fredholm integral equations of the first kind. Artalejo and Gómez-Corral [96] extended the $M/G/1$ queue with negative arrivals to handle situations where the positive customers follow a retrial policy. They proved that the distributions of the stationary queue length can still satisfy the Fredholm integral equation of the first kind, and provided an effective algorithm for numerically solving the Fredholm integral equation. Dudin and Nishimura [75] used the matrix-analytic method to analyze a $BMAP/SM/1$ queue with disasters. Li and Zhao [88] considered the $MAP/G/1$ queue with MAP arrivals of negative customers under two classes of removal rules.

7.5.4 The Retrial Queues

Retrial queues are an important mathematical model for telephone switch systems, digital cellular mobile networks, computer networks and so on. During the last two decades considerable attention has been paid to the study of retrial queues, which has been well documented, for example, by survey papers of Yang and Templeton [125], Falin [108], Kulkarni and Liang [116], Artalejo [96] and Gómez-Corral [110], and by two books of Falin and Templeton [109] and Artalejo and Gómez-Corral [98].

Retrial queues have been studied by some authors in terms of the matrix-analytic method. Readers may refer to Kulkarni [113,114], Kulkarni and Choi [115], Neuts and Rao [121], Liang and V. G. Kulkarni [118,119], Diamond and Alfa [104,105,106], Choi, Yang and Kim [102], He, Li and Zhao [111], Dudin and Klimenok [107], Choi, Chung and Dudin [103], Breuer, Dudin and Klimenok [99], Chakravarthy and Dudin [100,101], Li, Ying and Zhao [117], Shang, Liu and Li [122].

Retrial queues with unreliable servers have been discussed by some researchers. Readers may refer to Kulkarni and Choi [115], Yang and Li [124], Artalejo [95], Aissani [93], Aissani and Artalejo [94], Artalejo and Gómez-Corral [97], Wang, Cao and Li [123], and Li, Ying and Zhao [117].

Problems

7.1 Consider an $M/PH/1$ processor sharing queue, where the server is shared equally by all customers in the system. Compute the distribution of the sojourn time in this system. Further, provide a detailed analysis for a $MAP/PH/1$ processor sharing queue.

7.2 Consider an $M/M/c$ processor sharing queue with each server being shared equally by all customers in its working space. Compute the distribution of the sojourn time in this system.

7.3 Consider an infinite capacity buffer where the fluid input and output rates are controlled by the idle period and the busy period of a $MAP/G/1$ queue, respectively. Provide the stable condition of the fluid queue, and compute the stationary distribution of the buffer content.

7.4 Consider a finite capacity buffer where the fluid input and output rates are controlled by the idle period and the busy period of a $MAP/PH/1$ queue, respectively. Provide the stable condition of the fluid queue, and compute the stationary distribution of the buffer content.

7.5 Analyze a single-server FCFS queue with two types of independent arrivals, positive and negative. Positive arrivals correspond to customers who upon arrival, join the queue with the intention of being served and then leaving the system. At a negative arrival epoch, the system is affected if and only if customers are present. We assume that the arrivals of both positive and negative customers are of MAPs with matrix descriptors (C_1, D_1) and (C_2, D_2), respectively, where the infinitesimal generators $C_1 + D_1$ and $C_2 + D_2$ of sizes $m_1 \times m_1$ and $m_2 \times m_2$ are irreducible and positive recurrent. The service times of the positive arrivals are i.i.d. and of phase type with irreducible representation (β, S) of size n. For each of the following three cases, compute the stationary distribution of the queue length.

(1) The arrival of a negative customer removes all the customers in the system.

(2) The arrival of a negative customer removes the head customer (or the serving customer) in the system.

(3) The arrival of a negative customer removes the final customer in the queue line.

7.6 Analyze a single-server FCFS queue with a finite waiting room and two types of independent arrivals, positive and negative. Positive arrivals correspond to customers who upon arrival, join the queue with the intention of being served and then leaving the system. Each negative arrival is not served while it only occupies the waiting room until it is full. We assume that the arrivals of both positive and negative customers are of MAPs with matrix descriptors (C_1, D_1) and (C_2, D_2), respectively, where the infinitesimal generators $C_1 + D_1$ and $C_2 + D_2$ of sizes $m_1 \times m_1$ and $m_2 \times m_2$ are irreducible and positive recurrent. The service times of the positive arrivals are i.i.d. and of phase type with irreducible representation (β, S) of size n. Compute the transient distribution of the queue length.

7.7 Consider a $BMAP/G \oplus G/1/1$ retrial queue with no waiting space and the size of the orbit is infinite. If an arrival, either a primary or a retrial customer, finds that there is no customer in the server, then it enters the server immediately and receives service. Otherwise it enters the orbit and makes a retrial at a later time. Returning customers behave independently of each other and keep making retrials until they receive their requested service. Successive inter-retrial times are i.i.d. exponentially distributed random variables with retrial rate θ. The arrivals to the retrial queue are modeled by a BMAP with irreducible matrix descriptor $\{D_k, k \geqslant 0\}$ of size m; and the service times for the primary and retrial customers are assumed to be i.i.d. and have two different distribution functions as follows:

$$B_1(t) = 1 - \exp\left\{-\int_0^t \mu_1(v)\mathrm{d}v\right\}$$

and

$$B_2(t) = 1 - \exp\left\{-\int_0^t \mu_2(v)\mathrm{d}v\right\},$$

respectively. Provide the stable condition of the retrial queue, and compute the stationary distribution of the queue length.

7.8 Consider a $GI/PH/1/1$ retrial queue with no waiting space and the size of the orbit is infinite. If an arrival, either a primary or a retrial customer, finds that there is no customer in the server, then it enters the server immediately and receives service. Otherwise it enters the orbit and makes a retrial at a later time. Returning customers behave independently of each other and keep making retrials until they receive their requested service. Successive inter-retrial times are i.i.d. exponentially distributed random variables with retrial rate θ. The arrivals to the retrial queue are modeled by a renewal process with interarrival time distribution $F(x)$; and the service times for the primary and retrial customers are assumed to be i.i.d. distributed PH random variables with irreducible representation (β, S) of size n. Compute the transient distribution of the queue length.

7.9 Consider a $GI/G/1/1$ retrial queue with no waiting space and the size of the orbit is infinite. If an arrival, either a primary or a retrial customer, finds that there is no customer in the server, then it enters the server immediately and receives service. Otherwise it enters the orbit and makes a retrial at a later time. Returning customers behave independently of each other and keep making retrials until they receive their requested service. Successive inter-retrial times are i.i.d. exponentially distributed random variables with retrial rate θ. The arrivals to the retrial queue are modeled by a renewal process with interarrival time distribution $F(x)$; and the service times for the primary and retrial customers are assumed to be i.i.d. distributed random variables with service time distribution $G(x)$. Compute the transient distribution of the queue length.

7.10 Consider a $MAP/PH/1/1$ retrial queue with no waiting space and the size of the orbit is infinite. If an arrival, either a primary or a retrial customer, finds that there is no customer in the server, then it enters the server immediately and receives service. Otherwise it enters the orbit and makes a retrial at a later time. Returning customers behave independently of each other and keep making retrials until they receive their requested service. Successive inter-retrial times are i.i.d. PH distributed random variables with irreducible representation (α, T) of size n. The arrivals to the retrial queue are modeled by a MAP with irreducible matrix descriptor (C, D) of size m; and the service times for the primary and retrial customers are assumed to be i.i.d. PH distributed random variables with irreducible representation (β, S) of size k. Compute the transient distribution of the queue length.

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Processor-Sharing Queues

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8 Transient Solution

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Abstract In this chapter, we apply the *RG*-factorizations to provide a unified algorithmic framework for dealing with transient solution in stochastic models. The transient solution includes the transient probability, the first passage time, the sojourn time and time-inhomogeneous Markov chains. Based on the first passage time, we extends the PH distribution and the MAP to the GPH distribution and the GMAP from finite phases to infinite phases, respectively, and also study the time-inhomogeneous PH (PH(t)) distribution and the time-inhomogeneous MAP (MAP(t)). Finally, we analyze some queueing examples such as *GMAP*/*GPH*/1 and MAP(t)/PH(t)/1.

Keywords stochastic models, *RG*-factorization, time-inhomogeneous Markov chain, transient solution, the first passage time, the sojourn time, GPH distribution, GMAP, time-inhomogeneous PH distribution, time-inhomogeneous MAP.

In this chapter, we apply the *RG*-factorizations to provide a unified algorithmic framework for dealing with transient solution in stochastic models. The transient solution includes the transient probability, the first passage time, the sojourn time, and performance measures of time-inhomogeneous Markov chains. Note that the first passage time has been dealt with in Section 6.7. This chapter extends the PH distribution and the MAP to the generalized PH (GPH) distribution and the generalized MAP (GMAP) from finite phases to infinite phases, respectively. It also studies the time-inhomogeneous PH (PH(t)) distribution and the time-inhomogeneous MAP (MAP(t)), both of which are useful in the study of time-inhomogeneous stochastic models.

This chapter is organized as follows. Sections 8.1 studies the transient probability of an irreducible Markov chain which is either discrete-time or continuous-time. Also, this section and develops two effective algorithms for computing the transient probability. Sections 8.2 analyzes the first passage time of an Markov chain. This section gives two effective algorithms for computing the first passage time distribution with respect to the discrete-time and continuous-time cases, respectively. Based on this, we develop the GPH distribution and the GMAP. Sections 8.3 analyzes the sojourn times of an irreducible Markov chain by means of the PH distributions and the *RG*-factorizations. Section 8.4 discusses a time-inhomogeneous discrete-time Markov chain, and analyzes the asymptotic periodic distribution of a *d*-period Markov chain. Finally, Section 8.5 summarizes notes for the references related to the results of this chapter.

8.1 Transient Probability

In this section, we provide an algorithmic framework for computing the transient probability of an irreducible Markov chain which is either discrete-time or continuous-time. The algorithmic framework is based on the *RG*-factorizations.

8.1.1 Discrete-Time Markov Chains

Consider an irreducible *M*-state Markov chain $\{X_n, n \geqslant 0\}$ whose transition probability matrix is given by $P = (p_{i,j})_{1 \leqslant i,j \leqslant M}$, where M is either finite or infinite. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)$ be the initial probability vector of the Markov chain, that is, $P\{X_0 = k\} = \alpha_k$ for $1 \leqslant k \leqslant M$. We write $\pi_k(n) = P\{X_n = k\}$ and $\pi(n) = (\pi_1(n), \pi_2(n), \ldots, \pi_M(n))$. Thus

$$\pi(n) = \alpha P^n, \quad n \geqslant 0. \tag{8.1}$$

It is necessary to explain the probabilistic setting for $\pi(n)$ with $n \geqslant 0$ in terms of practical systems. The transient probability (8.1) can always be regarded as a forecast method for analyzing many practical issuses. For example, let us consider a market percentage prediction for five electric power companies. Due to dynamic competition, it is a crucial decision-making process to provide the market percentage prediction for each company. To achieve this, we can use the transient probability to compute the market percentage. Let $p_{i,j}$ be a probability that the last periodic customers of the ith electric power company purchase their electric power from the *j*th electric power company, we write the matrix $P = (p_{i,j})_{1 \leqslant i,j \leqslant 5}$. The five different electric power companies are denoted as A, B, C, D and E. Based on statistical analysis, we can obtain the initial market percentage vector as

$$\pi(0) = (0.20, 0.18, 0.16, 0.24, 0.22)$$

and the transition probability matrix P as

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0.60 & 0.10 & 0.05 & 0.15 & 0.10 \\ 0.05 & 0.70 & 0.05 & 0.10 & 0.10 \\ 0.10 & 0.20 & 0.50 & 0.15 & 0.05 \\ 0.05 & 0.10 & 0.05 & 0.75 & 0.05 \\ 0.10 & 0.05 & 0.10 & 0.10 & 0.65 \end{pmatrix} \end{array}.$$

Therefore, we can predict the market percentages of the five electric power companies in the current year as follows:

$$\pi(1) = \pi(0)P = (0.18, 0.22, 0.13, 0.27, 0.20);$$

and the market percentages in the next year as follows:

$$\pi(2) = \pi(1)P = (0.17, 0.24, 0.12, 0.28, 0.19).$$

When the Markov chain P is positive recurrent, we denote by $\pi = (\pi_1, \pi_2, \ldots, \pi_M)$ its stationary probability vector. In this case, if $\alpha = \pi$, then $\pi(n) = \pi P^n = \pi$ for each $n \geqslant 0$. At the same time, it is clear that $\lim_{n\to\infty} \pi(n) = \pi$, which is independent of any initial probability vector α.

Let $\pi^*(z) = \sum_{n=0}^{\infty} z^n \pi(n)$. Then $\pi^*(z) = \alpha(I - zP)^{-1}$. When $0 < z \leqslant 1$, it is clear that zP is either stochastic or substochastic. In this case, we can obtain the UL-type RG-factorization

$$I - zP = [I - R_U(z)][I - \varPsi_D(z)][I - G_L(z)],$$

which leads to

$$\pi^*(z) = \alpha[I - G_L(z)]^{-1}[I - \varPsi_D(z)]^{-1}[I - R_U(z)]^{-1};$$

and the LU-type RG-factorization

$$I - zP = [I - \overline{R}_L(z)][I - \overline{U}_D(z)][I - \overline{G}_U(z)],$$

which yields

$$\pi^*(z) = \alpha[I - \overline{G}_U(z)]^{-1}[I - \overline{U}_D(z)]^{-1}[I - \overline{R}_L(z)]^{-1}.$$

Let N_i be the return number of the Markov chain $\{X_n, n \geqslant 0\}$ to state i, and write $\gamma_i = E[N_i \mid X_0 = i]$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_M)$. It is easy to check that

$$\gamma = \sum_{n=1}^{\infty} n\pi(n) = \alpha P(I-P)^{-2} = \pi(1)[(I-P)^{-1}]^2.$$

Therefore,

(1) if the markov chain $\{X_n, n \geqslant 0\}$ is transient, then

$$\gamma = \pi(1)[(I-G_L)^{-1}(I-\Psi_D)^{-1}(I-R_U)^{-1}]^2;$$

(2) if $M = \infty$, then

$$\gamma = \pi(1)[(I-\overline{G}_U)^{-1}(I-\overline{U}_D)^{-1}(I-\overline{R}_L)^{-1}]^2.$$

8.1.2 An Approximate Algorithm

We provide an effective algorithm for computing the transient probability $\pi(n)$ for $n \geqslant 0$ of the Markov chains with infinite states. This algorithm is a key element for constructing the other three algorithms in this chapter.

For the vector α and the matrix P, there always exists a monotonously non-decreasing integer sequence $\{N_k\}$ such that for a sufficiently small $\varepsilon > 0$, we have

$$\sum_{j=N_0+1}^{\infty} \alpha_j < \varepsilon \tag{8.2}$$

and

$$\sum_{j=N_n+1}^{\infty} P_{i,j} < \varepsilon, \quad 1 \leqslant i \leqslant N_{n-1}, \quad n \geqslant 1. \tag{8.3}$$

Remark 8.1 *It is necessary to provide a detailed interpretation or a concrete procedure to determine the integer sequence* $\{N_k\}$ *based on the following three steps:*

Step 1 Since $\alpha e \leqslant 1$, i.e., $\sum_{i=1}^{\infty} \alpha_i \leqslant 1$, there exists an integer $N_0 > 1$ such that $\sum_{i=N_0+1}^{\infty} \alpha_i < \varepsilon$.

Step 2 Once determining N_0, using $\sum_{j=1}^{\infty} p_{i,j} \leqslant 1$, there exists an integer $N_1^{(i)} \geqslant N_0$ such that $\sum_{j=N_1^{(i)}+1}^{\infty} p_{i,j} < \varepsilon$ for $1 \leqslant i \leqslant N_0$. Let

$$N_1 = \max\{N_1^{(i)} : 1 \leqslant i \leqslant N_0\}.$$

Step 3 When determining N_k for $k\geqslant 1$, using $\sum_{j=1}^{\infty} p_{i,j}\leqslant 1$, there exists an integer $N_{k+1}^{(i)}\geqslant N_k$ such that $\sum_{j=N_{k+1}^{(i)}+1}^{\infty} p_{i,j}<\varepsilon$ for $1\leqslant i\leqslant N_k$. Let

$$N_{k+1}=\max\{N_{k+1}^{(i)}:1\leqslant i\leqslant N_k\}.$$

Lemma 8.1 *Let* $\pi(n)=\alpha P^n$ for $n\geqslant 0$. *Then for each* $n\geqslant 0$, *we have*

$$\sum_{j=N_n+1}^{\infty}\pi_j(n)<\varepsilon.$$

Proof Since $\pi(n)e=\alpha P^n e\leqslant 1$ for $n\geqslant 0$. we obtain that $\sum_{j=1}^{\infty}\pi_j(n)\leqslant 1.$ Hence, we have

$$\sum_{j=N_n+1}^{\infty}\pi_j(n)=\sum_{j=N_n+1}^{\infty}\sum_{l=1}^{\infty}\pi_l(n)p_{l,j}=\sum_{l=1}^{\infty}\pi_l(n-1)\sum_{j=N_n+1}^{\infty}p_{l,j}$$
$$<\varepsilon\sum_{l=1}^{\infty}\pi_l(n-1)\leqslant\varepsilon.$$

This completes the proof.

For a positive integer N, we write

$${}_N\pi(n)=({}_N\pi_1(n),{}_N\pi_2(n),{}_N\pi_3(n),\ldots,{}_N\pi_{N-1}(n),{}_N\pi_N(n)).$$

Specifically,

$${}_{N_0}\pi_j(0)=\alpha_j,\quad 1\leqslant j\leqslant N_0,\tag{8.4}$$

and

$${}_{N_n}\pi_j(n)=\sum_{i=1}^{N_{n-1}}{}_{N_{n-1}}\pi_i(n-1)p_{i,j},\quad 1\leqslant j\leqslant N_n, n\geqslant 1.\tag{8.5}$$

Theorem 8.1 *For* $1\leqslant j\leqslant N_n, n\geqslant 1$,

$$0\leqslant\pi_j(n)-{}_{N_n}\pi_j(n)\leqslant\sum_{j=1}^{N_n}[\pi_j(n)-{}_{N_n}\pi_j(n)]<\frac{1}{2}n(n+1)\varepsilon.$$

Proof This proof contains the following three steps.

Step 1 By induction, we first prove that $\pi_j(n)-{}_{N_n}\pi_j(n)\geqslant 0$ for $1\leqslant j\leqslant N_n$, $n\geqslant 1$.

When $n = 1$,

$$\pi_j(1) - {}_{N_1}\pi_j(1) = \sum_{i=1}^{\infty}[\pi_i(0) - {}_{N_0}\pi_i(0)]p_{i,j} + \sum_{i=N_0+1}^{\infty}\pi_i(0)p_{i,j}$$
$$= \sum_{i=N_0+1}^{\infty}\pi_i(0)p_{i,j} \geqslant 0,$$

since $\pi_i(0) = {}_{N_0}\pi_i(0) = \alpha_i$ for $1 \leqslant i \leqslant N_0$.

We assume that when $n = k, \pi_j(k) - {}_{N_n}\pi_j(k) \geqslant 0$. Then for $n = k+1$,

$$\pi_j(k+1) - {}_{N_{k+1}}\pi_j(k+1) = \sum_{i=1}^{N_k}[\pi_i(k) - {}_{N_k}\pi_i(k)]p_{i,j} + \sum_{i=N_k+1}^{\infty}\pi_i(k)p_{i,j} \geqslant 0.$$

Therefore, we obtain that $\pi_j(n) - {}_{N_n}\pi_j(n) \geqslant 0$ for $1 \leqslant j \leqslant N_n, n \geqslant 1$.

Step 2 Note that $\pi_j(n) - {}_{N_n}\pi_j(n) \geqslant 0$ for $1 \leqslant j \leqslant N_n$ and $n \geqslant 1$, it is clear that

$$\pi_j(n) - {}_{N_n}\pi_j(n) \leqslant \sum_{j=1}^{N_n}[\pi_j(n) - {}_{N_n}\pi_j(n)].$$

Step 3 We prove that $\sum_{j=1}^{N_n}[\pi_j(n) - {}_{N_n}\pi_j(n)] < \frac{1}{2}n(n+1)\varepsilon$ for $n \geqslant 1$ by induction.

When $n = 1$,

$$\sum_{j=1}^{N_1}[\pi_j(1) - {}_{N_n}\pi_j(1)] = \sum_{j=1}^{N_1}\sum_{i=N_0+1}^{\infty}\pi_i(0)p_{i,j} = \sum_{i=N_0+1}^{\infty}\pi_i(0)\sum_{j=1}^{N_1}p_{i,j}$$
$$\leqslant \sum_{i=N_0+1}^{\infty}\pi_i(0) = \sum_{i=N_0+1}^{\infty}\alpha_i < \varepsilon.$$

We assume that when $n = k, \sum_{j=1}^{N_k}[\pi_j(k) - {}_{N_n}\pi_j(k)] < \frac{1}{2}k(k+1)\varepsilon$. Then for $n=k+1$,

$$\sum_{j=1}^{N_{k+1}}[\pi_j(k+1) - {}_{N_n}\pi_j(k+1)] = \sum_{j=1}^{N_{k+1}}\left[\sum_{i=1}^{N_k}[\pi_i(k) - {}_{N_n}\pi_i(k)]p_{i,j} + \sum_{i=N_k+1}^{\infty}\pi_i(k)p_{i,j}\right]$$
$$\leqslant \sum_{i=1}^{N_k}[\pi_i(k) - {}_{N_n}\pi_i(k)] + \sum_{i=N_k+1}^{\infty}\pi_i(k)$$
$$< \frac{1}{2}k(k+1)\varepsilon + \varepsilon < \frac{1}{2}(k+1)(k+2)\varepsilon.$$

Therefore, we obtain that $\sum_{j=1}^{N_n}[\pi_j(n) - {}_{N_n}\pi_j(n)] < \frac{1}{2}n(n+1)\varepsilon$ for $n \geqslant 1$.

This completes the proof.

Note that the sequence $\{{}_{N_n}\pi_j(n)\}$ can be computed effectively based on both the vector α and the matrix P, thus ${}_{N_n}\pi_j(n)$ can be used to approximate the transient probability $\pi_j(n)$ under a sufficiently small error $\varepsilon > 0$.

Algorithm 8.1 Computation of Transient Probability

INPUT: The vector α, the matrix P, and the error $\varepsilon > 0$.

COMPUTATION:

Step 1 Generating the integer sequence $\{N_k\}$ according to (8.2) and (8.3).

Step 2 Iteratively computing $\{{}_{N_n}\pi_j(n)\}$ according to (8.4) and (8.5).

OUTPUT: The approximate transient probability $\{{}_{N_n}\pi_j(n)\}$.

As an illustration, we consider a *Geom*/*Geom*/1 queue, the customer arrivals form a Bernoulli process: During a time interval (n, n^+) there is an arrival with probability $p \in (0,1)$ while there is no arrival with probability $\overline{p} = 1 - p$. The service time S is geometric with distribution function $P\{S = k\} = \mu\overline{\mu}^{k-1}$ for $\mu \in (0,1), \overline{\mu} = 1 - \mu$ and $k \geqslant 1$. The customer departure can only occur in the time interval (n^-, n). In this system, there is a single server, the service discipline is FCFS, and all the service times and the interarrival times are independent of each other. Let L_n^- be the number of customers in the system at time n^- for $n \geqslant 0$. Then $\{L_n^- : n \geqslant 0\}$ is a discrete-time birth-death process whose transition probability matrix is given by

$$P = \begin{pmatrix} \overline{p} & p & & & \\ \overline{p}\mu & 1 - \overline{p}\mu - p\overline{\mu} & p\overline{\mu} & & \\ & \overline{p}\mu & 1 - \overline{p}\mu - p\overline{\mu} & p\overline{\mu} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

For the discrete-time birth-death process, it is easy to compute its transient probability $\pi(n) = \alpha P^n$ for $n \geqslant 0$ in terms of Algorithm 8.1.

8.1.3 Continuous-Time Markov Chains

We consider an irreducible continuous-time M-state Markov chain $\{X_t, t \geqslant 0\}$ whose infinitesimal generator is given by $Q = (q_{i,j})_{1 \leqslant i,j \leqslant M}$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)$ be the initial probability vector of the Markov chain. We write $h_k(t) = P\{X_t = k\}$ and $H(t) = (h_1(t), h_2(t), \ldots, h_M(t))$. It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = H(t) Q$$

with the initial condition $H(0) = \alpha$. Thus we obtain

$$H(t) = \alpha \exp\{Qt\}, \quad t \geqslant 0. \tag{8.6}$$

When the Markov chain Q is positive recurrent, we denote by $\pi=(\pi_1,\pi_2,...,\pi_M)$ its stationary probability vector. In this case, if $\alpha = \pi$, then $H(t) = \pi \exp\{Qt\} = \pi$ for each $t \geqslant 0$. At the same time, $\lim_{t\to\infty} H(t) = \pi$, which is independent of any initial probability vector α.

Remark 8.2 *Let* $V(t) = H(t)^{\mathrm{T}}$ *for* $t \geqslant 0$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) = QV(t)$$

with the initial condition $V(0) = e$ *,Where* e *is a column vector of ones with suitable size. Thus we obtain*

$$V(t) = \exp\{Qt\}e, \quad t \geqslant 0.$$

It is clear that $H(t)e = \alpha V(t)$ *for each* $t \geqslant 0$.

Let $\tilde{H}(s) = \int_0^{+\infty} \mathrm{e}^{-st} H(t)\mathrm{d}t$. Then $\tilde{H}(s) = \alpha(sI - Q)^{-1}$. When $s>0$, it is clear that $Q - sI$ is the infinitesimal generator of a continuous-time Markov chain. In this case, we can obtain the UL-type *RG*-factorization

$$sI - Q = [I - R_U(s)][-U_D(s)][I - G_L(s)],$$

which leads to

$$\tilde{H}(s) = \alpha[I - G_L(s)]^{-1}[-U_D(s)]^{-1}[I - R_U(s)]^{-1};$$

and the LU-type *RG*-factorization

$$sI - Q = [I - \bar{R}_L(s)][-\bar{U}_D(s)][I - \bar{G}_U(s)],$$

which yields

$$\tilde{H}(s) = \alpha[I - \bar{G}_U(s)]^{-1}[-\bar{U}_D(s)]^{-1}[I - \bar{R}_L(s)]^{-1}.$$

As an illustrating example, we consider a *BMAP*/*PH*(*M*/*PH*)/1 queue with a repairable server. The customer arrival process is a BMAP with irreducible matrix descriptor $\{D_k, k = 0,1,2,...\}$ of size *m*, the service times are i.i.d. and are of phase type with irreducible representation (β, S) of size *n*, the life time of the server is exponential with mean $1/\gamma > 0$ and its repair time is of phase type. Now, we need to compute the mean of the first failure time χ of the server. To this end, we denote by *N*(*t*), *I*(*t*) and *J* (*t*) the number of customers in the system and the two phases of the arrival and service processes at time *t*, respectively. At the same time, let the set of all the failed states be an absorbing state. Then $\{N(t), I(t), J(t) : t \geqslant 0\}$ is an irreducible continuous-time Markov chain whose infinitesimal generator is given by

$$Q = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ I \otimes S^0 & D_0 \oplus S - \gamma I & D_1 & D_2 & \cdots \\ & I \otimes (S^0 \beta) & D_0 \oplus S - \gamma I & D_1 & \cdots \\ & & I \otimes (S^0 \beta) & D_0 \oplus S - \gamma I & \cdots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Note that the Markov chain Q is transient due to $Q^0 = -Qe = (0, \gamma^{e^{\mathrm{T}}}, \gamma^{e^{\mathrm{T}}}, \ldots)^{\mathrm{T}}$. It is easy to see that the first failure time χ is of phase type with irreducible representation (ω, Q) of infinite size, where ω is the initial probability vector. For example, we may take $\omega = (\theta, 0, 0, \ldots)$ and θ is the stationary probability vector of the Markov chain $\sum_{k=0}^{\infty} D_k$. Let the reliability function $R(t) = P\{\chi > t\}$. Then

$$R(t) = \omega \exp\{Qt\} e.$$

Thus

$$E[\chi] = -\omega Q^{-1} e = -\omega (I - G_L)^{-1} U_D^{-1} (I - R_U)^{-1} e.$$

Now, we provide an effective algorithm for computing the transient probability vector $H(t)$ for $t \geqslant 0$ when $M = \infty$. Our algorithmic analysis is classified into the following two cases.

Case Ⅰ Bounded diagonal elements

We now discuss an irreducible infinite-state Markov chain Q with bounded diagonal elements, that is, $c = \sup_{k \geqslant 1} \{-q_{k,k}\} < +\infty$. In this case, we write

$$P = I + \frac{1}{c} Q.$$

It is easy to check that P is either stochastic or substochastic. At the same time, P is irreducible if and only if Q is irreducible.

Note that $H(t) = \alpha \exp\{Qt\}$ and $Q = cP - cI$, we obtain

$$H(t) = \alpha \exp\{(cP - cI)t\} = \mathrm{e}^{-ct} \sum_{n=0}^{\infty} \alpha P^n \frac{(ct)^n}{n!}.$$

For the term αP^n, we write

$$\pi(0) = \alpha,$$
$$\pi(n) = \alpha P^n, \quad n \geqslant 1.$$

Obviously, $\pi(n) = \pi(n-1)P$ for $n \geqslant 1$. At the same time, for $t \geqslant 0$ we have

$$H(t) = \mathrm{e}^{-ct} \sum_{n=0}^{\infty} \pi(n) \frac{(ct)^n}{n!} \tag{8.7}$$

It is seen from Eq. (8.7) that computations of $H(t)$ for $t \geqslant 0$ can be related to the approximate computation of $\pi(n) = \alpha P^n$ for $n \geqslant 0$.

We write

$$_{N_0}\pi_j(0) = \alpha_j, \quad 1 \leqslant j \leqslant N_0,$$

and

$$_{N_n}\pi_j(n) = \sum_{i=1}^{N_{n-1}} {}_{N_{n-1}}\pi_i(n-1) p_{i,j}, \quad 1 \leqslant j \leqslant N_n, n \geqslant 1.$$

For a sufficiently large positive integer M, we write

$$_M H(t) = \mathrm{e}^{-ct} \sum_{n=0}^{M} {}_{N_n}\pi(n) \frac{(ct)^n}{n!}. \tag{8.8}$$

Therefore, the function sequence $\{_M H(t)\}$ converges to the transient probability $H(t)$ for each $t \geqslant 0$, and this convergence is uniform for $t \in [0, T]$.

Algorithm 8.2 Computation of Transient Probability

INPUT: The vector α, the matrix Q, the error $\varepsilon > 0$.

COMPUTATION:

Step 1 Generating the integer sequence $\{N_k\}$.

Step 2 Iteratively computing $\{_{N_n}\pi(n)\}$.

Step 3 Determining the function sequence $\{_M H(t)\}$ according to Eq. (8.8).

OUTPUT: The approximate transient probability $\{_M H(t)\}$.

We consider a *BMAP*/*M*/1 processor-sharing queue, where the customer arrivals form a BMAP with irreducible matrix descriptor $\{D_k, k \geqslant 0\}$ of size m. The service times are i.i.d. and are exponential with mean $1/\mu$. The service discipline is that when there are n customers in the system, each customer receives service at rate $1/n$. We now compute the sojourn time distribution. To this end, we introduce the following description. A randomly chosen customer who finds n customers in the system on arrival is called Customer C_n. Let w_n denote the sojourn time of Customer C_n for $n \geqslant 1$. We write $W_n(x) = P\{w_n > x\}$. Then it is easy to check that

$$\frac{\mathrm{d}}{\mathrm{d}x} W_n(x) = W_n(x) Q$$

with the initial condition $W_n(0) = \alpha$, where a is a probability vector and

$$Q = \begin{pmatrix} D_0 - \mu I & D_1 & D_2 & D_3 & D_4 & \cdots \\ \frac{1}{2}\mu I & D_0 - \mu I & D_1 & D_2 & D_3 & \cdots \\ & \frac{2}{3}\mu I & D_0 - \mu I & D_1 & D_2 & \cdots \\ & & \frac{3}{4}\mu I & D_0 - \mu I & D_1 & \cdots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Further, we obtain

$$W_n(x) = a\exp\{Qx\}e$$

and

$$E[w_n] = \alpha(-Q)^{-1}_{\min} e = -\alpha(I - G_L)^{-1} U_D^{-1} (I - R_U)^{-1} e.$$

Let $d_{0,j}$ be the jth diagonal element of the matrix D_0 for $1 \leqslant j \leqslant m$. It is easy to see that $c = \max_{1\leqslant j\leqslant m}\{\mu - d_{0,j}\} = \mu - \min_{1\leqslant j\leqslant m}\{d_{0,j}\} < +\infty$. Thus we can use Algorithm 8.2 for computing the sequence $\{{}_M W(t)\}$ which approximates the sojourn time distribution $W(t)$ for each $t \geqslant 0$.

Case Ⅱ Unbounded diagonal elements

For the continuous-time Markov chains, there exist many infinitesimal generators whose diagonal elements are unbounded. For example, we consider an $M/M/1$ retrial queue, where the input is a Poisson process with rate $\lambda > 0$, the service times are i.i.d. and are exponential with mean $1/\mu > 0$. When there are n customers in the system, the retrial time of each customer in the orbit is exponential with mean $1/(n\theta) > 0$. Let $N(t)$ be the number of customers in the system at time t and

$$I(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is busy.} \end{cases}$$

Then $\{(N(t), I(t)) : t \geqslant 0\}$ is an irreducible continuous-time level-dependent QBD process whose infinitesimal generator is given by

$$Q = \begin{pmatrix} A_{0,0} & A_{0,1} & & & \\ A_{1,0} & A_{1,1} & A_{1,2} & & \\ & A_{2,1} & A_{2,2} & A_{2,3} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{8.9}$$

where for $k \geqslant 0$,

$$A_{k,k} = \begin{pmatrix} -\lambda - k\theta & \lambda \\ \mu & -\lambda - \mu \end{pmatrix},$$

$$A_{k,k+1} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$$

and

$$A_{k+1,k} = \begin{pmatrix} 0 & k\theta \\ 0 & 0 \end{pmatrix}.$$

It is clear that $c = \sup_{k\geqslant 0}\{\lambda + k\theta, \lambda + \mu\} = +\infty$.

When the diagonal elements of the matrix Q are unbounded, we introduce three different approximate methods for computing the transient probability $H(t)$ for each $t \geqslant 0$.

(1) The censored technique

For the state space $\Omega = \{1, 2, 3, \ldots\}$, we take two sets $E = \{1, 2, \ldots, N\}$ and $E^c = \{N+1, N+2, N+3, \ldots\}$. Based on the two state sets, the infinitesimal generator Q is partitioned as

$$Q = \begin{array}{c c} & \begin{array}{cc} E & E^c \end{array} \\ \begin{array}{c} E \\ E^c \end{array} & \begin{pmatrix} T & V \\ U & W \end{pmatrix} \end{array}$$

and the initial probability vector α is partitioned as $\alpha = (\alpha^{(1)}, \alpha^{(2)})$. Then the infinitesimal generator of the censored chain Q^E to set E is given by

$$Q^E = T + V(-W)_{\min}^{-1} U$$

and the initial probability vector of the censored chain Q^E is given by

$$\alpha^E = \alpha^{(1)} + \alpha^{(2)}(-W)_{\min}^{-1} U.$$

Thus the transient probability vector of the censored Markov chain Q^E is given by

$$H_N^{(C)}(t) = \alpha^E \exp\{Q^E t\}, \quad t \geqslant 0.$$

In order to show the approximate precision, we introduce a performance measure

$$c_N = \frac{\alpha^E(-Q^E)^{-1} e}{\alpha(-Q)_{\min}^{-1} e}.$$

As the value of c_N tends to one, the approximate precision tends to increase.

(2) The truncated approximation

We directly truncate the infinitesimal generator Q as the matrix T of size N. Let

$$H_N^{(T)}(t) = \alpha^{(1)} \exp\{Tt\}, \quad t \geqslant 0.$$

Then we can use the function $H_N^{(T)}(t)$ to approximate the transient probability $H(t)$ for each $t \geqslant 0$. In order to indicate the approximate precision, we introduce a performance measure

$$t_N = \frac{\alpha^{(1)}(-T)^{-1} e}{\alpha(-Q)_{\min}^{-1} e}.$$

As the value of c_N tends to one, the approximate precision tends to increase.

(3) The modified approximation

A modified approximation is a better idea that when changing some elements of the infinitesimal generator Q such that the modified matrix Q_{new} has the bounded diagonal elements, while the modified Markov chain Q_{new} is suitable to Algorithm 8.2. For example, a level-dependent Markov chain of $GI/M/1$ type may be modified as a level-independent Markov chain of $GI/M/1$ type, and a level-dependent Markov chain of $M/G/1$ type may be modified as a level-independent Markov chain of $M/G/1$ type. For example, we modify the QBD process Q given in (8.9) as a level-independent QBD process as follows:

$$Q_{\text{new}} = \begin{pmatrix} B_1 & B_0 & & & \\ B_2 & A_{K,K} & A_{K,K+1} & & \\ & A_{K-1,K} & A_{K,K} & A_{K,K+1} & \\ & & A_{K-1,K} & A_{K,K} & A_{K,K+1} \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where K is a sufficiently large integer,

$$B_1 = \begin{pmatrix} A_{0,0} & A_{0,1} & & & \\ A_{1,0} & A_{1,1} & A_{1,2} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{K-2,K-3} & A_{K-2,K-2} & A_{K-2,K-1} \\ & & & A_{K-1,K-2} & A_{K-1,K-1} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} \\ \\ \\ \\ A_{K-1,K} \end{pmatrix}, \quad B_2 = (A_{K-1,K}).$$

8.2 The First Passage Times

The first passage time for an irreducible discrete-time Markov chain with finite states is a discrete-time PH distribution, while the first passage time for an irreducible continuous-time Markov chain with finite states is a continuous-time PH distribution. Reader may refer to Chapter 2 in Neuts [39] for more details. In this section, we analyze the first passage time of an irreducible Markov chain with infinite states, which is either discrete-time or continuous-time. Note that the first passage time may be regarded as a generalized PH (GPH) distribution, which can be easily obtained from the N-state PH distribution as $N \to \infty$. Based on this, many useful properties of the N-state PH distribution can be formally extended to the GPH distribution. Also, we provide effective algorithms for computing the GPH distribution and its associated moments. Further, we discuss the GPH renewal processes and more generally, the generalized Markovian arrival processes (GMAPs), which have with infinite-many levels.

8.2.1 Discrete-Time GPH Distribution

Consider an irreducible discrete-time Markov chain with state space $\Omega = \{0,1,2,\ldots\}$ whose transition probability matrix is given by

$$P = \begin{array}{c} \\ 0 \\ E \end{array} \begin{array}{c} \begin{array}{cc} 0 & E \end{array} \\ \begin{pmatrix} 1 & 0 \\ T^0 & T \end{pmatrix} \end{array} \tag{8.10}$$

where $E = \{1,2,\ldots),T^0 \gneq 0$ and $Te + T^0 = e.$ We assume that state 0 is an absorbing state and all the others are transient.

Let (α_0, α) be the initial probability vector of the Markov chain, where $\alpha_0 + \alpha e = 1$. For the Markov chain given in Eq. (8.10), the probability distribution $\{p_k, k \geqslant 0\}$ of the number N of state transitions until absorption into the absorbing state 0 is called a discrete-time GPH distribution with representation (α, T). If $T + T^0\alpha$ is the transition probability matrix of an irreducible Markov chain, then this representation (α, T) is called an irreducible representation. In this case, we have

$$p_k = \begin{cases} \alpha_0, & k = 0, \\ \alpha T^{k-1} T^0, & k \geqslant 1. \end{cases}$$

Let $P^*(z) = \sum_{k=0}^{\infty} z^k p_k$. Then

$$P^*(z) = \alpha_0 + z\alpha (I - zT)^{-1} T^0.$$

When $0 < z \leqslant 1$, it is clear that zT is substochastic. In this case, we can obtain the UL-type RG-factorization

$$I - zT = [I - R_U(z)][I - \varPsi_D(z)][I - G_L(z)],$$

which leads to

$$P^*(z) = \alpha_0 + z\alpha [I - G_L(z)]^{-1} [I - \varPsi_D(z)]^{-1} [I - R_U(z)]^{-1};$$

and the LU-type RG-factorization

$$I - zT = [I - \overline{R}_L(z)][I - \overline{U}_D(z)][I - \overline{G}_U(z)],$$

which yields

$$P^*(z) = \alpha_0 + z\alpha[I - \overline{G}_U(z)]^{-1}[I - \overline{U}_D(z)]^{-1}[I - \overline{R}_L(z)]^{-1}.$$

Using the generating function $P^*(z)$, we can obtain the mean

$$E(N) = \alpha(I - T)^{-1} e$$

and its factorial moment of order k

$$E[N(N-1)\ldots(N-k+1)] = k!\alpha T^{k-1}(I-T)^{-k} e, \quad k \geqslant 2.$$

Note that

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} P^*(z)_{|z=1} = E[N(N-1)\ldots(N-k+1)].$$

It is easy to see that the RG-factorizations can be used to compute the factorial moments by using the UL-type RG-factorization

$$I - T = (I - R_U)(I - \varPsi_D)(I - G_L)$$

or the LU-type RG-factorization

$$I - T = (I - \overline{R}_L)(I - \overline{U}_D)(I - \overline{G}_U).$$

Now, we list some useful properties for the discrete-time GPH distribution. The proofs are easy, thus they are omitted here.

Property 8.1 Let $\{p_k\}$ and $\{q_k\}$ be two discrete-time GPH distributions with irreducible representations (α, T) and $(\beta, S), \alpha_0 = 1 - \alpha e, \beta_0 = 1 - \beta e.$ Then the discrete convolution $\{p_k\} * \{q_k\}$ is a discrete-time GPH distribution with irreducible representations (γ, L), where

$$\gamma = (\alpha, \alpha_0 \beta), \quad \gamma_0 = \alpha_0 \beta_0,$$
$$L = \begin{pmatrix} T & T^0 \beta \\ & S \end{pmatrix}, \quad L^0 = \begin{pmatrix} T^0 \beta_0 \\ S^0 \end{pmatrix}.$$

Property 8.2 Let $\{p_k\}$ be a discrete-time GPH distribution with irreducible representations (α, T) and mean μ. The residual distribution

$$q_k = \frac{1}{\mu} \sum_{j=k+1}^{\infty} p_k, \quad k \geqslant 0,$$

is a discrete-time GPH distribution with irreducible representations (γ, T), where

$$\gamma = \frac{1}{\mu} \alpha (I - T)^{-1} T, \quad \gamma_0 = \frac{1}{\mu}(1 - \alpha_0).$$

Property 8.3 Let $\{p_k\}$ and $\{q_k\}$ be two discrete-time GPH distributions with irreducible representations (α, T) and (β, S). Then the mixture $\sum_{v=0}^{\infty} p_v (q_k)^{v*}$ is of generalized phase type with irreducible representations (γ, L), where

$$\gamma = \alpha \otimes \beta (I - \alpha_0 S)^{-1},$$
$$\gamma_0 = \beta_0 + \alpha_0 \beta (I - \alpha_0 S)^{-1} S^0;$$
$$L = T \otimes I + (T^0 \alpha) \otimes [(I - \alpha_0 S)^{-1} S],$$
$$L^0 = T^0 \otimes [(I - \alpha_0 S)^{-1} S^0].$$

Property 8.4 Let

$$g_k = \sum_{n=k}^{\infty} p_n \binom{n}{k} p^k (1-p)^{n-k}, \quad k \geqslant 0.$$

If $\{p_k\}$ is a discrete-time GPH distribution with irreducible representations (α, T), then $\{g_k\}$ is also of generalized phase type with irreducible representations (γ, L), where

$$\gamma = \alpha p [I - (1-p)T]^{-1},$$
$$\gamma_0 = \alpha_0 + \alpha (1-p) [I - (1-p)T]^{-1} T^0;$$
$$L = pT[I - (1-p)T]^{-1},$$
$$L^0 = [I - (1-p)T]^{-1} T^0.$$

In what follows we provide an approximate algorithm for computing the probabilities p_k for $k \geqslant 0$.

For the vector α and the matrix $T = (T_{i,j})$, there always exists a monotonously non-decreasing integer sequence $\{N_k\}$ such that for a sufficiently small $\varepsilon > 0$, we have

$$\sum_{j=N_0+1}^{\infty} \alpha_j < \varepsilon$$

and

$$\sum_{j=N_n+1}^{\infty} T_{i,j} < \varepsilon, \quad 1 \leqslant i \leqslant N_{n-1}, n \geqslant 1.$$

Let $\pi(n) = \alpha T^n$ for $n \geqslant 0$. Then $\pi(0) = \alpha$ and $\pi(n) = \pi(n-1)T$ for $n \geqslant 1$. It is clear that $p_n = \pi(n) T^0$ for $n \geqslant 0$.

We write

$${}_{N_0}\pi_j(0) = \alpha_j, \quad 1 \leqslant j \leqslant N_0,$$

and

$$_{N_n}\pi_j(n) = \sum_{i=1}^{N_{n-1}} {}_{N_{n-1}}\pi_i(n-1)T_{i,j}, \quad 1 \leqslant j \leqslant N_n, n \geqslant 1.$$

It is clear that the sequence $\{_{N_n}\pi_j(n), n \geqslant 0\}$ is used to approximate the value $\pi_j(n)$. Based on this, we write

$$_{N_n}p(n) = \sum_{i=1}^{N_n} {}_{N_n}\pi_i(n-1)T_i^0. \tag{8.11}$$

It is easy to check that the sequence $\{_{N_n}p(n)\}$ can effectively approximate the probability p_n at time n for $n \geqslant 0$.

Algorithm 8.3 Computation of the GPH Distribution

INPUT: The vector α, T^0, the matrix T, the error $\varepsilon > 0$.

COMPUTATION:

Step 1 Generating the integer sequence $\{N_k\}$.

Step 2 Iteratively computing $\{_{N_n}\pi(n)\}$.

Step 3 Determining the function sequence $\{_{N_n}p(n)\}$ according to Eq. (8.11).

OUTPUT: The approximate transient probability $\{_{N_n}p(n)\}$.

We consider a *Geom*/*Geom*/*c* queue. The customer arrivals form a Bernoulli process during a time interval (n, n^+), there is an arrival with probability $p \in (0,1)$, while there is no arrival with probability $\overline{p} = 1 - p$. The service time S is geometric with distribution function $P\{S = k\} = \mu\overline{\mu}^{k-1}$ for $\mu \in (0,1)$, $\overline{\mu} = 1 - \mu$ and $k \geqslant 1$. The customer departure can only occur in the time interval (n^-, n). In this system, there are c identical servers, the service discipline is FCFS, and all the service times and the interarrival times are independent of each other. Let L_n^+ be the number of customers in the system at time n^+ for $n \geqslant 0$. We define

$$\tau = \inf\{n : L_n^+ \leqslant c\}.$$

Then it is clear that the random variable τ is the busy period of the system. It is easy to check that the busy period τ is of generalized phase type with irreducible representation (α, T), where

$$\alpha = (\alpha_{c+1}, \alpha_{c+2}, \alpha_{c+3}, \ldots),$$

$$T = \begin{pmatrix} t_{c+1} & t_c & t_{c-1} & \cdots & t_1 & t_0 & & & \\ & t_{c+1} & t_c & \cdots & t_2 & t_1 & t_0 & & \\ & & t_{c+1} & \cdots & t_3 & t_2 & t_1 & t_0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

and for $0 \leqslant k \leqslant c+1$,

$$t_k = \overline{p}\binom{c}{k-1}\mu^{k-1}\overline{\mu}^{c-k+1} + p\binom{c}{k}\mu^k\overline{\mu}^{c-k}.$$

For the discrete-time Markov chain T, it is easy to compute the GPH distribution p_n for $n \geqslant 0$ in terms of Algorithm 8.3.

8.2.2 Continuous-Time GPH Distribution

We consider an irreducible continuous-time Markov chain with state space $\Omega = \{0,1,2,...\}$ whose infinitesimal generator is given by

$$Q = \begin{array}{c c} & \begin{array}{cc} 0 & E \end{array} \\ \begin{array}{c} 0 \\ E \end{array} & \begin{pmatrix} 0 & 0 \\ T^0 & T \end{pmatrix} \end{array} \tag{8.12}$$

where $E = \{1,2,\ldots\}, T^0 \gneqq 0$ and $Te + T^0 = 0$. We assume that state 0 is an absorbing state and all the others are transient. Let (α_0, α) be the initial probability vector of the Markov chain, where $\alpha_0 + \alpha e = 1$. For the Markov chain given in Eq. (8.12), the probability distribution $F(x)$ of the time X until absorption into the absorbing state 0 is called a continuous-time GPH distribution with representation (α, T). If $T + T^0\alpha$ is the infinitesimal generator of an irreducible Markov chain, then this representation (α, T) is called an irreducible representation. In this case, it is easy to check that

$$F(x) = 1 - \alpha \exp\{Tx\}e$$

or

$$\overline{F}(x) = \alpha \exp\{Tx\}e.$$

Let the Laplace-Stieljes transform of $F(x)$ be $\tilde{f}^*(s) = \int_0^{+\infty} \exp\{-sx\} \mathrm{d}F(x)$. Then

$$\tilde{f}^*(s) = \alpha(sI - T)^{-1}T^0$$

for $s > 0$. It is clear that $T - sI$ is the infinitesimal generator of a Markov chain. In this case, we can obtain the UL-type RG-factorization

$$sI - T = [I - R_U(s)][-U_D(s)][I - G_L(s)],$$

which leads to

$$\tilde{f}^*(s) = \alpha[I - G_L(s)]^{-1}[-U_D(s)]^{-1}[I - R_U(s)]^{-1}T^0;$$

and the LU-type RG-factorization

$$sI - T = [I - \overline{R}_L(s)][-\overline{U}_D(s)][I - \overline{G}_U(s)],$$

which yields

$$\tilde{f}^*(s) = \alpha[I - \overline{G}_U(s)]^{-1}[-\overline{U}_D(s)]^{-1}[I - \overline{R}_L(s)]^{-1}T^0.$$

It is clear that the noncentral moment of order i

$$E[X^i] = i!(-1)^i \alpha T^{-i} e, \quad i \geqslant 1.$$

Note that we can use the RG-factorizations to compute the moments by using the UL-type RG-factorization

$$T = (I - R_U)\Psi_D(I - G_L)$$

or the LU-type RG-factorization

$$T = (I - \overline{R}_L)\overline{U}_D(I - \overline{G}_U).$$

Now, we list some useful properties for the continuous-time GPH distribution. Note that Properties 1.5 to 1.12 in Chapter 1 can directly be extended here, we only choose some of them under a new different notation. Note that the proofs of the properties are easy, and thus are omitted here.

Property 8.5 If $X \sim \mathrm{GPH}(\alpha, T)$ and $Y \sim \mathrm{GPH}(\beta, S)$, then $X + Y \sim \mathrm{PH}(\gamma, L)$, where $\gamma = (\alpha, \alpha_0 \beta)$ and

$$L = \begin{pmatrix} T & T^0\beta \\ 0 & S \end{pmatrix}.$$

Property 8.6 If $X \sim \mathrm{GPH}(\alpha, T)$ and $Y \sim \mathrm{GPH}(\beta, S)$, then min $\{X, Y\} \sim \mathrm{PH}(\alpha \otimes \beta, T \oplus S)$, and max $\{X, Y\} \sim \mathrm{PH}(\gamma, L)$, where

$$\gamma = (\alpha \otimes \beta, \beta_0 \alpha, \alpha_0 \beta),$$

$$L = \begin{pmatrix} T \oplus S & I \otimes S^0 & T^0 \otimes I \\ 0 & T & 0 \\ 0 & 0 & S \end{pmatrix}.$$

Property 8.7 If $\{s_k\} \sim \mathrm{GPH}(\beta, S)$ and $F(x) \sim \mathrm{GPH}(\alpha, T)$, then the infinite mixture $G(x) = \sum_{k=0}^{\infty} s_k F^{k*}(x) \sim \mathrm{GPH}(\gamma, L)$, where

$$\gamma = \alpha \otimes \beta(I - \alpha_0 S)^{-1},$$
$$L = T \otimes I + (1 - \alpha_0)T^0 \alpha \otimes (I - \alpha_0 S)^{-1} S.$$

Now, we provide an approximate algorithm for computing the function $\overline{F}(x) = \alpha \exp\{Tx\} e$ for $x \geqslant 0$.

Let $\pi(n)=\alpha T^n$ for $n \geqslant 0$. Then $\pi(0)=\alpha$ and $\pi(n)=\pi(n-1)T$ for $n \geqslant 1$. Thus we obtain

$$\overline{F}(x)=\alpha\sum_{k=0}^{\infty}T^n\frac{x^n}{n!}e=\sum_{n=0}^{\infty}\pi(n)\frac{x^n}{n!}e.$$

We write

$${}_{N_0}\pi_j(0)=\alpha_j,\quad 1\leqslant j\leqslant N_0,$$

and

$${}_{N_n}\pi_j(n)=\sum_{i=1}^{N_{n-1}}{}_{N_{n-1}}\pi_i(n-1)T_{i,j},\quad 1\leqslant j\leqslant N_n, n\geqslant 1.$$

It is clear that the sequence $\{{}_{N_n}\pi_j(n)\}$ is used to approximate the value $\pi_j(n)$. For a sufficiently large positive integer M, we write

$${}_M\overline{F}(x)=\sum_{n=0}^{M}{}_{N_n}\pi(n)\frac{x^n}{n!}. \tag{8.13}$$

It is easy to check that the sequence $\{{}_M\overline{F}(x)\}$ converges to the transient solution $\overline{F}(x)$ for each $x\geqslant 0$, and this convergence is uniform for $x\in[0,T]$.

Algorithm 8.4 Computation of the GPH Distribution

INPUT: The vector α, T^0, the matrix T, the error $\varepsilon>0$.

COMPUTATION:

Step 1 Generating the integer sequence $\{N_k\}$.

Step 2 Iteratively computing $\{{}_{N_n}\pi(n)\}$.

Step 3 Determining the function sequence $\{{}_M\overline{F}(x)\}$ according to Eq. (8.13).

OUTPUT: The approximate transient probability $\{{}_M\overline{F}(x)\}$.

We now compute the busy period distribution of a $PH/PH^X/1$ queue, where the interarrival and service times are all of phase type with irreducible representations (α,T) of size m and (β,S) of size n, respectively. The service batch size has the discrete probability distribution $\{p_n\}$. It is clear that the busy period χ is of generalized phase type with irreducible representations (γ,W), where

$$\gamma=(\alpha\otimes\beta,0,0,\dots)$$

and

$$W=\begin{pmatrix} A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$A_0=(T^0\alpha)\otimes I,\quad A_1=I\otimes S,\quad A_k=p_k I\otimes(S^0\beta),\quad k\geqslant 2.$$

Let

$$B(x) = P\{\chi > x\}.$$

Then using Algorithm 8.4 we can approximate $B(x)$ in terms of the sequence $\{B_M(x)\}$ for $x \in [0,a]$.

8.2.3 GMAPs

Similar to Section 5.1 we can provide a GPH renewal process based on the GPH distribution. Now, we consider a GMAP with irreducible matrix descriptor (C, D) of infinite size. Note the GMAP is a special case of the CMAP given in Section 5.7.

We write $P(k;t) = (P_{j,j'}(k;t))_{j,j' \geqslant 1}$, where $P_{j,j'}(k;t)$ is a conditional probability that the Markov chain $C + D$ is in the phase j' at time t and that k renewals occur in $[0,t)$, given that the Markov chain starts in the phase j at time 0. The matrix sequence $\{P(k;t)\}$ satisfies the forward Chapman-Kolmogorov differential equations

$$P'(0;t) = P(0;t)C,$$
$$P'(k;t) = P(k;t)C + P(k-1;t)D, \quad k \geqslant 1;$$

or the backward Chapman-Kolmogorov differential equations

$$P'(0;t) = CP(0;t),$$
$$P'(k;t) = CP(k;t) + DP(k-1;t), \quad k \geqslant 1.$$

At the same time, the initial condition is given by

$$P(k,0) = \begin{cases} I, & k = 0, \\ 0, & k \geqslant 1. \end{cases}$$

Let $P^*(z;t) = \sum_{k=0}^{\infty} z^k P(k;t)$. Then it is easy to see that

$$P^*(z;t) = \exp\{(C + zD)t\}.$$

If the Markov chain $C + D$ is irreducible and positive recurrent, we denote by θ the stationary probability vector of the Markov chain $C + D$. The stationary arrival rate of the GMAP is given by $\lambda = \theta De$.

In what follows we discuss the inter-dependent structure of the GMAP. To do this, we assume that an arrival occurs at time 0. Let τ_k be the kth arrival epoch of the GMAP for $k \geqslant 0$, where $\tau_0 = 0$. Let $X_n = \tau_n - \tau_{n-1}$. Then X_n is the nth interarrival time of the GMAP for $n \geqslant 1$. In general, these random variables X_n for $n \geqslant 1$ are not independent but they are identically distributed with marginal densities given by

$$f(t) = \theta \exp\{Ct\}De.$$

Define the matrix $W(t)$ with the (i, j) th element $W_{i,j}(t)$ which is a conditional probability density for an interarrival time $[0, t)$, terminating at phase j and beginning at the initial phase i. It is easy to check that $W(t) = \exp\{Ct\}D$. When the environment stochastic process $\{J(t); t \geqslant 0\}$ evolves from one arrival epoch to the next one, the transition probability matrix of the GMAP is given by

$$W = -\int_0^{+\infty} \exp\{Ct\}D\mathrm{d}t = -C_{\max}^{-1} D,$$

where

$$C_{\max}^{-1} = (I - G_L)^{-1} U_D^{-1} (I - R_U)^{-1}.$$

It is clear that $\theta W = \theta$. The joint probability density of the two random variables X_l and X_{l+k} is given by

$$f_{k,l}(x, y) = \theta \exp\{Ct\}DW^{k-1} \exp\{Ct\}De.$$

Therefore, we obtain

$$E[X_l X_{l+k}] = \int_0^{+\infty} \int_0^{+\infty} xyf_{k,l}(x, y)\mathrm{d}x\mathrm{d}y = \theta C_{\max}^{-1} W^k C_{\max}^{-1} e,$$

Note that

$$E[X_l]E[X_{l+k}] = (\theta C_{\max}^{-1} e)^2 = \theta C_{\max}^{-1} e\theta W^k C_{\max}^{-1} e,$$

the correlation between the two random variables X_l and X_{l+k} is given by

$$\rho_k = \frac{\theta C_{\max}^{-1} (I - e\theta) W^k C_{\max}^{-1} e}{\theta C_{\max}^{-1} (2I - e\theta) C_{\max}^{-1} e}.$$

8.2.4 Time-Inhomogeneous PH(t) Distribution

We consider a time-inhomogeneous continuous-time Markov chain with state space $\Omega = \{1, 2, \ldots, m, m+1\}$ whose infinitesimal generator is given by

$$Q = \begin{array}{c} \\ (1,2,\ldots,m) \\ m+1 \end{array} \begin{array}{c} \begin{array}{cc} (1,2,\ldots,m) & m+1 \end{array} \\ \begin{pmatrix} T(t) & T^0(t) \\ 0 & 0 \end{pmatrix} \end{array},$$

where $T^0(t) \gneq 0$ and $T(t)e + T^0(t) = 0$. It is clear that state $m+1$ is an absorbing state and all the others are transient. Let (α, α_{m+1}) be the initial probability vector of the Markov chain, where $\alpha e + \alpha_{m+1} = 1$. For the time-inhomogeneous Markov chain, the distribution $F(x)$ of the time X until absorption into the absorbing state $m+1$ is called a time-inhomogeneous PH distribution (PH (t)) with representation $(\alpha, T(t))$. If $T(t) + T^0(t)\alpha$ is the infinitesimal generator of an irreducible Markov chain, then this representation $(\alpha, T(t))$ is called an irreducible representation. In this case, we have

$$F(x) = 1 - \alpha \exp\left\{ \int_0^x T(t) \mathrm{d}t \right\} e.$$

8.2.5 Time-Inhomogeneous MAP (t)

we consider a MAP (t) with irreducible matrix descriptor $(C(t), D(t))$. We write $P(k;t) = (P_{j,j'}(k;t))_{j,j' \geqslant 1}$, where $P_{j,j'}(k;t)$ is a conditional probability that the Markov chain $C(t) + D(t)$ is in the phase j' at time t and that k renewals occur in $[0, t)$, given that the Markov chain starts in the phase j at time 0. The matrix sequence $\{P(k;t)\}$ satisfies the forward Chapman-Kolmogorov differential equations

$$P'(0;t) = P(0;t)C(t),$$
$$P'(k;t) = P(k;t)C(t) + P(k-1;t)D(t), \quad k \geqslant 1;$$

or the backward Chapman-Kolmogorov differential equations

$$P'(0;t) = C(t)P(0;t),$$
$$P'(k;t) = C(t)P(k;t) + D(t)P(k-1;t), \quad k \geqslant 1.$$

At the same time, the initial condition is given by

$$P(k,0) = \begin{cases} I, & k = 0, \\ 0, & k \geqslant 1. \end{cases}$$

Let $P^*(z;t) = \sum_{k=0}^{\infty} z^k P(k;t)$. Then it is easy to see that

$$P^*(z;t) = \exp\left\{ \int_0^t \left(C(u) + zD(u) \right) \mathrm{d}u \right\}.$$

8.2.6 A Time-Inhomogeneous MAP(t)/PH(t)/1 Queue

We consider a time-inhomogeneous MAP(t)/PH(t)/1 queue, where the arrival process is a MAP (t) with irreducible matrix descriptor $(C(t), D(t))$ and the service time distribution is of phase type with irreducible representation $(\alpha, T(t))$. The corresponding time-inhomogeneous QBD process is given by

$$Q(t)=\begin{pmatrix} C(t)\otimes I & D(t)\otimes I & & & \\ I\otimes[T^0(t)\alpha] & C(t)\oplus T(t) & D(t)\otimes I & & \\ & I\otimes[T^0(t)\alpha] & C(t)\oplus T(t) & D(t)\otimes I & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The transient probability vector of the QBD process is given by

$$\pi(t)=\pi(0)\exp\left\{\int_0^t Q(u)\mathrm{d}u\right\}.$$

8.3 The Sojourn Times

In this section, we study the sojourn times of an irreducible M-state Markov chain which is either discrete-time or continuous-time, and derive expressions for the probability distributions of the sojourn times. Note that M may be either finite or infinite.

8.3.1 Discrete-Time Markov Chains

Consider an irreducible discrete-time M-state Markov chain $\{X_n\}$ whose transition probability matrix $P=(p_{i,j})_{1\leqslant i,j\leqslant M}$. Let $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_M)$, where $\alpha_i=P\{X_0=i\}$ for $1\leqslant i\leqslant M$. Let $E=\{1,2,\ldots,K\}$ and $E^c=\{K+1,K+2,\ldots,M\}$.Then according to the two sets E and E^c, α and P are partitioned as

$$\alpha=(\alpha^{(1)},\alpha^{(2)})$$

and

$$P=\begin{pmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{pmatrix},$$

respectively. It is easy to see that the order of the matrix $P_{2,2}$ is finite if $M<\infty$ or is infinite if $M=\infty$. Since the Markov chain is irreducible, the two matrices $I-P_{1,1}$ and $I-P_{2,2}$ are both invertible. Specifically, this section only uses the minimal nonnegative inverse $(I-P_{2,2})^{-1}_{\min}$ when $M=\infty$. For simplicity of description, we still use the notation $(I-P_{2,2})^{-1}$ for the minimal nonnegative inverse $(I-P_{2,2})^{-1}_{\min}$.

Let

$$\Theta_1 = P_{1,2}(I - P_{2,2})^{-1} P_{2,1}(I - P_{1,1})^{-1}$$

and

$$\Theta_2 = P_{2,1}(I - P_{1,1})^{-1} P_{1,2}(I - P_{2,2})^{-1}.$$

Lemma 8.2 *If the Markov chain P is positive recurrent and $\pi = (\pi^{(1)}, \pi^{(2)})$ is its stationary probability vector, then $\pi^{(1)}\Theta_1 = \pi^{(1)}$ and $\pi^{(2)}\Theta_2 = \pi^{(2)}$.*

Proof Note that $\pi^{(1)}$ is the stationary probability vector of the censored chain $\Psi_1 = P_{1,1} + P_{1,2}(I - P_{2,2})^{-1} P_{2,1}$ to set E, we obtain that $\pi^{(1)}\Psi_1 = \pi^{(1)}$, that is

$$\pi^{(1)}\left[P_{1,1} + P_{1,2}(I - P_{2,2})^{-1} P_{2,1}\right] = \pi^{(1)},$$

which leads to $\pi^{(1)}\Theta_1 = \pi^{(1)}$. Similarly, we can yield $\pi^{(2)}\Theta_2 = \pi^{(2)}$. This completes the proof.

A sojourn of the Markov chain $\{X_n\}$ in set E is the state sequence X_m, $X_{m+1}, \ldots, X_{m+k}$ for $k \geqslant 1$, where $X_m, X_{m+1}, \ldots, X_{m+k-1} \in E$, while $X_{m-1}, X_{m+k} \notin E$. This sojourn lasts k time units, and begins at time m and finishes at time $m+k$.

Let V_k be a random variable representing the kth sojourn of the Markov chain $\{X_n\}$, $k \geqslant 1$. It is clear that the irreducibility of the Markov chain assures the existence of infinite sojourns in E with probability 1. For each $k \geqslant 1$, we write

$$v_k(i) = P\{V_k = i\},$$
$$v_k = (v_k(1), v_k(2), \ldots, v_k(K)), \quad E = \{1, 2, \ldots, k\}.$$

The following theorem provides expression for the vector v_k for $k \geqslant 1$.

Theorem 8.2 (1)

$$v_1 = \alpha^{(1)} + \alpha^{(2)}(I - P_{2,2})^{-1} P_{2,1}.$$

(2) *For* $k \geqslant 2$,

$$v_k = v_1 L^{k-1},$$

where

$$\begin{aligned} L &= (I - P_{1,1})^{-1} P_{1,2}(I - P_{2,2})^{-1} P_{2,1} \\ &= (I - P_{1,1})^{-1} \Theta_1 (I - P_{1,1}) . \end{aligned}$$

Proof (1) For $i \in E^c$ and $j \in E$, we write

$$h_{i,j} = P\{V_1 = j \mid X_0 = i\}$$

and $H=(h_{i,j})_{1\leqslant i,j\leqslant K}$. It is easy to see that H is the matrix of absorption probabilities from E^c to E, thus we obtain

$$H=(I-P_{2,2})^{-1}P_{2,1}.$$

Therefore,

$$P\{V_1=j\}=\alpha_j^{(1)}+\sum_{i\in E^c}P\{X_0=i\}h_{i,j}.$$

This gives, in matrix notation,

$$v_1=\alpha^{(1)}+\alpha^{(2)}(I-P_{2,2})^{-1}P_{2,1}.$$

(2) Let V_1^c be a random variable representing the state of E^c in which the first sojourn of the Markov chain $\{X_n\}$ begins. We write

$$w_{i,j}=P\{V_1^c=j \mid X_0=i\},\quad i\in E, j\in E^c,$$

and $W=(w_{i,j})_{i\in E,j\in E^c}$. Then

$$W=(I-P_{1,1})^{-1}P_{1,2}.$$

Let

$$l_{i,j}=P\{V_2=j\mid X_0=i\},\quad i\in E, j\in E,$$

and $L=(l_{i,j})_{1\leqslant i,j\leqslant K}$. Then

$$\begin{aligned}
l_{i,j}&=\sum_{k\in E^c}P\{V_1^c=k\mid X_0=i\}P\{V_2=j\mid V_1^c=k,X_0=i\}\\
&=\sum_{k\in E^c}P\{V_1^c=k\mid X_0=i\}P\{V_2=j\mid V_1^c=k\}\\
&=\sum_{k\in E^c}w_{i,k}h_{k,j},
\end{aligned}$$

thus we obtain

$$L=WH.$$

Note that

$$\begin{aligned}
P\{V_k=j\}&=\sum_{i\in E}P\{V_1=i\}P\{V_k=j\mid V_1=i\}\\
&=\sum_{i\in E}v_i l_{i,j}^{(k-1)},\quad k\geqslant 2,
\end{aligned}$$

hence we can obtain, in matrix notation,

$$v_k = v_1 L^{k-1}, \quad k \geqslant 2.$$

This completes the proof.

Remark 8.3 *When the size of the set* E *or* E^c *is large or infinite, we may use the RG-factorizations to compute the matrix L. We write*

$$I - P_{1,1} = \left[I - R_U^{(1)}\right]\left[I - \Psi_D^{(1)}\right]\left[I - G_L^{(1)}\right]$$

and

$$I - P_{2,2} = \left[I - R_U^{(2)}\right]\left[I - \Psi_D^{(2)}\right]\left[I - G_L^{(2)}\right].$$

Then

$$\begin{aligned} L = & \left[I - G_L^{(1)}\right]^{-1}\left[I - \Psi_D^{(1)}\right]^{-1}\left[I - R_U^{(1)}\right]^{-1} P_{1,2} \\ & \cdot \left[I - G_L^{(2)}\right]^{-1}\left[I - \Psi_D^{(2)}\right]^{-1}\left[I - R_U^{(2)}\right]^{-1} P_{2,1}. \end{aligned}$$

Now, we further simplify the two important matrices L and H according to the structure of the Markov chain P. Let

$$E_1 = \{j \in E : p_{i,j} > 0, i \in E^c\}.$$

Then E_1 is the state set of E directly accessible from E^c in one-step transition. We write $E_2 = E - E_1$. Then according to the two sets E_1 and E_2, the two matrices L and H are partitioned as

$$L = \begin{array}{c} \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{pmatrix} L_1 & 0 \\ L_2 & 0 \end{pmatrix} \end{array}$$

and

$$H = \begin{array}{c} \\ E^c \end{array} \begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ (H_1 \quad 0). \end{array}$$

At the same time, the matrix P is partitioned as

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E^c \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E^c \end{array} \\ \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} \\ T_{2,1} & T_{2,2} & T_{2,3} \\ T_{3,1} & 0 & T_{3,3} \end{pmatrix} \end{array}.$$

The following theorem provides expressions for the matrices L_1, L_2 and H_1 in terms of the matrices $T_{i,j}$ for $1 \leqslant i, j \leqslant 3$.

Theorem 8.3 *Based on the sets* E_1, E_2 *and* E^c, *we have*

$$\begin{aligned} L_1 &= [I - T_{1,1} - T_{1,2}(I - T_{2,2})^{-1} T_{2,1}]^{-1} \\ &\quad \cdot [T_{1,3} + T_{1,2}(I - T_{2,2})^{-1} T_{2,3}](I - T_{3,3})^{-1} T_{3,1}, \\ H_1 &= (I - T_{3,3})^{-1} T_{3,1} L_1 \end{aligned}$$

and

$$L_2 = (I - T_{2,2})^{-1} T_{2,1} L_1 + (I - T_{2,2})^{-1} T_{2,3} H_1.$$

Proof Note that

$$L = WH = (I - P_{1,1})^{-1} P_{1,3} H,$$

we obtain

$$(I - P_{1,1}) L = P_{1,3} H,$$

that is, in block-structured notation,

$$\begin{cases} L_1 = T_{1,1} L_1 + T_{1,2} L_2 + T_{1,3} H_1, \\ L_2 = T_{2,1} L_1 + T_{2,2} L_2 + T_{2,3} H_1, \\ H_1 = T_{3,1} L_1 + T_{3,3} H_1, \end{cases}$$

simple computation leads to the desired results. This completes the proof.

We denote by $N_{E,r}$ the time units spent in the rth sojourn of the Markov chain $\{X_n\}$ in the set E.

Theorem 8.4 *The random variable* $N_{E,r}$ *is of phase type with irreducible representation* $(v_r, P_{1,1})$ *for* $r \geqslant 1$.

Proof First, we derive the distribution of the random variable $N_{E,1}$.

For $i \in E$, we write

$$\beta_k(i) = P\{N_{E,1} = k \mid V_1 = i\}$$

and

$$\beta_k = (\beta_k(1), \beta_k(2), \ldots, \beta_k(K)).$$

Then

$$\beta_1(i) = \sum_{j \in E^c} p_{i,j}$$

and for $k \geqslant 2$,

$$\beta_k(i) = \sum_{l \in E} p_{i,l} P\{N_{E,1} = k \mid V_2 = l\}.$$

Hence, we obtain, in matrix notation,

$$\beta_1 = P_{1,2}e = (I - P_{1,1})e$$

and for $k \geqslant 2$,

$$\beta_k = P_{1,1}\beta_{k-1} = P_{1,1}^{k-1}\beta_1 = P_{1,1}^{k-1}(I - P_{1,1})e.$$

Therefore,

$$\begin{aligned} P\{N_{E,1} = k\} &= \sum_{i \in E} P\{V_1 = i\} P\{N_{E,1} = k \mid V_1 = i\} \\ &= v_1 P_{1,1}^{k-1}(I - P_{1,1})e, \end{aligned}$$

which indicates that the random variable $N_{E,1}$ is of phase type with irreducible representation $(v_1, P_{1,1})$.

Now, we derive the distribution of the random variable $N_{E,r}$ for $r \geqslant 2$. With similar analysis to that for $N_{E,1}$, we obtain

$$\begin{aligned} P\{N_{E,r} = k\} &= \sum_{i \in E} P\{V_r = i\} P\{N_{E,r} = k \mid V_r = i\} \\ &= v_r P_{1,1}^{k-1}(I - P_{1,1})e, \end{aligned}$$

which indicates that the random variable $N_{E,r}$ is of phase type with irreducible representation $(v_r, P_{1,1})$.

This completes the proof.

Using the moments of the discrete-time PH distribution, we have

$$E\left[N_{E,r}^m\right] = v_r \cdot m! P_{1,1}^{m-1}(I - P_{1,1})^{-m} e, \quad m \geqslant 1.$$

Remark 8.4 *The random vector* $(N_{E,1}, N_{E,2}, \ldots, N_{E,r})$ *is of r-dimensional phase type with the r-dimensional distribution function*

$$\begin{aligned} &P\{N_{E,1} = n_1, N_{E,2} = n_2, \ldots, N_{E,r} = n_r\} \\ =& v_1 P_{1,1}^{n_1 - 1} P_{1,2} \prod_{i=2}^{r} \left\{ (I - P_{2,2})^{-1} P_{2,1} P_{1,1}^{n_i - 1} P_{1,2} \right\} e \end{aligned}$$

for $n_1, n_2, \ldots, n_r \geqslant 1$.

8.3.2 Continuous-Time Markov Chains

This analysis is similar to that for the discrete-time case. Therefore, only a simple description for the sojourn times of a continuous-time Markov chain is provided.

Consider an irreducible continuous-time *M*-state Markov chain $\{X_t, t \geqslant 0\}$

whose infinitesimal generator $Q=(q_{i,j})_{1\leqslant i,j\leqslant M}$. Let $E=\{1,2,\ldots,K\}$ and $E^c=\{K+1,K+2,\ldots,M\}$. Then the matrix Q is partitioned as

$$Q=\begin{array}{c} \\ E \\ E^c \end{array}\!\!\begin{array}{c} \begin{array}{cc} E & E^c \end{array} \\ \begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{pmatrix} \end{array}$$

A sojourn of the Markov chain $\{X_t,t\geqslant 0\}$ in the state set E is a sequence $X_{t_m},X_{t_{m+1}},\ldots,X_{t_{m+k}}$ for $k\geqslant 1$, where t_i is an instant of the ith state transition; X_{t_m}, $X_{t_{m+1}},\ldots,X_{t_{m+k-1}}\in E, X_{t_{m-1}},X_{t_{m+k}}\notin E$. This sojourn lasts a time length $X_{t_{m+k}}-X_{t_m}$, and begins at time X_{t_m} and finishes at time $X_{t_{m+k}}$.

Let V_k be the state of E in which the kth sojourn of the Markov chain $\{X_t,t\geqslant 0\}$ begins. We write

$$v_k(i)=P\{V_k=i\}$$

and

$$v_k=(v_k(1),v_k(2),\ldots,v_k(K)).$$

Then we obtain

$$v_1=\alpha^{(1)}+\alpha^{(2)}(-Q_{2,2})^{-1}Q_{2,1}$$

and for $k\geqslant 2$,

$$v_k=v_1L^{k-1},$$

where

$$L=(-Q_{1,1})^{-1}Q_{1,2}(-Q_{2,2})^{-1}Q_{2,1}.$$

We denote by $T_{E,k}$ the time length spent during the kth sojourn of the Markov chain $\{X_t,t\geqslant 0\}$ in the state set E. Then for $k\geqslant 1$, the random variable $T_{E,k}$ is of phase type with irreducible representation $(v_k,Q_{1,1})$, and

$$E\left[T_{E,k}^m\right]=v_k\cdot m!(-Q_{1,1})^{-m}\,e.$$

Remark 8.5 *When the size of the set E or E^c is large or infinite, we may use the RG-factorizations to compute the matrix L and the mean $E[T_{E,k}]$. We write*

$$Q_{1,1}=\left[I-R_U^{(1)}\right]\left[U_D^{(1)}\right]\left[I-G_L^{(1)}\right]$$

and

$$Q_{2,2}=\left[I-R_U^{(2)}\right]\left[U_D^{(2)}\right]\left[I-G_L^{(2)}\right].$$

Then

$$L=\left[I-G_L^{(1)}\right]^{-1}\left[-U_D^{(1)}\right]^{-1}\left[I-R_U^{(1)}\right]^{-1} \cdot Q_{1,2}\left[I-G_L^{(2)}\right]^{-1}\left[-U_D^{(2)}\right]^{-1}\left[I-R_U^{(2)}\right]^{-1}Q_{2,1}$$

and

$$E\left[T_{E,k}\right]=v_k\left[I-G_L^{(1)}\right]^{-1}\left[-U_D^{(1)}\right]^{-1}\left[I-R_U^{(1)}\right]^{-1}e.$$

The random vector $(T_{E,1},T_{E,2},\ldots,T_{E,k})$ is of k-dimensional phase type with the m-dimensional distribution function

$$P\{T_{E,1}>t_1,T_{E,2}>t_2,\ldots,T_{E,k}>t_k\}=v_1(-Q_{1,1})^{-1}\left[I-\exp\{Q_{1,1}t_1\}\right]Q_{1,2} \cdot\prod_{i=2}^{k}\left\{(-Q_{2,2})^{-1}Q_{2,1}(-Q_{1,1})^{-1}\left[I-\exp\{Q_{1,1}t_i\}\right]Q_{1,2}\right\}e.$$

Now, we analyze a MAP/PH/1 queue, where the customer arrivals are a MAP with irreducible matrix descriptor (D_0,D_1) of size m and the service times are of phase type with irreducible representation (β,S) of size n. Let $N(t)$, $I(t)$ and $J(t)$ be the number of customers in the system, the phases of the arrival and service processes at time t, respectively. Then $\{(N(t),I(t),J(t)):t\geqslant 0\}$ is a level-independent QBD process whose infinitesimal generator is given by

$$Q=\begin{pmatrix} D_0 & D_1\otimes\beta & & & \\ I\otimes S^0 & D_0\oplus S & D_1\otimes I & & \\ & I\otimes(S^0\beta) & D_0\oplus S & D_1\otimes I & \\ & & I\otimes(S^0\beta) & D_0\oplus S & D_1\otimes I \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We consider the following two cases:

Case Ⅰ $E=\{\text{Level 0, Level 1}\}$ and $\alpha=(\alpha_0,\alpha_1,\ldots)$. Then

$$v_1=(\alpha_0,\alpha_1)+(\alpha_2,\alpha_3,\ldots)(-Q_{2,2})^{-1}Q_{2,1}$$

and

$$v_k=v_1\left[(-Q_{1,1})^{-1}Q_{1,2}(-Q_{2,2})^{-1}Q_{2,1}\right]^{k-1},\quad k\geqslant 2.$$

Case Ⅱ $E=\{\text{Level 1, Level 2}\}$ and $\alpha=(\alpha_0,\alpha_1,\alpha_2,\ldots)$. We take $E_1^c=\{\text{Level 0}\}$ and $E_2^c=\{\text{Level 3, Level 4},\ldots\}$. Then the matrix Q is partitioned as

$$Q = \begin{array}{c} \\ E_1^c \\ E \\ E_2^c \end{array} \begin{array}{c} \begin{array}{ccc} E_1^c & E & E_2^c \end{array} \\ \begin{pmatrix} Q_{1,1} & Q_{1,2} & Q_{1,3} \\ Q_{2,1} & Q_{2,2} & Q_{2,3} \\ Q_{3,1} & Q_{3,2} & Q_{3,3} \end{pmatrix} \end{array}.$$

In this case, we write

$$Q = \begin{array}{c} \\ E \\ E_1^c \\ E_2^c \end{array} \begin{array}{c} \begin{array}{ccc} E & E_1^c & E_2^c \end{array} \\ \begin{pmatrix} Q_{2,2} & Q_{2,1} & Q_{2,3} \\ Q_{1,2} & Q_{1,1} & Q_{1,3} \\ Q_{3,2} & Q_{3,1} & Q_{3,3} \end{pmatrix} \end{array}.$$

Hence, we obtain

$$v_1 = (\alpha_1, \alpha_2) + (\alpha_0, \alpha_3, \alpha_4, \ldots) \begin{pmatrix} -Q_{1,1} & -Q_{1,3} \\ -Q_{3,1} & -Q_{3,3} \end{pmatrix}^{-1} \begin{pmatrix} Q_{1,2} \\ Q_{3,2} \end{pmatrix}$$

and for $k \geqslant 2$

$$v_k = v_1 \left[(-Q_{2,2})^{-1} (Q_{2,1} \quad Q_{2,3}) \begin{pmatrix} -Q_{1,1} & -Q_{1,3} \\ -Q_{3,1} & -Q_{3,3} \end{pmatrix}^{-1} \begin{pmatrix} Q_{1,2} \\ Q_{3,2} \end{pmatrix} \right]^{k-1}$$

8.4 Time-Inhomogeneous Discrete-Time Models

In this section, we discuss an irreducible time-inhomogeneous discrete-time Markov chain, and provide expression for its transient probability. Further, when the Markov chain is periodic, we obtain a new expression for the asymptotic periodic distribution.

Consider an irreducible time-inhomogeneous discrete-time Markov chain $\{(X_n, J_n) : n \geqslant 0\}$ whose transition probability matrix is given by

$$P(n) = \begin{pmatrix} P_{0,0}^{(n)} & P_{0,1}^{(n)} & P_{0,2}^{(n)} & \cdots \\ P_{1,0}^{(n)} & P_{1,1}^{(n)} & P_{1,2}^{(n)} & \cdots \\ P_{2,0}^{(n)} & P_{2,1}^{(n)} & P_{2,2}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad n \geqslant 0,$$

where $P_{k,k}^{(n)}$ is a matrix of size m_k for $k \geqslant 0$, the sizes of all the other matrices can be determined accordingly. The (i, j) th entry of the matrix $P_{l,k}^{(n)}$ is given by

$$\left(P_{l,k}^{(n)}\right)_{i,j} = P\{X_{n+1} = k, J_{n+1} = j \mid X_n = l, J_n = i\}$$

which depends on the time n for $n \geqslant 0$.

8.4.1 The Transient Probability Vector

We write

$$\pi_{k,j}(n) = P\{X_n = k, J_n = j\},$$
$$\pi_k(n) = (\pi_{k,1}(n), \pi_{k,2}(n), \ldots, \pi_{k,m_k}(n)),$$
$$\pi(n) = (\pi_0(n), \pi_1(n), \pi_2(n), \ldots), \quad n \geqslant 0.$$

It is clear that if $\pi(0) = \alpha$ is the initial probability vector, then

$$\pi(1) = \pi(0)P(0)$$

and for $n \geqslant 1$,

$$\pi(n) = \pi(n-1)P(n-1). \tag{8.14}$$

Hence we obtain that for $1 \leqslant m \leqslant n$

$$\pi(n) = \pi(n-m)P(n-m)P(n-m+1)\ldots P(n-1), \tag{8.15}$$

we have

$$\pi(n) = \alpha P(0)P(1)P(2)\ldots P(n-1), \quad n \geqslant 1. \tag{8.16}$$

Based on (8.15), we introduce a notation

$$\mathcal{P}(m,n) = P(n-m)P(n-m+1)\ldots P(n-1)P(n), \quad 1 \leqslant m \leqslant n. \tag{8.17}$$

Then

$$\pi(n) = \pi(n-m)\mathcal{P}(m,n-1), \quad 1 \leqslant m \leqslant n. \tag{8.18}$$

The following two propositions provide some closure properties for the matrix $\mathcal{P}(m,n)$ for $1 \leqslant m \leqslant n$.

Proposition 8.1 *For $1 \leqslant m \leqslant n$, we have*

(1) *if $P(n)$ is stochastic for each $n \geqslant 1$, then $\mathcal{P}(m,n)$ is also stochastic;*

(2) *if $P(n)$ is irreducible for each $n \geqslant 1$, then $\mathcal{P}(m,n)$ is also irreducible.*

Proposition 8.2 *For $1 \leqslant m \leqslant n$, we have*

(1) *If the Markov chain $P(n)$ is recurrent for each $n \geqslant 1$, then the Markov chain $\mathcal{P}(m,n)$ is also recurrent.*

(2) *If the Markov chain $P(n)$ is positive recurrent for each $n \geqslant 1$, then the Markov chain $\mathcal{P}(m,n)$ is also positive recurrent.*

(3) *If the Markov chain $P(n)$ is recurrent for each $n \geqslant 1$, and there exists a Markov chain $P(n-k)$ which is null recurrent for $0 \leqslant k \leqslant m$, then the Markov*

chain $\mathcal{P}(m,n)$ *is null recurrent.*

(4) *If there exists a Markov chain* $P(n-k)$ *which is transient for* $0 \leqslant k \leqslant m$*, then the Markov chain* $\mathcal{P}(m,n)$ *is transient.*

In what follows we analyze the block structure for the matrix $\mathcal{P}(m,n)$ with respect to some special Markov chains such as the QBD processes, the Markov chains of $GI/M/1$ type and the Markov chains of $M/G/1$ type.

(1) The QBD processes

If the Markov chain $P(n)$ is a QBD process for each $n \geq 1$, then the Markov chain $\mathcal{P}(m,n)$ is also a QBD process for $1 \leqslant m \leqslant n$.

We provide a computational illustration. Let

$$P(n) = \begin{pmatrix} A_{0,0}^{(n)} & A_{0,1}^{(n)} & & \\ A_{1,0}^{(n)} & A_{1,1}^{(n)} & A_{1,2}^{(n)} & \\ & A_{2,1}^{(n)} & A_{2,2}^{(n)} & A_{2,3}^{(n)} \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad n \geqslant 1.$$

Then

$$P(n-1)P(n) = \begin{pmatrix} \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & \\ * & * \end{pmatrix} & & \\ \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & \\ * & * \end{pmatrix} & \\ & \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & \\ * & * \end{pmatrix} \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

where $*$ represents a non-zero block which can be obtained easily.

(2) Markov chains of $GI/M/1$ type

If the Markov chain $P(n)$ is a Markov chain of $GI/M/1$ type for each $n \geqslant 0$, then the Markov chain $\mathcal{P}(m,n)$ is also a Markov chain of $GI/M/1$ type for $1 \leqslant m \leqslant n$.

(3) Markov chains of $M/G/1$ type

If the Markov chain $P(n)$ is a Markov chain of $M/G/1$ type for each $n \geqslant 0$, then the Markov chain $\mathcal{P}(m,n)$ is also a Markov chain of $M/G/1$ type for $1 \leqslant m \leqslant n$.

8.4.2 The Asymptotic Periodic Distribution

We assume that the irreducible time-inhomogeneous discrete-time Markov chain

$\{(X_n, J_n) : n \geqslant 0\}$ is d-periodic, that is,

$$P(n) = P(n + kd), \quad k \geqslant 1, 0 \leqslant n \leqslant d - 1.$$

In this case, if $m = n + kd$ for $k \geqslant 1$ and $0 \leqslant n \leqslant d - 1,$ then

$$P(m) = P(n).$$

This indicates that the dynamical behavior of the Markov chain $\{(X_n, J_n) : n \geqslant 0\}$ is completely determined by the d matrices $P(0), P(1), \ldots, P(d-1).$

Let $m = n + kd$ for $k \geqslant 1$ and $0 \leqslant n \leqslant d - 1.$ Then

$$\begin{aligned}\pi(m) &= \pi(m-1)P(m-1) \\ &= \pi(n)P(n)P(n+1)\ldots P(n+d-1) \\ &\quad \cdot P(n+d)P(n+d+1)\ldots P(n+2d-1) \\ &\quad \ldots \\ &\quad \cdot P(n+(k-1)d)P(n+(k-1)d+1)\ldots P(n+kd-1) \\ &= \pi(n)[P(n)P(n+1)\ldots P(n+d-1)]^k.\end{aligned}$$

Hence, we obtain

$$\pi(m) = \alpha P(0)P(1)P(2)\ldots P(n-1)\left[P(n)P(n+1)\ldots P(n+d-1)\right]^k. \tag{8.19}$$

Let

$$\Im(n,d) = \lim_{k\to\infty} \pi(n + kd).$$

If $P(k)$ is stochastic for each $k \geqslant 1,$ then

$$\mathcal{P}(n, n+d-1) = P(n)P(n+1)\ldots P(n+d-1)$$

is also stochastic. Under the stochastic condition, the Cesaro limit

$$\mathcal{P}^*(n, n+d-1) = \lim_{N\to\infty} \frac{1}{N+1} \sum_{l=0}^{N} [\mathcal{P}(n, n+d-1)]^l$$

always exists. Thus, we obtain

$$\begin{aligned}\Im(n,d) &= \pi(n)\mathcal{P}^*(n, n+d-1) \\ &= \alpha P(0)P(1)P(2)\ldots P(n-1)\mathcal{P}^*(n, n+d-1).\end{aligned} \tag{8.20}$$

Specifically, if the Markov chain $\mathcal{P}(n, n+d-1)$ is positive recurrent, then

$$\theta(n,d)P(n, n+d-1) = \theta(n,d)$$

and $\theta(n,d)e = 1.$ Hence we have

$$\mathcal{P}^*(n, n+d-1) = e\theta(n,d).$$

In this case, it is clear that

$$\Im(n,d) = \alpha P(0)P(1)P(2)\dots P(n-1)e\theta(n,d) = \theta(n,d). \tag{8.21}$$

This shows that $\Im(n,d)$ can be expressed by the stationary probability vector $\theta(n,d)$ of the Markov chain $\mathcal{P}(n,n+d-1)$.

For the limit $\Im(n,d)$ for $0 \leqslant n \leqslant d-1$, Eq. (8.20) provides expression of $\Im(n,d)$ under a weaker condition in which the matrix $P(k)$ is stochastic for each $k \geqslant 1$; while Eq. (8.21) provides expression of $\Im(n,d)$ under a stronger condition under which the Markov chain $\mathcal{P}(n,n+d-1)$ is positive recurrent. Therefore, it is necessary that we discuss the conditions under which the Markov chain $\mathcal{P}(n,n+d-1)$ is positive recurrent, and provide effective algorithms for computing the stationary probability vector $\theta(n,d)$. Here, we only discuss the Markov chain of $GI/M/1$ type. The other Markov chains can be similarly dealt with by the results of Chapter 2.

If the Markov chain $P(n)$ is a level-independent Markov chain of $GI/M/1$ type for each $n \geqslant 1$, then the Markov chain $\mathcal{P}(m,n)$ is also a level-independent Markov chain of $GI/M/1$ type for $1 \leqslant m \leqslant n.$ Let

$$P(n) = \begin{pmatrix} B_1^{(n)} & B_0^{(n)} & & & \\ B_2^{(n)} & A_1^{(n)} & A_0^{(n)} & & \\ B_3^{(n)} & A_2^{(n)} & A_1^{(n)} & A_0^{(n)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad n \geqslant 1.$$

Then

$$P(n-1)P(n) = \begin{pmatrix} \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & \\ * & * \end{pmatrix} & & & \\ \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & \\ * & * \end{pmatrix} & & \\ \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \begin{pmatrix} * & \\ * & * \end{pmatrix} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The state space of the Markov chain $\mathcal{P}(1,n) = P(n-1)P(n),$ in $GI/M/1$ type, is given by

$$\Omega(n-1 \text{ to } n) = \left\{ \bigcup_{i=k\times 2}^{(k+1)\times 2-1} (\text{Level } i) : k = 0,1,2,\dots \right\}.$$

By induction, the state space of the Markov chain

$$P(m,n) = P(n-m)P(n-m+1)\dots P(n),$$

in $GI/M/1$ type, is given by

$$\Omega(n-m \text{ to } n)=\left\{\bigcup_{i=k\times 2^m}^{(k+1)\times 2^m-1}(\text{Level } i): k=0,1,2,\dots\right\},\quad 1\leqslant m\leqslant n.$$

That is, for the level-independent Markov chain $\mathcal{P}(m,n)$ of $GI/M/1$ type, the new kth level of $\Omega(n-m \text{ to } n)$ is given by

$$\widetilde{\text{Level } k}=\bigcup_{i=k\times 2^m}^{(k+1)\times 2^m-1}(\text{Level } i).$$

In this case, we have

$$P(m,n)=\begin{pmatrix} B_1^{(m,n)} & B_0^{(m,n)} & & & \\ B_2^{(m,n)} & A_1^{(m,n)} & A_0^{(m,n)} & & \\ B_3^{(m,n)} & A_2^{(m,n)} & A_1^{(m,n)} & A_0^{(m,n)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},\quad 1\leqslant m\leqslant n.$$

When the Markov chain $P(k)$ is d-periodic for each $k\geqslant 0$, we are interested in the matrix

$$\mathcal{P}(n,n+d-1)=P(n)P(n+1)\dots P(n+d-1),\quad 0\leqslant n\leqslant d-1.$$

We assume that the Markov chain $\mathcal{P}(n,n+d-1)$ is irreducible and positive recurrent. Let $\omega(n,n+d-1)$ be the stationary probability vector of the Markov chain. Then

$$\omega(n,n+d-1)=(\omega_0(n,n+d-1),\omega_1(n,n+d-1),\omega_2(n,n+d-1),\dots)$$

partitioned according to $\widetilde{\text{Level}\,k}$ for $k\geqslant 0$, and $\theta(n,n+d-1)$ its stationary probability vector

$$\theta(n,n+d-1)=(\theta_0(n,n+d-1),\theta_1(n,n+d-1),\theta_2(n,n+d-1),\dots)$$

partitioned according to $\widetilde{\text{Level}\,k}$ for $k\geqslant 0$. Obviously, $\omega(n,n+d-1)=\theta(n, n+d-1)$. Let $\eta=2^{d-1}$. Then for $k\geqslant 0$,

$$\omega_k(n,n+d-1)=(\theta_{k\eta}(n,n+d-1),\theta_{k\eta+1}(n,n+d-1),\\ \dots,\theta_{(k+1)\eta-1}(n,n+d-1)).$$

Let $R(n,n+d-1)$ be the minimal nonnegative solution to the matrix equation

$$R(n,n+d-1)=\sum_{k=0}^{\infty}[R(n,n+d-1)]^k A_k^{(n,n+d-1)}.$$

Then

$$\omega_k(n,n+d-1)=\omega_0(n,n+d-1)[R(n,n+d-1)]^k, \quad k\geqslant 1,$$

where $\omega_0(n,n+d-1)$ is the stationary probability vector of the censored chain

$$U_0(n,n+d-1)=\sum_{k=0}^{\infty}[R(n,n+d-1)]^k B_{k+1}^{(n,n+d-1)}$$

to $\widetilde{\text{Level } 0}$. In this case, we have

$$\Im(n,d)=\omega(n,n+d-1), \quad 0\leqslant n\leqslant d-1.$$

8.5 Notes in the Literature

Transient solution is always useful in the study of stochastic models such as the first passage time, the sojourn time and the fundamental period. The exact transient solution for the queue length and the busy period in an $M/M/1$ queue was studied in terms of the modified Bessel functions, e.g., see, Lederman and Reuter [30], Champernowne [13], Clarke [15], Grassmann [19], Massey [36], Massey and Whitt [37,38], Abate and Whitt [1 – 2], Parthasarathy [42], Baccelli and Massey [5], and Zhang and Coyle [55]. Griffiths, Leonenko and Williams [20] considered the transient solution to the $M/E_k/1$ queue. Lucantoni, Choudhury and Whitt [32] analyzed the transient performance measures of the $BMAP/G/1$ queue. Choudhury, Lucantoni and Whitt [14] analyzed the $M_t/G_t/1$ queue. Taaffe and Ong [53] discussed a nonstationary $PH(t)/M(t)/s/c$ queue, and Ong and Taaffe [41] analyzed the $PH(t)/PH(t)/1/c$ queue. Dormuth and Alfa [17] discussed the $MAP(t)/PH(t)/1/K$ queue.

Hsu and He [22] analyzed the distribution of the first passage time for Markov chains of $GI/M/1$ type. Hsu and Yuan [23] studied the transient solution of denumerable Markov chains. Hsu and Yuan [24] discussed the first passage times of Markov chains and the associated algorithms. Hsu, Yuan and Li [25] discussed the first passage time of Markov renewal processes.

Aggregated Markov chains provide natural models of partially observable stochastic models, e.g., see Kemeny and Snell [28], Stewart [51], Fredkin and Rice [18], Ball and Sansom [9], Sumita and Rieders [52], Iordache, Bucurescu and Pascu [26], Ball [6], Ball, Milne and Yeo [7], Rubino and Sericola [45,47], Jalali and Hawkes [27], Ball and Yeo [10], Rydén [48], Ball, Milne, Tame and Yeo [8], Larget [29], and Stadje [50]. Based on the block-structured analysis, the sojourn times of Markov chains were discussed in, such as, Rubino and Sericola [46], Sericola [49] and Csenki [16].

Harrison and Lemoine [21] first studied the limits of the periodic queues. Readers may further refer to Asmussen and Thorisson [4], Bambos and Walrand [11], Lemoine [31], Rolski [43,44], Whillie [54], Breuer [12], and Alfa and Margolius [3].

Baccelli and Massey [5] and Margolius [33] provided a sample path analysis for the $M_t/M_t/1$ queue. Margolius [34] analyzed transient and periodic solution to the time-inhomogeneous continuous-time QBD processes, and Margolius [35] further introduced the two crucial matrices $R(s,t)$ and $\widetilde{R}(s,t)$ to express the transient and periodic solution.

This chapter is based on Hsu and Yuan [23,24], Hsu, Yuan and Li [25], Rubino and Sericola [46], Sericola [49], Csenki [16] and Alfa and Margolius [3].

Problems

8.1 Consider the classical $M/M/1$ queue with arrival rate λ and service rate μ, compute the following transient performance measures:

(1) The queue length distribution,

(2) the waiting time distribution, and

(3) the busy period distribution.

8.2 Consider the classical $M/M/1$ queue with arrival rate λ and service rate μ. Let $\rho = \lambda / \mu$. For the three cases: $\rho < 1$, $\rho = 1$ and $\rho > 1$, apply the censoring technique to compute the following transient performance measures:

(1) The queue length distribution, and

(2) the busy period distribution.

8.3 Consider the classical $M/M/1$ queue with arrival rate λ and service rate μ. Let $\rho = \lambda / \mu$. For the three cases: $\rho < 1$, $\rho = 1$ and $\rho > 1$, apply the directed truncation to compute the busy period distribution.

8.4 Consider the classical $M/M^X/1$ queue with arrival rate λ, service rate μ and the service batch size distribution $\{a_k\}$ with $a = \sum_{k=1}^{\infty} k a_k$. Let $\rho = \lambda/(\mu a)$. For the three cases: $\rho < 1$, $\rho = 1$ and $\rho > 1$, apply the censoring technique to compute the busy period distribution under each following condition:

(1) $a_k = p^{k-1}(1-p)$ for $k \geqslant 1$ and $0 < p < 1$.

(2) $a_k = \dfrac{\varsigma}{k^2}$ for $k \geqslant 1$ and $\varsigma = \sum_{k=1}^{\infty} \dfrac{1}{k^2}$.

8.5 Consider an irreducible continuous-time birth-death process $\{X_t, t \geqslant 0\}$ whose infinitesimal generator is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & \mu & -(\lambda+\mu) & \lambda & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ with $\alpha_i = P\{X_0 = i\}$ for $i \geqslant 0$, and $B(x) = P\{\tau > x\}$ where $\tau = \inf\{X_t = 0\}$. Numerically compute the function $B(x)$ under each following condition:

(1) $\alpha_k = \delta_{k,1}$ for $k \geqslant 0$.

(2) $\alpha_k = p^{k-1}(1-p)$ for $k \geqslant 1$ and $0 < p < 1$.

(3) $a_k = \dfrac{\varsigma}{k^2}$ for $k \geqslant 1$ and $\varsigma = \sum\limits_{k=1}^{\infty} \dfrac{1}{k^2}$.

8.6 Consider the classical MAP/PH/1 queue with the MAP irreducible matrix descriptor (C, D) of size m and the PH irreducible representation (β, S) of size n. Let θ be the stationary probability vector of the Markov chain $C + D$. We write $\rho = (\theta D e)/(-\beta S^{-1} e)$. For the three cases: $\rho < 1$, $\rho = 1$ and $\rho > 1$, analyze the departure process of this queue by means of the GMAP.

8.7 Consider the classical $M^X/M^X/1$ queue with the arrival rate λ, the arrival batch size distribution $\{a_k\}$ with $a = \sum\limits_{k=1}^{\infty} k a_k < +\infty$, the service rate μ, and the service batch size distribution $\{b_k\}$ with $b = \sum\limits_{k=1}^{\infty} k b_k < +\infty$. Let $\rho = (\lambda a)/(\mu b)$. For the three cases: $\rho < 1$, $\rho = 1$ and $\rho > 1$, compute the probability vectors v_1 of the first sojourn of this queue in the state set $E = \{2, 3, 4\}$.

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9 Quasi-Stationary Distributions

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Abstract In this chapter, we study quasi-stationary distributions for block-structured Markov chains with either finitely-many levels or infinitely-many levels. We derive the *RG*-factorizations for the β-discounted block-structured transition matrix. We provide conditions for the state α-classification, and derive two sets of expressions for the quasi-stationary distribution. As important examples, we analyze Markov chains of *M*/*G*/1 type, Markov chains of *GI*/*M*/1 type, Markov chains of *GI*/*G*/1 type and level-dependent QBD processes.

Keywords Stochastic models, *RG*-factorization, quasi-stationary distribution, β-discounted block-structured transition matrix, state α-classification, decay parameter.

In this chapter, we study quasi-stationary distributions for block-structured Markov chains with either finitely-many levels or infinitely-many levels. To achieve this, we give a detailed analysis for β-discounted block-structured transition matrices in terms of the *RG*-factorizations. We provide conditions for the state α-classification, and derive two sets of expressions for the quasi-stationary distribution based on the UL-type *RG*-factorization. As important examples, we analyze Markov chains of *M*/*G*/1 type, Markov chains of *GI*/*M*/1 type, Markov chains of *GI*/*G*/1 type and level-dependent QBD processes. It is worthwhile to note that this chapter provides a new theoretical interpretation for the matrix-geometric solution for Markov chains of *GI*/*M*/1 type and the matrix-iterative solution for Markov chains of *M*/*G*/1 type, e.g., see Neuts [21,22]. Also, this chapter extends the *RG*-factorizations to a more general class: The β-discounted transition matrices. The results of this chapter are useful in many applied areas such as Markov reward processes, Markov decision processes, and stochastic game theory.

This chapter is organized as follows. Section 9.1 uses the Perron-Frobenius theorem to compute the quasi-stationary distributions for a block-structured Markov chain with finitely-many levels. Based on the *RG*-factorizations, Section 9.2

provides conditions for the state α-classification and expressions for the quasi-stationary distributions of a block-structured Markov chain with infinitely-many levels. Sections 9.3 to 9.6 discuss Markov chains of *M*/*G*/1 type, Markov chains of *GI*/*M*/1 type, Markov chains of *GI*/*G*/1 type, and level-dependent QBD processes, respectively. Sections 9.7 analyzes the quasi-stationary distributions of continuous-time Markov chains. Sections 9.8 and 9.9 provide two important applications: Determining the decay parameter for the GPH distribution, and tailed analysis for the QBD processes with infinitely-many phases. Finally, Section 9.10 summarizes notes for the references related to the results of this chapter.

Throughout this chapter, the block-structured transition probability matrix $P=(p_{(i,r),(j,s)})$, given in Eq. (2.1), is assumed to be irreducible. Let α be the radius of convergence for the matrix P. Then

$$\alpha=\sup\left\{z\geqslant 1:\sum_{n=0}^{\infty}z^n p_{(i,r),(j,s)}^{(n)}<\infty\right\},$$

where $p_{(i,r),(j,s)}^{(n)}$ is the n-step transition probability of the Markov chain P from state (i,r) to state (j,s). Note that the radius α of convergence is independent of states (i,r) and (j,s). It is clear that $\alpha=1$ if P is recurrent, while $\alpha>1$ if P is transient.

When P is transient, the states of P can be further classified as α-recurrent or α-transient according to that $\widehat{\alpha P}=\sum_{k=0}^{\infty}\alpha^k P^k$ is infinite or finite, respectively. The matrix $\widehat{\alpha P}=\sum_{k=0}^{\infty}(\alpha P)^k$ is referred as the fundamental matrix of αP. If P is α-recurrent, either $\lim_{n\to\infty}\alpha^n p_{(i,r),(j,s)}^{(n)}>0$ for all states (i,r) and (j,s), or $\lim_{n\to\infty}\alpha^n p_{(i,r),(j,s)}^{(n)}=0$ for all states (i,r) and (j,s). In the former case, P is called α-positive recurrent and in the latter case, α-null recurrent.

For $1<\beta\leqslant\alpha$, a nonnegative non-zero row vector $\pi(\beta)$ is said to be the quasi-stationary distribution of P if $\pi(\beta)=\pi(\beta)\beta P$ and $\pi(\beta)e=1$, where e is a column vector of ones with suitable size. This chapter focuses on the following three issues:

(1) How to determine the radius α of convergence,

(2) how to provide conditions for the state α-classification if $\alpha>1$, and

(3) how to express the quasi-stationary distribution $\pi(\beta)$ for $1<\beta\leqslant\alpha$.

9.1 Finitely-Many Levels

In this section, we consider an irreducible block-structured Markov chain with finitely-many levels, provide conditions for the state α-classification and give expression for the quasi-stationary distribution in terms of the *RG*-factorizations.

We consider an irreducible block-structured Markov chain with finitely-many levels whose transition probability matrix is given by

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,M} \\ P_{1,0} & P_{1,1} & \cdots & P_{1,M} \\ \vdots & \vdots & & \vdots \\ P_{M,0} & P_{M,1} & \cdots & P_{M,M} \end{pmatrix}$$

Then the β-discounted transition matrix βP is given by

$$\beta P = \begin{pmatrix} \beta P_{0,0} & \beta P_{0,1} & \cdots & \beta P_{0,M} \\ \beta P_{1,0} & \beta P_{1,1} & \cdots & \beta P_{1,M} \\ \vdots & \vdots & & \vdots \\ \beta P_{M,0} & \beta P_{M,1} & \cdots & \beta P_{M,M} \end{pmatrix} \tag{9.1}$$

Similar to the analysis in Section 2.6, for the discounted transition matrix βP we can define the *R*-, *U*- and *G*-measures which lead to the *RG*-factorizations.

9.1.1 The UL-Type *RG*-Factorization

For $0 \leqslant i, j \leqslant$ k and $0 \leqslant k \leqslant M$, it is clear from Section 2.6 that

$$P_{i,j}^{[\leqslant k]}(\beta) = \beta P_{i,j} + \sum_{n=k+1}^{M} P_{i,n}^{[\leqslant n]}(\beta)\{I - P_{n,n}^{[\leqslant n]}(\beta)\}^{-1} P_{n,j}^{[\leqslant n]}(\beta).$$

Note that $P_{i,j}^{[\leqslant M]}(\beta) = \beta P_{i,j}$ and $P_{i,j}^{[\leqslant 0]}(\beta) = P_{i,j}^{[0]}(\beta)$.

Let

$$\Psi_n(\beta) = P_{n,n}^{[\leqslant n]}(\beta), \quad 0 \leqslant n \leqslant M,$$

$$R_{i,j}(\beta) = P_{i,j}^{[\leqslant j]}(\beta)[I - \Psi_j(\beta)]^{-1}, \quad 0 \leqslant i < j \leqslant M$$

and

$$G_{i,j}(\beta) = [I - \Psi_i(\beta)]^{-1} P_{i,j}^{[\leqslant i]}(\beta), \quad 0 \leqslant j < i \leqslant M.$$

Then the UL-type *RG*-factorization is given by

$$I - \beta P = [I - R_U(\beta)][I - \Psi_D(\beta)][I - G_L(\beta)],$$

where

$$R_U(\beta)=\begin{pmatrix} 0 & R_{0,1}(\beta) & R_{0,2}(\beta) & R_{0,3}(\beta) & \cdots & R_{0,M-1}(\beta) & R_{0,M}(\beta) \\ & 0 & R_{1,2}(\beta) & R_{1,3}(\beta) & \cdots & R_{1,M-1}(\beta) & R_{1,M}(\beta) \\ & & 0 & R_{2,3}(\beta) & \cdots & R_{2,M-1}(\beta) & R_{2,M}(\beta) \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & R_{M-2,M-1}(\beta) & R_{M-2,M}(\beta) \\ & & & & & 0 & R_{M-1,M}(\beta) \\ & & & & & & 0 \end{pmatrix},$$

$$\Psi_D(\beta)=\mathrm{diag}(\Psi_0(\beta),\Psi_1(\beta),\Psi_2(\beta),\Psi_3(\beta),\ldots,\Psi_{M-1}(\beta),\Psi_M(\beta))$$

and

$$G_L(\beta)=\begin{pmatrix} 0 & & & & & & \\ G_{1,0}(\beta) & 0 & & & & & \\ G_{2,0}(\beta) & G_{2,1}(\beta) & 0 & & & & \\ G_{3,0}(\beta) & G_{3,1}(\beta) & G_{3,2}(\beta) & 0 & & & \\ G_{4,0}(\beta) & G_{4,1}(\beta) & G_{4,2}(\beta) & G_{4,3}(\beta) & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ G_{M,0}(\beta) & G_{M,1}(\beta) & G_{M,2}(\beta) & G_{M,3}(\beta) & \cdots & G_{M,M-1}(\beta) & 0 \end{pmatrix}.$$

For the UL-type *RG*-factorization, the following proposition provides properties for the diagonal block matrix $\Psi_D(\beta)$. The proof is clear, and thus is omitted here.

Proposition 9.1 (1) *If* $1\leqslant \beta<\alpha$, *then* $I-\Psi_i(\beta)$ *is invertible for* $0\leqslant i\leqslant M$.

(2) *If P is* α*-transient, then* $I-\Psi_i(\alpha)$ *is invertible for* $0\leqslant i\leqslant M$.

(3) *If* P *is* α*-recurrent, then* $I-\Psi_i(\alpha)$ *is invertible for* $1\leqslant i\leqslant M$ *but* $I-\Psi_0(\alpha)$ *is singular.*

Now, we provive a new expression for the radius α of convergence. Let

$$\Psi_0(\beta)=\beta P_{0,0}+\beta^2(P_{0,1},P_{0,2},\ldots,P_{0,M})[I-\beta P^{[\geqslant 1]}]^{-1} \cdot (P_{1,0}^{\mathrm{T}},P_{2,0}^{\mathrm{T}},\ldots,P_{M,0}^{\mathrm{T}})^{\mathrm{T}}$$

Then

$$\alpha=\sup\{\beta\geqslant 1:\det(I-\Psi_0(\beta))\neq 0\}.$$

9.1.2 The LU-Type *RG*-Factorization

For $k\leqslant i,j\leqslant M$ and $0\leqslant k\leqslant M$, it is clear from Section 2.6 that

$$P_{i,j}^{[\geqslant k+1]}(\beta)=\beta P_{i,j}+\sum_{n=0}^{k}P_{i,n}^{[\geqslant n]}(\beta)\{I-P_{n,n}^{[\geqslant n]}(\beta)\}^{-1}P_{n,j}^{[\geqslant n]}(\beta).$$

Note that $P_{i,j}^{[\geqslant M]}(\beta)=P_{i,j}^{[M]}(\beta)$ and $P_{i,j}^{[\geqslant 0]}(\beta)=\beta P_{i,j}$.

Let

$$\Phi_n(\beta)=P_{n,n}^{[\geqslant n]}(\beta), \quad 0\leqslant n\leqslant M,$$
$$\overline{R}_{i,j}(\beta)=P_{i,j}^{[\geqslant j]}(\beta)[I-\Phi_j(\beta)]^{-1}, \quad 0\leqslant j<i\leqslant M,$$

and

$$\overline{G}_{i,j}(\beta)=[I-\Phi_i(\beta)]^{-1}P_{i,j}^{[\geqslant i]}(\beta), \quad 0\leqslant i<j\leqslant M.$$

Then the LU-type *RG*-factorization is given by

$$I-\beta P=[I-\overline{R}_L(\beta)][I-\Phi_D(\beta)][I-\overline{G}_U(\beta)],$$

where

$$\overline{R}_L(\beta)=\begin{pmatrix} 0 & & & & & & \\ \overline{R}_{1,0}(\beta) & 0 & & & & & \\ \overline{R}_{2,0}(\beta) & \overline{R}_{2,1}(\beta) & 0 & & & & \\ \overline{R}_{3,0}(\beta) & \overline{R}_{3,1}(\beta) & \overline{R}_{3,2}(\beta) & 0 & & & \\ \overline{R}_{4,0}(\beta) & \overline{R}_{4,1}(\beta) & \overline{R}_{4,2}(\beta) & \overline{R}_{4,3}(\beta) & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ \overline{R}_{M,0}(\beta) & \overline{R}_{M,1}(\beta) & \overline{R}_{M,2}(\beta) & \overline{R}_{M,3}(\beta) & \cdots & \overline{R}_{M,M-1}(\beta) & 0 \end{pmatrix},$$

$$\Phi_D(\beta)=\operatorname{diag}(\Phi_0(\beta),\Phi_1(\beta),\Phi_2(\beta),\Phi_3(\beta),\ldots,\Phi_{M-1}(\beta),\Phi_M(\beta))$$

and

$$\overline{G}_U(\beta)=\begin{pmatrix} 0 & \overline{G}_{0,1}(\beta) & \overline{G}_{0,2}(\beta) & \overline{G}_{0,3}(\beta) & \cdots & \overline{G}_{0,M-1}(\beta) & \overline{G}_{0,M}(\beta) \\ & 0 & \overline{G}_{1,2}(\beta) & \overline{G}_{1,3}(\beta) & \cdots & \overline{G}_{1,M-1}(\beta) & \overline{G}_{1,M}(\beta) \\ & & 0 & \overline{G}_{2,3}(\beta) & \cdots & \overline{G}_{2,M-1}(\beta) & \overline{G}_{2,M}(\beta) \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & \overline{G}_{M-2,M-1}(\beta) & \overline{G}_{M-2,M}(\beta) \\ & & & & & 0 & \overline{G}_{M-1,M}(\beta) \\ & & & & & & 0 \end{pmatrix}.$$

For the LU-type *RG*-factorization, the following proposition provides properties for the diagonal block matrix $\Phi_D(\beta)$. The proof is clear and is omitted here.

Proposition 9.2 (1) *If* $1\leqslant \beta<\alpha$, *then* $I-\Phi_i(\beta)$ *is invertible for* $0\leqslant i\leqslant M$.

(2) *If* P *is* α*-transient, then* $I-\Phi_i(\alpha)$ *is invertible for* $0\leqslant i\leqslant M$.

(3) *If* P *is* α*-recurrent, then* $I-\Phi_i(\alpha)$ *is invertible for* $0\leqslant i\leqslant M-1$ *but*

$I-\Phi_M(\alpha)$ *is singular.*

Let

$$\Phi_M(\beta)=\beta P_{M,M}+\beta^2(P_{M,0},P_{M,1},\dots,P_{M,M-1})[I-\beta P^{[\leqslant M-1]}]^{-1}$$
$$\cdot(P_{0,M}^{\mathrm{T}},P_{1,M}^{\mathrm{T}},\dots,P_{M-1,M}^{\mathrm{T}})^{\mathrm{T}}.$$

Then

$$\alpha=\sup\{\beta\geqslant 1:\det(I-\Phi_M(\beta))\neq 0\}.$$

The following corollary provides conditions for the state α-classification of the Markov chain P. The proof is clear, and thus is omitted here.

Corollary 9.1 (1) *P is α-recurrent if and only if det $(I-\Psi_0(\alpha))=0,$ and if and only if det $(I-\Phi_M(\alpha))=0.$*

(2) *P is α-transient if and only if det $(I-\Psi_0(\alpha))\neq 0,$ and if and only if det $(I-\Phi_M(\alpha))\neq 0.$*

Since the Markov chain P is irreducible and finite-state, the condition under which P is α-recurrent indicates that P is α-positive recurrent. This is due to the fact that there does not exist the α-null recurrence for any finite-state, irreducible and α-recurrent Markov chain.

9.1.3 State α-Classification and Quasi-stationary Distribution

When $\pi(\beta)$ is the quasi-stationary distribution of the Markov chain P, then

$$\pi(\beta)(I-\beta P)=0.$$

Using the UL-type RG-factorization, we obtain

$$\pi(\beta)[I-R_U(\beta)][I-\Psi_D(\beta)][I-G_L(\beta)]=0.$$

Note that the Markov chain P is either α-positive recurrent or α-transient, we need to consider the following two cases:

Case Ⅰ $1\leqslant\beta<\alpha$ or $\beta=\alpha$ under which P is α-transient. In this case, the three matrices $I-R_U(\beta), I-\Psi_D(\beta)$ and $I-G_L(\beta)$ are all invertible, hence $\pi(\beta)=0.$ That is, the quasi-stationary distribution of P does not exist.

Case Ⅱ $\beta=\alpha$ under which P is α-positive recurrent. In this case, we write the quasi-stationary distribution $\pi(\alpha)=(\pi_0(\alpha),\pi_1(\alpha),\dots,\pi_M(\alpha)).$

Using the UL-type RG-factorization, we obtain

$$\begin{cases}\pi_0(\alpha)=\tau x_0(\alpha),\\ \pi_k(\alpha)=\displaystyle\sum_{i=0}^{k-1}\pi_i(\alpha)R_{i,k}(\alpha), \quad 1\leqslant k\leqslant M,\end{cases}\tag{9.2}$$

where $x_0(\alpha)$ is the left Perron-Frobenius eigenvector of the matrix $\Psi_0(\alpha)$ and the scalar τ is uniquely determined by $\sum_{k=0}^{M}\pi_k(\alpha)e = 1.$

By means of the LU-type RG-factorization, we obtain

$$\begin{cases}\pi_M(\alpha) = \tau x_M(\alpha), \\ \pi_k(\alpha) = \sum_{i=k+1}^{M}\pi_i(\alpha)\overline{R}_{i,k}(\alpha), \quad 0 \leqslant k \leqslant M-1,\end{cases}$$

where $x_M(\alpha)$ is the left Perron-Frobenius eigenvector of the matrix $\Psi_M(\alpha)$ and the scalar τ is uniquely determined by $\sum_{k=0}^{M}\pi_k(\alpha)e = 1.$

Remark 9.1 *If P is α-positive recurrent and $\pi(\alpha)$ is the quasi-stationary distribution of P, then $\pi(\alpha) = \pi(\alpha)\alpha P$. Thus we obtain.*

$$\gamma\pi(\alpha) = \pi(\alpha)P,$$

where $\gamma = \alpha^{-1}$ which is called the decay parameter of the Markov chain P. Since $0<\gamma<1$ if $\alpha>1$ and $\pi(\alpha)P^n = \gamma^n\pi(\alpha)$ for all $n \geqslant 1$, the decay parameter γ showes the convergence velocity of the matrix $P^n \to 0$, as $n \to \infty$.

In the rest of this chapter, we shall analyze the quasi-stationary distributions of Markov chains with infinitely-many levels, which are different from those in this section.

9.2 Infinitely-Many Levels

In this section, we consider an irreducible block-structured Markov chain with infinitely-many levels, provide conditions for the state α-classification, and give expressions for the quasi-stationary distributions based on the state α-classification and the RG-factorizations.

We consider a β-discounted transition matrix as follows:

$$\beta P = \begin{pmatrix} \beta P_{0,0} & \beta P_{0,1} & \beta P_{0,2} & \cdots \\ \beta P_{1,0} & \beta P_{1,1} & \beta P_{1,2} & \cdots \\ \beta P_{2,0} & \beta P_{2,1} & \beta P_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \tag{9.3}$$

where the transition probability matrix P is given in Eq. (2.1).

Let E be a subset of the state space Ω, and $E^c = \Omega - E$. According to the subsets E and E^c, the β-discounted transition matrix βP is partitioned as

$$\beta P = \begin{array}{c} \\ E \\ E^c \end{array} \begin{array}{c} \begin{array}{cc} E & E^c \end{array} \\ \begin{pmatrix} \beta T & \beta U \\ \beta V & \beta W \end{pmatrix} \end{array} \tag{9.4}$$

The censored β-discounted transition matrix $P^E(\beta)$ to the censored set E is given by

$$P^E(\beta) = \beta T + \beta U \widehat{W}(\beta) \beta V, \tag{9.5}$$

where $\widehat{W}(\beta) = \sum_{k=0}^{\infty} (\beta W)^k = (I - \beta W)^{-1}$. Similarly, we also have

$$P^{E^c}(\beta) = \beta W + \beta V (I - \beta T)^{-1} \beta U. \tag{9.6}$$

By using the two different censored β-discounted transition matrices $P^E(\beta)$ and $P^{E^c}(\beta)$, we can organize the UL-and LU-types RG-factorizations, respectively, both of which are described by means of the censoring invariance.

9.2.1 The UL-Type *RG*-Factorization

Let

$$P^{[\leqslant n]}(\beta) = \begin{pmatrix} \phi_{0,0}^{(n)}(\beta) & \phi_{0,1}^{(n)}(\beta) & \dots & \phi_{0,n}^{(n)}(\beta) \\ \phi_{1,0}^{(n)}(\beta) & \phi_{1,1}^{(n)}(\beta) & \dots & \phi_{1,n}^{(n)}(\beta) \\ \vdots & \vdots & & \vdots \\ \phi_{n,0}^{(n)}(\beta) & \phi_{n,1}^{(n)}(\beta) & \dots & \phi_{n,n}^{(n)}(\beta) \end{pmatrix}, \quad n \geqslant 0.$$

Then for $n \geqslant 0, 0 \leqslant i,j \leqslant n$,

$$\phi_{i,j}^{(n)}(\beta) = \beta P_{i,j} + \sum_{k=n+1}^{\infty} \phi_{i,k}^{(k)}(\beta) \sum_{l=0}^{\infty} [\phi_{k,k}^{(k)}(\beta)]^l \phi_{k,j}^{(k)}(\beta).$$

We write

$$\Psi_n(\beta) = \phi_{n,n}^{(n)}(\beta), \quad n \geqslant 0,$$

$$R_{i,j}(\beta) = \phi_{i,j}^{(j)}(\beta) \sum_{l=0}^{\infty} [\phi_{j,j}^{(j)}(\beta)]^l, \quad 0 \leqslant i < j,$$

and

$$G_{i,j}(\beta) = \sum_{l=0}^{\infty} [\phi_{i,i}^{(i)}(\beta)]^l \phi_{i,j}^{(i)}(\beta), \quad 0 \leqslant j < i.$$

Thus we obtain

$$R_{i,j}(\beta)=\phi_{i,j}^{(j)}(\beta)[I-\Psi_j(\beta)]^{-1}, \quad 0\leqslant i<j,$$
$$G_{i,j}(\beta)=[I-\Psi_i(\beta)]^{-1}\phi_{i,j}^{(i)}(\beta), \quad 0\leqslant j<i.$$

The UL-type *RG*-factorization is given by

$$I-\beta P=[I-R_U(\beta)][I-\Psi_D(\beta)][I-G_L(\beta)], \tag{9.7}$$

where

$$R_U(\beta)=\begin{pmatrix} 0 & R_{0,1}(\beta) & R_{0,2}(\beta) & R_{0,3}(\beta) & \cdots \\ & 0 & R_{1,2}(\beta) & R_{1,3}(\beta) & \cdots \\ & & 0 & R_{2,3}(\beta) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$\Psi_D(\beta)=\text{diag}\,(\Psi_0(\beta),\Psi_1(\beta),\Psi_2(\beta),\Psi_3(\beta),\dots)$$

and

$$G_L(\beta)=\begin{pmatrix} 0 & & & & \\ G_{1,0}(\beta) & 0 & & & \\ G_{2,0}(\beta) & G_{2,1}(\beta) & 0 & & \\ G_{3,0}(\beta) & G_{3,1}(\beta) & G_{3,2}(\beta) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From the UL-type *RG*-factorization, the following proposition provides properties for the matrix $\Psi_D(\beta)$. The proof is clear, and thus is omitted here.

Proposition 9.3 (1) *If* $1\leqslant\beta<\alpha$, *then* $I-\Psi_i(\beta)$ *is invertible for* $i\geqslant 0$.

(2) *If* P *is* α*-transient, then* $I-\Psi_i(\alpha)$ *is invertible for* $i\geqslant 0$.

(3) *If* P *is* α*-recurrent, then* $I-\Psi_i(\alpha)$ *is invertible for* $i\geqslant 1$ *but* $I-\Psi_0(\alpha)$ *is singular.*

Using Proposition 9.3, the following corollary for the state α-classification can be obtained.

Corollary 9.2 (1) P *is* α*-recurrent if and only if* $\det(I-\Psi_0(\alpha))=0$.

(2) P *is* α*-transient if and only if* $\det(I-\Psi_0(\alpha))\neq 0$.

9.2.2 Two Sets of Expressions

In the *RG*-factorization Eq. (9.7), the three matrices, $[I-R_U(\beta)]$, $[I-\Psi_D(\beta)]$

and $[I-G_L(\beta)]$ are associative. Corollary 1 – 9 of Kemeny, Snell and Knapp [13] indicates that they are associative with any nonnegative row vector π, which will lead to solutions for the β-invariant measure.

Lemma 9.1 *Let π be any nonnegative row vector. Then*

$$\begin{aligned}\pi(I-\beta P) &= \{\pi[I-R_U(\beta)]\}\{[I-\varPsi_D(\beta)][I-G_L(\beta)]\}\\ &= \{\pi[I-R_U(\beta)][I-\varPsi_D(\beta)]\}[I-G_L(\beta)].\end{aligned}$$

According to the two different cases for the α-recurrent and the α-transient, we derive two sets of expressions for the quasi-stationary distribution of the Markov chain P.

The first set: $\beta=\alpha$ under which the Markov chain P is α-recurrent.

In this case, det $(I-\varPsi_0(\alpha))=0$. Thus, the system of linear equations $x_0(\alpha)[I-\varPsi_0(\alpha)]=0$ and $x_0(\alpha)e=1$ must exist a nonnegative non-zero solution based on the left Perron-Frobeniusthe vector of the matrix $\varPsi_0(\alpha)$. The following theorem expresses the quasi-stationary distribution of the Markov chain P. The proof is clear by the RG-factorization.

Theorem 9.1 *If the Markov chain P is α-recurrent, then the quasi-stationary distribution $\pi(\alpha)=(\pi_0(\alpha),\pi_1(\alpha),\ldots)$ is given by*

$$\begin{cases}\pi_0(\alpha)=\tau x_0(\alpha),\\ \pi_k(\alpha)=\displaystyle\sum_{i=0}^{k-1}\pi_i(\alpha)R_{i,k}(\alpha), \quad k\geqslant 1,\end{cases}$$

where $x_0(\alpha)$ is the left Perron-Frobeniusthe vector of the matrix $\varPsi_0(\alpha)$ and the scalar τ is uniquely determined by $\sum_{k=0}^{\infty}\pi_k(\alpha)e=1$.

When the Markov chain P is α-recurrent, Theorem 9.1 provides a general expression for the quasi-stationary distribution for $\beta=\alpha$. However, if $1\leqslant\beta<\alpha$ or $\beta=\alpha$ under which the Markov chain P is α-transient, the study of the quasi-stationary distribution is different. In what follows we provide a detailed analysis for the second case.

The second set: $1\leqslant\beta<\alpha$ or $\beta=\alpha$ under which the Markov chain P is α-transient.

In this case, det $(I-\varPhi_0(\alpha))\neq 0$. Obviously, the system of linear equations $x_0(\alpha)[I-\varPhi_0(\alpha)]=0$ and $x_0(\alpha)e=1$ exists only a zero solution. Therefore, expression for the quasi-stationary distribution can not be derived by means of a similar method to Theorem 9.1.

To derive expression of the quasi-stationary distribution, we first need to solve the linear equation $y(\beta)[I-G_L(\beta)]=0$, that is

$$(y_0(\beta), y_1(\beta), \ldots)\begin{pmatrix} I & & & & \\ -G_{1,0}(\beta) & I & & & \\ -G_{2,0}(\beta) & -G_{2,1}(\beta) & I & & \\ -G_{3,0}(\beta) & -G_{3,1}(\beta) & -G_{3,2}(\beta) & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0. \qquad (9.8)$$

In general, it is not easy to provide an effective method to solve Eq. (9.8). For this reason, finding a nonnegative non-zero solution to Eq. (9.8) should be a key for expressing the quasi-stationary distribution.

As illustration, we consider some special cases such as Markov chains of *M*/*G*/1 type, Markov chains of *GI*/*M*/1 type, Markov chains of *GI*/*G*/1 type and level-dependent QBD processes. Note that Sections 9.3 to 9.6 will organize how to solve Eq. (9.8) for the special cases.

Suppose a nonnegative non-zero solution to Eq. (9.8) has been obtained. The following theorem provides the quasi-stationary distribution of the Markov chain.

Theorem 9.2 *For* $1 < \beta < \alpha$ or $\beta = \alpha$ *under which* P *is* α*-transient, if the linear equation* $y(\beta)[I - G_L(\beta)] = 0$ *exists a nonnegative non-zero solution* $y(\beta) = (y_0(\beta), y_1(\beta), \ldots)$, *then the quasi-stationary distribution* $\pi(\beta) = (\pi_0(\beta), \pi_1(\beta), \ldots)$ *is given by*

$$\begin{cases} \pi_0(\beta) = \tau y_0(\beta)[I - \Psi_0(\beta)]^{-1}, \\ \pi_k(\beta) = y_k(\beta)[I - \Psi_k(\beta)]^{-1} + \sum_{i=0}^{k-1} \pi_i(\beta) R_{i,k}(\beta), \quad k \geqslant 1, \end{cases}$$

where the scalar τ *is uniquely determined by* $\sum_{k=0}^{\infty} \pi_k(\beta) e = 1$.

Proof Since

$$\pi(\beta)[I - R_U(\beta)][I - \Psi_D(\beta)][I - G_L(\beta)] = 0$$

and

$$y(\beta)[I - G_L(\beta)] = 0,$$

we only need to solve the equation

$$(\pi_0(\beta), \pi_1(\beta), \ldots)[I - R_U(\beta)][I - \Psi_D(\beta)] = (y_0(\beta), y_1(\beta), \ldots)$$

which leads to

$$(\pi_0(\beta), \pi_1(\beta), \ldots)[I - R_U(\beta)] = (y_0(\beta), y_1(\beta), \ldots)[I - \Psi_D(\beta)]^{-1}.$$

Some simple computations can yield the desired result.

9.2.3 The LU-Type *RG*-Factorization

Let

$$P^{[\geqslant n]}(\beta)=\begin{pmatrix} \eta_{n,n}^{(n)}(\beta) & \eta_{n,n+1}^{(n)}(\beta) & \eta_{n,n+2}^{(n)}(\beta) & \cdots \\ \eta_{n+1,n}^{(n)}(\beta) & \eta_{n+1,n+1}^{(n)}(\beta) & \eta_{n+1,n+2}^{(n)}(\beta) & \cdots \\ \eta_{n+2,n}^{(n)}(\beta) & \eta_{n+2,n+1}^{(n)}(\beta) & \eta_{n+2,n+2}^{(n)}(\beta) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Then for $n \geqslant 0, i, j \geqslant n+1$,

$$\eta_{i,j}^{(n+1)}(\beta)=\beta P_{i,j}+\sum_{k=0}^{n}\eta_{i,k}^{(k)}(\beta)[I-\eta_{k,k}^{(k)}(\beta)]^{-1}\eta_{k,j}^{(k)}(\beta).$$

We write

$$\Phi_n(\beta)=\eta_{n,n}^{(n)}(\beta), \quad n \geqslant 0,$$

$$\overline{R}_{i,j}(\beta)=\eta_{i,j}^{(j)}(\beta)[I-\eta_{j,j}^{(j)}(\beta)]^{-1}, \quad 0 \leqslant j < i,$$

and

$$\overline{G}_{i,j}(\beta)=[I-\eta_{i,i}^{(i)}(\beta)]^{-1}\eta_{i,j}^{(i)}(\beta), \quad 0 \leqslant i < j.$$

The LU-type *RG*-factorization is given by

$$I-P(\beta)=[I-\overline{R}_L(\beta)][I-\Phi_D(\beta)][I-\overline{G}_U(\beta)], \tag{9.9}$$

where

$$\overline{R}_L(\beta)=\begin{pmatrix} 0 & & & & \\ \overline{R}_{1,0}(\beta) & 0 & & & \\ \overline{R}_{2,0}(\beta) & \overline{R}_{2,1}(\beta) & 0 & & \\ \overline{R}_{3,0}(\beta) & \overline{R}_{3,1}(\beta) & \overline{R}_{3,2}(\beta) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Phi_D(\beta)=\operatorname{diag}(\Phi_0(\beta),\Phi_1(\beta),\Phi_2(\beta),\Phi_3(\beta),\ldots)$$

and

$$\overline{G}_U(\beta)=\begin{pmatrix} 0 & \overline{G}_{0,1}(\beta) & \overline{G}_{0,2}(\beta) & \overline{G}_{0,3}(\beta) & \cdots \\ & 0 & \overline{G}_{1,2}(\beta) & \overline{G}_{1,3}(\beta) & \cdots \\ & & 0 & \overline{G}_{2,3}(\beta) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

From the LU-type *RG*-factorization, the following proposition provides an important property for the matrix $\Phi_D(\beta)$ which is a key to understand the LU-type *RG*-factorization.

Proposition 9.4 *If* $1 \leqslant \beta \leqslant \alpha$, *then* $I - \Phi_i(\beta)$ *is invertible for* $i \geqslant 0$.

It is interesting to express the quasi-stationary distribution $\pi(\beta) = (\pi_0(\beta), \pi_1(\beta), \ldots)$ by means of the LU-type *RG*-factorization. Since $I - \Phi_i(\beta)$ is invertible for $1 \leqslant \beta \leqslant \alpha$ and $i \geqslant 0$, it is easy to check that the equation $z(\beta)(I - \Phi_D(\beta)) \cdot (I - \overline{G}_U(\beta)) = 0$ only exists a zero solution: $z(\beta) = 0$. Therefore, in order to express the quasi-stationary distribution, we have to solve the equation $\pi(\beta)[I - \overline{R}_L(\beta)] = 0$. That is,

$$(\pi_0(\beta), \pi_1(\beta), \ldots) \begin{pmatrix} I & & & & \\ -\overline{R}_{1,0}(\beta) & I & & & \\ -\overline{R}_{2,0}(\beta) & -\overline{R}_{2,1}(\beta) & I & & \\ -\overline{R}_{3,0}(\beta) & -\overline{R}_{3,1}(\beta) & -\overline{R}_{3,2}(\beta) & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0. \qquad (9.10)$$

It is complicated to provide an effective method to solve Eq. (9.10). Therefore, the challenge is to find a nonnegative non-zero solution for Eq. (9.10) to obtain the quasi-stationary distribution. To achieve this, we consider a special case: A level-dependent Markov chain of *M*/*G*/1 type, it is easy to check that

$$\overline{R}_{i,j}(\beta) = 0, \text{ for } i \geqslant j + 2.$$

In this case, Eq. (9.10) becomes a special case as follows:

$$(\pi_0(\beta), \pi_1(\beta), \ldots) \begin{pmatrix} I & & & & \\ -\overline{R}_{1,0}(\beta) & I & & & \\ & -\overline{R}_{2,1}(\beta) & I & & \\ & & -\overline{R}_{3,2}(\beta) & I & \\ & & & \ddots & \ddots \end{pmatrix} = 0, \qquad (9.11)$$

which yields

$$\pi_k(\beta) = \pi_{k+1}(\beta)\overline{R}_{k+1,k}(\beta), \quad k \geqslant 0. \qquad (9.12)$$

It is worthwhile to note that the solution structure of Eq. (9.11) or Eq. (9.12) is important in the study of the quasi-stationary distribution of an irreducible Markov chain with infinitely-many levels.

Lemma 9.2 *If the level-dependent Markov chain of M/G/*1 *type is irreducible, then for the R-measure:* $\overline{R}_k(\beta) = \overline{R}_{k,k-1}(\beta)$ *for* $k \geqslant 1$, *there exists a sequence of* *probability vectors* $\{y_k(\beta), k \geqslant 0\}$ *and a sequence of positive scalars* $\{\eta_k, k \geqslant 0\}$ *such that*

$$y_{k+1}(\beta)\overline{R}_{k+1}(\beta) = \eta_k y_k(\beta).$$

Proof Let $y_{N,N}(\beta)$ be an arbitrary probability vector on level $N \geqslant 1$. Since the QBD process P is irreducible, each state on level k has a path to level $k-1$, which shows that each row of the matrix $\overline{R}_k(\beta)$ is non-zero and nonnegative. Hence $y_{N,N}(\beta)\overline{R}_N(\beta)$ is a convex combination of the rows of $\overline{R}_N(\beta)$. We take

$$\eta_{N-1} y_{N,N-1}(\beta) = y_{N,N}(\beta)\overline{R}_N(\beta)e.$$

Then $\eta_{N-1} > 0$ and

$$y_{N,N-1}(\beta) = \frac{1}{\eta_{N-1}} y_{N,N}(\beta)\overline{R}_N(\beta)$$

is a probability vector on level $N-1$. Proceeding inductively, we can obtain a sequence of probability vector $\{y_{N,k}(\beta), 0 \leqslant k \leqslant N-1\}$ and a sequence of positive scalars $\{\eta_{N,k}, 0 \leqslant k \leqslant N-1\}$ such that for $0 \leqslant k \leqslant N-1$,

$$y_{N,k+1}(\beta)\overline{R}_{k+1}(\beta) = \eta_{N,k} y_{N,k}(\beta).$$

By repeating the above procedure, for each $k \geqslant 0$ we can obtain a sequence of probability vector $\{y_{N,k}(\beta), N \geqslant k\}$. Since $y_{N,k}(\beta)e = 1$ for $N \geqslant k$, there exists a subsequence $y_{N_r,k}(\beta)$ such that this limit: $\lim_{r \to \infty} y_{N_r,k}(\beta) = y_k(\beta)$, exists and is a probability vector. Fix $k = k^*$, we can obtain a sequence of stochastic vectors $\{y_k(\beta), 0 \leqslant k \leqslant k^*\}$ and a sequence of positive numbers $\{\eta_k, 0 \leqslant k \leqslant k^*\}$ such that

$$y_{k+1}(\beta)\overline{R}_{k+1}(\beta) = \eta_k y_k(\beta).$$

Again by the compactness of the probability vector set, there exists a subsequence $\{N_r'\}$ of $\{N_r\}$ such that this limit: $\lim_{r \to \infty} y_{N_r', k^*+1}(\beta) = y_{k^*+1}(\beta)$, and we have $\eta_{k^*} = y_{k^*+1}(\beta)\overline{R}_{k^*+1}(\beta)e$. Therefore, by induction the desired result follows by repeating this argument infinitely-many times, which can be done by means of the Axiom of Choice. This completes the proof.

The following theorem provides expression for the quasi-stationary distribution of the level-dependent Markov chain of $M/G/1$ type.

Theorem 9.3 *If* $\sum_{k=0}^{\infty} \frac{\eta_{k+1}}{\eta_0 \eta_1 \cdots \eta_{k-1}} < +\infty$, *then the quasi-stationary distribution for* $1 \leqslant \beta \leqslant \alpha$ *is given by*

$$\pi_0(\beta) = \tau y_0(\beta)$$

and for $k \geqslant 1$,

$$\pi_k(\beta) = \frac{\tau}{\eta_0 \eta_1 \cdots \eta_{k-1}} y_k(\beta),$$

where

$$\tau = \frac{1}{\sum_{k=0}^{\infty} \frac{\eta_{k+2}}{\eta_0 \eta_1 \cdots \eta_k}}.$$

Proof Note that for $k \geqslant 0$

$$y_{k+1}(\beta) \overline{R}_{k+1}(\beta) = \eta_k y_k(\beta),$$

we obtain

$$\tau y_0(\beta) = \frac{\tau}{\eta_0} y_1(\beta) \overline{R}_1(\beta),$$

which leads to

$$\pi_0(\beta) = \pi_1(\beta) \overline{R}_1(\beta);$$

$$\frac{\tau}{\eta_0} y_1(\beta) = \frac{\tau}{\eta_0 \eta_1} y_2(\beta) \overline{R}_2(\beta),$$

which leads to

$$\pi_1(\beta) = \pi_2(\beta) \overline{R}_2(\beta).$$

We assume that when $n = k$, we have $\pi_k(\beta) = \pi_{k+1}(\beta) \overline{R}_{k+1}(\beta)$. Then when $n = k + 1$, we have

$$\begin{aligned}
\pi_{k+1}(\beta) &= \frac{\tau}{\eta_0 \eta_1 \cdots \eta_k} y_{k+1}(\beta) \\
&= \frac{\tau}{\eta_0 \eta_1 \cdots \eta_k \eta_{k+1}} y_{k+2}(\beta) \overline{R}_{k+2}(\beta) \\
&= \pi_{k+2}(\beta) \overline{R}_{k+2}(\beta).
\end{aligned}$$

Therefore, by induction it is easy to see that for any nonnegative integer n we have

$$\pi_n(\beta) = \pi_{n+1}(\beta) \overline{R}_{n+1}(\beta),$$

which leads to

$$(\pi_0(\beta), \pi_1(\beta), \ldots) \begin{pmatrix} I & & & & \\ -\overline{R}_{1,0}(\beta) & I & & & \\ & -\overline{R}_{2,1}(\beta) & I & & \\ & & -\overline{R}_{3,2}(\beta) & I & \\ & & & \ddots & \ddots \end{pmatrix} = 0.$$

Thus we obtain

$$\pi(\beta)[I-\overline{R}_L(\beta)][I-\Phi_D(\beta)][I-\overline{G}_U(\beta)]=0$$

or

$$\pi(\beta)(I-\beta P)=0.$$

Since

$$\pi(\beta)e=\left(\tau y_0(\beta),\frac{\tau}{\eta_0}y_1(\beta),\frac{\tau}{\eta_0\eta_1}y_2(\beta),\dots\right)e=1,$$

it is clear that $\pi(\beta)=\left(\tau y_0(\beta),\frac{\tau}{\eta_0}y_1(\beta),\frac{\tau}{\eta_0\eta_1}y_2(\beta),\dots\right)$ is the quasi-stationary distribution of the level-dependent Markov chain of $M/G/1$ type. This completes the proof.

9.3 Markov Chains of *M*/*G*/1 Type

In this section, we consider the quasi-stationary distribution of an irreducible Markov chain of $M/G/1$ type. The conditions for the state α-classification and the expressions for the quasi-stationary distribution are derived in detail.

We consider an irreducible aperiodic Markov chain of $M/G/1$ type whose transition matrix P is given by

$$P=\begin{pmatrix} D_1 & D_2 & D_3 & D_4 & \cdots \\ D_0 & A_1 & A_2 & A_3 & \cdots \\ & A_0 & A_1 & A_2 & \cdots \\ & & A_0 & A_1 & \cdots \\ & & & \ddots & \ddots \end{pmatrix}, \tag{9.13}$$

where D_1 is a matrix of size $m_0\times m_0$, all A_i are square matrices of finite size m, the sizes of the other block-entries are determined accordingly and all empty entries are zero.

9.3.1 The UL-Type *RG*-Factorization

We write

$$Q=\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & \cdots \\ A_0 & A_1 & A_2 & A_3 & \cdots \\ & A_0 & A_1 & A_2 & \cdots \\ & & A_0 & A_1 & \cdots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Let the fundamental matrix $\widehat{\beta Q} = \sum_{n=0}^{\infty} (\beta Q)^n = (\widehat{Q}_{i,j}(\beta))_{i,j\geqslant 1}$ and $N(\beta) = \widehat{Q}_{1,1}(\beta)$. We first define the G-measure: The matrices $G(\beta)$ and $G_{1,0}(\beta)$. It is clear from Eq. (2.29) that

$$G(\beta) = N(\beta)\beta A_0. \tag{9.14}$$

The (r,s)th entry of $G(\beta)$ can be interpreted as the total expected discounted reward with rate β induced by hitting state (i,s) upon the process entering $L_{\leqslant i}$ for the first time, given that the process starts in state $(i+1,r)$.

The following lemma expresses the matrices $\widehat{Q}_{j,1}(\beta)$ for $j \geqslant 1$, which are key to define the R-, U- and G-measures later.

Lemma 9.3

$$\widehat{Q}_{j,1}(\beta) = G(\beta)^{j-1} N(\beta), \quad j \geqslant 1.$$

Proof It follows from the skip-free-to-left property of the transition matrix βQ that

$$(\widehat{Q}_{2,1}(\beta)^{\mathrm{T}}, \widehat{Q}_{3,1}(\beta)^{\mathrm{T}}, \ldots)^{\mathrm{T}} = (N(\beta)^{\mathrm{T}}, \widehat{Q}_{2,1}(\beta)^{\mathrm{T}}, \ldots)^{\mathrm{T}} [\beta A_0 N(\beta)]^{\mathrm{T}}.$$

Using the recursive expressions and $N(\beta)\beta A_0 = G(\beta)$ repeatedly, we can obtain the desired result.

Though the matrix $G(\beta)$ is defined as the product of $N(\beta)$ and βA_0, we usually first compute $G(\beta)$ and then determine $N(\beta)$ in terms of $G(\beta)$. The following theorem expresses $N(\beta)$ in terms of $G(\beta)$ and $N_0(\beta)$, the (1,1)st block-entry of the matrix $\widehat{\beta P} = \sum_{k=0}^{\infty} (\beta P)^k$, in terms of $N(\beta)$.

Theorem 9.4 *For the transition matrix P of $M/G/1$ type, we have*

(1) *the matrix $N(\beta)$ can be expressed as*

$$N(\beta) = \left[I - \sum_{k=1}^{\infty} \beta A_k G(\beta)^{k-1} \right]^{-1}, \tag{9.15}$$

or $N(\beta)$ is the fundamental matrix for $\Psi(\beta) = \beta \sum_{k=1}^{\infty} A_k G(\beta)^{k-1}$;

(2) *the matrix $N_0(\beta)$ can be expressed as*

$$N_0(\beta) = [I - \Psi_0(\beta)]^{-1}, \tag{9.16}$$

where

$$\Psi_0(\beta) = \beta D_1 + \sum_{k=1}^{\infty} \beta D_{k+1} G(\beta)^{k-1} N(\beta) \beta D_0, \tag{9.17}$$

or $N_0(\beta)$ *is the fundamental matrix for* $\Psi_0(\beta)$.

Proof We only prove (1), while (2) can be proved similarly.

Let

$$\mathcal{P}=\begin{pmatrix} T & H \\ L & Q \end{pmatrix},$$

where

$$T=A_1,\quad H=(A_2,A_3,A_4,\ldots),\quad L=(A_0^{\mathrm{T}},0,0,\ldots)^{\mathrm{T}}.$$

Then it is clear from $\mathcal{P}=Q$, hence $N(\beta)=\widehat{Q}_{1,1}(\beta)$ is the fundamental matrix for $\beta T+\beta H\widehat{\beta Q}\beta L$. Thus we have

$$\begin{aligned}\beta T+\beta H\widehat{\beta Q}\beta L &= \beta A_1+\beta H(\widehat{Q}_{1,1}(\beta),\widehat{Q}_{2,1}(\beta),\ldots)^{\mathrm{T}}\beta A_0 \\ &= \beta A_1+\sum_{k=2}^{\infty}\beta A_k\widehat{Q}_{k-1,1}(\beta)\beta A_0.\end{aligned}$$

Note that $N(\beta)\beta A_0=G(\beta)$ and using Lemma 9.3, this easily completes the proof.

It follows from Eq. (9.15) that the matrix $G(\beta)$ satisfies the following nonlinear matrix equation:

$$G(\beta)=\sum_{k=0}^{\infty}\beta A_k G(\beta)^k. \tag{9.18}$$

At the same time, $G(\beta)$ is the minimal nonnegative solution to Eq. (9.18).

The G-measure for the matrix βP of $M/G/1$ type consists of two matrices, $G(\beta)$ defined in Eq. (9.15) and $G_{1,0}(\beta)$ defined by

$$G_{1,0}(\beta)=\widehat{Q}_{1,1}(\beta)\beta D_0=N(\beta)\beta D_0. \tag{9.19}$$

The (r, s)th entry of $G_{1,0}(\beta)$ can be interpreted as the total expected discounted reward with rate β induced by hitting state $(0, s)$ upon the process entering level 0 for the first time, given that the process starts in state $(1, r)$.

Consider the fundamental matrix $\widehat{\beta Q}$ of βQ. Let the first block-column of $\widehat{\beta Q}$ be $(\widehat{Q}_{1,1}(\beta)^{\mathrm{T}},\widehat{Q}_{2,1}(\beta)^{\mathrm{T}},\ldots)^{\mathrm{T}}$. The R-measure for the matrix βP in Eq. (9.13) consists of two sequences of matrices $R_{0,k}(\beta)$ and $R_k(\beta)$ for $k\geqslant 1$, defined by

$$R_{0,k}(\beta)=\sum_{l=1}^{\infty}\beta D_{k+l}\widehat{Q}_{l,1}(\beta) \tag{9.20}$$

and

$$R_k(\beta)=\sum_{l=1}^{\infty}\beta A_{k+l}\widehat{Q}_{l,1}(\beta). \tag{9.21}$$

Applying Lemma 9.3 to Eq. (9.20) and Eq. (9.21), the R-measure can then be expressed as for $k \geqslant 1$,

$$R_{0,k}(\beta) = \sum_{i=1}^{\infty} \beta D_{k+i} G(\beta)^{i-1} N(\beta) \tag{9.22}$$

and

$$R_k(\beta) = \sum_{i=1}^{\infty} \beta A_{k+i} G(\beta)^{i-1} N(\beta). \tag{9.23}$$

For the transition matrix P of $M/G/1$ type in Eq. (9.13), the UL-type RG-factorization is given by

$$I - \beta P = [I - R_U(\beta)][I - \Psi_D(\beta)][I - G_L(\beta)], \tag{9.24}$$

where

$$[I - R_U(\beta)] = \begin{pmatrix} I & -R_{0,1}(\beta) & -R_{0,2}(\beta) & -R_{0,3}(\beta) & \cdots \\ & I & -R_1(\beta) & -R_2(\beta) & \cdots \\ & & I & -R_1(\beta) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$\Psi_D(\beta) = \mathrm{diag}(\Psi_0(\beta), \Psi(\beta), \Psi(\beta), \ldots)$$

and

$$[I - G_L(\beta)] = \begin{pmatrix} I & & & & \\ -G_{1,0}(\beta) & I & & & \\ & -G(\beta) & I & & \\ & & -G(\beta) & I & \\ & & & \ddots & \ddots \end{pmatrix}.$$

9.3.2 The State α-Classification

For the Markov chain of $M/G/1$ type, the determination of the radius α of convergence and the conditions on the state α-classification are based on the combination of the state α-classification for the corresponding matrix without boundaries and the special treatment of the boundary. For convenience of description, here we state two results in Kijima [14] again.

Let

$$A^*(z) = \sum_{k=0}^{\infty} A_k z^k, \quad 0 \leqslant z < z_0. \tag{9.25}$$

We denote by $\chi(z)$ the Perron-Frobenius eigenvalue of the nonnegative matrix $A^*(z)$ for $0 \leqslant z < z_0$.

Lemma 9.4 *If* $z_0>1$, *then there always exists the unique* γ *such that* $\chi(z) \geqslant \gamma z$ *for all* $0<z<z_0$, *and there exists some* θ *with* $0<\theta \leqslant z_0$ *such that* $\chi(\theta)=\theta\gamma$. *If* $\theta = z_0$, *then* $\gamma = \chi(z_0)/z_0$. *Otherwise,* γ *and* θ *can be determined by solving the system of equations*

$$\begin{cases} \chi(\theta) = \gamma\theta, \\ \chi'(\theta) = \gamma. \end{cases} \tag{9.26}$$

By using this lemma, Kijima [14] was able to show the following result.

Theorem 9.5 *For the transition matrix P of M/G/*1 *type without boundaries* ($D_k = A_k$ *for all* $k \geqslant 0$), *if* γ *is determined in Lemma 9.4, then the radius* $\overline{\alpha}$ *of convergence of P satisfies* $\overline{\alpha} = 1/\gamma$ *and P is* $\overline{\alpha}$*-transient.*

In fact, θ is the maximal eigenvalue of the matrix $G(\overline{\alpha})$. The transition matrix of *M*/*G*/1 type without boundaries is always $\overline{\alpha}$-transient, However, the transition matrix of *M*/*G*/1 type with boundaries can be either α-transient, α-positive recurrent or α-null recurrent.

For the transition matrix *P* of *M*/*G*/1 type given in Eq. (9.13) with boundaries, we can perform spectral analysis on the censored matrix $\Psi_0(\beta)$ to level 0 to obtain conditions on the state α-classifications and a determination of the radius of convergence. However, it seems more convenient to reach this goal by considering the relationship between the censored matrix $\Psi_0(\beta)$ and its fundamental matrix $N_0(\beta)$.

Let $u_0(\beta)$ and $n_0(\beta)$ be the maximal eigenvalues of the censored matrix $\Psi_0(\beta)$ and its fundamental matrix $N_0(\beta)$, respectively. Then $n_0(\beta)=1/[1-u_0(\beta)]$. It follows from results of linear algebra that the first two statements of the following lemma are true, for example, Seneta [32], and the other two follow from the definitions of the radius of convergence and $N_0(\beta)$.

Lemma 9.5 *Let* $\overline{\alpha}$ *and* α *be the radiu of convergence of Q and P, respectively. In* (1) *and* (2), *assume* $0 < \beta \leqslant \overline{\alpha}$.

(1) *Both* $u_0(\beta)$ *and* $n_0(\beta)$ *are strictly increasing in* β, *and*

(2) $u_0(\beta)<1$ *if and only if* $N_0(\beta) < \infty$.

(3) $N_0(\beta) < \infty$ *if* $\beta < \alpha$ *and* $N_0(\beta) = \infty$ *if* $\beta > \alpha$.

(4) $\alpha \leqslant \overline{\alpha}$.

The state α-classification is characterized by the following conditions.

Theorem 9.6

(1) *If for all* $0 < \beta \leqslant \overline{\alpha}, u_0(\beta) < 1$, *then* $N_0(\overline{\alpha}) < \infty$ *and* $\alpha = \overline{\alpha}$. *Therefore, P is* α*-transient;*

(2) *If there exists a* β^* *with* $0 < \beta^* \leqslant \overline{\alpha}$ *such that* $u_0(\beta^*) = 1$, *then* $\alpha = \beta^*$

and $N_0(\alpha) = \infty$. *Therefore, P is* α*-recurrent.*

Proof Based on the facts: $n_0(\beta) = 1/[1 - u_0(\beta)]$ and $n_0(\beta) < \infty$ if and only if $N_0(\beta) < \infty$, we discuss the following two cases:

Case Ⅰ There exists no solution to $1 - u_0(\beta) = 0$ for $0 < \beta \leqslant \overline{\alpha}$. In this case, $n_0(\overline{\alpha}) < \infty$. Hence $N_0(\overline{\alpha}) < \infty$. Therefore, $\alpha \geqslant \overline{\alpha}$. This, together with (4) of Lemma 9.5, implies $\alpha = \overline{\alpha}$. Hence, P is α-transient.

Case Ⅱ There exists a solution β^* to $1 - u_0(\beta) = 0$ for $0 < \beta^* \leqslant \overline{\alpha}$. In this case, $n_0(\beta^*) = \infty$, hence there exists at least one infinite entry of $N_0(\beta)$. This leads to $\alpha = \beta^* \leqslant \overline{\alpha}$. Therefore, P is α-recurrent. This completes the proof.

9.3.3 Two Sets of Expressions

Now, we use the UL-type *RG*-factorization to express the quasi-stationary distributions for the transition matrix P of $M/G/1$ type with boundary. We present two sets of expressions, one for an α-recurrent matrix with $\beta = \alpha$ and the other for all the other cases.

Case Ⅰ $\beta = \alpha$ under which P is α-recurrent.

In this case, the quasi-stationary distribution is expressed in the following theorem.

Theorem 9.7 *If P is* α*-recurrent, then the quasi-stationary distribution is given by*

$$\pi_0(\alpha) = \kappa x_0$$

and

$$\pi_k(\alpha) = \pi_0(\alpha) R_{0,k}(\alpha) + \sum_{i=1}^{k-1} \pi_i(\alpha) R_{k-i}(\alpha), \quad k \geqslant 1,$$

where x_0 *is the left Perron-Frobeniusthe vector of the matrix* $\Psi_0(\alpha)$ *and the constant* κ *satisfies* $\sum_{k=0}^{\infty} \pi_k(\alpha) e = 1$.

Proof We solve $\pi(\alpha)(I - \alpha P) = 0$ by two steps. In the first step, we write

$$x = \pi(\alpha)[I - R_U(\alpha)]. \tag{9.27}$$

Then

$$x[I - \Psi_D(\alpha)][I - G_L(\alpha)] = 0, \tag{9.28}$$

which is equivalent to

$$x_0[I - \Psi_0(\alpha)] - x_1[I - \Psi(\alpha)]G_{1,0}(\alpha) = 0,$$
$$x_k[I - \Psi(\alpha)] - x_{k+1}[I - \Psi(\alpha)]G(\alpha) = 0, \quad k \geqslant 1.$$

Since P is α-recurrent, it follows from Theorem 9.6 that the maximal eigenvalue of $\Psi_0(\alpha)$ is $u_0(\alpha)=1$. Therefore, for the censored discounted transition matrix $\Psi_0(\alpha)$, there exists a nonnegative non-zero x_0 such that

$$x_0[I-\Psi_0(\alpha)]=0.$$

Hence, $(\kappa x_0, 0, 0, \ldots)$ is a solution to Eq. (9.28).

It follows from Eq. (9.27) that

$$\kappa x_0 = \pi_0(\alpha),$$
$$-\pi_0(\alpha)R_{0,k}(\alpha)-\sum_{i=1}^{k-1}\pi_i(\alpha)R_{k-i}(\alpha)+\pi_k(\alpha)=0, \quad k \geqslant 1.$$

Some simple computations lead to the desired results. This completes the proof.

Case Ⅱ $\beta < \alpha$ for the α-recurrent or $\beta \leqslant \alpha$ for the α-transient.

In this case, we need to construct a nonnegative non-zero solution y to $y[I-G_L(\beta)]=0$, which is described as the following lemma.

Lemma 9.6 *For every* $0<\beta\leqslant\alpha$, *there exist a* $\theta_\beta>0$ *and a nonnegative non-zero vector z such that*

$$zG(\beta)=\theta_\beta z.$$

Proof Since $G(\beta)\geqslant 0$, the maximal eigenvalue θ_β of $G(\beta)$ is nonnegative. If $\theta_\beta>0$, then the lemma is proved by choosing z as the left eigenvector of $G(\beta)$ associated with θ_β.

It follows from Neuts [22], by using the irreducibility of the Markov chain P, that $\theta_1>0$. Therefore, $\theta_\beta\geqslant\theta_1>0$ for all $\beta\geqslant 1$ since $G(\beta)$ is increasing in β.

For $0<\beta<1$, the proof also relies on the irreducibility of P. Suppose that there were an s with $0<s<1$ such that $\theta_s=0$. Then, $\theta_\beta=0$ for all $0<\beta\leqslant s$. Therefore, all the eigenvalues of $G(\beta)$, when $0<\beta\leqslant s$, would be zero according to the Perron-Frobenius theorem for nonnegative matrices. It follows from the Cayley-Hamilton theorem that

$$G^m(\beta)=0, \quad 0<\beta\leqslant s, \tag{9.29}$$

where m is the size of matrix $G(\beta)$. On the other hand, according to the probabilistic interpretation of $G^m(\beta)$ and the assumption of irreducibility on P, $G^m(\beta)\neq 0$, which contradicts (9.29). This completes the proof.

Let

$$y=\pi(\alpha)[I-R_U(\beta)][I-\Psi_D(\beta)]. \tag{9.30}$$

This is equivalent to

$$y_0 = \pi_0[I - \Psi_0(\beta)],$$
$$y_1 = [-\pi_0 R_{0,1}(\beta) + \pi_1][I - \Psi(\beta)],$$
$$y_k = \left[-\pi_0 R_{0,k}(\beta) - \sum_{i=1}^{k-1} \pi_i R_{k-i}(\beta) + \pi_k\right][I - \Psi(\beta)], \quad k \geqslant 2.$$

Since both $[I - \Psi_0(\beta)]$ and $[I - \Psi(\beta)]$ are invertible in this case, we can express π_k in terms of y_k as follows:

$$\pi_0 = y_0[I - \Psi_0(\beta)]^{-1}, \tag{9.31}$$

$$\pi_1 = \pi_0 R_{0,1}(\beta) + y_1[I - \Psi(\beta)]^{-1}, \tag{9.32}$$

$$\pi_k = \pi_0 R_{0,k}(\beta) + \sum_{i=1}^{k-1} \pi_i R_{k-i}(\beta) + \pi_k[I - \Psi(\beta)]^{-1}, \quad k \geqslant 2. \tag{9.33}$$

Now, we solve

$$y[I - G_L(\beta)] = 0 \tag{9.34}$$

for nonnegative non-zero y. If such a solution exists, then π calculated by Eq. (9.31), Eq. (9.32) and Eq. (9.33) is nonnegative and non-zero. Equation (9.34) is equivalent to

$$y_0 - y_1 G_{1,0}(\beta) = 0,$$
$$y_k - y_{k+1} G(\beta) = 0, \quad k \geqslant 1.$$

By using Lemma 9.6 and letting $y_0 = zG_{1,0}(\beta)$, we can easily check that $y = (y_0, z, z/\theta_\beta, z/\theta_\beta^2, ...)$ is a nonnegative non-zero solution to Eq. (9.34). Substituting y into Eq. (9.31), Eq. (9.32) and Eq. (9.33), the quasi-stationary distribution is given in the following theorem.

Theorem 9.8 *For* $1 < \beta < \alpha$ *or* $\beta = \alpha$ *under which P is* α*-transient, the quasi-stationary distribution of P is given by*

$$\pi_0 = y_0 N_0(\beta)$$

and for $k \geqslant 1$,

$$\pi_k = \frac{z}{\theta_\beta^{k-1}} \left\{ N(\beta) + \sum_{i=1}^{k-1} G(\beta)^i N(\beta) \sum_{\substack{0 \leqslant j_1 \leqslant \ldots \leqslant j_i \leqslant i \\ \sum_t j_t = i}} R_{j_1}(\beta) R_{j_2}(\beta) \ldots R_{j_i}(\beta) \right.$$
$$\left. + G(\beta)^{k-1} G_{1,0}(\beta) N_0(\beta) \sum_{i=1}^{k} R_{0,i}(\beta) \sum_{\substack{0 \leqslant j_1 \leqslant \ldots \leqslant j_{k-i} \leqslant k-i \\ \sum_t j_t = k-i}} R_{j_1}(\beta) R_{j_2}(\beta) \ldots R_{j_{k-1}}(\beta) \right\},$$

where $R_0(\beta) = I$.

9.3.4 Conditions for α-Positive Recurrence

The expressions for the quasi-stationary distribution are only based on the α-recurrence and the α-transience. However, it is necessary to classify the α-recurrence as the α-positive recurrence or the α-null recurrence. Now, we provide conditions for classifying the α-recurrence as either an α-positive recurrence or an α-null recurrence.

For simplicity of description, we assume that the matrix $A^*(1)$ is irreducible and stochastic. Besides the quasi-stationary distribution π provided in Theorem 9.7, we also need to similarly express the α-invariant vector v according to the system of equations $[I-\alpha P]v=0$ and $\pi v=1$. It is easy to check by the RG-factorization that

$$v_0 = w_0, \quad v_k = G(\alpha)^{k-1} G_{1,0}(\alpha) w_0, \quad k \geqslant 1, \tag{9.35}$$

where w_0 is the unique, up to a multiplication of a positive constant, solution of $[I-\Psi_0(\alpha)]w_0 = 0$.

It follows from Theorem 9.7 that

$$\Pi^*(z) = x_0 R_0^*(z)[I - R^*(z)]^{-1} = x_0 R_0^*(z) \sum_{n=0}^{\infty} [R^*(z)]^n,$$

which gives

$$\pi_k = x_0 R_{0,k}(\alpha) \circledast \sum_{n=0}^{\infty} R_k(\alpha)^{\circledast n}, \quad k \geqslant 1. \tag{9.36}$$

It follows from Eq. (9.35) and Eq. (9.36) that

$$\sum_{i=0}^{\infty} \pi_i v_i = x_0 v_0 + x_0 \sum_{k=1}^{\infty} \left[R_{0,k}(\alpha) * \sum_{n=0}^{\infty} R_k(\alpha)^{*n} \right] G(\alpha)^{k-1} G_{1,0}(\alpha) v_0. \tag{9.37}$$

Clearly, $\sum_{i=0}^{\infty} \pi_i v_i < \infty$ if and only if

$$\sum_{k=1}^{\infty} \left[R_{0,k}(\alpha) * \sum_{n=0}^{\infty} R_k(\alpha)^{*n} \right] G(\alpha)^{k-1} < \infty. \tag{9.38}$$

Let g_α and $H(\alpha)$ be the maximal eigenvalue and the associated right eigenvector of $G(\alpha)$, respectively. Since $A^*(1)$ is irreducible, we have $H(\alpha)>0$. It follows from Eq. (9.38) that

$$\sum_{k=1}^{\infty} \left[R_{0,k}(\alpha) * \sum_{n=0}^{\infty} R_k(\alpha)^{*n} \right] G(\alpha)^{k-1} H(\alpha) = \frac{1}{g_\alpha} R_0^*(g_\alpha) \sum_{n=0}^{\infty} [R^*(g_\alpha)]^n H(\alpha).$$

Then, Eq. (9.38) is true if and only if, (1) $R_0^*(g_\alpha) < \infty$, (2) $R^*(g_\alpha) < \infty$, and (3) the matrix $I - R^*(g_\alpha)$ is invertible.

The following lemma provides the conditions under which, (1) $R_0^*(g_\alpha) < \infty$ and (2) $R^*(g_\alpha) < \infty$.

Lemma 9.7 (1) $R_0^*(g_\alpha) < \infty$ *if and only if* $\sum_{k=1}^{\infty} kD_k G(\alpha)^{k-1} < \infty$.

(2) $R^*(g_\alpha) < \infty$ *if and only if* $\sum_{k=1}^{\infty} kA_k G(\alpha)^{k-1} < \infty$.

Proof We only prove (1), while (2) can be proved similarly.

It is clear that

$$R_0^*(g_\alpha) = \sum_{k=1}^{\infty} g_\alpha^k R_{0,k}(\alpha) = \sum_{k=1}^{\infty} g_\alpha^k \sum_{i=1}^{\infty} \alpha D_{k+i} G(\alpha)^{i-1} N(\alpha).$$

Hence we obtain

$$R_0^*(g_\alpha) N(\alpha)^{-1} H(\alpha) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \alpha g_\alpha^{k+i-1} D_{k+i} H(\alpha) = \alpha \sum_{k=1}^{\infty} kg(\alpha)^{k-1} D_k H(\alpha),$$

which illustrates that $R_0^*(g_\alpha) < \infty$ if and only if $\sum_{k=1}^{\infty} kg(\alpha)^{k-1} D_k < \infty$, and if and only if $\sum_{k=1}^{\infty} kD_k G(\alpha)^{k-1} < \infty$. This completes the proof.

In what follows we provide a condition under which the matrix $I - R^*(g_\alpha)$ is invertible. To this end, we need to use the *RG*-factorization for the repeated row as follows:

$$zI - \alpha A^*(z) = [I - R^*(z)][I - \Psi(\alpha)][zI - G(\alpha)]. \tag{9.39}$$

Let $\chi(z)$ be the maximal eigenvalue of the matrix $A^*(z)$ for $z>0$. It is clear that property 7 about $\chi(z)$ in Bean, Pollett and Taylor [3] (p. 393 – 394) also holds for the transition matrix of *M*/*G*/1 type. Note that the matrix $A^*(1)$ is irreducible and stochastic, then the equation $z = \alpha\chi(z)$ has two different roots in $(0,z_0)$ if $1 \leqslant \alpha < \overline{\alpha}$, and one root repeated twice in $(0,z_0)$ if $\alpha = \overline{\alpha}$, where z_0 is the radius of convergence of $A^*(z)$. Furthermore, the equation det $(zI - \alpha A^*(z)) = 0$ has two different roots in $(0,z_0)$ if $1 \leqslant \alpha < \overline{\alpha}$, and one root repeated twice in $(0,z_0)$ if $\alpha = \overline{\alpha}$.

Lemma 9.8 (1) *If* $\alpha < \overline{\alpha}$, *then the matrix* $I - R^*(g_\alpha)$ *is invertible.*

(2) *If* $\alpha = \overline{\alpha}$, *then the matrix* $I - R^*(g_\alpha)$ *is singular.*

Proof (1) If $\alpha < \overline{\alpha}$, then the equation det $(zI - \alpha A^*(z)) = 0$ has two different roots in $(0,z_0)$. Since

$$\{0<z<z_0 : \det(zI-\alpha A^*(z))=0\} = \{0<z<z_0 : \det(I-R^*(z))=0\} \\ \bigcup \{0<z<z_0 : \det(zI-G(\alpha))=0\}$$

and $z=g_\alpha$ is a positive root to the equation $\det(zI-G(\alpha))=0$, it is not a positive root to the equation $\det(I-R^*(z))=0.$ Thus, $I-R^*(g_a)$ is invertible.

(2) If $\alpha=\overline{\alpha}$, then the equation $\det(zI-\alpha A^*(z))=0$ has one root repeated twice in $(0,z_0)$. Since $z=g_\alpha$ is a positive and simple root to the equation $\det(zI-G(\alpha))=0$, it must be a positive and simple root to the equation $\det(I-R^*(z))=0.$ Thus, $I-R^*(g_\alpha)$ is singular.

This completes the proof.

In the proof of Theorem 9.9, we need a result in Theorem 6.4 of Seneta (1981), which is restated in the following lemma in the block-partitioned form.

Lemma 9.9 *Suppose* $\pi=(\pi_0,\pi_1,\pi_2,\ldots)$ *is a* β*-invariant measure and* $v=(v_0^{\mathrm{T}},v_1^{\mathrm{T}},v_2^{\mathrm{T}},\ldots)^{\mathrm{T}}$ *is a* β*-invariant vector of the transition matrix P, partitioned according to levels. Then, P is* α*-positive recurrent if* $\pi v=\sum_{k=0}^{\infty}\pi_i v_i<+\infty,$ *in which case* $\beta=\alpha,\pi$ *is (a multiple of) the unique* α*-invariant measure of P and v is (a multiple of) the unique* α*-invariant vector of P. Conversely, if P is* α*-positive recurrent, and* π *and v are respectively an invariant measure and vector, then* $\pi v<+\infty$.

For an α-recurrent P, the following theorem further provides conditions under which P is α-positive recurrent or α-null recurrent. The proof follows the above discussions.

Theorem 9.9 *If* $\sum_{k=1}^{\infty}kD_kG(\alpha)^{k-1}<\infty, \sum_{k=1}^{\infty}kA_kG(\alpha)^{k-1}<\infty$ and $\alpha<\overline{\alpha}$, *then the* α*-recurrent Markov chain is* α*-positive recurrent; otherwise, it is* α*-null recurrent.*

Remark 9.2 *If* $\alpha=1$ and $\overline{\alpha}>1$, *then the three conditions in Theorem 9.9 are the same conditions as those in Remark b of Neuts [22] (p. 140 – 141). This is because in this situation,* $G(1)$ *is stochastic. Therefore,* (1) $\sum_{k=1}^{\infty}kD_kG(1)^{k-1}<\infty$ *if and only if* $\sum_{k=1}^{\infty}kD_k<\infty$; (2) $\sum_{k=1}^{\infty}kA_kG(1)^{k-1}<\infty$ *if and only if* $\sum_{k=1}^{\infty}kA_k<\infty$; *and* (3) *for the recurrent matrix,* $\alpha<\overline{\alpha}$ *if and only if* $I-R^*(1)$ *is invertible, which is equivalent to* $\rho<1$.

9.4 Markov Chains of *GI*/*M*/1 Type

In this section, we consider the quasi-stationary distribution of an irreducible Markov chain of *GI*/*G*/1 type, and derive conditions for the state α-classification and expressions for the quasi-stationary distribution.

We consider an irreducible Markov chain of *GI*/*M*/1 type whose transition matrix is given by

$$P=\begin{pmatrix} D_1 & D_0 & & & \\ D_2 & A_1 & A_0 & & \\ D_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{9.40}$$

where D_1 is a square matrix of finite size m_0, all A_i are square matrices of finite size m, the sizes of the other block-entries are determined accordingly and all empty entries are zero.

Let $A=\sum_{k=0}^{\infty} A_k$ and

$$Q=\begin{pmatrix} A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{9.41}$$

Let $\bar{\alpha}$ be the radius of convergence of Q. It is clear that $\alpha \leqslant \bar{\alpha}$.

Let $\widehat{\beta Q}=(\widehat{Q}_{i,j}(\beta))_{i,j\geqslant 1}$ be the fundamental matrix for the matrix βQ and write $N(\beta)=\widehat{Q}_{1,1}(\beta)$. Also, write the (1,1)st block-entry in $\widehat{\beta P}$ as $N_0(\beta)$.

The R-measure for βP of *GI*/*M*/1 type consists of two matrices, $R(\beta)$ and $R_{0,1}(\beta)$ defined by

$$R(\beta)=\beta A_0 N(\beta) \tag{9.42}$$

and

$$R_{0,1}(\beta)=\beta D_0 N(\beta). \tag{9.43}$$

Clearly, $R(\beta)$ and $R_{0,1}(\beta)$ are matrices of size $m\times m$ and $m_0\times m$, respectively.

The G-measure for the matrix βP consists of two sequences of matrices $G_{k,0}(\beta)$ and $G_k(\beta)$ for $k=1,2,\ldots$, which are of size $m\times m_0$ and $m\times m$ respectively, and is defined by

$$G_{k,0}(\beta)=\sum_{i=1}^{\infty}\widehat{Q}_{1,i}(\beta)\beta D_{k+i} \tag{9.44}$$

and

$$G_k(\beta)=\sum_{i=1}^{\infty}\widehat{Q}_{1,i}(\beta)\beta A_{k+i}. \tag{9.45}$$

The following lemma says that all the block-entries in the first block-row in $\widehat{\beta Q}$ can be explicitly expressed in terms of the two matrices $R(\beta)$ and $N(\beta)$.

Lemma 9.10

$$\widehat{Q}_{1,j}(\beta) = N(\beta)R(\beta)^{j-1}, \quad j \geqslant 1. \tag{9.46}$$

Proof It is clear that

$$(\widehat{Q}_{1,2}(\beta), \widehat{Q}_{1,3}(\beta), \ldots) = N(\beta)\beta H \widehat{\beta Q},$$

where $\beta H = (\beta A_0, 0, 0, \ldots)$. The repeating structure and the property of skip-free-to-right of the transition matrix βQ lead to

$$(\widehat{Q}_{1,2}(\beta), \widehat{Q}_{1,3}(\beta), \ldots) = N(\beta)\beta A_0 (N(\beta), \widehat{Q}_{1,2}(\beta), \widehat{Q}_{1,3}(\beta), \ldots).$$

The proof is completed by the above recursive expression and repeatedly using $\beta A_0 N(\beta) = R(\beta)$.

It follows from the definition Eq. (9.44) and Eq. (9.45) of the G-measure and Lemma 9.10 that for $k = 1, 2, \ldots,$

$$G_{k,0}(\beta) = \sum_{i=1}^{\infty} N(\beta)R(\beta)^{i-1} \beta D_{k+i} \tag{9.47}$$

and

$$G_k(\beta) = \sum_{i=1}^{\infty} N(\beta)R(\beta)^{i-1} \beta A_{k+i}. \tag{9.48}$$

The following theorem says that both $N(\beta)$ and $N_0(\beta)$ can be expressed in terms of the matrix $R(\beta)$.

Theorem 9.10 *For the transition matrix of GI/M/1 type, the matrix* $N(\beta)$ *is expressed as*

$$N(\beta) = [I - \Psi(\beta)]^{-1}, \tag{9.49}$$

where

$$\Psi(\beta) = \sum_{k=1}^{\infty} R(\beta)^{k-1} \beta A_k. \tag{9.50}$$

The matrix $N_0(\beta)$ *is expressed as*

$$N_0(\beta) = [I - \Psi_0(\beta)]^{-1}, \tag{9.51}$$

where

$$\Psi_0(\beta) = \beta D_1 + \sum_{k=1}^{\infty} \beta D_0 N(\beta) R(\beta)^{k-1} \beta D_{k+1}. \tag{9.52}$$

Proof We only prove Eq. (9.49) and Eq. (9.50), while Eq. (9.51) and Eq. (9.52) can be proved similarly.

Let

$$\mathcal{P} = \begin{pmatrix} T & H \\ L & Q \end{pmatrix},$$

where

$$T = A_1, \quad H = (A_0, 0, 0, \ldots), \quad L = (A_2^{\mathrm{T}}, A_3^{\mathrm{T}}, A_4^{\mathrm{T}}, \ldots)^{\mathrm{T}}.$$

It is clear that

$$N(\beta) = \widehat{Q}_{1,1}(\beta) = [I - \Psi(\beta)]^{-1},$$

where for $\mathcal{P} = Q$ and $T = A_1$,

$$\begin{aligned} \Psi(\beta) &= \beta T + \beta H \widehat{\beta Q} \beta L = \beta A_1 + \beta A_0 \sum_{k=2}^{\infty} \widehat{Q}_{1,k-1}(\beta) \beta A_k \\ &= \sum_{k=1}^{\infty} R(\beta)^{k-1} \beta A_k. \end{aligned}$$

The last equation follows from $\beta A_0 N(\beta) = R(\beta)$ by using Lemma 9.10.

It follows from Eq. (9.42) and Eq. (9.49) that $R(\beta)$ satisfies the matrix equation

$$R(\beta) = \sum_{k=0}^{\infty} R(\beta)^k \beta A_k. \tag{9.53}$$

It is clear that $R(\beta)$ is the minimal nonnegative solution to Eq. (9.53).

For the matrix P of $GI/M/1$ type, the UL-type RG-factorization is given by

$$I - \beta P = [I - R_U(\beta)][I - \Psi_D(\beta)][I - G_L(\beta)], \tag{9.54}$$

where

$$[I - R_U(\beta)] = \begin{pmatrix} I - R_{0,1}(\beta) & & & \\ & I & -R(\beta) & \\ & & I & -R(\beta) \\ & & & \ddots & \ddots \\ & & & & \end{pmatrix},$$

$$\Psi_D(\beta) = \mathrm{diag}(\Psi_0(\beta), \Psi(\beta), \Psi(\beta), \ldots)$$

and

$$[I - G_L(\beta)] = \begin{pmatrix} I & & & & \\ -G_{1,0}(\beta) & I & & & \\ -G_{2,0}(\beta) & -G_1(\beta) & I & & \\ -G_{3,0}(\beta) & -G_2(\beta) & -G_1(\beta) & I & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

9.4.1 Spectral Analysis

Now, we provide spectral analysis for the R-and G-measures. The spectral analysis is a key for expressing the quasi-stationary distribution for the Markov chain of $GI/M/1$ type.

For convenience of description, we first assume that the matrix A is stochastic, and then extend our discussions to the strictly substochastic case. Let $\rho = \omega\xi$, where $\xi = \sum_{k=1}^{\infty} kA_k e$ and ω is the stationary probability vector of the matrix A.

Let

$$A^*(z) = \sum_{k=0}^{\infty} A_k z^k, \quad 0 \leqslant z < z_0, \tag{9.55}$$

where z_0 is the radius of convergence of function $A^*(z)$. Let $\chi(z)$, $u(z)$ and $v(z)$ be the maximal eigenvalue and the corresponding Perron-Frobenius left and right eigenvectors of the matrix $A^*(z)$, respectively, and $r(\beta)$, $l(\beta)$ and $h(\beta)$ the maximal eigenvalue and the corresponding Perron-Frobenius left and right eigenvectors of the matrix $R(\beta)$, respectively. Using the same proof as in Lemma 1.3.2 of Neuts [21], we can show from Eq. (9.53) that $r(\beta) = \beta\chi(r(\beta))$ and $l(\beta) = cu\,(r(\beta))$, where c is a non-zero constant. Without loss of generality, we set $c = 1$. It means that $z = r(\beta)$ is a positive solution of $z = \beta\chi(z)$. Actually, based on the discussion in Bean, Pollett and Taylor [3], we know from the irreducibility of the matrix A that for the equation $z = \beta\chi(z)$, there exist exactly two different positive solutions if $0 < \beta < \overline{\alpha}$ or only one (repeated twice) if $\beta = \overline{\alpha}$. We denote the two solutions by $\tilde{\theta}_\beta$ and θ_β and assume that $0 < \tilde{\theta}_\beta \leqslant \theta_\beta$. The following facts are a simple summarization on $\tilde{\theta}_\beta$ and θ_β based on Lemma 2.1 in Kijima [14].

We now consider the following three cases:

(1) Assume $\rho > 1$ and $\overline{\alpha} > 1$. In this case, $\tilde{\theta}_{\overline{\alpha}} < 1$. In fact, for $1 < \beta < \overline{\alpha}$,

$$0 < \tilde{\theta}_1 < \tilde{\theta}_\beta < \tilde{\theta}_{\overline{\alpha}} = \theta_{\overline{\alpha}} < \theta_\beta < \theta_1 = 1;$$

and for $0 < \beta < 1$,

$$0 < \tilde{\theta}_\beta < \tilde{\theta}_1 < \theta_1 = 1 < \theta_\beta < z_0.$$

(2) Assume $\rho < 1$ and $\overline{\alpha} > 1$. In this case, $\tilde{\theta}_{\overline{\alpha}} > 1$. In fact, for $1 < \beta < \overline{\alpha}$,

$$1 = \tilde{\theta}_1 < \tilde{\theta}_\beta < \tilde{\theta}_{\overline{\alpha}} = \theta_{\overline{\alpha}} < \theta_\beta < \theta_1 < z_0;$$

and for $0 < \beta < 1$,

$$0 < \tilde{\theta}_\beta < \tilde{\theta}_1 = 1 < \theta_1 < \theta_\beta < z_0.$$

(3) Assume $\rho = 1$. In this case, $\tilde{\theta}_{\overline{\alpha}} = 1$. Note that $\tilde{\theta}_{\overline{\alpha}} = \overline{\alpha}\chi(\tilde{\theta}_{\overline{\alpha}})$, we obtain that $\overline{\alpha} = \dfrac{1}{\chi(1)} = 1$, since the matrix A is stochastic. For $0 < \beta < 1$,

$$0 < \tilde{\theta}_\beta < 1 < \theta_\beta < z_0.$$

Figure 9.1 and Fig. 9.2 intuitively depict the above three cases, respectively. Let

$$G_\beta^*(z) = \sum_{k=1}^{\infty} z^k G_k(\beta).$$

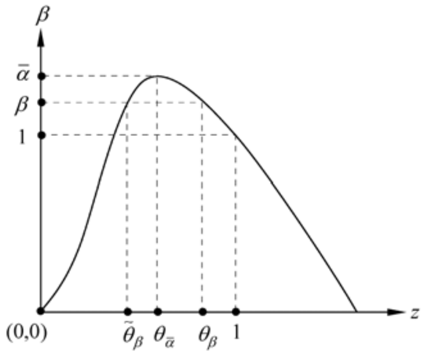

Figure 9.1 The case with $\rho > 1$

Lemma 9.11 *For* $0 < \beta \leqslant \overline{\alpha}$,

$$zI - \beta A^*(z) = [zI - R(\beta)][I - \Psi(\beta)][I - G_\beta^*(z)]. \tag{9.56}$$

In what follows we show that θ_β is the maximal eigenvalue of the matrix $R_{\max}(\beta)$ defined in Eq. (9.57), which is another nonnegative solution to the matrix equation $X = \sum_{k=0}^{\infty} X^k \beta A_k$. We show $\tilde{\theta}_\beta = r(\beta)$. We define

$$R_{\max}(\beta) = R(\beta) + h(\beta)u(\theta_\beta) \cdot [\theta_\beta I - R(\beta)], \tag{9.57}$$

where $u(\theta_\beta)$ and $h(\beta)$ are selected such that $u(\theta_\beta)h(\beta) = 1$, since $u(\theta_\beta) > 0$ according to the irreducibility of the matrix A.

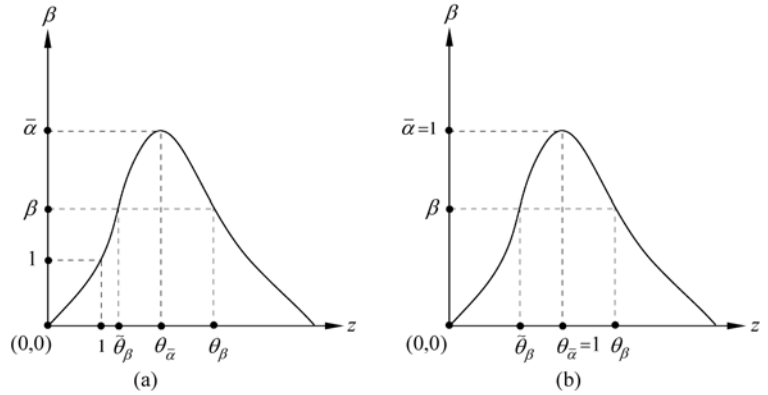

Figure 9.2 (a) The case with $\rho<1$; (b) The case with $\rho=1$

Lemma 9.12 (1) *θ_β is the maximal eigenvalue of $R_{\max}(\beta)$, and $u(\theta_\beta)$ and $h(\beta)$ are the associated left and right eigenvectors of $R_{\max}(\beta)$, respectively.*

(2) *$R_{\max}(\beta)$ is a solution to the equation $X=\sum_{k=0}^{\infty}X^k\beta A_k$ for $0<\beta\leqslant\bar{\alpha}$.*

(3) *$R_{\max}(\beta)$ is nonnegative for satisfying $R_{\max}(\beta)\gneqq R(\beta)$ for $0<\beta<\bar{\alpha}$ and $R_{\max}(\bar{\alpha})=R(\bar{\alpha})$.*

Proof (1) This can be directly verified.

(2) By induction, we can obtain that for all $k\geqslant 0$,

$$R_{\max}(\beta)^k=R(\beta)^k+h(\beta)u(\theta_\beta)\cdot[\theta_\beta^k I-R(\beta)^k].$$

Note that $\beta\chi(\theta_\beta)=\theta_\beta$, we have $u(\theta_\beta)\sum_{k=0}^{\infty}\theta_\beta^k\beta A_k=\theta_\beta u(\theta_\beta)$. Therefore,

$$\begin{aligned}\sum_{k=0}^{\infty}R_{\max}(\beta)^k\beta A_k&=\sum_{k=0}^{\infty}R(\beta)^k\beta A_k+h(\beta)u(\theta_\beta)\cdot\left[\sum_{k=0}^{\infty}\theta_\beta^k\beta A_k-\sum_{k=0}^{\infty}R(\beta)^k\beta A_k\right]\\&=R(\beta)+h(\beta)u(\theta_\beta)\cdot[\theta_\beta I-R(\beta)]=R_{\max}(\beta).\end{aligned}$$

(3) Since $h(\beta)\geqslant 0$ and $u(\theta_\beta)\geqslant 0$, it follows from Eq. (9.57) that in order to prove that $R_{\max}(\beta)\geqslant R(\beta)$, we only need to check that $u(\theta_\beta)R(\beta)\leqslant\theta_\beta u(\theta_\beta)$. To do this, we write

$$R_0(\beta)=0$$

and

$$R_{N+1}(\beta)=\sum_{k=0}^{\infty}R_N(\beta)^k\beta A_k,\quad N\geqslant 1.$$

It is clear that

$$u(\theta_\beta)R_0(\beta)=0\leqslant\theta_\beta u(\theta_\beta).$$

In what follows we assume that for any given $N=l$,

$$u(\theta_\beta)R_l(\beta) \leqslant \theta_\beta u(\theta_\beta),$$

then for $N=l+1$ we obtain

$$u(\theta_\beta)R_{l+1}(\beta) = u(\theta_\beta)\sum_{k=0}^{\infty} R_l(\beta)^k \beta A_k \leqslant u(\theta_\beta)\sum_{k=0}^{\infty}\theta_\beta^k \beta A_k = \theta_\beta u(\theta_\beta).$$

Therefore, by induction we know that for all $N \geqslant 0$ we have

$$u(\theta_\beta)R_N(\beta) \leqslant \theta_\beta u(\theta_\beta). \tag{9.58}$$

Similar analysis to Lemma 1.2.3 in Neuts [21] leads to

$$R(\beta) = \lim_{N\to\infty} R_N(\beta),$$

it follows from Eq. (9.58) that $u(\theta_\beta)R(\beta) \leqslant \theta_\beta u(\theta_\beta)$.

For $0<\beta<\overline{\alpha}$, since

$$h(\beta)u(\theta_\beta)\cdot[\theta_\beta I - R(\beta)]\cdot h(\beta) = h(\beta)[\theta_\beta - r(\beta)] \gneqq 0,$$

we obtain

$$h(\beta)u(\theta_\beta)\cdot[\theta_\beta I - R(\beta)] \gneqq 0,$$

hence $R_{\max}(\beta) \gneqq R(\beta)$.

For $\beta = \overline{\alpha}$, since the equation $z = \overline{\alpha}\chi(z)$ has only one solution, we obtain $\theta_{\overline{\alpha}} = r(\overline{\alpha})$ and $u(\theta_{\overline{\alpha}}) = Al(\theta_{\overline{\alpha}})$ is also a left eigenvector of the matrix $R(\overline{\alpha})$. Clearly,

$$h(\overline{\alpha})u(\theta_{\overline{\alpha}})\cdot[\theta_{\overline{\alpha}} I - R(\overline{\alpha})] = 0.$$

Thus, $R_{\max}(\overline{\alpha}) = R(\overline{\alpha})$. This completes the proof.

The following corollary follows from the proof of the above result.

Corollary 9.3 *For* $0<\beta \leqslant \overline{\alpha}, \tilde{\theta}_\beta(\beta) = r(\beta)$.

Further, we show that the larger solution θ_β to the equation $z = \beta\chi(z)$ is also a solution to the equation det $(I - G_\beta^*(z)) = 0$.

Theorem 9.11 *For all* $0<\beta \leqslant \overline{\alpha}, z = \theta_\beta$ *is a positive solution to equation det* $(I - G_\beta^*(z)) = 0$.

Proof Since the matrix $I - \Psi(\beta)$ is invertible for $0<\beta\leqslant\overline{\alpha}$, it follows from Lemma 9.11 that

$$\{z>0 : \det\,(zI - \beta A^*(z)) = 0\} = \{z>0 : \det\,(zI - R(\beta)) = 0\} \\ \bigcup \{z>0 : \det\,(I - G_\beta^*(z)) = 0\}.$$

Since for $0<\beta<\overline{\alpha},\theta_\beta>r(\beta)$ and $r(\beta)$ is the maximal eigenvalue of $R(\beta)$, we obtain that det $(\theta_\beta I-R(\beta))\neq 0$ Note that $\det(\theta_\beta I-\beta A^*(\theta_\beta))=0$, we obtain that $z=\theta_\beta$ must be a positive solution to equation $\det(I-G_\beta^*(z))=0$. Finally, since $f(\beta)=\det(I-G_\beta^*(\theta_\beta))$ is left-continuous at $\beta=\overline{\alpha}$, $\det(I-G_{\overline{\alpha}}^*(\theta_{\overline{\alpha}}))=\lim_{\beta\nearrow\overline{\alpha}} f(\beta)=0$.

Lemma 9.13 *For all $0<\beta\leqslant\overline{\alpha}$, the maximal eigenvalue of the nonnegative matrix $G_\beta^*(\theta_\beta)$ is equal to one.*

Proof Let $g(z)$ be the maximal eigenvalue of the nonnegative matrix $G_\beta^*(z)$ for $z\in[0,z_0)$, where z_0 is the radius of convergence of the matrix $A^*(z)$. When $z=\theta_\beta$, we write $\lambda=g(\theta_\beta)$. Therefore, $\frac{1}{\lambda}g(\theta_\beta)=1$. Suppose that $\lambda>1$. Noting that $g(z)$ is an increasing and continuous function for $z\in[0,z_0),g(0)=0$ and $g(\theta_\beta)=\lambda>1$, there must exist at least one point $\theta^*(\beta)\in(0,\theta_\beta)$ such that $g(\theta^*(\beta))=1$ according to the Mean Value Theorem for continuous functions. Thus, det $(I-G_\beta^*(\theta^*(\beta)))=0$, and so det $(\theta^*(\beta)I-A^*(\theta^*(\beta)))=0$. Therefore, $z=\theta^*(\beta)$ is a positive solution to the equation $z=\beta\chi(z)$. As shown earlier in this section, the equation $z=\beta\chi(z)$ has two different positive solutions θ_β and $r(\beta)$ for $0<\beta<\overline{\alpha}$. For $0<\beta<\overline{\alpha}$, it follows from Theorem 9.11 that det $(I-G_\beta^*(r(\beta)))\neq 0$. Therefore, $\theta^*(\beta)\neq r(\beta)$. Based on this, $z=\theta^*(\beta)$ is a third positive solution to the equation $z=\beta\chi(z)$, which is a contradiction. For $\beta=\overline{\alpha},z=\theta_{\overline{\alpha}}=r(\overline{\alpha})$ is the unique positive solution to the equation $z=\overline{\alpha}\chi(z)$. This contradicts that $\theta^*(\overline{\alpha})<\theta_{\overline{\alpha}}$ is also a positive solution to the equation $z=\overline{\alpha}\chi(z)$. This completes the proof.

The conclusion given below will be used in constructing a solution for the quasi-stationary distribution. This conclusion is a direct consequence of Theorem 9.11 and Lemma 9.13.

Corollary 9.4 *For $0<\beta\leqslant\overline{\alpha}$, there always exists a nonnegative non-zero row vector x_1 such that $x_1=x_1G_\beta^*(\theta_\beta)$, that is,*

$$x_1=x_1\sum_{i=1}^{\infty}\theta_\beta^i G_i(\beta). \tag{9.59}$$

In what follows we first construct a formal solution for the quasi-stationary distribution $\{\pi_k\}$. To guarantee π_k for each $k\geqslant 0$ is finite, we need to know whether or not $\sum_{i=1}^{\infty}\theta_\beta^{i-1}G_{i,0}(\beta)$ is finite, which is discussed in the following theorem.

Theorem 9.12 *Suppose that the matrix A is stochastic. For* $0<\beta\leqslant\overline{\alpha}$,

(1) *if* $\rho\neq 1$, *then* $\sum_{i=1}^{\infty}\theta_{\beta}^{i-1}G_{i,0}(\beta)$ *is finite*

(2) *if* $\rho=1$ *and* $\sum_{i=1}^{\infty}i^2A_i$ *is finite, then* $\sum_{i=1}^{\infty}\theta_{\beta}^{i-1}G_{i,0}(\beta)$ *is finite*

Proof It follows from Eq. (9.47) that

$$\sum_{k=1}^{\infty}\theta_{\beta}^{k-1}G_{k,0}(\beta)=N(\beta)\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\theta_{\beta}^{k-1}R(\beta)^{i-1}\beta D_{k+i}$$
$$=N(\beta)\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}\theta_{\beta}^{i}R(\beta)^{k-1-i}\beta D_{k+1}.$$

Since $u(\theta_\beta)>0$, the convergence of the above sum is equivalent to the convergence of

$$u(\theta_\beta)N(\beta)^{-1}\sum_{k=1}^{\infty}\theta_{\beta}^{k-1}G_{k,0}(\beta)=u(\theta_\beta)\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}\theta_{\beta}^{i}R(\beta)^{k-1-i}\beta D_{k+1}\geqslant 0.$$

Notice that $u(\theta_\beta)R(\beta)\leqslant\theta_\beta u(\theta_\beta)$, we obtain

$$u(\theta_\beta)\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}\theta_{\beta}^{i}R(\beta)^{k-1-i}\beta D_{k+1}\leqslant u(\theta_\beta)\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}\theta_{\beta}^{k-1}\beta D_{k+1}$$
$$=u(\theta_\beta)\sum_{k=1}^{\infty}k\theta_{\beta}^{k-1}\beta D_{k+1}.$$

Since θ_β is strictly increasing for $0<\beta\leqslant\overline{\alpha}$ according to the irreducibility of the matrix A,

$$\sum_{k=1}^{\infty}k\theta_{\beta}^{k-1}D_{k+1}\leqslant\sum_{k=1}^{\infty}k\theta_{\overline{\alpha}}^{k-1}D_{k+1}.$$

Therefore, the convergence of $\sum_{k=1}^{\infty}k\theta_{\overline{\alpha}}^{k-1}D_{k+1}$ is a sufficient condition for the convergence of $\sum_{k=1}^{\infty}\theta_{\beta}^{k-1}G_{k,0}(\beta)$ for $0<\beta\leqslant\overline{\alpha}$.

Since P is either stochastic or strictly substochastic and the matrix A is stochastic, we obtain

$$D_{k+1}e\leqslant e-\sum_{i=0}^{k}A_ie=\sum_{i=k+1}^{\infty}A_ie,$$

which means that

$$\sum_{k=1}^{\infty} k\theta_{\overline{\alpha}}^{k-1} D_{k+1} e \leqslant \sum_{k=1}^{\infty}\sum_{i=k+1}^{\infty} k\theta_{\overline{\alpha}}^{k-1} A_i e = \sum_{i=2}^{\infty}\left(\sum_{k=1}^{i-1} k\theta_{\overline{\alpha}}^{k-1}\right) A_i e. \tag{9.60}$$

Consider the following two cases.

Case Ⅰ $\rho \neq 1$. In this case, $\theta_{\overline{\alpha}} \neq 1$ by Lemma 2.1 in Kijima [14]. Hence,

$$\sum_{k=1}^{i-1} k\theta_{\overline{\alpha}}^{k-1} = \frac{1 - i\theta_{\overline{\alpha}}^{i-1} + (i-1)\theta_{\overline{\alpha}}^{i}}{(1-\theta_{\overline{\alpha}})^2} \tag{9.61}$$

It is clear from Eq. (9.60) and Eq. (9.61) that if $\sum_{i=1}^{\infty} i\theta_{\overline{\alpha}}^{i-1} A_i e$ is finite, then $\sum_{i=1}^{\infty} \theta_{\overline{\alpha}}^{i-1} G_{i,0}(\overline{\alpha})$ is finite. Notice that the matrix function $A^*(z)$ is analytic at $\theta_{\overline{\alpha}} \in (0, z_0)$, we obtain

$$\sum_{i=1}^{\infty} i\theta_{\overline{\alpha}}^{i-1} A_i = \frac{\mathrm{d}}{\mathrm{d}z} A^*(z)_{|z=\theta_{\overline{\alpha}}}$$

is finite. Therefore, $\sum_{i=1}^{\infty} i\theta_{\overline{\alpha}}^{i-1} A_i e$ is finite.

Case Ⅱ $\rho = 1$. In this case, $\theta_{\overline{\alpha}} = 1$ according to Lemma 2.1 in Kijima [14]. Therefore,

$$\sum_{k=1}^{i-1} k\theta_{\overline{\alpha}}^{k-1} = \sum_{k=1}^{i-1} k = \frac{i(i-1)}{2}.$$

The rest of the proof is obvious now.

Remark 9.3 *When the matrix A is irreducible and strictly substochastic, the equation* $z = \beta\chi(z)$ *still has two positive solutions if* $0 < \beta < \overline{\alpha}$ *or only one solution* (*repeated twice*) *if* $\beta = \overline{\alpha}$. *However, the series* $\sum_{i=1}^{\infty} \theta_{\beta}^{i-1} G_{i,0}(\beta)$ *may be infinite even under the assumption* $\sum_{i=1}^{\infty} i^2 A_i < +\infty$. *For example, let* $A_k = \lambda b^k$, $0 < b < 1, 0 < \lambda < 1 - b, k \geqslant 0,$ *and* $D_l = \dfrac{1-b-\lambda}{2(1-b)}$ for $l \geqslant 2$. *We can obtain that* $\overline{\alpha} = \dfrac{1}{4b\lambda}$ *and* $\tilde{\theta}_{\overline{\alpha}} = \dfrac{1}{2b}$. *It follows from Eq.* (9.47) *that*

$$\sum_{k=1}^{\infty} \theta_{\overline{\alpha}}^{k-1} G_{k,0}(\overline{\alpha}) = \overline{\alpha} N(\overline{\alpha}) \frac{1-b-\lambda}{2(1-b)} \sum_{k=1}^{\infty} k\left(\frac{1}{2b}\right)^{k-1} = +\infty$$

whenever $0 < b \leqslant \dfrac{1}{2}$.

Remark 9.4 *Let*

$$\delta = \sup\left\{\beta \geqslant 1 : \sum_{i=1}^{\infty} \theta_{\beta}^{i-1} G_{i,0}(\beta) < +\infty\right\}.$$

Then $\delta = \alpha \leqslant \overline{\alpha}$. *If* $\sum_{k=1}^{\infty} \theta_{\alpha}^{k-1} G_{k,0}(\alpha) < +\infty$, *then P is* α*-transient; if* $\sum_{k=1}^{\infty} \theta_{\alpha}^{k-1} G_{k,0}(\alpha)$ *is infinite, then P is* α*-recurrent. It is clear that if* $0<\beta<\alpha$, *then* $\sum_{i=1}^{\infty} \theta_{\beta}^{i-1} G_{i,0}(\beta)<+\infty$; *otherwise* $\sum_{i=1}^{\infty} \theta_{\beta}^{i-1} G_{i,0}(\beta)$ *is infinite.*

Now, we discuss the state α-classification of the Markov chain. To do this, let $u_0(\beta)$ and $n_0(\beta)$ be the maximal eigenvalues of the censored matrix $\Psi_0(\beta)$ and its fundamental matrix $N_0(\beta)$, respectively. Then $n_0(\beta) = 1/[1-u_0(\beta)]$.

Theorem 9.13 (1) *If for all* $0 < \beta \leqslant \overline{\alpha}, u_0(\beta) < 1$, *then* $N_0(\overline{\alpha}) < \infty$ *and* $\alpha = \overline{\alpha}$. *In this case, P is* α*-transient.*

(2) *If there exists a* β^* *with* $0 < \beta^* \leqslant \overline{\alpha}$ *such that* $u_0(\beta^*) = 1$, *then* $\alpha = \beta^*$ *and* $N_0(\alpha)$ *is infinite. In this case, P is* α*-recurrent.*

Proof Based on the facts: $n_0(\beta) = 1/[1-u_0(\beta)]$ and $n_0(\beta) < \infty$ if and only if $N_0(\beta) < \infty$, we discuss the following two cases:

Case Ⅰ There exists no solution to $1-u_0(\beta) = 0$ for $0<\beta \leqslant \overline{\alpha}$. In this case, $n_0(\overline{\alpha}) < \infty$, Hence $N_0(\overline{\alpha}) < \infty$. Therefore, $\alpha \geqslant \overline{\alpha}$. This, together with the fact $\alpha \leqslant \overline{\alpha}$, implies $\alpha = \overline{\alpha}$. Hence, *P* is α-transient.

Case Ⅱ There exists a solution β^* to $1-u_0(\beta) = 0$ for $0 < \beta^* \leqslant \overline{\alpha}$. In this case, $n_0(\beta^*) = \infty$, hence there exists at least one infinite entry of $N_0(\beta)$. This leads to $\alpha = \beta^* \leqslant \overline{\alpha}$ Therefore, *P* is α-recurrent.

This completes the proof.

9.4.2 Two Sets of Expressions

We use the UL-type *RG*-factorization to express the quasi-stationary distribution $\pi(\beta)=\{\pi_0(\beta),\pi_1(\beta),\pi_2(\beta),\ldots\}$ for $1 < \beta \leqslant \alpha$. We present two sets of expressions: One for an α-recurrent matrix with $\beta = \alpha$ and another for all the other cases. To compute the quasi-stationary distribution, we need to verify $\pi = \pi \beta P$, that is,

(1) $\pi_0 = \sum_{k=0}^{\infty} \pi_k \beta D_{k+1}$,

(2) $\pi_1 = \pi_0 \beta D_0 + \sum_{k=1}^{\infty} \pi_k \beta A_k$, and

(3) $\pi_l = \sum_{k=0}^{\infty} \pi_{k+l-1} \beta A_k$ for $l \geqslant 2$.

The first set: α-recurrent with $\beta = \alpha$

In this case, we solve $\pi(I - \alpha P) = 0$ according to Eq. (9.54) by two steps. Let

$$x = \pi[I - R_U(\alpha)]. \tag{9.62}$$

Then

$$x[I - \Psi_D(\alpha)][I - G_L(\alpha)] = 0. \tag{9.63}$$

In the first step, if $x = (x_0, x_1, \ldots)$ and $\pi = (\pi_0, \pi_1, \ldots)$ are partitioned according to levels, then Eq. (9.63) is equivalent to

$$x_0[I - \Psi_0(\alpha)] - \sum_{i=1}^{\infty} x_i [I - \Psi(\alpha)] G_{i,0}(\alpha) = 0,$$

$$x_k[I - \Psi(\alpha)] - \sum_{i=k+1}^{\infty} x_i [I - \Psi(\alpha)] G_{i-k}(\alpha) = 0, \quad k \geqslant 1.$$

Since P is α-recurrent, it follows from (2) in Theorem 9.13 that the maximal eigenvalue of $\Psi_0(\alpha)$ is $u_0(\alpha) = 1$. Therefore, for the nonnegative and irreducible $\Psi_0(\beta)$, there exists a positive x_0 such that

$$x_0[I - \Psi_0(\alpha)] = 0.$$

Hence, $(\tau x_0, 0, 0, \ldots)$ is a solution to Eq. (9.63).

In the second step, to express π_k in terms of x_k we solve Eq. (9.62) and obtain

$$\pi_0 = \tau x_0$$

and

$$\pi_k = \pi_0 R_{0,1}(\alpha) R^{k-1}(\alpha), \quad k \geqslant 1.$$

We prove the following theorem.

Theorem 9.14 *If P is α-recurrent, then the quasi-stationary distribution for $\beta = \alpha$ is given by*

$$\pi_0 = \tau x_0 \tag{9.64}$$

and

$$\pi_k = \pi_0 R_{0,1}(\alpha) R^{k-1}(\alpha), \quad k \geqslant 1, \tag{9.65}$$

where x_0 *is the left Perron-Frobeniusthe vector of the matrix* $\varPsi_0(\alpha)$, *and the constant* τ *makes* $\sum_{k=0}^{\infty}\pi_k e = 1$.

Proof We show that π_k for $k \geqslant 0$ given in Theorem 9.14 satisfy equations (1), (2) and (3).

Equation (3) holds because of Eq. (9.53),

$$\begin{aligned}\sum_{k=0}^{\infty}\pi_{k+l-1}\alpha A_k &= \pi_0 R_{0,1}(\alpha)R(\alpha)^{l-2}\sum_{k=0}^{\infty}R(\alpha)^k \alpha A_k \\ &= \pi_0 R_{0,1}(\alpha)R(\alpha)^{l-1} = \pi_l.\end{aligned}$$

To see that (1) holds, we use Eq. (9.52) and the fact $\pi_0 \varPsi_0(\alpha) = \pi_0$,

$$\sum_{k=0}^{\infty}\pi_k \alpha D_{k+1} = \pi_0\left[\alpha D_1 + R_{0,1}(\alpha)\sum_{k=1}^{\infty}R(\alpha)^{k-1}\alpha D_{k+1}\right] = \pi_0\varPsi_0(\alpha) = \pi_0.$$

Finally, (2) holds by noticing $R_{0,1}(\alpha) = \alpha D_0 N(\alpha)$ and $N(\alpha) = (I - \varPsi(\alpha))^{-1}$, and using Eq. (9.50),

$$\begin{aligned}\pi_0\alpha D_0 + \sum_{k=1}^{\infty}\pi_k\alpha A_k &= \pi_0\left[\alpha D_0 + R_{0,1}(\alpha)\sum_{k=1}^{\infty}R(\alpha)^{k-1}\alpha A_k\right] \\ &= \pi_0\alpha D_0[I + N(\alpha)\varPsi(\alpha)] = \pi_0 R_{0,1}(\alpha) = \pi_1.\end{aligned}$$

Therefore, the expression in Theorem 9.14 is unique, up to multiplication by a positive constant, α-invariant measure. This completes the proof.

The second set: α-recurrent with $\beta < \alpha$ or α-transient with $\beta \leqslant \alpha$.

In this case, we need to proceed in two steps. Let

$$y = \pi[I - R_U(\beta)][I - \varPsi_D(\beta)]. \tag{9.66}$$

Then

$$y[I - G_L(\beta)] = 0. \tag{9.67}$$

In the first step, we need to solve the Eq. (9.67) for a nonnegative non-zero row vector y. If such a solution exists, then π can be calculated by y. Equation (9.67) is equivalent to

$$y_0 = \sum_{i=1}^{\infty} y_i G_{i,0}(\beta) \tag{9.68}$$

and

$$y_k = \sum_{i=1}^{\infty} y_{k+i} G_i(\beta), \quad k \geqslant 1. \tag{9.69}$$

We first consider a special solution to Eq. (9.69) and Eq. (9.68). Let

$$y_k = \theta_\beta^{k-1} x_1, \quad k \geqslant 1, \tag{9.70}$$

where x_1 is determined as the unique, up to constant multiplications, nonnegative solution of the equation $x_1 = x_1 \sum_{i=1}^{\infty} \theta_\beta^i G_i(\beta)$. Then it follows from Eq. (9.69) and Eq. (9.68) that

$$y_0 = x_1 \sum_{i=1}^{\infty} \theta_\beta^{i-1} G_{i,0}(\beta) \tag{9.71}$$

and from Eq. (9.69) and Eq. (9.70) that

$$x_1 = x_1 \sum_{i=1}^{\infty} \theta_\beta^i G_i(\beta),$$

which is the same as Eq. (9.59) and it follows from Theorem 9.12 that y_0 is finite. Therefore, Eq. (9.71) and Eq. (9.70) are a nonnegative non-zero solution to Eq. (9.68) and Eq. (9.69), or Eq. (9.67).

In the second step, solving $y = \pi[I - R_U(\beta)][I - \Psi_D(\beta)]$ in Eq. (9.66) we obtain

$$\begin{aligned} y_0 &= \pi_0[I - \Psi_0(\beta)], \\ y_1 &= [-\pi_0 R_{0,1}(\beta) + \pi_1][I - \Psi(\beta)], \\ y_k &= [-\pi_{k-1} R(\beta) + \pi_k][I - \Psi(\beta)], \quad k \geqslant 2. \end{aligned}$$

Since both $[I - \Psi_0(\beta)]$ and $[I - \Psi(\beta)]$ are invertible in this case, we can express π_k in terms of y_k:

$$\pi_0 = y_0 N_0(\beta), \tag{9.72}$$

$$\pi_1 = \pi_0 R_{0,1}(\beta) + y_1 N(\beta) \tag{9.73}$$

and

$$\pi_k = \pi_{k-1} R(\beta) + y_k N(\beta), \quad k \geqslant 2. \tag{9.74}$$

Based on Eq. (9.72), Eq. (9.73), Eq. (9.74), Eq. (9.70) and Eq. (9.71), we can construct a formal solution for the β-invariant measure: The construction of π_0 is obvious. For the case of $0 < \beta \leqslant \alpha$ with $k \geqslant 1$, we have

$$\begin{aligned} \pi_k = {} & x_1 \sum_{i=1}^{\infty} \theta_\beta^{i-1} G_{i,0}(\beta) N_0(\beta) R_{0,1}(\beta) R^{k-1}(\beta) \\ & + x_1 N(\beta) \sum_{\substack{i+j=k-1 \\ i \geqslant 0, j \geqslant 0}} \theta_\beta^i R^j(\beta). \end{aligned} \tag{9.75}$$

Note that the matrix $\theta_\beta I - R(\beta)$ is invertible for $0<\beta<\overline{\alpha}$ and singular for $\beta=\overline{\alpha}$, the expression for $\beta=\overline{\alpha}$ is slightly different from the case for $0<\beta<\overline{\alpha}$, so we need to present them separately.

When $0<\beta<\overline{\alpha}$,

$$\sum_{\substack{i+j=k-1\\ i\geqslant 0, j\geqslant 0}} \theta_\beta^i R^j(\beta) = [\theta_\beta I - R(\beta)]^{-1}[\theta_\beta^k I - R^k(\beta)],$$

since $\theta_\beta I - R(\beta)$ is invertible. Therefore we give the following theorem.

Theorem 9.15 *For $0<\beta<\alpha$, the quasi-stationary distribution of P is given by*

$$\pi_0 = x_1 \sum_{i=1}^{\infty} \theta_\beta^{i-1} G_{i,0}(\beta) N_0(\beta) \tag{9.76}$$

and

$$\pi_k = V_k(\beta) + W_k(\beta), \quad k \geqslant 1, \tag{9.77}$$

where

$$V_k(\beta) = x_1 N(\beta)[\theta_\beta I - R(\beta)]^{-1}\theta_\beta^k$$

and

$$W_k(\beta) = x_1 \left\{ \sum_{i=1}^{\infty} \theta_\beta^{i-1} G_{i,0}(\beta) N_0(\beta) R_{0,1}(\beta) - N(\beta)[\theta_\beta I - R(\beta)]^{-1} R(\beta) \right\} R^{k-1}(\beta).$$

Proof Let

$$g_1(\beta) = x_1 N(\beta)[\theta_\beta I - R(\beta)]^{-1}.$$

It follows from Eq. (9.56) that

$$\theta_\beta I - \sum_{k=0}^{\infty} \theta_\beta^k \beta A_k = [\theta_\beta I - R(\beta)][I - \Psi(\beta)]\left[I - \sum_{i=1}^{\infty} \theta_\beta^i G_i(\beta) \right],$$

we obtain

$$g_1(\beta)\left[\theta_\beta I - \sum_{k=0}^{\infty} \theta_\beta^k \beta A_k \right] = x_1 \left[I - \sum_{i=1}^{\infty} \theta_\beta^i G_i(\beta) \right] = 0,$$

that is

$$g_1(\beta) \sum_{k=0}^{\infty} \theta_\beta^k \beta A_k = \theta_\beta g_1(\beta).$$

Hence, we obtain $g_1(\beta) = \kappa u(\theta_\beta)$, where κ is a positive constant. We set $\kappa = 1$.

Let

$$g_2(\beta) = x_1 \left\{ N(\beta)[\theta_\beta I - R(\beta)]^{-1} R(\beta) - \sum_{i=1}^{\infty} \theta_\beta^{i-1} G_{i,0}(\beta) N_0(\beta) R_{0,1}(\beta) \right\}.$$

Then it follows from Eq. (9.77) that

$$\pi_k = \theta_\beta g_1(\beta) \theta_\beta^{k-1} - g_2(\beta) R(\beta)^{k-1}, \quad k \geqslant 1.$$

We first check (3) as follows:

$$\begin{aligned}\sum_{k=0}^{\infty} \pi_{k+l-1} \beta A_k &= \theta_\beta g_1(\beta) \sum_{k=0}^{\infty} \theta_\beta^{k+l-2} \beta A_k - g_2(\beta) \sum_{k=0}^{\infty} R(\beta)^{k+l-2} \beta A_k \\ &= \theta_\beta g_1(\beta) \theta_\beta^{l-1} - g_2(\beta) R(\beta)^{l-1} = \pi_l.\end{aligned}$$

Then we check (1) as follows:

$$\begin{aligned}\sum_{k=0}^{\infty} \pi_k \beta D_{k+1} &= x_1 \sum_{i=1}^{\infty} \theta_\beta^{i-1} G_{i,0}(\beta) N_0(\beta) \beta D_1 + g_1(\beta) \sum_{k=1}^{\infty} [\theta_\beta^k - R(\beta)^k] \beta D_{k+1} \\ &\quad + x_1 \sum_{i=1}^{\infty} \theta_\beta^{i-1} G_{i,0}(\beta) N_0(\beta) R_{0,1}(\beta) \sum_{k=1}^{\infty} R(\beta)^{k-1} \beta D_{k+1},\end{aligned}$$

note that

$$R_{0,1}(\beta) = \beta D_0 N(\beta), \quad N_0(\beta) \Psi_0(\beta) = -I + N_0(\beta),$$

$$\Psi_0(\beta) = \beta D_1 + \beta D_0 N(\beta) \sum_{k=1}^{\infty} R(\beta)^{k-1} \beta D_{k+1},$$

we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \pi_k \beta D_{k+1} &= \pi_0 + x_1 N(\beta) \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \theta_\beta^i R(\beta)^{k-1-i} \beta D_{k+1} \\ &\quad - x_1 N(\beta) \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \theta_\beta^{i-1} R(\beta)^{k-1} \beta D_{k+i}.\end{aligned}$$

It is easy to check that

$$\sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \theta_\beta^i R(\beta)^{k-1-i} \beta D_{k+1} = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \theta_\beta^{i-1} R(\beta)^{k-1} \beta D_{k+i}.$$

Hence (1) is correct. Finally, we prove (2). To do this, we compute

$$\begin{aligned}\pi_0 \beta D_0 + \sum_{k=1}^{\infty} \pi_k \beta A_k &= x_1 \sum_{i=1}^{\infty} \theta_\beta^{i-1} G_{i,0}(\beta) N_0(\beta) \beta D_0 \\ &\quad + g_1(\beta) \sum_{k=1}^{\infty} \theta_\beta^k \beta A_k - g_2(\beta) \sum_{k=1}^{\infty} R(\beta)^{k-1} \beta A_k,\end{aligned}$$

note that

$$g_1(\beta)\sum_{k=1}^{\infty}\theta_\beta^k \beta A_k = g_1(\beta)\theta_\beta - g_1(\beta)\beta A_0$$

and

$$g_2(\beta)\sum_{k=1}^{\infty} R(\beta)^{k-1}\beta A_k = g_2(\beta)\Psi(\beta) = g_1(\beta)R(\beta) - g_1(\beta)\beta A_0$$
$$- x_1\sum_{i=1}^{\infty}\theta_\beta^{i-1} G_{i,0}(\beta)N_0(\beta)R_{0,1}(\beta)\Psi(\beta),$$

we obtain

$$\pi_0\beta D_0 + \sum_{k=1}^{\infty}\pi_k\beta A_k = g_1(\beta)[\theta_\beta I - R(\beta)] + x_1\sum_{i=1}^{\infty}\theta_\beta^{i-1}G_{i,0}(\beta)N_0(\beta)R_{0,1}(\beta)$$
$$= g_1(\beta)\theta_\beta - g_2(\beta) = \pi_1.$$

This implies that the expression in Theorem 9.97 is a solution for the β-invariant measure. This completes the proof.

When P is α-transient with $\beta = \alpha$ (in this case, $\alpha = \overline{\alpha}$), a similar proof to that in Theorem 9.15 leads to that a nonnegative non-zero α-invariant measure of P is given by

$$\pi_0 = x_1\sum_{i=1}^{\infty}\theta_\alpha^{i-1}G_{i,0}(\alpha)N_0(\alpha)$$

and

$$\pi_k = V_k(\alpha) + W_k(\alpha), \quad k \geqslant 1,$$

where

$$V_k(\alpha) = x_1 N(\alpha)\sum_{\substack{i+j=k-1\\ i\geqslant 0, j\geqslant 0}}\theta_\alpha^i R^j(\alpha) \tag{9.78}$$

and

$$W_k(\alpha) = x_1\sum_{i=1}^{\infty}\theta_\alpha^{i-1}G_{i,0}(\alpha)N_0(\alpha)R_{0,1}(\alpha)R^{k-1}(\alpha).$$

Under the assumption of the irreducibility of the matrix A, the maximal eigenvalue θ_α of the matrix $R(\alpha)$ is simple. Let $J(\alpha) = \operatorname{diag}(\theta_\alpha, \Lambda(\alpha))$ be the Jordan's normal form of $R(\alpha)$. Then the real part of each diagonal entry of $\Lambda(\alpha)$ is strictly less than θ_α, and there always exists an invertible matrix $T(\alpha)$ such that

$$T^{-1}(\alpha)R(\alpha)T(\alpha) = J(\alpha) = \operatorname{diag}(\theta_\alpha, \Lambda(\alpha)).$$

We write

$$T^{-1}(\alpha)=\begin{pmatrix} t_{11}(\alpha) & T_{12}(\alpha) \\ T_{21}(\alpha) & T_{22}(\alpha) \end{pmatrix},$$

where $t_{11}(\alpha), T_{12}(\alpha), T_{21}(\alpha)$ and $T_{22}(\alpha)$ are a scalar, row and column vectors of size $(m-1)$ and a matrix of size $(m-1)\times(m-1)$, respectively.

Lemma 9.14 *For all $k \geqslant 0$,*

$$\Lambda(\alpha)^k (T_{21}(\alpha), T_{22}(\alpha)) = (T_{21}(\alpha), T_{22}(\alpha)) R(\alpha)^k.$$

Proof Simple matrix computation can lead to that for all $k \geqslant 0$,

$$\mathrm{diag}(\theta_\alpha^k, \Lambda(\alpha)^k) T^{-1}(\alpha) = \begin{pmatrix} \theta_\alpha^k (t_{11}(\alpha), T_{12}(\alpha)) \\ \Lambda(\alpha)^k (T_{21}(\alpha), T_{22}(\alpha)) \end{pmatrix}$$

and

$$T^{-1}(\alpha) R(\alpha)^k = \begin{pmatrix} (t_{11}(\alpha), T_{12}(\alpha)) R(\alpha)^k \\ (T_{21}(\alpha), T_{22}(\alpha)) R(\alpha)^k \end{pmatrix}.$$

Since $T^{-1}(\alpha) R(\alpha)^k = \mathrm{diag}(\theta_\alpha^k, \Lambda(\alpha)^k) T^{-1}(\alpha)$, we obtain that for all $k \geqslant 0$,

$$\Lambda(\alpha)^k (T_{21}(\alpha), T_{22}(\alpha)) = (T_{21}(\alpha), T_{22}(\alpha)) R(\alpha)^k.$$

This completes the proof.

Let

$$V_k(\alpha) = x_1 N(\alpha) \sum_{i=0}^{k-1} \theta_\alpha^i R(\alpha)^{k-1-i}$$

and

$$x_1 N(\alpha) T(\alpha) = (\psi, \Psi).$$

Then

$$\begin{aligned}
V_k(\alpha) &= x_1 N(\alpha) \sum_{i=0}^{k-1} \theta_\alpha^i R(\alpha)^{k-1-i} \\
&= (\psi, \Psi) \begin{pmatrix} k\theta_\alpha^{k-1} & 0 \\ 0 & [\theta_\alpha I - \Lambda(\alpha)]^{-1} [\theta_\alpha^k I - \Lambda(\alpha)^k] \end{pmatrix} \begin{pmatrix} t_{11}(\alpha) & T_{12}(\alpha) \\ T_{21}(\alpha) & T_{22}(\alpha) \end{pmatrix} \\
&= k\theta_\alpha^{k-1} \psi (t_{11}(\alpha), T_{12}(\alpha)) + \Psi [\theta_\alpha I - \Lambda(\alpha)]^{-1} [\theta_\alpha^k I - \Lambda(\alpha)^k] (T_{21}(\alpha), T_{22}(\alpha)) \\
&= k\theta_\alpha^{k-1} \psi (t_{11}(\alpha), T_{12}(\alpha)) + \theta_\alpha^k \Psi [\theta_\alpha I - \Lambda(\alpha)]^{-1} (T_{21}(\alpha), T_{22}(\alpha)) \\
&\quad - \Psi [\theta_\alpha I - \Lambda(\alpha)]^{-1} (T_{21}(\alpha), T_{22}(\alpha)) R(\alpha)^k.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\pi_k &= W_k(\alpha)+V_k(\alpha)\\ &= k\theta_\alpha^{k-1}\psi(t_{11}(\alpha),T_{12}(\alpha))+\theta_\alpha^k\Psi[\theta_\alpha I-\Lambda(\alpha)]^{-1}(T_{21}(\alpha),T_{22}(\alpha))\\ &\quad-\{\Psi[\theta_\alpha I-\Lambda(\alpha)]^{-1}(T_{21}(\alpha),T_{22}(\alpha))R(\alpha)\\ &\quad-x_1\sum_{i=1}^{\infty}\theta_\alpha^{i-1}G_{i,0}(\alpha)N_0(\alpha)R_{0,1}(\alpha)\}R(\alpha)^{k-1}.\end{aligned}$$

We summarize the above result in the following theorem.

Theorem 9.16 *If P is α-transient and $\beta=\alpha$, then the quasi-stationary distribution is given by*

$$\pi_0 = x_1\sum_{i=1}^{\infty}\theta_\alpha^{i-1}G_{i,0}(\alpha)N_0(\alpha)$$

and for $k\geqslant 1$,

$$\begin{aligned}\pi_k &= k\theta_\alpha^{k-1}\psi(t_{11}(\alpha),T_{12}(\alpha))+\theta_\alpha^k\Psi[\theta_\alpha I-\Lambda(\alpha)]^{-1}(T_{21}(\alpha),T_{22}(\alpha))\\ &\quad-\{\Psi[\theta_\alpha I-\Lambda(\alpha)]^{-1}(T_{21}(\alpha),T_{22}(\alpha))R(\alpha)\\ &\quad-x_1\sum_{i=1}^{\infty}\theta_\alpha^{i-1}G_{i,0}(\alpha)N_0(\alpha)R_{0,1}(\alpha)\}R(\alpha)^{k-1}.\end{aligned} \tag{9.79}$$

Now, we consider the special case where $D_i = A_i$ for all $i\geqslant 0$, which shows the transition matrix without the boundary. In this case,

$$N_0(\beta)=N(\beta),\quad R_{0,1}(\beta)=R(\beta),\quad G_{i,0}(\beta)=G_i(\beta),\quad i\geqslant 1.$$

Let $y_k=\theta_\beta^k y_0$ for $k\geqslant 0$. Then $y_0=y_0\sum_{i=1}^{\infty}\theta_\beta^i G_i(\beta)$ and $y_0=x_1/\theta_\beta$. Hence,

$$\begin{aligned}W_k(\beta) &= x_1\sum_{i=1}^{\infty}\theta_\beta^{i-1}G_{i,0}(\beta)N_0(\beta)R_{0,1}(\beta)R(\beta)^{k-1}\\ &= \frac{x_1}{\theta_\beta}N(\beta)R(\beta)^k = y_0N(\beta)R(\beta)^k\end{aligned}$$

and

$$V_k(\beta)=x_1N(\beta)\sum_{i=0}^{k-1}\theta_\beta^i R(\beta)^{k-1-i}=y_0N(\beta)\sum_{i=0}^{k-1}\theta_\beta^{i+1}R(\beta)^{k-1-i}.$$

Therefore, we obtain

$$\pi_k = W_k(\beta)+V_k(\beta)=y_0N(\beta)\sum_{i=0}^{k}\theta_\beta^i R(\beta)^{k-i}. \tag{9.80}$$

Corollary 9.5 *For the transition matrix P of GI/M/*1 *type without the boundary,*
(1) *if* $0<\beta<\alpha$,

$$\pi_k = y_0 N(\beta)[\theta_\beta I - R(\beta)]^{-1}[\theta_\beta^k I - R(\beta)^k], \quad k \geqslant 1; \tag{9.81}$$

(2) *if* $\beta=\alpha$,

$$\begin{aligned}\pi_k &= k\theta_\alpha^{k-1}\psi(t_{11}(\alpha),T_{12}(\alpha)) + \theta_\alpha^k \Psi[\theta_\alpha I - A(\alpha)]^{-1}(T_{21}(\alpha),T_{22}(\alpha)) \\ &\quad - \Psi[\theta_\alpha I - A(\alpha)]^{-1}(T_{21}(\alpha),T_{22}(\alpha))R(\alpha)^k, \quad k \geqslant 1.\end{aligned} \tag{9.82}$$

For an irreducible level-independent QBD process, we have provided two different types of expressions from Markov chains of *M/G/*1 type and Markov chains of *GI/M/*1 type. It is necessary to indicate that the two expressions are identical. For simplicity of interpretation, we only consider the QBD process without the boundary, and can directly obtain the other from either of the two expressions.

Lemma 9.15 *For a QBD process without the boundary, let* γ_α *and* θ_α *be the maximal eigenvalues of the matrices* $G(\alpha)$ *and* $R(\alpha)$, *respectively. Then* $\gamma_\alpha = 1/\theta_\alpha$.

Proof For the QBD process and $z>0$, we denote by $\chi(z)$ and $\tilde{\chi}(z)$ the maximal eigenvalues of the matrices $A_0 + zA_1 + z^2 A_2$ and $z^2 A_0 + zA_1 + A_2$, respectively. Since

$$z^2 A_0 + zA_1 + A_2 = z^2\left[A_0 + \frac{1}{z}A_1 + \left(\frac{1}{z}\right)^2 A_2\right],$$

we obtain

$$\tilde{\chi}(z) = z^2 \chi\left(\frac{1}{z}\right). \tag{9.83}$$

Noting that γ_α is the unique positive solution (repeated twice) to the equation $z=\alpha\tilde{\chi}(z)$, it follows from Eq. (9.83) that γ_α is a positive solution to the equation $1/z = \alpha\chi(1/z)$, that is, $1/\gamma_\alpha$ is a positive solution to the equation $z=\alpha\chi(z)$. Since θ_α is the unique positive solution (repeated twice) to the equation $z=\alpha\chi(z)$, we obtain that $\theta_\alpha = 1/\gamma_\alpha$. This completes the proof.

Using the Markov chain of *M/G/*1 type, we have

$$\pi_k = \frac{z}{\gamma_\alpha^{k-1}} \sum_{i=0}^{k-1} G(\alpha)^i N(\alpha) R(\alpha)^i, \tag{9.84}$$

where z is the nonnegative non-zero left eigenvector of the matrix $G(\alpha)$. Noting that $\gamma_\alpha = 1/\theta_\alpha$, $y_0 = y_0\theta_\alpha G(\alpha)$ and $\gamma_\alpha z = zG(\alpha)$, we have $y_0 = Az$. Without loss of generality, we set $c = 1$, i.e., $z = y_0$. Thus, it follows from Eq. (9.84) that

$$\pi_k = y_0 N(\alpha)\sum_{i=0}^{k-1}\theta_\alpha^{k-1-i} R(\alpha)^i,$$

which is the same as Eq. (9.80). Therefore, it follows from $\alpha = \bar{\alpha}$ that

$$\pi_k = k\theta_\alpha^{k-1}\psi(t_{11}(\alpha), T_{12}(\alpha)) + \theta_\alpha^k \Psi[\theta_\alpha I - A(\alpha)]^{-1}(T_{21}(\alpha), T_{22}(\alpha))$$
$$-\Psi[\theta_\alpha I - A(\alpha)]^{-1}(T_{21}(\alpha), T_{22}(\alpha))R(\alpha)^k, \quad k \geqslant 1.$$

This leads to the explicit expression Eq. (9.82) based on the Markov chain of *GI*/*M*/1 type.

9.4.3 Conditions for α-Positive Recurrence

We now provide conditions under which the Markov chain P of *GI*/*M*/1 type is either α-positive recurrent or α-null recurrent.

When the matrix P of *GI*/*M*/1 type is α-recurrent, the quasi-stationary distribution is matrix-geometric, as shown in Theorem 9.14. At the same time, the α-invariant vector, which is a nonnegative non-zero column vector v such that $v = \alpha P v$, can be expressed as

$$v_0 = w_0, \tag{9.85}$$

$$v_k = G_{k,0}(\alpha)v_0 + \sum_{i=1}^{k-1} G_{k-i}(\alpha)v_i, \quad k \geqslant 1, \tag{9.86}$$

where w_0 is the unique, up to multiplication of a positive constant, solution of $[I - \Psi_0(\alpha)]w_0 = 0.$ The expression does not rely on whether or not the matrix A is stochastic. Moreover, when the matrix A is strictly substochastic, the equation $z = \beta\chi(z)$ has still two positive solutions if $0 < \beta < \bar{\alpha}$ or only one positive solution (repeated twice) if $\beta = \bar{\alpha}.$ Based on the fact, the following lemma provides conditions for classifying the α-positive recurrent and the α-null recurrent.

For simplicity of description, we assume that the matrix $A^*(1)$ is irreducible and stochastic.

Lemma 9.16 *If* $\sum_{k=1}^{\infty} kR(\alpha)^{k-1}D_k < \infty, \sum_{k=1}^{\infty} kR(\alpha)^{k-1}A_k < \infty$ *and* $\alpha < \bar{\alpha}$, *then an* α*-recurrent Markov chain P is* α*-positive recurrent; otherwise, it is* α*-null recurrent.*

Proof According to Lemma 9.9, both expressions for the α-invariant measure π and the α-invariant vector v are needed. Let $V^*(z) = \sum_{k=1}^{\infty} z^k v_k$, $G^*(z) = \sum_{k=1}^{\infty} z^k G_k(\alpha)$ and $G_0^*(z) = \sum_{k=1}^{\infty} z^k G_{k,0}(\alpha)$. For convenience of description, we express the α-invariant vector in the following way. It follows from Eq. (9.85) and Eq. (9.86) that

$$V^*(z) = [I - G^*(z)]^{-1} G_0^*(z) v_0 = \sum_{n=0}^{\infty} [G^*(z)]^n G_0^*(z) v_0,$$

which gives

$$v_k = \sum_{n=0}^{\infty} G_k(\alpha)^{\circledast n} \circledast G_{k,0}(\alpha) v_0, \quad k \geqslant 1. \tag{9.87}$$

It follows from Theorem 9.14 and Eq. (9.87) that

$$\sum_{i=0}^{\infty} \pi_i v_i = x_0 v_0 + x_0 R_{0,1}(\alpha) \sum_{k=1}^{\infty} R(\alpha)^{k-1} \left(\sum_{n=0}^{\infty} G_k(\alpha)^{\circledast n} \circledast G_{k,0}(\alpha) \right) v_0. \tag{9.88}$$

Note that the matrix $\Psi_0(\alpha)$ is nonnegative and irreducible, the invariant measure x_0 and vector v_0 are all positive. Since each column of the nonnegative matrix $R_{0,1}(\alpha)$ is non-zero, it is clear that $x_0 R_{0,1}(\alpha) > 0$. Therefore, $\sum_{i=0}^{\infty} \pi_i v_i < \infty$ if and only if

$$\sum_{k=1}^{\infty} R(\alpha)^{k-1} \left[\sum_{n=0}^{\infty} G_k(\alpha)^{\circledast n} \circledast G_{k,0}(\alpha) \right] < \infty. \tag{9.89}$$

It is known that $r(\alpha)$ and $l(\alpha)$ are the maximal eigenvalue of $R(\alpha)$ and the associated left eigenvector, respectively. Since $A^*(1)$ is irreducible, $l(\alpha)>0$. It follows from Eq. (9.89) that

$$l(\alpha) \sum_{k=1}^{\infty} R(\alpha)^{k-1} \left[\sum_{n=0}^{\infty} G_k(\alpha)^{\circledast n} \circledast G_{k,0}(\alpha) \right] = \frac{1}{r(\alpha)} l(\alpha) \sum_{n=0}^{\infty} [G^*(r(\alpha))]^n G_0^*(r(\alpha)).$$

Clearly, Eq. (9.89) is true if and only if, (1) $G_0^*(r(\alpha)) < \infty$, (2) $G^*(r(\alpha)) < \infty$, and (3) the matrix $I - G^*(r(\alpha))$ is invertible.

We first analyze the condition in (3). Note that the equation $z = \beta \chi(z)$ has exactly two different positive solutions $z = r(\alpha)$ and $z = \theta_\alpha$ if $0 < \alpha < \overline{\alpha}$ or only one solution $z = r(\overline{\alpha}) = \theta_{\overline{\alpha}}$ (repeated twice) if $\alpha = \overline{\alpha}$, also noting that $z = \theta_\alpha$ is the unique positive solution of the equation det $(I - G^*(z)) = 0$, we obtain that if $\alpha < \overline{\alpha}$, then the matrix $I - G^*(r(\alpha))$ is invertible; and if $\alpha = \overline{\alpha}$, then the matrix $I - G^*(r(\alpha))$ is singular. Therefore, the matrix $I - G^*(r(\alpha))$ is invertible if $\alpha < \overline{\alpha}$.

We then analyze the conditions in (1) and (2). We show that (1) $G_0^*(r(\alpha)) < \infty$ if and only if $\sum_{k=1}^{\infty} k r(\alpha)^{k-1} D_k < \infty$; and (2) $G^*(r(\alpha)) < \infty$ if and only if $\sum_{k=1}^{\infty} k r(\alpha)^{k-1} A_k < \infty$. We only provide details for (1), while (2) can be discussed similarly.

It follows from Eq. (9.44) that

$$G_0^*(r(\alpha)) = \sum_{k=1}^{\infty} r(\alpha)^k G_{k,0}(\alpha) = \sum_{k=1}^{\infty} r(\alpha)^k \sum_{i=1}^{\infty} N(\alpha) R(\alpha)^{i-1} \alpha D_{k+i}.$$

Hence,

$$l(\alpha) N(\alpha)^{-1} G_0^*(r(\alpha)) = l(\alpha) \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} r_{\alpha}^{k+i-1} \alpha D_{k+i} = l(\alpha) \sum_{k=1}^{\infty} k r(\alpha)^{k-1} \alpha D_k,$$

which means that $G_0^*(r(\alpha)) < \infty$ if and only if $\sum_{k=1}^{\infty} k r(\alpha)^{k-1} D_k < \infty$. This completes the proof.

The following theorem and corollary provide simple conditions for classifying the α-positive recurrent and α-null recurrent for the Markov chain of $GI/M/1$ type. At the same time, the proof is not difficult according to Lemma 9.16 and is omitted here.

Theorem 9.17 *Suppose the Markov chain P of $GI/M/1$ type is α-recurrent and the matrix A is stochastic or strictly substochastic. P is α-positive if and only if*

(1) $\alpha < \bar{\alpha}$ *when* $r(\alpha) < 1$;

(2) $\alpha < \bar{\alpha}, \sum_{k=1}^{\infty} k r(\alpha)^{k-1} A_k < \infty$ *and* $\sum_{k=1}^{\infty} k r(\alpha)^{k-1} D_k < \infty$ *when* $r(\alpha) > 1$; *or*

(3) $\alpha < \bar{\alpha}$ *and* $\sum_{k=1}^{\infty} k A_k < \infty$ *and* $\sum_{k=1}^{\infty} k D_k < \infty$ *when* $r(\alpha) = 1$.

When the matrix A is stochastic, the conditions in the above theorem for the cases of $r(\alpha) > 1$ and $r(\alpha) = 1$ can be further simplified as in the following corollary.

Corollary 9.6 *Suppose the Markov chain P of $GI/M/1$ type is α-recurrent and the matrix A is stochastic.*

(1) *If* $r(\alpha) > 1$, *P is α-positive if and only if* $\alpha < \bar{\alpha}$ *and* $\sum_{k=1}^{\infty} k r(\alpha)^{k-1} A_k < \infty$.

(2) *If* $r(\alpha) = 1$, *P is α-positive if* $\alpha < \bar{\alpha}$ *and* $\sum_{k=1}^{\infty} k^2 A_k < \infty$.

Proof If the matrix A is stochastic, then $D_k e \leqslant \sum_{i=k}^{\infty} A_i e$ for all $k \geqslant 2$. Therefore,

$$\sum_{k=2}^{\infty} k r(\alpha)^{k-1} D_k e \leqslant \sum_{k=2}^{\infty} \sum_{i=k}^{\infty} k r(\alpha)^{k-1} A_i e = \sum_{i=2}^{\infty} \sum_{k=2}^{i} k r(\alpha)^{k-1} A_i e$$
$$= \begin{cases} \displaystyle\sum_{i=2}^{\infty} \frac{i^2 + i - 2}{2} A_i e, & \text{if } r(\alpha) = 1, \\ \displaystyle\sum_{i=2}^{\infty} \frac{2r(\alpha) - r(\alpha)^2 - (i+1) r(\alpha)^i + i r(\alpha)^{i+1}}{(1 - r(\alpha))^2} A_i e, & \text{if } r(\alpha) \neq 1. \end{cases}$$

The rest of this proof is clear.

If the matrix A is stochastic, then Corollary 9.6 and (1) of Theorem 9.17 imply that the state α-classification of P is independent of the boundary matrices D_k for $k \geqslant 0$. This is similar to Theorem 1.3.2 in Neuts [21].

9.5 Markov Chains of *GI*/*G*/1 Type

In this section, we consider the quasi-stationary distribution of an irreducible Markov chain of $GI/G/1$ type, and provide conditions for the state α-classification and expressions for the quasi-stationary distribution.

From Eq. (3.1), the β-discounted transition matrix of $GI/G/1$ type is written as

$$\beta P = \begin{pmatrix} \beta D_0 & \beta D_1 & \beta D_2 & \beta D_3 & \cdots \\ \beta D_{-1} & \beta A_0 & \beta A_1 & \beta A_2 & \cdots \\ \beta D_{-2} & \beta A_{-1} & \beta A_0 & \beta A_1 & \cdots \\ \beta D_{-3} & \beta A_{-2} & \beta A_{-1} & \beta A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{9.90}$$

For the UL-type measures, we write that for $1 \leqslant i, j \leqslant n$,

$$\Phi_0(\beta) = P_{n,n}^{[\leqslant n]}(\beta),$$
$$\Phi_i(\beta) = P_{n-i,n}^{[\leqslant n]}(\beta),$$
$$\Phi_{-j}(\beta) = P_{n,n-j}^{[\leqslant n]}(\beta),$$

which are independent of the number $n \geqslant 1$ according to the censoring invariance.

We define the R-measure

$$R_{0,i}(\beta) = P_{0,i}^{[\leqslant i]}(\beta)[I - \Phi_0(\beta)]^{-1},$$
$$R_i(\beta) = \Phi_i(\beta)[I - \Phi_0(\beta)]^{-1}, \quad i \geqslant 1; \tag{9.91}$$

the G-measure

$$G_{j,0}(\beta) = [I - \Phi_0(\beta)]^{-1} P_{j,0}^{[\leqslant j]}(\beta),$$
$$G_j(\beta) = [I - \Phi_0(\beta)]^{-1} \Phi_{-j}(\beta), \quad j \geqslant 1; \tag{9.92}$$

and the U-measure

$$\Psi_0(\beta) = \beta D_0 + \sum_{k=1}^{\infty} R_{0,k}(\beta)[I - \Phi_0(\beta)] G_{k,0}(\beta)$$

and

$$\Phi_0(\beta) = \beta A_0 + \sum_{k=1}^{\infty} R_k(\beta)[I - \Phi_0(\beta)] G_k(\beta).$$

For the Markov chain of *GI*/*G*/1 type, the UL-type *RG*-factorization can be simplified as

$$I-\beta P=[I-R_U(\beta)][I-\Phi_D(\beta)][I-G_L(\beta)], \tag{9.93}$$

where

$$R_U(\beta)=\begin{pmatrix} 0 & R_{0,1}(\beta) & R_{0,2}(\beta) & R_{0,3}(\beta) & \cdots \\ & 0 & R_1(\beta) & R_2(\beta) & \cdots \\ & & 0 & R_1(\beta) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$\Phi_D(\beta)=\mathrm{diag}(\Psi_0(\beta),\Phi_0(\beta),\Phi_0(\beta),\Phi_0(\beta),\dots)$$

and

$$G_L(\beta)=\begin{pmatrix} 0 & & & & \\ G_{1,0}(\beta) & 0 & & & \\ G_{2,0}(\beta) & G_1(\beta) & 0 & & \\ G_{3,0}(\beta) & G_2(\beta) & G_1(\beta) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let

$$A^*(z,\beta)=\sum_{i=-\infty}^{\infty} z^i\beta A_i=\beta A^*(z),$$

$$R^*(z,\beta)=\sum_{i=1}^{\infty} z^i R_i(\beta),$$

$$G^*(z,\beta)=\sum_{j=1}^{\infty} z^{-j} G_j(\beta).$$

Using a similar analysis to that in Theorem 3.5, the *RG*-facorization for the repeated row is given by

$$I-A^*(z,\beta)=[I-R^*(z,\beta)][I-\Phi_0(\beta)][I-G^*(z,\beta)]. \tag{9.94}$$

Suppose $A=\sum_{k=-\infty}^{\infty} A_k$ is irreducible and stochastic. Let θ be the stationary probability vector of the Markov chain A, and

$$\rho=\frac{\theta\sum_{k=1}^{\infty} kA_k e}{\theta\sum_{k=1}^{\infty} kA_{-k} e}.$$

Now, we provide some useful properties for the positive root $z(\beta)$ to equation det $(I - A^*(z,\beta)) = 0$.

Let $z_1(\beta)$ and $z_2(\beta)$ be the positive solutions to the equations det $(I - R^*(z,\beta)) = 0$ and det $(I - G^*(z,\beta)) = 0$, respectively. It is clear that $z_1(\beta) \geqslant z_2(\beta)$ based on the definitions of $R^*(z,\beta)$ and $G^*(z,\beta)$. Note that

$$\{z(\beta) : \det (I - A^*(z,\beta)) = 0\} = \{z(\beta)) : \det (I - R^*(z,(\beta)) = 0\} \\ \bigcup \{z(\beta) : \det (I - G^*(z,(\beta)) = 0\} \qquad (9.95)$$

thus $z_1(\beta)$ and $z_2(\beta)$ must be two positive solutions to the equations det $(I - A^*(z,\beta)) = 0$.

Let $r(z,\beta)$ and $g(z,\beta)$ be the maximal eigenvalues of the matrices $R^*(z,\beta)$ and $G^*(z,\beta)$, respectively. It follows from Eq. (9.95) that

$$\{z(\beta) : \beta\chi(z) = 1\} = \{z(\beta) : r(z,\beta) = 1\} \bigcup (z(\beta) : g(z,\beta) = 1\}.$$

Thus, it is easy to see that $z_1(\beta)$ and $z_2(\beta)$ are the positive solutions to equations $r(z,\beta) = 1$ and $g(z,\beta) = 1$, respectively.

Lemma 9.17 *For* $0 < \beta \leqslant \alpha, z_1(\beta)$ *is strictly decreasing, while* $z_2(\beta)$ *is strictly increasing. Specifically,* $z_1(\alpha) = z_2(\alpha)$.

Proof Note that $R^*(z,\beta) = \sum_{i=1}^{\infty} z^i R_i(\beta)$ and

$$R_i(\beta) = \Phi_i(\beta) \sum_{n=0}^{\infty} [\Phi_0(\beta)]^n,$$

it is easy to see that $R_i(\beta)$ is monotonely increasing in $\beta \geqslant 0$ for each $i \geqslant 1$, and so is $R^*(z,\beta)$. Hence $r(z,\beta)$ is monotonely increasing in $\beta \geqslant 0$. It is easy to check that the equation $r(z,\beta) = 1$ in dicates that $z_1(\beta)$ is monotonely decreasing in $\beta \geqslant 0$. Similarly, $z_2(\beta)$ is monotonely increasing in $\beta \geqslant 0$. Based on this, we can obtain that for $0 \leqslant \beta \leqslant \alpha$,

$$z_1(\beta) \geqslant z_1(\alpha) \geqslant z_2(\alpha) \geqslant z_2(\beta).$$

Since there is the unique positive solution $z(\alpha)$ to det $(I - A^*(z,\beta)) = 0$, we have that $z_1(\alpha) = z_2(\alpha) = z(\alpha)$. This completes the proof.

Theorem 9.18 (1) *If* $\rho < 1$, *then* $z_1(\beta) \geqslant z_2(\beta) > 1$ *for* $1 < \beta \leqslant \alpha$, *while* $z_2(\beta) \leqslant 1 < z_1(\beta)$ *for* $0 < \beta \leqslant 1$.

(2) *If* $\rho > 1$, *then* $z_2(\beta) \leqslant z_1(\beta) < 1$ *for* $1 < \beta \leqslant \alpha$, *while* $z_2(\beta) < z_2(\alpha) < 1 \leqslant z_1(\beta)$ *for* $0 < \beta \leqslant 1$.

(3) *If* $\rho=1$, *then* $\alpha=1$ *and* $z_1(\alpha)=z_2(\alpha)=1, z_2(\beta)\leqslant 1\leqslant z_1(\beta)$ *for* $0<\beta\leqslant\alpha$.

Proof We only prove (1), while (2) and (3) can be proved similarly.

We construct a Markov chain of *GI*/*G*/1 type whose transition matrix is given by

$$P=\begin{pmatrix} D_0 & \sum\limits_{i=1}^{\infty} D_i & 0 & 0 & \cdots \\ D_{-1} & A_0 & A_1 & A_2 & \cdots \\ D_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ D_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

If $\rho<1$, then it is easy to see from the special boundary that P is positive recurrent, hence $sp(R)<1$ and $sp(G)=1$ by Theorem 3.17. Applying Lemma 9.17, we obtain that for $1<\beta\leqslant\alpha$,

$$z_1(\beta)\geqslant z_1(\alpha)=z_2(\alpha)\geqslant z_2(\beta)>z_2(1)=sp(G)=1,$$

and for $1<\beta\leqslant 1$,

$$z_1(\beta)\geqslant z_1(\alpha)>z_2(1)\geqslant sp(G)=1$$

and

$$z_2(\beta)<z_2(1)=sp(G)=1.$$

This completes the proof.

For intuitively understanding Lemma 9.17 and Theorem 9.18, Fig. 9.3 to Fig. 9.5 describe the structure of $z_1(\beta)$ and $z_2(\beta)$ for $0<\beta\leqslant\alpha$, which are related to each of the three cases: $0<\rho<1$, $\rho=1$ and $\rho>1$, respectively.

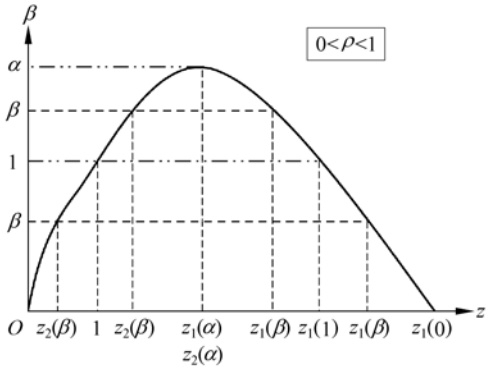

Figure 9.3 The case with $0<\rho<1$

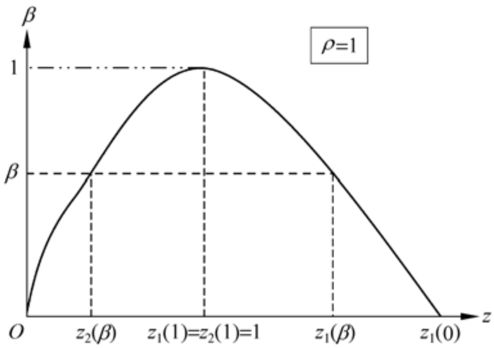

Figure 9.4 The case with $\rho = 1$

Applying Lemma 9.17 and Theorem 9.18, the following three corollaries are easy to be deriven. Note that the last one is the main drift result for Markov chains of $GI/G/1$ type.

Corollary 9.7 (1) $\rho < 1$ *if and only if* $z_1(\alpha) = z_2(\alpha) > 1$.

(2) $\rho > 1$ *if and only if* $z_1(\alpha) = z_2(\alpha) < 1$.

(3) $\rho = 1$ *if and only if* $z_1(\alpha) = z_2(\alpha) = 1$.

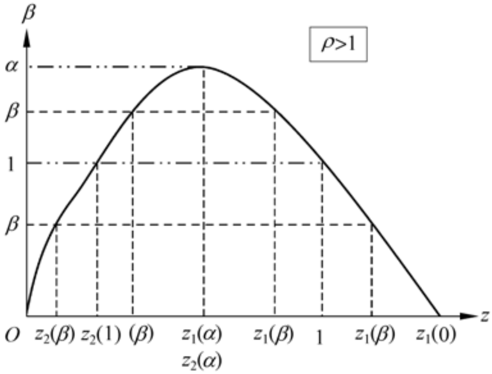

Figure 9.5 The case with $\rho > 1$

Corollary 9.8 *The radius* α *of convergence of the Markov chain of* $GI/G/1$ *type is the minimal positive solution to the equation* $z_1(x) = z_2(x)$ *for* $x > 0$.

Corollary 9.9 *For* $\beta = 1$, *we have*

(1) $\rho < 1$ *if and only if* $z_1(1) > 1$ *and* $z_2(1) = 1$.

(2) $\rho < 1$ *if and only if* $z_1(1) = 1$ *and* $z_2(1) < 1$.

(3) $\rho = 1$ *if and only if* $z_1(1) = z_2(1) = 1$.

We write

$$\Psi(\beta) = \beta D_0 + \beta^2 (D_1, D_2, D_3, \ldots)(I - \beta W)^{-1}_{\min} (D^{\mathrm{T}}_{-1}, D^{\mathrm{T}}_{-2}, D^{\mathrm{T}}_{-3}, \ldots)^{\mathrm{T}}.$$

Let $\xi(\beta)$ be the maximal eigenvalue of the censored β-discounted transition matrix $\Psi_0(\beta)$ to level 0. Then the solution to the eigenvalue equation $\xi(\beta)=1$ can determine the state α-classification of the Markov chain of *GI/G/*1 type. The following theorem provides conditions for the state α-classification.

Theorem 9.19 (1) *If there exists a minimal positive solution* ζ *to the equation* $\xi(\beta)=1$ *for* $\beta>0$, *then* $\alpha=\zeta$ *and the Markov chain P is* α*-recurrent.*

(2) *If there does not exist any positive solution to the equation* $\xi(\beta)=1$ *for* $\beta>0$, *then the Markov chain P is* α*-transient.*

Proof We first prove (1)

Let

$$\begin{pmatrix} N_{0,0}(\beta) & N_{0,1}(\beta) & N_{0,2}(\beta) & \cdots \\ N_{1,0}(\beta) & N_{1,1}(\beta) & N_{1,2}(\beta) & \cdots \\ N_{2,0}(\beta) & N_{2,1}(\beta) & N_{2,2}(\beta) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} = \sum_{k=0}^{\infty} (\beta P)^k .$$

Then

$$N_{0,0}(\beta)=[I-\Psi_0(\beta)]^{-1}.$$

Let $\nu(\beta)$ be the the maximal eigenvalue of the matrix $N_{0,0}(\beta)$. Then

$$\nu(\beta)=\frac{1}{1-\xi(\beta)}, \quad \beta>0.$$

If $\xi(\zeta)=1$, then

$$\nu(\zeta)=\frac{1}{1-\xi(\zeta)}=+\infty, \tag{9.96}$$

while

$$\nu(\beta)=\frac{1}{1-\xi(\beta)}<+\infty, \quad 0<\beta>\zeta. \tag{9.97}$$

Note that

$$\min_{1\leqslant i\leqslant m_0}\left\{\sum_{j=1}^{m_0}[N_{0,0}(\beta)]_{i,j}\right\}\leqslant \nu(\beta)\leqslant \max_{1\leqslant i\leqslant m_0}\left\{\sum_{j=1}^{m_0}[N_{0,0}(\beta)]_{i,j}\right\}. \tag{9.98}$$

By using Eq. (9.96) and Eq. (9.97), the right inequation of Eq. (9.98) indicates that there exists at least a pair (i_0, j_0) such that

$$[N_{0,0}(\zeta)]_{i_0,j_0}=+\infty$$

while the left inequality of Eq. (9.98) indicates that for any pair (i,j),

$$[N_{0,0}(\beta)]_{i,j} < +\infty, \quad 0 < \beta < \zeta.$$

Based on the definition of the state α-classification, it is seen that $\alpha = \zeta$ and the Markov chain P is α-recurrent.

Now, we prove (2).

If there does not exist any positive solution to the equation $\xi(\beta) = 1$ for $0 < \beta \leqslant \alpha$, then

$$[N_{0,0}(\beta)]_{i,j} < +\infty, \quad 0 < \beta \leqslant \alpha,$$

and

$$[N_{0,0}(\alpha + \varepsilon)]_{i,j} = +\infty, \quad \text{for any } \varepsilon > 0.$$

Based on the definition of the state α-classification, it is seen that the Markov chain P is α-transient. This completes the proof.

Let

$$W = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 & \cdots \\ A_{-1} & A_0 & A_1 & A_2 & \cdots \\ A_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ A_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The following corollary provides the state α-classification for the Markov chain W. Note that it is a generalization of the state α-classification for Markov chains of $M/G/1$ type or $GI/M/1$ type by Kijima [14].

Corollary 9.10 *The Markov chain W is always α-transient.*

Proof Let

$$\Psi_0(\beta) = \beta A_0 + \beta^2 (A_1, A_2, A_3, \ldots)(I - \beta W)^{-1}_{\min} (A^{\mathrm{T}}_{-1}, A^{\mathrm{T}}_{-2}, A^{\mathrm{T}}_{-3}, \ldots)^{\mathrm{T}}.$$

Then it is clear that $\xi(\beta) < 1$ for all $\beta > 0$. Hence the Markov chain W is always α-transient.

In what follows, we provide expressions for the quasi-stationary distribution.

Applying the RG-factorization Eq. (9.93), the following theorem expresses the quasi-stationary distribution of the Markov chain P of $GI/G/1$ type which is α-recurrent. The proof is clear, and thus is omitted here.

Theorem 9.20 *If the Markov chain P of $GI/G/1$ type is α-recurrent, then the quasi-stationary distribution $\pi(\alpha) = (\pi_0(\alpha), \pi_1(\alpha), \pi_2(\alpha), \ldots)$ is given by*

$$\begin{cases} \pi_0(\alpha) = \tau x_0(\alpha), \\ \pi_k(\alpha) = \pi_0(\alpha) R_{0,k}(\alpha) + \sum\limits_{i=1}^{k-1} \pi_i(\alpha) R_{k-i}(\alpha), \quad k \geqslant 1, \end{cases}$$

where $x_0(\alpha)$ is the left Perron-Frobeniusthe vector of the matrix $\Psi_0(\alpha)$ and the scalar τ is uniquely determined by $\sum_{k=0}^{M} \pi_k(\alpha)\, e = 1.$

Now, we consider $1 \leqslant \beta < \alpha$ or $\beta = \alpha$ under which *P* is α-transient. In this case, it is a key to solve the equation

$$(y_0(\beta), y_1(\beta), y_2(\beta), \ldots) \begin{pmatrix} I & & & & \\ -G_{1,0}(\beta) & I & & & \\ -G_{2,0}(\beta) & -G_1(\beta) & I & & \\ -G_{3,0}(\beta) & -G_2(\beta) & -G_1(\beta) & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (0, 0, 0, \ldots),$$

which leads to

$$\begin{cases} y_0(\beta) - \sum_{k=1}^{\infty} y_k(\beta)\, G_{k,0}(\beta) = 0, \\ y_k(\beta) - \sum_{i=k+1}^{\infty} y_i(\beta)\, G_{i-k}(\beta) = 0, \quad k \geqslant 1. \end{cases} \tag{9.99}$$

The following lemma provides a spectral property for the matrices $G^*(z,\beta)$ and $G_0^*(z,\beta) = \sum_{j=1}^{\infty} z^{-j} G_{j,0}(\beta)$. Thus, the quasi-stationary distribution can be directly expressed by means of the eigenvalues and eigenvectors of the matrices $G^*(z,\beta)$ and $G_0^*(z,\beta)$.

Lemma 9.18 *There exists a constant* $\gamma > 0$, *and two nonnegative non-zero row vectors* y *and* y_0 *such that*

$$\begin{cases} y = y \sum_{k=1}^{\infty} \gamma^{-k} G_k(\beta), \\ y_0 = y \sum_{k=1}^{\infty} \gamma^{-k} G_{k,0}(\beta). \end{cases} \tag{9.100}$$

Proof For the equation det $(I - G^*(z,\beta)) = 0$, there exists a positive root $\gamma = z_2(\beta)$. That is, det $(I - G^*(\gamma,\beta)) = 0$. In this case, there must exist a nonnegative non-zero row vector y such that $y = yG^*(\gamma,\beta)$, i.e., $y = y\sum_{k=1}^{\infty} \gamma^{-k} G_k(\beta)$. Once γ and y are given, we write

$$y_0 = y \sum_{k=1}^{\infty} \gamma^{-k} G_{k,0}(\beta).$$

by means of the first equality of Eq. (9.100). This completes the proof.

The following lemma provides a nonnegative non-zero solution to the equation Eq. (9.99). Such a solution is a key for expressing the quasi-stationary distribution.

Lemma 9.19 *The row vector* $Y=(y_0, y\gamma^{-1}, y\gamma^{-2}, y\gamma^{-3},\dots)$ *is a nonnegative non-zero solution to the equation Eq.* (9.99).

Proof Let $y_0(\beta)=y_0$ and $y_k(\beta)=y\gamma^{-k}$ for $k\geqslant 1$. It is easy to see that

$$y_0(\beta)-\sum_{k=1}^{\infty} y_k(\beta)\, G_{k,0}(\beta)=y_0-\sum_{k=1}^{\infty} y\gamma^{-k}G_{k,0}(\beta)=0$$

in terms of the second equality of Eq. (9.100). At the same time, for $k\geqslant 1$ we have

$$\begin{aligned}
y_k(\beta)-\sum_{i=k+1}^{\infty} y_i(\beta)G_{i-k}(\beta)&=y\gamma^{-k}-\sum_{i=k+1}^{\infty} y\gamma^{-i}G_{i-k}(\beta)\\
&=\gamma^{-k}\left[y-y\sum_{k=1}^{\infty}\gamma^{-k}G_k(\beta)\right]\\
&=0
\end{aligned}$$

in terms of the first equality of Eq. (9.100). This completes the proof.

Theorem 9.21 *If* $1\leqslant\beta<\alpha$ *or* $\beta=\alpha$ *under which P is* α*-transient, then the quasi-stationary distribution* $\pi(\beta)=(\pi_0(\beta),\pi_1(\beta),\pi_2(\beta),\dots)$ *is given by*

$$\begin{cases}\pi_0(\beta)=\kappa y_0 N_0(\beta),\\ \pi_k(\beta)=\kappa\gamma^{-k}yN(\beta)+\pi_0(\beta)R_{0,k}(\beta)+\displaystyle\sum_{i=1}^{k}\pi_i(\beta)R_{k-i}(\beta), \quad k\geqslant 1,\end{cases}$$

where $N_0(\beta)=[I-\Psi_0(\beta)]^{-1}, N(\beta)=[I-\Phi_0(\beta)]^{-1}$ *and the positive constant* κ *satisfies* $\sum_{k=0}^{\infty}\pi_k(\beta)\, e=1$.

Proof Since $\pi(\beta)(I-\beta P)=0$, it is clear from the UL-type *RG*-factorization that

$$\pi(\beta)[I-R_U(\beta)][I-\Phi_D(\beta)][I-G_L(\beta)]=0.$$

Let

$$Y=\pi(\beta)[I-R_U(\beta)][I-\Phi_D(\beta)]. \tag{9.101}$$

Then

$$Y[I-G_L(\beta)]=0,$$

which leads to $Y=\kappa(y_0, y\gamma^{-1}, y\gamma^{-2}, y\gamma^{-3},\dots)$ in terms of Lemma 9.19. It follows from Eq. (9.101) that

$$\pi(\beta)[I-R_U(\beta)][I-\Phi_D(\beta)]=\kappa(y_0, y\gamma^{-1}, y\gamma^{-2}, y\gamma^{-3},\dots).$$

Thus, we obtain

$$\pi(\beta)[I - R_U(\beta)] = \kappa(y_0, y\gamma^{-1}, y\gamma^{-2}, y\gamma^{-3}, \ldots)[I - \Phi_D(\beta)]^{-1}$$
$$= \kappa(y_0 N_0(\beta), \gamma^{-1} yN(\beta), \gamma^{-2} yN(\beta), \gamma^{-3} yN(\beta), \ldots).$$

Some simple computations lead to the desired result.

9.6 Level-Dependent QBD Processes

In this section, we consider the quasi-stationary distribution of an irreducible level-dependent QBD process, and provide conditions for the state α-classification and expressions for the quasi-stationary distribution.

Consider an irreducible discrete-time QBD process whose transition probability matrix is given by

$$P = \begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & & \\ & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & \\ & & A_2^{(3)} & A_1^{(3)} & A_0^{(3)} \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let α be the radius of convergence of the matrix P. It is clear that $\alpha \geqslant 1$.

We write

$$P = \begin{pmatrix} T^{(k)} & V^{(k)} \\ U^{(k)} & Q^{(k)} \end{pmatrix}, \quad k \geqslant 0,$$

where

$$Q^{(k)} = \begin{pmatrix} A_1^{(k)} & A_0^{(k)} & & & \\ A_2^{(k+1)} & A_1^{(k+1)} & A_0^{(k+1)} & & \\ & A_2^{(k+2)} & A_1^{(k+2)} & A_0^{(k+2)} & \\ & & A_2^{(k+3)} & A_1^{(k+3)} & A_0^{(k+3)} \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

the other matrices can be determined accordingly. Obviously, $Q^{(0)} = P$. We further assume that the Markov chain $Q^{(k)}$ is irreducible for each $k \geqslant 0$. Let α_k be the radius of convergence of the matrix $Q^{(k)}$ for $k \geqslant 0$. Since

$$\alpha = \sup\left\{ z \geqslant 1 : \sum_{n=0}^{\infty} z^n p_{(i,r),(j,s)}^{(n)} < \infty, i, j \geqslant 0 \right\}$$

and

$$\alpha_k = \sup\left\{ z \geqslant 1 : \sum_{n=0}^{\infty} z^n p_{(i,r),(j,s)}^{(n)} < \infty, i, j \geqslant k \geqslant 1 \right\},$$

we obtain $\alpha_k \geqslant \alpha$ for $k \geqslant 1$.

9.6.1 The UL-Type *RG*-Factorization

Now, we provide the UL-type R-, U- and G-measures for the matrix βP for $0 < \beta \leqslant \alpha$.

(1) The R-measure

The R-measure: $R_k(\beta)$ for $k \geqslant 0$, is the minimal nonnegative solution to the system of equations

$$\beta A_0^{(k)} + R_k(\beta)\beta A_1^{(k+1)} + R_k(\beta)R_{k+1}(\beta)\beta A_2^{(k+2)} = R_k(\beta), \quad k \geqslant 0,$$

or

$$A_0^{(k)} + R_k(\beta)A_1^{(k+1)} + R_k(\beta)R_{k+1}(\beta)A_2^{(k+2)} = \frac{1}{\beta}R_k(\beta), \quad k \geqslant 0.$$

Specifically, since $\gamma = 1/\alpha$ is the decay parameter of the Markov chain P, we have

$$A_0^{(k)} + R_k(\gamma)A_1^{(k+1)} + R_k(\gamma)R_{k+1}(\gamma)A_2^{(k+2)} = \gamma R_k(\gamma), \quad k \geqslant 0.$$

(2) The G-measure

The G-measure: $G_l(\beta)$ for $l \geqslant 1$, is the minimal nonnegative solution to the system of equations

$$\beta A_0^{(l)} G_{l+1}(\beta) G_l(\beta) + \beta A_1^{(l)} G_l(\beta) + \beta A_2^{(l)} = G_l(\beta), \quad l \geqslant 1,$$

or

$$A_0^{(l)} G_{l+1}(\beta) G_l(\beta) + A_1^{(l)} G_l(\beta) + A_2^{(l)} = \frac{1}{\beta} G_l(\beta), \quad l \geqslant 1.$$

Specifically, for the decay parameter γ we have

$$A_0^{(l)} G_{l+1}(\gamma) G_l(\gamma) + A_1^{(l)} G_l(\gamma) + A_2^{(l)} = \gamma G_l(\gamma), \quad l \geqslant 1.$$

(3) The U-measure

The U-measure: $\Psi_k(\beta)$ for $k \geqslant 0$, is given by

$$\Psi_k(\beta) = \beta A_1^{(k)} + R_k(\beta)\beta A_2^{(k+1)}$$

or

$$\Psi_k(\beta) = \beta A_1^{(k)} + \beta A_0^{(k)} G_{k+1}(\beta).$$

The following theorem provides useful properties for the U-measure: $\Psi_k(\beta)$ for $k \geqslant 0$.

Theorem 9.22 (1) *For $k \geqslant 1$, the matrices $I - \Psi_k(\beta)$ are all invertible for $0 < \beta \leqslant \alpha$.*

(2) *the matrix $I - \Psi_0(\beta)$ is invertible for $0 < \beta < \alpha$. The matrix $I - \Psi_0(\alpha)$ is invertible if the Markov chain P is α-transient; while $I - \Psi_0(\alpha)$ is singular if the Markov chain P is α-recurrent.*

Proof (1) According to the censoring technique, it is easy to see that for $k \geqslant 1$, the Markov chains $\Psi_k(\beta)$ are all α-transient. Thus, the matrices $I - \Psi_k(\alpha)$ for $k \geqslant 1$ are all invertible for $0 < \beta \leqslant \alpha$.

(2) Based on the censoring technique, it is clear that if the Markov chain P is α-transient, then the Markov chain $\Psi_0(\alpha)$ is α-transient. Hence, the matrix $I - \Psi_0(\alpha)$ is invertible; if the Markov chain P is α-recurrent, then the Markov chain $\Psi_0(\alpha)$ is α-recurrent, hence the matrix $I - \Psi_0(\alpha)$ is singular.

This completes the proof.

Note that the Markov chain P always has some related properties to the censored Markov chain $\Psi_0(\beta)$ to level 0, it is necessary to provide a detailed analysis for the matrix $\Psi_0(\beta)$ for $0 < \beta \leqslant \alpha$. Let

$$N_0(\beta) = \sum_{n=0}^{\infty} [\Psi_0(\beta)]^n.$$

Then

$$N_0(\beta) = [I - \Psi_0(\beta)]^{-1}.$$

Based on (2) in Theorem 9.22, it is easy to see that

$$\alpha = \sup\{\beta \geqslant 1 : N_0(\beta) < +\infty\}$$

or

$$\alpha = \sup\{\beta \geqslant 1 : \det\,(I - \Psi_0(\beta)) \neq 0\}.$$

The following theorem provides conditions for the state α-classification of the QBD process P. The proof is clear, and thus is omitted here.

Theorem 9.23 (1) *If the matrix $N_0(\alpha)$ is finite, then P is α-transient.*

(2) *If the matrix $N_0(\alpha)$ is infinite, then P is α-recurrent.*

Based on Theorem 9.23, we provide other conditions for the state α-classification of the QBD process P as follows:

(1) If det $(I - \Psi_0(\alpha)) \neq 0$, then P is α-transient.

(2) If det $(I-\Psi_0(\alpha))=0,$ then P is α-recurrent.

(3) Let $u_0(\beta)$ be the maximal eigenvalue of the matrix $\Psi_0(\beta).$ If there does not exist a positive solution to the equation $u_0(\beta)=1$ for $\beta\geqslant 1,$ then P is α-transient.

(4) If there exists a positive solution η to the equation $u_0(\beta)=1$ for $\beta\geqslant 1,$ then P is α-recurrent.

Based on the R-, U- and G-measures, the UL-type RG-factorization is given by

$$I-\beta P=[I-R_U(\beta)][I-\Psi_D(\beta)][I-G_L(\beta)],$$

where

$$R_U(\beta)=\begin{pmatrix} 0 & R_0(\beta) & & & \\ & 0 & R_1(\beta) & & \\ & & 0 & R_2(\beta) & \\ & & & \ddots & \ddots \end{pmatrix},$$

$$\Psi_D(\beta)=\text{diag}\,(\Psi_0(\beta),\Psi_1(\beta),\Psi_2(\beta),\ldots)$$

and

$$G_L(\beta)=\begin{pmatrix} 0 & & & & \\ G_1(\beta) & 0 & & & \\ & G_2(\beta) & 0 & & \\ & & G_3(\beta) & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Now, we provide expressions for the quasi-stationary distribution of the level-dependent QBD process. Our analysis is classified in two sets of expressions: α-recurrent with $\beta=\alpha,$ and α-recurrent with $\beta<\alpha$ or α-transient with $\beta\leqslant\alpha.$

The first set: α-recurrent with $\beta=\alpha$

Note that $\pi(\alpha)[I-\alpha P]=0,$ we have

$$\pi(\alpha)[I-R_U(\alpha)][I-\Psi_D(\alpha)][I-G_L(\alpha)]=0.$$

Let $x(\alpha)=\pi(\alpha)[I-R_U(\alpha)].$ Then

$$x(\alpha)[I-\Psi_D(\alpha)][I-G_L(\alpha)]=0.$$

Since the QBD process P is α-recurrent, we can obtain

$$x(\alpha)=(\tau x_0(\alpha),0,0,\ldots),$$

where $x_0(\alpha)$ is the left Perron-Frobeniusthe vector of the matrix $\Psi_0(\alpha).$ Therefore,

$$\pi(\alpha)[I-R_U(\alpha)]=(\tau x_0(\alpha),0,0,...),$$

which leads to the quasi-stationary distribution of the QBD process P as follows:

$$\pi_0(\alpha)=\tau x_0(\alpha), \tag{9.102}$$

$$\pi_k(\alpha) = \pi_0(\alpha) R_0(\beta) R_1(\beta) \ldots R_{k-1}(\beta), \quad k \geqslant 1, \tag{9.103}$$

where the constant τ makes $\sum_{k=0}^{\infty} \pi_k(\alpha) e = 1$.

The second set: α-recurrent with $\beta < \alpha$ or α-transient with $\beta \leqslant \alpha$

For $\beta \leqslant \alpha$ under which P is α-transient or $\beta < \alpha$ under which P is α-recurrent, it is clear that the matrix $I - \Psi_0(\beta)$ is invertible. In this case, we need to use the following lemma to compute the quasi-stationary distribution.

Lemma 9.20 *For the G-measure: $G_l(\beta)$ for $l \geqslant 1$, there exists a sequence of stochastic vectors $\{z_k(\beta), k \geqslant 0\}$ and a sequence of positive numbers $\{\rho_k, k \geqslant 0\}$ such that*

$$z_{k+1}(\beta) G_{k+1}(\beta) = \rho_k z_k(\beta), \quad k \geqslant 0.$$

Proof Let $z_{N,N}(\beta)$ be an arbitrary probability vector on level $N \geqslant 1$. Since the QBD process P is irreducible, each state on level k has a path to level $k-1$, which shows that each row of the matrix $G_k(\beta)$ is non-zero and nonnegative. Hence $z_{N,N}(\beta) G_N(\beta)$ is a convex combination of the rows of $G_k(\beta)$. We take

$$\rho_{N-1} z_{N,N-1}(\beta) = z_{N,N}(\beta) G_N(\beta) e.$$

Then $\rho_{N-1} > 0$ and

$$z_{N,N-1}(\beta) = \frac{1}{\rho_{N-1}} z_{N,N}(\beta) G_N(\beta)$$

is a probability vector on level $N-1$. Proceeding inductively, we can obtain a sequence of probability vector $\{z_{N,k}(\beta), 0 \leqslant k \leqslant N-1\}$ and a sequence of positive scalars $\{\rho_{N,k}, 0 \leqslant k \leqslant N-1\}$ such that for $0 \leqslant k \leqslant N-1$,

$$z_{N,k+1}(\beta) G_{k+1}(\beta) = \rho_{N,k} z_{N,k}(\beta).$$

By repeating the above procedure, for each $k \geqslant 0$ we can obtain a sequence of probability vector $\{z_{N,k}(\beta), N \geqslant k\}$. Since $z_{N,k}(\beta) e = 1$ for $N \geqslant k$, there exists a subsequence $\{z_{N_r,k}(\beta),\ r \geqslant 1\}$ such that $\lim_{r \to \infty} z_{N_r,k}(\beta) = z_k(\beta)$, and $z_k(\beta)$ is a probability vector. $k = k^*$, we can obtain a sequence of stochastic vectors $\{z_k(\beta), 0 \leqslant k \leqslant k^*\}$ and a sequence of positive numbers $\{\rho_k, 0 \leqslant k \leqslant k^*\}$ such that

$$z_{k+1}(\beta) G_{k+1}(\beta) = \rho_k z_k(\beta).$$

Again by the compactness of the probability vector set, there exists a subsequence $\{N_r'\}$ of $\{N_r\}$ such that $\lim_{r \to \infty} z_{N_r', k^*+1}(\beta) = z_{k^*+1}(\beta)$, and $\rho_{k^*} = z_{k^*+1}(\beta) G_{k^*+1}(\beta) e$.

Therefore, by induction the desired result follows by repeating this argument infinitely-many times, which can be done by means of the Axiom of Choice. This completes the proof.

From $\pi(\beta)[I-\beta P]=0,$ we have

$$\pi(\beta)[I-R_U(\beta)][I-\Psi_D(\beta)][I-G_L(\beta)]=0.$$

Let $x(\beta)=\pi(\beta)[I-R_U(\beta)][I-\Psi_D(\beta)].$ Then

$$x(\beta)[I-G_U(\beta)]=0,$$

which is, in block entries,

$$x_k(\beta)=x_{k+1}(\beta)\,G_{k+1}(\beta),\quad k\geqslant 0.$$

Using Lemma 9.20, we write

$$x_0(\beta)=z_0(\beta)$$
$$x_k(\beta)=\frac{1}{\rho_0\rho_1\cdots\rho_{k-1}}z_k(\beta),\quad k\geqslant 1.$$

It is easy to check that

$$x_0\,(\beta)=z_0(\beta)=\frac{1}{\rho_0}z_1(\beta)G_1(\beta)=x_1(\beta)G_1(\beta),$$
$$x_1\,(\beta)=\frac{1}{\rho_0}z_1(\beta)=\frac{1}{\rho_0\rho_1}z_2(\beta)G_2(\beta)=x_2(\beta)G_2(\beta),$$

we assume that for $n=k$ we have

$$\begin{aligned}x_k(\beta)&=\frac{1}{\rho_0\rho_1\cdots\rho_{k-1}}z_k(\beta)=\frac{1}{\rho_0\rho_1\cdots\rho_{k-1}\rho_k}z_{k+1}(\beta)G_{k+1}(\beta)\\&=x_{k+1}(\beta)G_{k+1}(\beta),\end{aligned}$$

then for $n=k+1$ we obtain

$$\begin{aligned}x_{k+1}(\beta)&=\frac{1}{\rho_0\rho_1\cdots\rho_k}z_{k+1}(\beta)=\frac{1}{\rho_0\rho_1\cdots\rho_k\rho_{k+1}}z_{k+2}(\beta)G_{k+2}(\beta)\\&=x_{k+2}(\beta)G_{k+2}(\beta).\end{aligned}$$

Therefore, by induction, for any nonnegative integer n we can obtain

$$x_n(\beta)=\frac{1}{\rho_0\rho_1\cdots\rho_{n-1}}z_n(\beta)=x_{n+1}(\beta)G_{n+1}(\beta).$$

It is clear that the vector

$$x(\beta)=\left(z_0(\beta),\frac{1}{\rho_0}z_1(\beta),\frac{1}{\rho_0\rho_1}z_2(\beta),\dots\right)$$

is a non-zero nonnegative solution to the equation $x(\beta)[I-G_L(\beta)]=0.$ Thus, we have

$$\pi(\beta)[I-R_U(\beta)][I-\Psi_D(\beta)]=x(\beta),$$

which leads to

$$\pi(\beta)=x(\beta)[I-\Psi_D(\beta)]^{-1}[I-R_U(\beta)]^{-1}.$$

Let

$$N_l(\beta)=[I-\Psi_l(\beta)]^{-1}$$

and

$$X_k^{(l)}(\beta)=R_l(\beta)R_{l+1}(\beta)\dots R_{l+k-1}(\beta),\quad k\geqslant 1, l\geqslant 0.$$

Then

$$[I-\Psi_D(\beta)]^{-1}=\mathrm{diag}(N_0(\beta),N_1(\beta),N_2(\beta),\dots)$$

and

$$[I-R_U(\beta)]^{-1}=\begin{pmatrix} I & X_1^{(0)}(\beta) & X_2^{(0)}(\beta) & X_3^{(0)}(\beta) & \dots \\ & I & X_1^{(1)}(\beta) & X_2^{(1)}(\beta) & \dots \\ & & I & X_1^{(2)}(\beta) & \dots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Therefore, we obtain

$$\pi_0(\beta)=x_0(\beta)N_0(\beta)=z_0(\beta)N_0(\beta) \tag{9.104}$$

and for $k\geqslant 1$,

$$\begin{aligned}\pi_k(\beta)&=x_k(\beta)N_k(\beta)+\sum_{l=0}^{k-1}x_l(\beta)N_l(\beta)X_{k-l}^{(l)}(\beta)\\&=\frac{1}{\rho_0\rho_1\cdots\rho_{k-1}}z_k(\beta)N_k(\beta)+\sum_{l=0}^{k-1}\frac{1}{\rho_0\rho_1\cdots\rho_{l-1}}z_l(\beta)N_l(\beta)\\&\quad\cdot R_l(\beta)R_{l+1}(\beta)\dots R_{k-1}(\beta).\end{aligned} \tag{9.105}$$

Based on Eq. (9.102) to Eq. (9.105), we summarize the expressions for the quasi-stationary distribution of the level-dependent QBD process as the following theorem.

Theorem 9.24 (1) *For* $\beta = \alpha$ *under which* P *is* α*-recurrent, the quasi-stationary distribution of the level-dependent QBD process is given by*

$$\pi_0(\alpha) = \tau x_0(\alpha),$$
$$\pi_k(\alpha) = \pi_0(\alpha) R_0(\beta) R_1(\beta) \ldots R_{k-1}(\beta), \quad k \geqslant 1,$$

where $x_0(\alpha)$ *is the left Perron-Frobeniusthe vector of the matrix* $\Psi_0(\alpha)$ *and the constant* τ *makes* $\sum_{k=0}^{\infty} \pi_k(\alpha) e = 1$.

(2) *For* $\beta \leqslant \alpha$ *under which* P *is* α*-transient or* $\beta < \alpha$ *under which* P *is* α*-recurrent, the quasi-stationary distribution of the level-dependent QBD process is given by*

$$\pi_0(\beta) = x_0(\beta) N_0(\beta) = z_0(\beta) N_0(\beta)$$

and for $k \geqslant 1$,

$$\pi_k(\beta) = \frac{1}{\rho_0 \rho_1 \cdots \rho_{k-1}} z_k(\beta) N_k(\beta) + \sum_{l=0}^{k-1} \frac{1}{\rho_0 \rho_1 \cdots \rho_{l-1}} z_l(\beta) N_l(\beta) R_l(\beta) R_{l+1}(\beta) \ldots R_{k-1}(\beta).$$

Remark 9.5 *Since for* $l \geqslant 0$

$$z_l(\beta) = \frac{1}{\rho_l} z_{l+1}(\beta) G_{l+1}(\beta) = \frac{1}{\rho_l \rho_{l+1} \cdots \rho_{k-1}} z_k(\beta) G_k(\beta) \ldots G_{l+2}(\beta) G_{l+1}(\beta),$$

we obtain

$$\pi_k(\beta) = \frac{1}{\rho_0 \rho_1 \cdots \rho_{k-1}} z_k(\beta) \left[N_k(\beta) + \sum_{l=0}^{k-1} G_k(\beta) \ldots G_{l+2}(\beta) G_{l+1}(\beta) N_l(\beta) \cdot R_l(\beta) R_{l+1}(\beta) \ldots R_{k-1}(\beta) \right]. \tag{9.106}$$

9.6.2 Conditions for α-Positive Recurrence

Now, we provide a condition under which the QBD process P is α-positive recurrent. Let

$$N(\alpha) = \sum_{k=0}^{\infty} (\alpha P)^k$$

and $N_{i,j}(\alpha)$ the (i,j)th block-entry of the matrix $N(\alpha)$. It is clear that

$$N_{0,0}(\alpha)=\sum_{k=0}^{\infty}\Psi_0^k(\alpha)$$

and

$$N_{1,1}(\alpha)=\sum_{k=0}^{\infty}\Psi_1^k(\alpha),$$

where

$$\Psi_0(\alpha)=\alpha A_1^{(0)}+\alpha^2 A_0^{(0)}N_{1,1}(\alpha)A_2^{(1)}.$$

The following theorem provides a condition under which the QBD process P is α-positive recurrent.

Theorem 9.25 *Suppose the QBD process P is α-recurrent.*

(1) *If the matrix* $\frac{\mathrm{d}}{\mathrm{d}\alpha}N_{1,1}(\alpha)$ *is finite, then P is α-positive recurrent.*

(2) *If the matrix* $\frac{\mathrm{d}}{\mathrm{d}\alpha}N_{1,1}(\alpha)$ *is infinite, then P is α-null recurrent.*

Proof We only prove (1), while (2) can be proved similarly.

Since

$$\Psi_0(\alpha)=\alpha A_1^{(0)}+\alpha^2 A_0^{(0)}N_{1,1}(\alpha)A_2^{(1)},$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\Psi_0(\alpha)=A_1^{(0)}+2\alpha A_0^{(0)}N_{1,1}(\alpha)A_2^{(1)}+\alpha^2 A_0^{(0)}\frac{\mathrm{d}}{\mathrm{d}\alpha}N_{1,1}(\alpha)A_2^{(1)}.$$

Note that $I-\Psi_1(\alpha)$ is invertible, it is clear that the matrix $N_{1,1}(\alpha)=[I-\Psi_1(\alpha)]^{-1}$ is finite. If the matrix $\frac{\mathrm{d}}{\mathrm{d}\alpha}N_{1,1}(\alpha)$ is finite, then the matrix $\frac{\mathrm{d}}{\mathrm{d}\alpha}\Psi_0(\alpha)$ is finite.

For the QBD process $\{X_n, n\geqslant 0\}$, we write

$$f_{k,i;k,i}^{(n)}=P\{X_n=(k,i),X_l\neq(k,i)\text{ for }1\leqslant l\leqslant n-1\mid X_0=(k,i)\}$$

and

$$\mu_{k,i}(\alpha)=\sum_{n=0}^{\infty}n\alpha^n f_{k,i;k,i}^{(n)}.$$

Note that the irreducible QBD process $\{X_n, n\geqslant 0\}$ is α-positive recurrent if $\mu_{k,i}(\alpha)<+\infty$ and α-null recurrent otherwise.

Note that $\left(\Psi_0^{(n)}\right)_{i,j}$, the (i,j)th entry of $\Psi_0^{(n)}$, is the probability that the QBD process arrives at state (k,i) at time n for the first time, given its initial state (k,i) at time 0. Let

$$F_{0,i;0,i}(\alpha)=\sum_{n=0}^{\infty}\alpha^n f_{0,i;0,i}^{(n)}.$$

Then

$$\mu_{0,i}(\alpha)=\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}F_{0,i;0,i}(\alpha).$$

Since

$$\begin{aligned}F_{0,i;0,i}(\alpha)&=(\Psi_0(\alpha))_{i,i}+\sum_{j_1\neq i}(\Psi_0(\alpha))_{i,j_1}(\Psi_0(\alpha))_{j_1,i}\\&\quad+\sum_{j_1,j_2\neq i}(\Psi_0(\alpha))_{i,j_1}(\Psi_0(\alpha))_{j_1,j_2}(\Psi_0(\alpha))_{j_2,i}+\dots\\&=\sum_{t=1}^{\infty}\sum_{\substack{j_0,j_t=i\\ j_k\neq i,1\leqslant k\leqslant t-1}}\prod_{s=1}^{t}(\Psi_0(\alpha))_{j_{s-1},j_s},\end{aligned}$$

we obtain

$$\begin{aligned}\mu_{0,i}(\alpha)=\alpha\sum_{t=1}^{\infty}&\sum_{\substack{j_0,j_t=i\\ j_k\neq i,1\leqslant k\leqslant t-1}}\sum_{r=1}^{t}\prod_{s=1}^{r-1}(\Psi_0(\alpha))_{j_{s-1},j_s}\\&\cdot\frac{\mathrm{d}}{\mathrm{d}\alpha}(\Psi_0(\alpha))_{j_{r-1},j_r}\prod_{s=r+1}^{t}(\Psi_0(\alpha))_{j_{s-1},j_s}.\end{aligned}$$

Let $x=(x_1,x_2,\dots,x_{m_0})$ be the left eigenvector of the matrix $\Psi_0(\alpha)$ with the maximal eigenvalue $\eta(\alpha)=1$ due to the condition under which the QBD process P is α-recurrent. We write

$$U=\Delta\Psi_0(\alpha)^{\mathrm{T}}\Delta^{-1}$$

where $\Delta=\mathrm{diag}\,(x_1,x_2,\dots,x_{m_0})$. It is clear that the matrix U is stochastic. The expected first return time for state i of the Markov chain U is given by

$$\begin{aligned}E[D_i]&=\sum_{t=1}^{\infty}t\sum_{\substack{j_0,j_t=i\\ j_k\neq i,1\leqslant k\leqslant t-1}}\prod_{s=1}^{t}U_{j_{s-1},j_s}\\&=\sum_{t=1}^{\infty}t\sum_{\substack{j_0,j_t=i\\ j_k\neq i,1\leqslant k\leqslant t-1}}\prod_{s=1}^{t}(\Psi_0(\alpha)^{\mathrm{T}})_{j_{s-1},j_s}<+\infty.\end{aligned}$$

If the matrix $\frac{\mathrm{d}}{\mathrm{d}\alpha}N_{1,1}(\alpha)$ is finite, then there exists a larger positive number K such that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}(\varPsi_0(\alpha))_{j_{r-1},j_r} \leqslant K(\varPsi_0(\alpha))_{j_{r-1},j_r}.$$

Thus, we have

$$\begin{aligned}\mu_{0,i}(\alpha) &\leqslant \alpha K\sum_{t=1}^{\infty} t \sum_{\substack{j_0,j_t=i\\ j_k\neq i, 1\leqslant k\leqslant t-1}} \prod_{s=1}^{t}(\varPsi_0(\alpha)^{\mathrm{T}})_{j_{s-1},j_s}\\ &= \alpha K E[D_i] < +\infty,\end{aligned}$$

which indicates that the QBD process P is α-positive recurrent. This completes the proof.

Remark 9.6 *Theorem 9.9 illustrates that the α-recurrent QBD process is α-positive recurrent if and only if $\alpha < \overline{\alpha}$. A similar analysis shows that the α-recurrent level-dependent QBD process is α-positive recurrent if and only if $\alpha < \overline{\alpha}$. Now, we compare this result with Theorem 9.25. If $\alpha < \overline{\alpha}$, then since $\alpha < \overline{\alpha} \leqslant \alpha_2$, where $\overline{\alpha} = \alpha_1, N^{(2)}(z)$ is analytic at $z = \alpha$. Thus, $\frac{\mathrm{d}}{\mathrm{d}\alpha}N^{(2)}(\alpha) = \frac{\mathrm{d}}{\mathrm{d}z}N^{(2)}(z)_{|z=\alpha} < \infty$. On the contrary, if $\frac{\mathrm{d}}{\mathrm{d}\alpha}N^{(2)}(\alpha) < \infty$, then $\alpha < \overline{\alpha}$, since* $N^{(2)}(\overline{\alpha}) = \infty$ and $N^{(2)}(z)$ *is increasing for* $1 \leqslant z < \alpha_2$.

9.7 Continuous-Time Markov Chains

In this section, similar to the analysis in the discrete-time case, we study the quasi-stationary distribution of an irreducible continuous-time Markov chain with block structure, and provide conditions for the state α-classification and expressions for the quasi-stationary distribution.

We consider an irreducible continuous-time Markov chain whose infinitesimal generator is given by

$$Q = \begin{pmatrix} Q_{0,0} & Q_{0,1} & Q_{0,2} & \cdots \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & \cdots \\ Q_{2,0} & Q_{2,1} & Q_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

where the size of the matrix $Q_{k,k}$ is $m_k \times m_k$ for $k \geqslant 0$, and the sizes of the other matrices are determined accordingly.

We write

$$P(t) = \exp\{Qt\}$$

and let $P_{l,i;k,j}(t)$ be the $(l,i;k,j)$th entry of the matrix $P(t)$, where l and k are the level numbers and i and j are the phase numbers. We define

$$\alpha = \sup\left\{\beta \geqslant 1: \int_0^{+\infty} \exp\left\{-\frac{1}{\beta}t\right\} P_{l,i;k,j}(t)\mathrm{d}t < +\infty\right\}.$$

It is easy to check that if the Markov chain Q is irreducible, then the positive number α is independent of state (l,i) and (k,j). In this case,

$$\int_0^{+\infty} \exp\left\{-\frac{1}{\beta}t\right\} \exp\{Q_t\}\mathrm{d}t = \int_0^{+\infty} \exp\left\{\left[Q - \frac{1}{\beta}I\right]t\right\}\mathrm{d}t$$
$$= \left[\frac{1}{\beta}I - Q\right]_{\min}^{-1}.$$

If the matrix $[\alpha^{-1}I - Q]_{\min}^{-1}$ is finite, then the Markov chain Q is α-transient, otherwise it is α-recurrent. Let $\gamma = \alpha^{-1}$. Then γ is the decay parameter of the Markov chain Q.

For $1 \leqslant \beta \leqslant \alpha$, if there exists a nonnegative non-zero row vector $\pi(\beta)$ such that

$$\pi(\beta)\beta Q = \pi(\beta) \tag{9.107}$$

and

$$\pi(\beta)e = 1, \tag{9.108}$$

then $\pi(\beta)$ is called the quasi-stationary distribution of the continuous-time Markov chain Q.

The following proposition provides a useful relation between the matrix $P(t)$ and the quasi-stationary distribution $\pi(\beta)$. The proof is easy, and thus is omitted here.

Proposition 9.5 *If the vector $\pi(\beta)$ is the quasi-stationary distribution of the continuous-time Markov chain Q, then*

$$\pi(\beta)P(t) = \pi(\beta)\exp\left\{\frac{t}{\beta}\right\}$$

or

$$\pi(\beta)\left[\exp\left\{-\frac{t}{\beta}\right\}P(t)\right] = \pi(\beta).$$

It follows from Eq. (9.107) that

$$\pi(\beta)Q = \frac{1}{\beta}\pi(\beta)$$

or

$$\pi(\beta)\left(Q - \frac{1}{\beta}I\right) = 0. \tag{9.109}$$

Let

$$\mathcal{Q}(\beta) = Q - \frac{1}{\beta}I.$$

Note that the matrix $\mathcal{Q}(\beta)$ is the infinitesimal generator of an irreducible continuous-time Markov chain, thus it is necessary to provide the *RG*-factorizations for computing the quasi-stationary distribution.

9.7.1 The UL-Type *RG*-Factorization

For the continuous-time Markov chain $\mathcal{Q}(\beta)$, we write $E = \{0,1,2,\ldots,n\}$ and $E^c = \{n+1, n+2, \ldots\}$. Based on the two sets E and E^c, the matrix $\mathcal{Q}(\beta)$ is partitioned as

$$\mathcal{Q}(\beta) = \begin{pmatrix} T(\beta) & V(\beta) \\ H(\beta) & W(\beta) \end{pmatrix}.$$

Then

$$\mathcal{Q}^{[\leqslant n]}(\beta) = T(\beta) + V(\beta)[-W(\beta)]_{\min}^{-1} H(\beta).$$

The block-entry expression of the matrix $\mathcal{Q}^{[\leqslant n]}(\beta)$ is written as

$$\mathcal{Q}^{[\leqslant n]}(\beta) = \begin{pmatrix} f_{0,0}^{(n)}(\beta) & f_{0,1}^{(n)}(\beta) & \cdots & f_{0,n}^{(n)}(\beta) \\ f_{1,0}^{(n)}(\beta) & f_{1,1}^{(n)}(\beta) & \cdots & f_{1,n}^{(n)}(\beta) \\ \vdots & \vdots & & \vdots \\ f_{n,0}^{(n)}(\beta) & f_{n,1}^{(n)}(\beta) & \cdots & f_{n,n}^{(n)}(\beta) \end{pmatrix}.$$

For $i, j \leqslant n-1$, we have

$$f_{i,j}^{(n-1)}(\beta) = \beta Q_{i,j} + \sum_{k=n}^{\infty} f_{i,k}^{(k)}(\beta)\left[-f_{k,k}^{(k)}(\beta)\right]^{-1} f_{k,j}^{(k)}(\beta).$$

We define the *U*-measure as

$$U_n(\beta) = f_{n,n}^{(n)}(\beta), \quad n \geqslant 0,$$

the R-measure as

$$R_{i,j}(\beta) = f_{i,j}^{(j)}(\beta)[-U_j\beta)]^{-1}, \quad 0 \leqslant i < j,$$

and the G-measure as

$$G_{i,j} = [-U_i(\beta)]^{-1} f_{i,j}^{(i)}(\beta), \quad 0 \leqslant j < i.$$

The UL-type RG-factorization is given by

$$Q(\beta) = [I - R_U(\beta)]U_D(\beta)[I - G_U(\beta)],$$

where

$$R_U(\beta) = \begin{pmatrix} 0 & R_{0,1}(\beta) & R_{0,2}(\beta) & R_{0,3}(\beta) & \cdots \\ & 0 & R_{1,2}(\beta) & R_{1,3}(\beta) & \cdots \\ & & 0 & R_{2,3}(\beta) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$U_D(\beta) = \mathrm{diag}\ (U_0(\beta), U_1(\beta), U_2(\beta), U_3(\beta), \ldots)$$

and

$$G_L(\beta) = \begin{pmatrix} 0 & & & & \\ G_{1,0}(\beta) & 0 & & & \\ G_{2,0}(\beta) & G_{2,1}(\beta) & 0 & & \\ G_{3,0}(\beta) & G_{3,1}(\beta) & G_{3,2}(\beta) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In what follows we study some useful properties for the censored discounted Markov chain $U_k(\beta)$ for $k \geqslant 0$, which are necessary in analysis of the quasi-stationary distribution.

Proposition 9.6 (1) *For* $1 \leqslant \beta < \alpha$, *the matrix* $U_k(\beta)$ *is invertible for* $k \geqslant 0$.

(2) *The matrix* $U_k(\alpha)$ *is invertible for* $k \geqslant 1$.

(3) *The matrix* $U_0(\alpha)$ *is invertible if the Markov chain* Q *is* α*-transient; while* $U_0(\alpha)$ *is singular if the Markov chain* Q *is* α*-recurrent.*

Based on (3) in Proposition 9.6, we have

$$\alpha = \sup\{\beta \geqslant 1 : \det(U_0(\beta)) \neq 0\}.$$

Let

$$N_0(\beta) = -U_0^{-1}(\beta).$$

Then

$$\alpha = \sup\{\beta \geqslant 1 : N_0(\beta) < +\infty\}.$$

When the Markov chain Q is α-recurrent, the following theorem provides expressions for the quasi-stationary distribution, which is similar to that in the discrete-time case.

Theorem 9.26 *For $\beta = \alpha$ under which the continuous-time Markov chain Q is α-recurrent, the quasi-stationary distribution is expressed as*

$$\pi_0(\alpha) = \tau x_0(\alpha)$$

and

$$\pi_k(\alpha) = \sum_{i=0}^{k-1} \pi_i(\alpha) R_{i,k}(\alpha), \quad k \geqslant 1,$$

where $x_0(\alpha)$ is a nonnegative non-zero solution to the system of equations $x_0(\alpha)U_0(\alpha) = 0$ and $x_0(\alpha)e = 1$, and the constant τ makes $\sum_{k=0}^{\infty} \pi_k(\alpha)e = 1$.

Proof Since

$$\pi(\alpha)[Q - \alpha^{-1}I] = 0,$$

we obtain

$$\pi(\alpha)[I - R_U(\alpha)]U_D(\alpha)[I - G_U(\alpha)] = 0.$$

Let

$$x = \pi(\alpha)[I - R_U(\alpha)].$$

Then

$$xU_D(\alpha)[I - G_U(\alpha)] = 0.$$

Based on the censoring technique, it is clear that $x = (\tau x_0(\alpha), 0, 0, \ldots)$ is a nonnegative non-zero solution to $xU_D(\alpha)[I - G_U(\alpha)] = 0$. Thus we obtain

$$\pi(\alpha)[I - R_U(\alpha)] = (\tau x_0(\alpha), 0, 0, \ldots),$$

which leads to the desired result. This completes the proof.

In what follows we consider another type of expression for the quasi-stationary distribution for either $\beta < \alpha$ or $\beta = \alpha$ under which the continuous-time Markov chain Q is α-transient. Note that in this case, a general irreducible Markov chain with infinitely-many levels can not be dealt with, thus we only simply study a continuous-time Markov chain of *GI*/*G*/1 type.

We consider an irreducible continuous-time Markov chain of *GI*/*G*/1 type whose infinitesimal generator is given by

$$Q = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ D_{-1} & A_0 & A_1 & A_2 & \cdots \\ D_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ D_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The RG-factorization of the matrix $Q(\beta)$ is given by

$$Q(\beta) = [I - R_U(\beta)]U_D(\beta)[I - G_L(\beta)],$$

where

$$R_U(\beta) = \begin{pmatrix} 0 & R_{0,1}(\beta) & R_{0,2}(\beta) & R_{0,3}(\beta) & \cdots \\ & 0 & R_1(\beta) & R_2(\beta) & \cdots \\ & & 0 & R_1(\beta) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix},$$

$$U_D(\beta) = \text{diag}\,(U_0(\beta), \Phi_0(\beta), \Phi_0(\beta), \Phi_0(\beta), \ldots)$$

and

$$G_L(\beta) = \begin{pmatrix} 0 & & & & \\ G_{1,0}(\beta) & 0 & & & \\ G_{2,0}(\beta) & G_1(\beta) & 0 & & \\ G_{3,0}(\beta) & G_2(\beta) & G_1(\beta) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For the G-measure: $G_{k,0}(\beta)$ and $G_k(\beta)$ for $k \geqslant 1$, there exist a constant $\gamma > 0$, and two nonnegative non-zero row vectors y and y_0 such that

$$\begin{cases} y = y \sum_{k=1}^{\infty} \gamma^{-k} G_k(\beta), \\ y_0 = y \sum_{k=1}^{\infty} \gamma^{-k} G_{k,0}(\beta). \end{cases}$$

Thus, using a similar analysis to that of the discrete-time case, we can obtain the following theorem whose proof is easy, and thus is omitted here.

Theorem 9.27 *If $1 \leqslant \beta < \alpha$ or $\beta = \alpha$ under which the continuous-time Markov chain Q of GI/G/1 type is α-transient, then the quasi-stationary distribution $\pi(\beta) = (\pi_0(\beta), \pi_1(\beta), \pi_2(\beta), \ldots)$ is given by*

$$\begin{cases} \pi_0(\beta) = \kappa y_0 N_0(\beta), \\ \pi_k(\beta) = \kappa \gamma^{-k} y N(\beta) + \pi_0(\beta) R_{0,k}(\beta) + \sum_{i=1}^{k} \pi_i(\beta) R_{k-i}(\beta), \quad k \geqslant 1, \end{cases}$$

where $N_0(\beta) = [-U_0(\beta)]^{-1}, N(\beta) = [-\Phi_0(\beta)]^{-1}$ and the positive constant κ satisfies $\sum_{k=0}^{\infty} \pi_k(\beta) e = 1$.

9.7.2 The LU-Type *RG*-Factorization

Now, we construct the LU-type *RG*-factorization of the continuous-time Markov chain $Q(\beta)$.

Let

$$Q^{[\geqslant n]}(\beta) = W(\beta) + H(\beta)[-T(\beta)]^{-1} V(\beta).$$

The block-entry expression of the matrix $Q^{[\geqslant n]}(\beta)$ is written as

$$Q^{[\geqslant n]}(\beta) = \begin{pmatrix} h_{n,n}^{(n)}(\beta) & h_{n,n+1}^{(n)}(\beta) & h_{n,n+2}^{(n)}(\beta) & \cdots \\ h_{n+1,n}^{(n)}(\beta) & h_{n+1,n+1}^{(n)}(\beta) & h_{n+1,n+2}^{(n)}(\beta) & \cdots \\ h_{n+2,n}^{(n)}(\beta) & h_{n+2,n+1}^{(n)}(\beta) & h_{n+2,n+2}^{(n)}(\beta) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

For $i, j \geqslant n+1$, we obtain

$$h_{i,j}^{(n+1)}(\beta) = \beta Q_{i,j} + \sum_{k=0}^{n} h_{i,k}^{(k)}(\beta)[-h_{k,k}^{(k)}(\beta)]^{-1} h_{k,j}^{(k)}(\beta).$$

We define the *U*-measure as

$$\overline{U}_n(\beta) = h_{n,n}^{(n)}(\beta), \quad n \geqslant 0,$$

the *R*-measure as

$$\overline{R}_{i,j}(\beta) = h_{i,j}^{(j)}(\beta)[-\overline{U}_j(\beta)]^{-1}, \quad 0 \leqslant j < i,$$

and the *G*-measure as

$$\overline{G}_{i,j}(\beta) = (-\overline{U}_i(\beta)]^{-1} h_{i,j}^{(i)}(\beta), \quad 0 \leqslant i < j.$$

The LU-type *RG*-factorization is given by

$$Q(\beta) = [I - \overline{R}_L(\beta)]\overline{U}_D(\beta)[I - \overline{G}_U(\beta)],$$

where

$$\overline{R}_L(\beta) = \begin{pmatrix} 0 & & & & \\ \overline{R}_{1,0}(\beta) & 0 & & & \\ \overline{R}_{2,0}(\beta) & \overline{R}_{2,1}(\beta) & 0 & & \\ \overline{R}_{3,0}(\beta) & \overline{R}_{3,1}(\beta) & \overline{R}_{3,2}(\beta) & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\overline{U}_D(\beta) = \operatorname{diag}(\overline{U}_0(\beta), \overline{U}_1(\beta), \overline{U}_2(\beta), \overline{U}_3(\beta), \ldots)$$

and

$$\overline{G}_U(\beta) = \begin{pmatrix} 0 & \overline{G}_{0,1}(\beta) & \overline{G}_{0,2}(\beta) & \overline{G}_{0,3}(\beta) & \cdots \\ & 0 & \overline{G}_{1,2}(\beta) & \overline{G}_{1,3}(\beta) & \cdots \\ & & 0 & \overline{G}_{2,3}(\beta) & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

9.8 Decay Rate for the GPH Distribution

In this section, we use the quasi-stationary distribution and the *RG*-factorization to study the decay rate of a GPH distribution which is either continuous-time or discrete-time, and show that the decay rate can be determined by both the transition matrix and the initial probability vector.

9.8.1 The Discrete-Time PH Distribution with Finitely Many Phases

We first consider a discrete-time PH distribution with irreducible representation (α, T) of order m whose random variable X has the following distribution:

$$p_k = P\{X = k\} = \begin{cases} \alpha_0, & k = 0, \\ \alpha T^{k-1} T^0, & k \geqslant 1, \end{cases} \tag{9.110}$$

where $\alpha_0 + \alpha e = 1$ and $T^0 + Te = e.$ For simplicity of exposition, we assume that the transition matrix T is irreducible. Note that the matrix T is irreducible if and only if $(I+T)^{m-1} > 0.$

It is clear from Eq. (9.110) that

$$P^*(z) = E[z^X] = \alpha_0 + z\alpha(I - zT)^{-1} T^0. \tag{9.111}$$

We now are interested in the tailed behavior of the PH random variable X as follows:

$$\lim_{k\to\infty} P\{X = k\} = \lim_{k\to\infty} p_k$$

or

$$\lim_{k\to\infty} P\{X > k\} = \lim_{k\to\infty} \overline{p}_{\leqslant k}$$

where $\overline{p}_{\leqslant k} = \sum_{l=k+1}^{\infty} p_l.$ Note that

$$\overline{p}_{\leqslant k}=\sum_{l=k+1}^{\infty}\alpha T^{l-1}T^{0}=\alpha T^{k}e,$$

thus we only need to analyze the tailed behavior $\lim_{k\to\infty} p_k$.

For the irreducible nonnegative matrix T, we define

$$\alpha(T)=\sup\left\{\beta\geqslant 1:\sum_{n=0}^{\infty}\beta^{n}t_{i,j}^{(n)}<+\infty\right\},$$

which is independent of i and j, where $t_{i,j}^{(n)}$ is the (i,j) th element of the matrix T^n for $n\geqslant 0$. We denote by $\rho(T)$ the maximal eigenvalue of the matrix T.

Note that the matrix T is of order m, thus we have

$$\alpha(T)=\frac{1}{\rho(T)}. \tag{9.112}$$

In this case, we only need to analyze the spectral radius $\rho(T)$ for studying the tailed behavior $\lim_{k\to\infty} p_k$. To this end, the following two lemmas, stated in 8.4.4 and 8.4.6 in Horn and Johnson [11], are useful for describing the spectral radius $\rho(T)$.

Lemma 9.21 *For the irreducible nonnegative matrix T, we have*

(1) $\rho(T)>0$ *is an algebraically (and thus geometrically) simple eigenvalue of T.*

(2) *There must exist a positive row vector u and a positive column vector v such that*

$$uT=\rho(T)u,\quad Tv=\rho(T)v,$$
$$ue=uv=1.$$

Lemma 9.22 *If the irreducible nonnegative matrix T has r eigenvalues of maximal modulus* $\rho(T)$, *then the* 1*st eigenvalue is given by*

$$\lambda_l=\rho(T)\exp\left\{\frac{2\pi l}{r}i\right\},$$

where $i^2=-1$ *and* $0\leqslant l\leqslant r-1$. *At the same time, the eigenvalue* λ_l *is an algebraically (and thus geometrically) simple eigenvalue of T.*

If the irreducible nonnegative matrix T has r eigenvalues of maximal modulus $\rho(T)$, then there exists an invertible matrix D such that

$$T=D^{-1}JD,$$

where J is the Jordan canonical form of the matrix T, $J=\text{daig}\,(J_1,J_2)$ and

$$J_1=\rho(T)\ \text{diag}\left(\exp\left\{\frac{2\pi l}{r}i\right\},l=0,1,\ldots,r-1\right).$$

Hence, we obtain

$$\begin{aligned}T^{k-1} &= D^{-1}\mathrm{diag}(J_1^{k-1}, J_2^{k-1})D \\ &= \rho^{k-1}(T)\cdot D^{-1}\mathrm{diag}(\tilde{J}_1^{k-1}, [\rho^{-1}(T)J_2]^{k-1})D,\end{aligned}$$

where

$$\tilde{J}_1^{k-1} = \mathrm{diag}\left(\exp\left\{k\frac{2\pi l}{r}i\right\}, l = 0, 1, \ldots, r-1\right).$$

It is clear that

$$\lim_{k\to\infty}[\rho^{-1}(T)J_2]^{k-1} = 0,$$

but this limit $\lim_{k\to\infty}\tilde{J}_1^{k-1}$ does not exist for $r \geqslant 2$. In this case, we have

$$\begin{aligned}p_k &= \alpha T^{k-1}T^0 \\ &= \rho^{k-1}(T)\cdot \alpha D^{-1}\mathrm{diag}\left(\tilde{J}_1^{k-1}, [\rho^{-1}(T)J_2]^{k-1}\right)DT^0 \\ &\leqslant \rho^{k-1}(T)\cdot \alpha \| D^{-1}\| \|\mathrm{diag}\left(\tilde{J}_1^{k-1}, [\rho^{-1}(T)J_2]^{k-1}\right)\| \| D\| T^0 \\ &= \rho^{k-1}(T)\cdot \alpha \| D^{-1}\| \|\mathrm{diag}\left(I, [\rho^{-1}(T)J_2]^{k-1}\right)\| \| D\| T^0.\end{aligned}$$

Let

$$\phi = \alpha \| D^{-1} \| \| \mathrm{diag}(I, 0) \| \| D \| T^0.$$

Then as $k \to \infty$, we have

$$p_k \leqslant \rho^{k-1}(T)\cdot \alpha \| D^{-1} \| \| \mathrm{diag}(I, [\rho^{-1}(T)J_2]^{k-1}) \| \| D \| T^0 \to \rho^{k-1}(T)\phi. \quad (9.113)$$

Specifically, if the irreducible nonnegative matrix T has a single eigenvalue of maximal modulus $\rho(T)$, then it is easy to check that

$$T^{k-1} = \rho^{k-1}(T)vu,$$

thus we obtain

$$p_k \to \rho^{k-1}(T)\varphi, \quad \text{as } k \to \infty, \quad (9.114)$$

where

$$\varphi = \alpha v u T^0.$$

Definition 9.1 *A nonnegative matrix A of order m is said to be primitive if it is irreducible and has only one eigenvalue of maximal modulus.*

Obviously, if the matrix T is primitive, then the tailed behavior $\lim_{k\to\infty} p_k$ is completely determined by the spectral radius $\rho(T)$ in terms of Eq. (9.114).

The following proposition provides conditions under which the matrix T is primitive. Their proofs may refer to 8.5.2 and 8.5.9 in Horn and Johnson [1].

Proposition 9.7 (1) *The matrix T of order m is primitive if and only if* $T^n > 0$ *for some* $n \geqslant 1$.

(2) *The matrix T of order m is primitive if and only if* $T^{m^2-2m+2} > 0$.

9.8.2 The Discrete-Time GPH Distribution with Infinitely-many Phases

When the size of the matrix T is infinite, and even though there do not exist the eigenvalues and eigenvectors for the matrix T, we have a similar concept: the decay rate $1/\alpha_T$ and the quasi-stationary distribution $\pi(\alpha_T)$, is given by

$$\pi(\alpha_T)\alpha_T T = \pi(\alpha_T)$$

or

$$\pi(\alpha_T)T = \frac{1}{\alpha_T}\pi(\alpha_T)$$

which has been analyzed in the previous sections. Therefore, the tailed behavior $\lim_{k\to\infty} p_k$ can be described by the radius α_T of convergence of the matrix T.

Note that

$$I - zT = [I - R_U(z)][I - \Psi_D(z)][I - G_L(z)],$$

it follows from Eq. (9.111) that

$$P^*(z) = \alpha_0 + z\alpha[I - G_L(z)]^{-1}[I - \Psi_D(z)]^{-1}[I - R_U(z)]^{-1}T^0.$$

Let

$$\gamma = \sup\{\beta \geqslant 1 : P^*(\beta) < +\infty\}.$$

It is clear that the discrete-time GPH distribution $\{p_k\}$ is heavy-tailed if and only if $\gamma = 1$, and $\{p_k\}$ is light-tailed if and only if $\gamma > 1$.

In general, it is difficult to determine the decay rate $1/\gamma$ for a general GPH distribution with irreducible representation $\alpha(T)$. In what follows we analyze two special cases for a level-dependent discrete-time QBD process.

9.8.3 The Level-Dependent QBD Processes

Let

$$T=\begin{pmatrix} A_1^{(1)} & A_0^{(1)} & & & \\ A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & & \\ & A_2^{(3)} & A_1^{(3)} & A_0^{(3)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then we consider the following two cases:

Case Ⅰ $A_0^{(1)}e+A_1^{(1)}e \lneqq e$ and $A_0^{(k)}e+A_1^{(k)}e+A_2^{(k)}e=e$ for $k\geqslant 2$.

In this case, it is clear that

$$T^0=\left([A_2^{(1)}e]^{\mathrm{T}},0^{\mathrm{T}},0^{\mathrm{T}},\dots\right)^{\mathrm{T}}.$$

We introduce the notation

$$\prod_{j=M}^{N}A_j=A_MA_{M-1}\dots A_N.$$

Note that

$$[I-\Psi_D(z)]^{-1}[I-R_U(z)]^{-1}T^0=\begin{pmatrix}[I-\Psi_1(z)]^{-1}A_2^{(1)}e\\ 0\\ 0\\ \vdots\end{pmatrix},$$

$$[I-G_L(z)]^{-1}=\begin{pmatrix} I & & & & \\ Y_2^{(2)}(z) & I & & & \\ Y_3^{(3)}(z) & Y_2^{(3)}(z) & I & & \\ Y_4^{(4)}(z) & Y_3^{(4)}(z) & Y_2^{(4)}(z) & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with for $1\leqslant l\leqslant k$,

$$Y_1^{(l)}(z)=I,$$

$$Y_k^{(l)}(z)=G_l(z)G_{l-1}(z)\dots G_{l-k+1}(z)=\prod_{i=l}^{l-k+1}G_i(z),$$

and

$$\alpha[I-G_L(z)]^{-1}=\left(\sum_{k=1}^{\infty}\alpha_kY_k^{(k)}(z),*,*,\dots\right)$$

$$=\left(\sum_{k=1}^{\infty}\left[\alpha_k\prod_{j=k}^{1}G_j(z)\right],*,*,\dots\right),$$

we obtain

$$\begin{aligned}P^*(z) &= \alpha_0 + z\alpha[I - G_L(z)]^{-1}[I - \Psi_D(z)]^{-1}[I - R_U(z)]^{-1}T^0 \\ &= \alpha_0 + z\sum_{k=1}^{\infty}\left[\alpha_k \prod_{j=k}^{1} G_j(z)\right]\cdot[I - \Psi_1(z)]^{-1} A_2^{(1)} e.\end{aligned}$$

We write

$$\eta = \sup\left\{\beta \geqslant 1 : \sum_{k=1}^{\infty}\alpha_k \prod_{j=k}^{1} G_j(\beta) e < +\infty\right\}$$

and

$$\alpha_T = \sup\{\beta \geqslant 1 : [I - \Psi_1(\beta)]^{-1} < +\infty\}.$$

It is clear that

$$\gamma = \min\{\eta, \alpha_T\}.$$

If the QBD process is level-independent, then $G_k(\beta) = G(\beta)$ for $k \geqslant 1$. Hence we obtain

$$\eta = \sup\left\{\beta \geqslant 1 : \sum_{k=1}^{\infty}\alpha_k G^k(\beta) < +\infty\right\}.$$

Case Ⅱ $\alpha = (\alpha_1, 0, 0, \ldots)$.

In this case, note that

$$\alpha[I - G_L(z)]^{-1}[I - \Psi_D(z)]^{-1} = (\alpha_1[I - \Psi_1(z)]^{-1}, 0, 0, \ldots),$$

$$[I - R_U(z)]^{-1} = \begin{pmatrix} I & X_1^{(1)}(z) & X_2^{(1)}(z) & X_3^{(1)}(z) & \cdots \\ & I & X_1^{(2)}(z) & X_2^{(2)}(z) & \cdots \\ & & I & X_1^{(3)}(z) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix}$$

with

$$\begin{aligned}X_0^{(l)}(z) &= I, \\ X_k^{(l)}(z) &= R_l(z)R_{l+1}(z)\ldots R_{l+k-1}(z) = \prod_{i=l}^{l+k-1} R_i(z)\end{aligned}$$

and

$$[I - R_U(z)]^{-1}T^0 = \begin{pmatrix} \sum_{k=1}^{\infty} X_k^{(1)}(z)T_k^0 \\ * \\ * \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{\infty}\prod_{i=1}^{k} R_i(z)T_k^0 \\ * \\ * \\ \vdots \end{pmatrix},$$

we obtain

$$P^*(z) = \alpha_0 + z\alpha_1[I - \Psi_1(z)]^{-1} \cdot \sum_{k=1}^{\infty}\prod_{i=1}^{k} R_i(z)T_k^0.$$

Let

$$\eta = \sup\left\{\beta \geqslant 1 : \sum_{k=1}^{\infty}\prod_{i=1}^{k} R_i(\beta)T_k^0 < +\infty\right\}$$

and

$$\alpha_T = \sup\{\beta \geqslant 1 : [I - \Psi_1(\beta)]^{-1} < +\infty\}.$$

Then $\gamma = \min\{\eta, \alpha_T\}$.

9.8.4 The Continuous-Time GPH Distribution

We consider a continuous-time GPH distribution with irreducible representation (α, T) whose analysis is similar to the discrete-time case. Note that the size of the matrix T is infinite, we can determine the decay rate $1/\alpha_T$ of the matrix T, and thus the quasi-stationary distribution $\pi(\alpha_T)$ of the Markov chain T is given by

$$\pi(\alpha_T) = \pi(\alpha_T)\alpha_T T$$

or

$$\pi(\alpha_T)T = \frac{1}{\alpha_T}\pi(\alpha_T).$$

Therefore, the tailed behavior $\lim_{k\to\infty} p_k$ can be described by only the radious α_T of convergence of the matrix T.

Note that

$$F(x) = 1 - \alpha\exp\{Tx\}e$$

and the Laplace-Stieltjes transform

$$f^*(s) = \alpha_0 + \alpha(sI - T)^{-1}T^0.$$

Since

$$T - sI = [I - R_U(s)]U_D(s)[I - G_L(s)],$$

it follows from Eq. (9.111) that

$$f^*(s) = \alpha_0 + \alpha[I - G_L(s)]^{-1}[-U_D(z)]^{-1}[I - R_U(s)]^{-1}T^0.$$

Let

$$\gamma = \sup\{s \geqslant 0 : f^*(-s) < +\infty\}.$$

It is clear that the continuous-time GPH distribution $F(x)$ is heavy-tailed if and only if $\gamma = 0; F(x)$ is light-tailed if and only if $\gamma > 0$.

In general, it is difficult to determine the decay rate γ for a general GPH distribution with irreducible representation (α, T). In what follows we analyze a level-dependent Markov chain of *M*/*G*/1 type and a level-dependent Markov chain of *GI*/*M*/1 type, respectively.

9.8.5 The Level-Dependent Markov Chains of *M*/*G*/1 Type

Let

$$T = \begin{pmatrix} A_1^{(1)} & A_2^{(1)} & A_3^{(1)} & A_4^{(1)} & \cdots \\ A_0^{(2)} & A_1^{(2)} & A_2^{(2)} & A_3^{(2)} & \cdots \\ & A_0^{(3)} & A_1^{(3)} & A_2^{(3)} & \cdots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We assume that $\sum_{k=1}^{\infty} A_k^{(1)} e \lneqq 0$ and $\sum_{k=0}^{\infty} A_k^{(l)} e = 0$ for $l \geqslant 2$. In this case, it is clear that

$$T^0 = \left(-[A_0^{(1)}e]^{\mathrm{T}}, 0^{\mathrm{T}}, 0^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}.$$

Note that

$$[-U_D(s)]^{-1}[I - R_U(s)]^{-1}T^0 = \begin{pmatrix} [-U_1(s)]^{-1} A_0^{(1)} e \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

$$[I-G_L(s)]^{-1}=\begin{pmatrix} I & & & & \\ Y_2^{(2)}(s) & I & & & \\ Y_3^{(3)}(s) & Y_2^{(3)}(s) & I & & \\ Y_4^{(4)}(s) & Y_3^{(4)}(s) & Y_2^{(4)}(s) & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with for $1 \leqslant l \leqslant k$,

$$Y_1^{(l)}(s)=I$$

$$Y_k^{(l)}(s)=G_l(s)G_{l-1}(s)\dots G_{l-k+1}(s)=\prod_{i=l}^{l-k+1} G_i(s),$$

and

$$\begin{aligned}\alpha[I-G_L(s)]^{-1}&=\left(\sum_{k=1}^{\infty}\alpha_k Y_k^{(k)}(s),*,*,\dots\right)\\&=\left(\sum_{k=1}^{\infty}\left[\alpha_k\prod_{j=k}^{1}G_j(s)\right],*,*,\dots\right),\end{aligned}$$

we obtain

$$\begin{aligned}f^*(s)&=\alpha_0+\alpha[I-G_L(s)]^{-1}[-U_D(s)]^{-1}[I-R_U(s)]^{-1}T^0\\&=\alpha_0+\sum_{k=1}^{\infty}\left[\alpha_k\prod_{j=k}^{1}G_j(s)\right]\cdot[-U_1(s)]^{-1}A_0^{(1)}e.\end{aligned}$$

We write

$$\eta=\sup\left\{s\geqslant 0:\sum_{k=1}^{\infty}\alpha_k\prod_{j=k}^{1}G_j(-s)e<+\infty\right\}$$

and

$$\alpha_T=\sup\{s\geqslant 0:[-U_1(-s)]^{-1}<+\infty\}.$$

It is clear that

$$\gamma=\min\{\eta,\alpha_T\},$$

which depends on both the matrix T and the initial probability vector $\alpha=(\alpha_0,\alpha_1,\alpha_2,\dots)$.

9.8.6 The Level-Dependent Markov Chains of *GI*/*M*/1 Type

Let

$$T = \begin{pmatrix} A_1^{(1)} & A_0^{(1)} & & & \\ A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & & \\ A_3^{(3)} & A_2^{(3)} & A_1^{(3)} & A_0^{(3)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We assume that $\alpha = (\alpha_1, 0, 0, \ldots)$. In this case, note that

$$\alpha[I - G_L(s)]^{-1}[-U_D(s)]^{-1} = (\alpha_1[-U_1(s)]^{-1}, 0, 0, \ldots),$$

$$[I - R_U(s)]^{-1} = \begin{pmatrix} I & X_1^{(1)}(s) & X_2^{(1)}(s) & X_3^{(1)}(s) & \cdots \\ & I & X_1^{(2)}(s) & X_2^{(2)}(s) & \cdots \\ & & I & X_1^{(3)}(s) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix}$$

with

$$X_0^{(l)}(s) = I,$$

$$X_k^{(l)}(s) = R_l(s)R_{l+1}(s)\ldots R_{l+k-1}(s) = \prod_{i=l}^{l+k-1} R_i(s).$$

Note that $T^0 = ((T_1^0)^{\mathrm{T}}, (T_2^0)^{\mathrm{T}}, (T_3^0)^{\mathrm{T}}, \ldots)^{\mathrm{T}}$, we have

$$[I - R_U(s)]^{-1}T^0 = \begin{pmatrix} \sum_{k=0}^{\infty} X_k^{(1)}(s)T_{k+1}^0 \\ * \\ * \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty}\prod_{i=1}^{k-1} R_i(s)T_{k+1}^0 \\ * \\ * \\ \vdots \end{pmatrix},$$

we obtain

$$P^*(z) = \alpha_0 + \alpha_1[-U_1(s)]^{-1} \cdot \sum_{k=0}^{\infty}\prod_{i=1}^{k} R_i(s)T_{k+1}^0.$$

Let

$$\eta = \sup\left\{ s \geqslant 0 : \sum_{k=0}^{\infty}\prod_{i=1}^{k} R_i(s)T_{k+1}^0 < +\infty \right\}$$

and

$$\alpha_T = \sup\{s \geqslant 0 : [-U_1(s)]^{-1} < +\infty\}.$$

Then $\gamma = \min\{\eta, \alpha_T\}$.

9.9 QBD Processes with Infinitely-Many Phases

In this section, we apply the quasi-stationary distribution and the UL-type RG-factorization to study the decay rate of the stationary probability vector of a discrete-time level-independent QBD process with infinitely-many phases.

Consider an irreducible discrete-time level-independent QBD process with infinitely-many phases whose transition probability matrix is given by

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ C_0 & B & A & & \\ & C & B & A & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{9.115}$$

where the sizes of matrices B_0 and B are infinite and the sizes of all other blocks are determined accordingly. We assume that the QBD process is irreducible and positive recurrent. Let $\pi = (\pi_0, \pi_1, \pi_2, \dots)$ be the stationary probability vector of the QBD process. Then $\pi_k = \pi_1 R^{k-1}$ for $k \geqslant 1$, where the matrix R is the minimal nonnegative solution to the matrix equation $A + RB + R^2 C = R$. In addition, let the matrix G be the minimal nonnegative solution to the matrix equation $AG^2 + BG + C = G$. Let U be the transition probability matrix of the censored chain of the Markov chain $\mathcal{P}$ to level 0, where

$$\mathcal{P} = \begin{pmatrix} B & A & & & \\ C & B & A & & \\ & C & B & A & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then

$$\begin{aligned} U &= B + (A, 0, 0, \dots)[I - \mathcal{P}]_{\min}^{-1} (C^{\mathrm{T}}, 0, 0, \dots)^{\mathrm{T}} \\ &= B + A(I - U)^{-1} C. \end{aligned}$$

It is easy to see from the censoring technique that

$$R = A(I-U)^{-1}$$

and

$$G = (I-U)^{-1} C.$$

It is easy to check from the RG-factorization for the repeated row that

$$I - (\eta^{-1} A + B + \eta C) = (\eta I - R)(I - U)(\eta^{-1} I - G).$$

We assume that there exists a positive constant $\eta < 1$ and two positive vectors x and y such that

Condition 1 $x(\eta^{-1}A+B+\eta C)=x$ and $(\eta^{-1}A+B+\eta C)y=y$.
Condition 2 $xe=1$ and $xy<+\infty$.
Condition 3 $\eta^{-1}xAy \neq \eta xCy$.

Then $xR=\eta x$ and $Rz=\eta z$, where $z=(I-U)(\eta^{-1}I-G)y$.

For the positive recurrent QBD process Q, if Conditions 1 to 3 hold, $c=\pi_1 z<+\infty$ and the QBD process $\mathcal{P}$ is irreducible, then

$$R^n \sim \eta^n zx, \quad \text{as } n\to\infty.$$

Thus, we obtain that as $n\to\infty$

$$\pi_n=\pi_1 R^n \sim \eta^n cx.$$

We consider a double QBD process P given in Eq. (9.115) whose repeated blocks are given by

$$A=\begin{pmatrix} \overline{a}_1 & \overline{a}_0 & & & & \\ \overline{a}_2 & a_1 & a_0 & & & \\ & a_2 & a_1 & a_0 & & \\ & & a_2 & a_1 & a_0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$B=\begin{pmatrix} \overline{b}_1 & \overline{b}_0 & & & & \\ \overline{b}_2 & b_1 & b_0 & & & \\ & b_2 & b_1 & b_0 & & \\ & & b_2 & b_1 & b_0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

and

$$C=\begin{pmatrix} \overline{c}_1 & \overline{c}_0 & & & & \\ \overline{c}_2 & c_1 & c_0 & & & \\ & c_2 & c_1 & c_0 & & \\ & & c_2 & c_1 & c_0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We write

$$\begin{aligned} D(\eta) &= A+\eta B+\eta^2 C \\ &= \begin{pmatrix} \overline{\gamma}(\eta) & \overline{\lambda}(\eta) & & & & \\ \overline{\mu}(\eta) & \gamma(\eta) & \lambda(\eta) & & & \\ & \mu(\eta) & \gamma(\eta) & \lambda(\eta) & & \\ & & \mu(\eta) & \gamma(\eta) & \lambda(\eta) & \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \end{aligned}$$

and

$$E(\eta)=\begin{pmatrix} \gamma(\eta) & \lambda(\eta) & & & \\ \mu(\eta) & \gamma(\eta) & \lambda(\eta) & & \\ & \mu(\eta) & \gamma(\eta) & \lambda(\eta) & \\ & & \mu(\eta) & \gamma(\eta) & \lambda(\eta) \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let $\alpha(\eta)$ be the radius of convergence for the matrix $D(\eta)$. We denote by x and y the $\alpha(\eta)$-invariant measure and the $\alpha(\eta)$-invariant vector of the matrix $D(\eta)$. To determine $\alpha(\eta)$, we first need to compute the radius $\overline{\alpha}(\eta)$ of convergence for the matrix $E(\eta)$. Let $v=1/\overline{\alpha}(\eta)$ and $\chi(z)=\mu(\eta)+z\gamma(\eta)+z^2\lambda(\eta)$. Then solving the system of equations

$$\begin{cases}\chi(\theta)=v\theta,\\ \chi'(\theta)=v,\end{cases}$$

we obtain

$$\begin{cases}\theta=\sqrt{\dfrac{\mu(\eta)}{\lambda(\eta)}},\\ v=\gamma(\eta)+2\sqrt{\lambda(\eta)\mu(\eta)}.\end{cases}$$

Hence we have

$$\overline{\alpha}(\eta)=\frac{1}{\gamma(\eta)+2\sqrt{\lambda(\eta)\mu(\eta)}}.$$

To provide the state classification of the matrix $D(\eta)$, we consider the censored matrix of $D(\eta)$ to level 0 which is given by

$$U_0(\eta)=\frac{\overline{\gamma}(\eta)}{\eta}+\frac{\overline{\lambda}(\eta)\overline{\mu}(\eta)}{2\sqrt{\lambda(\eta)\mu(\eta)}}\left(1-\frac{\gamma(\eta)}{\eta}-\sqrt{\left(\frac{\gamma(\eta)}{\eta}-1\right)^2-\frac{4\lambda(\eta)\mu(\eta)}{\eta^2}}\right).$$

Thus, we obtain

$$q(\eta)=U_0(\eta)-1=\sum_{k=0}^{\infty}D_k\eta^k,$$

Note that, in Condition 2: $xe=1$ and $xy<+\infty$, this indicates that the matrix $D(\eta)$ is $\alpha(\eta)$-positive recurrent. That is, we consider the QBD process with infinitely-many phases in which the matrix $D(\eta)$ must be $\alpha(\eta)$-positive recurrent.

The following theorem provides a sufficient condition under which the matrix $D(\eta)$ is $\alpha(\eta)$-positive recurrent. The proof is clear, and thus is omitted here.

Theorem 9.28 *If $\alpha(\eta) < \overline{\alpha}(\eta)$, then the matrix $D(\eta)$ must be $\alpha(\eta)$-positive recurrent. In this case, $\alpha = 1/\eta$.*

If the matrix $D(\eta)$ is $\alpha(\eta)$-positive recurrent, then the $\alpha(\eta)$-invariant measure is given by

$$x_0 = z_0$$

and

$$x_k = z_0 R_{0,1}(\alpha) R^{k-1}(\alpha), \quad k \geqslant 1,$$

where $z_0 > 0$ such that $\sum_{k=0}^{\infty} x_k = 1$, or

$$z_0 = \frac{1}{1 + R_{0,1}(\alpha)[1 - R(\alpha)]^{-1}},$$

$$R(\alpha) = \frac{1 - \alpha\gamma(\eta) - \sqrt{(\alpha\gamma(\eta) - 1)^2 - 4\alpha^2 \lambda(\eta)\mu(\eta)}}{2\alpha\mu(\eta)}$$

and

$$R_{0,1}(\alpha) = \frac{\overline{\lambda}(\eta)}{\lambda(\eta)} R(\alpha);$$

and the $\alpha(\eta)$-invariant vector is given by

$$y_0 = w_0$$

and

$$y_k = w_0 G_{1,0}(\alpha) G^{k-1}(\alpha), \quad k \geqslant 1,$$

where $w_0 > 0$ such that $\sum_{k=0}^{\infty} x_k y_k < +\infty$, or

$$w_0 = \frac{1}{1 + G_{0,1}(\alpha)[1 - G(\alpha)]^{-1}},$$

$$G(\alpha) = \frac{1 - \alpha\gamma(\eta) - \sqrt{(\alpha\gamma(\eta) - 1)^2 - 4\alpha^2 \lambda(\eta)\mu(\eta)}}{2\alpha\lambda(\eta)}$$

and

$$G_{1,0}(\alpha) = \frac{\overline{\mu}(\eta)}{\mu(\eta)} G(\alpha).$$

Note that

$$\sum_{k=0}^{\infty} x_k y_k = z_0 w_0 \left(1 + \frac{\overline{\lambda}(\eta)\overline{\mu}(\eta)}{\lambda(\eta)\mu(\eta)} \sum_{n=1}^{\infty} \left(\frac{\mu(\eta)}{\lambda(\eta)} \right)^n R^{2n}(\alpha) \right),$$

we obtain that $\sum_{k=0}^{\infty} x_k y_k < +\infty.$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{\mu(\eta)}{\lambda(\eta)} \right)^n R^{2n}(\alpha) < +\infty.$$

Thus we have

$$\frac{\mu(\eta)}{\lambda(\eta)} R^2(\alpha) < 1.$$

For the double QBD process, if the matrix $D(\eta)$ is $\alpha(\eta)$-positive recurrent with $\alpha = 1/\eta,$ then

$$R^n \sim \eta^n zx, \quad \text{as } n \to \infty.$$

Thus, we obtain that as $n \to \infty.$

$$\pi_n = \pi_1 R^n \sim \eta^n cx,$$

where

$$c = \pi_1 z < +\infty$$

and

$$z = (I - U)(\eta^{-1} I - G) y.$$

9.10 Notes in the Literature

Quasi-stationary behaviour for block-structured Markov chains is not only theoretically important, but also it is found to have many interesting applications in practical areas, including biology by Scheffer [30], Holling [10], Pakes [24] and Pollett [26]; chemistry by Oppenheim, Shuler and Weiss[23], Parsons and Pollett [25] and Pollett [27]; telecommunications by Schrijner [31]; queues by Makimoto [20], Kijima and Makimoto [15], among others. Three excellent overviews on the quasi-stationarity behaviour can be found in Schrijiner [31], Kijima and Makimoto [15] and Pollett [29].

The study of the quasi-stationary behavior was initiated by Yaglom[33]. Since then, significant advances have been made through the efforts of many researchers. For the block-structured transition matrices, those works were centered in obtaining probabilistic measures for expressing the radius of convergence, the state α-classification, and the quasi-stationary distributions. Kijima [14] first analyzed the quasi-stationary distributions of the Markov chains of *GI*/*M*/1 type without the boundary and the Markov chains of *M*/*G*/1 type without the boundary through the matrix *R* and the matrix *G*, respectively. Some preliminary results for expressing the quasi-stationary distributions of the Markov chains of *GI*/*M*/1 type and *M*/*G*/1 type were obtained in Li [17]. A complete work for the quasi-stationary distributions of the Markov chains of *GI*/*M*/1 type and *M*/*G*/1 type were obtained in Li [18,19]. On the other hand, the quasi-stationarity distributions of the QBD processes can be found in Makimoto [20],Bean, Bright, Latouche, Pearce, Pollett and Taylor [1], and Bean, Pollett and Taylor [3,4].

The concept of β-invariant measures is a generalization of the quasi-stationarity distributions, e.g., see Derman [5], Harris [9], Latouche, Pearce and Taylor [16], and Gail, Hantler and Taylor [7]. In this case, π can still be interpreted probabilistically in terms of the movement of particles whose initial states are governed by Poisson distributions. Readers may refer to Derman [5], Kelly [12], Latouche, Pearce and Taylor [16], and Gail, Hantler and Taylor [17].

This chapter is mainly based on Li and Zhao [18,19], Kijima [14], Bean, Pollett and Taylor [3], Fujimoto, Takahashi and Makimoto [6], Bean and Nielsen [2] and Haque, and Zhao and Liu [8]. At the same time, we have also added some new results without publication for a more systematical organization.

Problems

9.1 For a Markov chain *P* of *GI*/*G*/1 type, provide probabilistic interpretation on the *R*-, *U*- and *G*-measures of the matrix βP.

9.2 Consider a continuous-time Markov chain *Q* of *GI*/*G*/1 type without the boundary, prove that *Q* is α-transient and provide expression for the quasi-stationary distribution.

9.3 Construct an irreducible Markov chain with finitely many levels, which is α-transient.

9.4 For an α-recurrent Markov chain of *GI*/*G*/1 type, provide conditions under which *P* is α-positive recurrent.

9.5 For a *PH*/*PH*/1/*N* queue, compute the quasi-stationary distribution when the server is in a busy period.

9.6 For a *MAP*/*PH*/1 queue with a repairable server, compute the quasi-stationary distribution before the server fails for the first time.

9.7 For a *BMAP*/M^X/1 queue with a repairable server, compute the quasi-stationary distribution before the server fails for the first time.

9.8 For a level-dependent QBD process, provide expression for the quasi-stationary distribution in terms of the LU-type *RG*-factorization.

9.9 Consider a two-demand queueing system, where there are three types of customers. The first and second types of customers can enter server one and server two for their services, respectively. Each arrival of the third type of customers simultaneously places two service demands to server one and server two. The arrivals of the three types of customers form Poisson processes with arrival rates λ_1, λ_2 and λ_3, respectively. The service times at the two servers are exponential with rates μ_1 and μ_2, respectively. The waiting rooms before the two servers are infinite. Use the QBD process with infinitely-many phases to analyze the decay parameter for the stationary probability vector.

9.10 Consider a pre-emptive priority queue with an infinite waiting room. Two types of Poisson customers arrive independently at rate λ_1 for lower priority customers and at rate λ_2 for higher priority customers. The two types of customers require the same exponential service time at rate μ. with the preemptive rule, a higher priority customer, upon arrival, passes all lower priority customers in the queue or takes over the service if a lower priority customer is currently being served. Use the QBD process with infinitely-many phases to analyze the decay parameter for the stationary probability vector.

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10 Markov Reward Processes

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Abstract In this chapter, we consider reward processes of an irreducible continuous-time block-structured Markov chain. By using the *RG*-factorizations, we provide a unified algorithmic framework to derive expressions for conditional distributions and conditional moments of the reward processes. As an important example, we study the reward processes for an irreducible continuous-time level-dependent QBD process with either finitely-many levels or infinitely-many levels. At the same time, we provide a simple introduction to the reward processes of an irreducible discrete-time block-structured Markov chain.

Keywords stochastic models, *RG*-factorization, reward process, accumulated reward, reward rate, the first accumulated time.

In this chapter, we consider various reward processes of an irreducible continuous-time block-structured Markov chain. By using the UL- and LU-types of *RG*-factorizations, we provide a unified algorithmic framework to derive expressions for conditional distributions and conditional moments of the reward processes. As an important example, we study the reward processes for an irreducible continuous-time level-dependent QBD process with either finitely-many levels or infinitely-many levels. At the same time, we provide a simple introduction to the reward processes of an irreducible discrete-time block-structured Markov chain.

This chapter is organized as follows. Section 10.1 considers reward processes of an irreducible continuous-time block-structured Markov chain in terms of the UL- and LU-types of *RG*-factorizations. Section 10.2 deals with the transient accumulated rewards in terms of the partial differential equations. Section 10.3 computes moments of both the accumulated reward and the first accumulated time to a given reward. Section 10.4 analyzes an accumulated reward process for an irreducible continuous-time level-dependent QBD process in terms of a system of infinite-dimensional linear equations. Sections 10.5 and 10.6 study an up-type reward process of the QBD process with finitely-many levels and with infinitely-many levels, respectively. Section 10.7 discusses a down-type reward process and a return-type reward process for the QBD process. Section 10.8 provides a simple introduction to the reward processes for an irreducible discrete-time block-structured Markov chain. Finally, Section 10.9 summarizes notes for the references related to the results of this chapter.

10.1 Continuous-Time Markov Reward Processes

In this section, we consider various reward processes of an irreducible continuous-time block-structured Markov chain in terms of the UL- and LU-types of RG-factorizations, and provide expressions for the conditional distributions and onditioncel moments of the reward processes.

We consider an irreducible continuous-time block-structured Markov chain $\{x_t, t \geqslant 0\}$ on the state space $\Omega = \{(k, j) : k \geqslant 0, 1 \leqslant j \leqslant m_k\}$ whose infinitesimal generator is given by

$$Q = \begin{pmatrix} D_{0,0} & D_{0,1} & D_{0,2} & \cdots \\ D_{1,0} & D_{1,1} & D_{1,2} & \cdots \\ D_{2,0} & D_{2,1} & D_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Let

$$\pi_{k,j}(t) = P\{x_t = (k, j)\},$$
$$\pi_k(t) = (\pi_{k,1}(t), \pi_{k,2}(t), \ldots, \pi_{k,m_k}(t))$$

and

$$\pi(t) = (\pi_0(t), \pi_1(t), \pi_2(t), \ldots).$$

Then it is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(t) = \pi(t)Q, \tag{10.1}$$

which leads to

$$\pi(t) = \pi(0)\exp\{Qt\},$$

where $\pi(0)$ is the initial probability vector of the Markov chain Q.

If the Markov chain Q is positive recurrent, then the limit $\pi = \lim_{t \to \infty} \pi(t)$ exists, and it is clear that $\pi Q = 0$ and $\pi e = 1$, Where e is a column vector of ones with suitable size. Further, if $\pi(0) = \pi$, then $\pi(t) = \pi$ for all $t \geqslant 0$.

Let $L(t)=\int_0^t \pi(x)\mathrm{d}x$. We call $L(t)$ a cumulative state probability vector of the Markov chain Q. At the same time, $L_{k,i}(t)$ is the expected total time spent by the Markov chain Q at state (k,i) during the time interval $[0,t)$. It is easy to check that

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t)=L(t)Q+\pi(0). \tag{10.2}$$

Thus we have

$$L(t)=\pi(0)\int_0^t \exp\{Qx\}\mathrm{d}x. \tag{10.3}$$

Specifically, we write $\tau=\int_0^{+\infty}\pi(x)\mathrm{d}x$. Note that $\lim\limits_{t\to\infty}\mathrm{d}L(t)/\mathrm{d}t=0$, it follows from Eq. (10.2) that

$$\tau Q=-\pi(0) \tag{10.4}$$

Using the UL-type RG-factorization, we obtain

$$\tau=-\pi(0)Q_{\max}^{-1}=-\pi(0)(I-G_L)^{-1}U_D^{-1}(I-R_U)^{-1}.$$

For

$$U_D^{-1}=\mathrm{diag}(U_0^{-1},U_1^{-1},U_2^{-1},\ldots),$$

it is worthwhile to note that U_0^{-1} is the ordinary inverse if the Markov chain Q is transient; U_0^{-1} is the group inverse $U_0^{\#}$ if the Markov chain Q is recurrent. Note that the group inverse is given by

$$U_0^{-1}=(U_0-ex_0)^{-1}-ex_0,$$

where x_0 is the sectionary probability vector of the Markov chain U_0.

On the other hand, applying the LU-type RG-factorization we obtain

$$\tau=-\pi(0)Q_{\max}^{-1}=-\pi(0)(I-\overline{G}_U)^{-1}\,\overline{U}_D^{-1}(I-\overline{R}_L)^{-1}.$$

Note that $\overline{U}_0^{-1}$ is the ordinary inverse of the censored matrix $\overline{U}_0$ for any irreducible Markov chain Q.

Remark 10.1 (1) *If $E\subset\Omega$, then $\tau e=\sum\limits_{(k,i)\in E}\tau_{k,i}$ is the expected total time spent by the Markov chain Q in the state set E.*

(2) *If $Qe\lneqq 0$, then the Markov chain Q contains at least one absorbing state, hence $\tau e=\sum\limits_{(k,i)\in\Omega}\tau_{k,i}$ is the mean time to absorption.*

Let $f(x)$ be a real function which guarantees that the random variable $f(X_t)$ is finite for $t \geqslant 0$ a.s.. Then $f(X_t)$ is an instantaneous reward rate at time *t*. We now consider the instantaneous reward rate $f(X_t)$ through Eq. (10.5) to Eq. (10.9), which can be easily proved by the law of total probability.

10.1.1 The Expected Instantaneous Reward Rate at Time *t*

$$E[f(X_t)] = \sum_{(k,i)\in\Omega} \pi_{k,i}(t) f(k,i) = \sum_{k=0}^{\infty} \pi_k(t) f_k = \pi(t) f, \tag{10.5}$$

where

$$f_k = (f(k,1), f(k,2), \dots, f(k,m_k))^{\mathrm{T}}, \quad f = (f_0^{\mathrm{T}}, f_1^{\mathrm{T}}, f_2^{\mathrm{T}}, \dots)^{\mathrm{T}}.$$

If the Markov chain Q is stable, then the limit of the expected instantaneous reward rate $f(X_t)$ is given by

$$\lim_{t\to+\infty} E[f(X_t)] = \sum_{(k,i)\in\Omega} \pi_{k,i} f(k,i) = \sum_{k=0}^{\infty} \pi_k f_k = \pi f. \tag{10.6}$$

10.1.2 The *n*th Moment of the Instantaneous Reward Rate at Time *t*

$$E[f(X_t)^n] = \sum_{(k,i)\in\Omega} \pi_{k,i}(t) f(k,i)^n. \tag{10.7}$$

If the Markov chain Q is stable, then the limit of the nth moment of the instantaneous reward rate $f(X_t)$ is given by

$$\lim_{t\to+\infty} E\left[f(X_t)^n\right] = \sum_{(k,i)\in\Omega} \pi_{k,i} f(k,i)^n. \tag{10.8}$$

10.1.3 The Distribution of the Instantaneous Reward Rate at Time *t*

Note that

$$\begin{aligned} P\{f(X_t) \leqslant x\} &= \sum_{(k,i)\in\Omega} P\{f(X_t) \leqslant x, X_t = (k,i)\} \\ &= \sum_{(k,i)\in\Omega} P\{f(k,i) \leqslant x, X_t = (k,i)\} \\ &= \sum_{\substack{f(k,i)\leqslant x \\ (k,i)\in\Omega}} \pi_{k,i}(t). \end{aligned}$$

If the Markov chain Q is stable, then

$$\lim_{t \to +\infty} P\{f(X_t) \leqslant x\} = \sum_{\substack{f(k,i) \leqslant x \\ (k,i) \in \Omega}} \pi_{k,i}. \tag{10.9}$$

10.1.4 The Accumulated Reward Over [0,*t*)

For the instantaneous reward rate $f(X_t)$, an accumulated reward over the time interval $[0,t)$ is defined as

$$\Phi(t) = \int_0^t f(X_u)\mathrm{d}u. \tag{10.10}$$

We assume that the real function $f(x)$ makes the random variable $\Phi(t) < +\infty$ for $t \geqslant 0$, a.s..

The accumulated reward process $\Phi(t)$ may represent many interesting performance measures of a stochastic model. For example, we analyze an irreducible QBD process: (1) if $f(k,j)=1$ for $m \leqslant k \leqslant n$, then $\Phi(t)$ represents the sojourn time of the QBD process on levels m to n in the time interval $[0,t]$. (2) If $f(k,j)$ is a reward associated with operating at state (k,j) for $(k,j) \in \Omega$, then $\Phi(t)$ represents the total reward of the QBD process in the time interval $[0,t]$.

10.1.5 The Expected Accumulated Reward *Φ*(*t*) Over [0,*t*)

Note that

$$\begin{aligned} E[\Phi(t)] &= E\left[\int_0^t f(X_u)\mathrm{d}u\right] \\ &= \sum_{(k,i)\in\Omega} f(k,i)\int_0^t P\{X_u = (k,i)\}\mathrm{d}u \\ &= \sum_{(k,i)\in\Omega} L_{k,i}(t) f(k,i), \end{aligned}$$

we obtain

$$E[\Phi(t)] = \sum_{(k,i)\in\Omega} L_{k,i}(t) f(k,i) = \sum_{k=0}^{\infty} L_k(t) f_k = L(t) f. \tag{10.11}$$

10.1.6 The *n*th Moment of the Accumulated Reward *Φ*(*t*) Over [0,*t*)

$$E[\Phi(t)^n] = \sum_{(k,i)\in\Omega} L_{k,i}(t) f(k,i)^n. \tag{10.12}$$

As a practical application, we use the Markov reward process to analyze a finite-buffer $M/M/1/m$ queue with server breakdowns and repairs. A reward rate of 1 is assigned to all the system operational states and a reward rate of 0 is assigned to all the system failure states. The instantaneous availability of the system is $E[f(X(t))]$ and the cumulative operational time of the system in the time interval $[0,t)$ is $E[\Phi(t)]$. The interval availability of the system in the time interval $[0,t)$ is $E[\Phi(t)]/t$, and the stationary availability of the system is $\lim_{t\to+\infty} E[\Phi(t)]/t$. On the other hand, the measures related to the time to the system first failure are of interest. To compute these measures, all the failure states are regarded as an absorbing state. In this case, the reliability function is $E[f(X(t))]$. The lifetime of the system in the time interval $[0,t)$ is $E[\Phi(t)]$ and the mean time to the system first failure is $E[\Phi(+\infty)]$.

10.2 The Transient Accumulated Rewards

In this section, we analyze the transient accumulated rewards of an irreducible continuous-time block-structured Markov chain in terms of a useful method devoloped by the partial differential equations.

Based on the real function $f(x)$ and the Markov chain Q, the transient probability distribution of the accumulated reward $\Phi(t)$ is defined by

$$\Theta(t,x) = P\{\Phi(t) \leqslant x\}.$$

We write

$$\begin{cases} H_{k,j}(t,x) = P\{\Phi(t) \leqslant x, X(t) = (k,j)\}, \\ H_k(t,x) = (H_{k,1}(t,x), H_{k,2}(t,x), \ldots, H_{k,m_k}(t,x)) \end{cases} \tag{10.13}$$

and

$$H(t,x) = (H_0(t,x), H_1(t,x), H_2(t,x), \ldots);$$
$$\varDelta(f_k) = \mathrm{diag}(f(k,1), f(k,1), \ldots, f(k,m_k))$$

and

$$\varDelta = \mathrm{diag}\left(\varDelta(f_0), \varDelta(f_1), \varDelta(f_2), \ldots\right).$$

It is clear that

$$\Theta(t,x) = H(t,x)e.$$

For the Markov reward process $\{\Phi(t), t \geqslant 0\}$, it is easy to obtain the Kolmogorov's forward equation as follows:

$$\frac{\partial H_{k,j}(t,x)}{\partial t}+\frac{\partial H_{k,j}(t,x)}{\partial x}f(k,j)=\sum_{(l,i)\in\Omega}H_{l,i}(t,x)D_{(l,i),(k,j)},\tag{10.14}$$

or

$$\frac{\partial H_k(t,x)}{\partial t}+\frac{\partial H_k(t,x)}{\partial x}\Delta(f_k)=\sum_{l=0}^{\infty}H_l(t,x)D_{l,k},$$

Therefore, we have

$$\frac{\partial H(t,x)}{\partial t}+\frac{\partial H(t,x)}{\partial x}\Delta=H(t,x)Q\tag{10.15}$$

with the initial conditions

$$H(t,0)=\pi(0)\delta(t)\tag{10.16}$$

and

$$H(0,x)=\pi(0)\delta(x).\tag{10.17}$$

Remark 10.2 *Let* $L_{k,j}(t,x)=P\{\Phi(t)\leqslant x,X(t)=(k,j)\}$ *and* $L(t,x)=H^{\mathrm{T}}(t,x)$. *Then the Kolmogorov's backward equation is given by*

$$\frac{\partial L_{k,j}(t,x)}{\partial t}+f(k,j)\frac{\partial L_{k,j}(t,x)}{\partial x}=\sum_{(l,i)\in\Omega}D_{(k,j),(l,i)}L_{l,i}(t,x),\tag{10.18}$$

which leads to

$$\frac{\partial L(t,x)}{\partial t}+\Delta\frac{\partial L(t,x)}{\partial x}=QL(t,x),\tag{10.19}$$

with the initial conditions

$$L(t,0)=e\delta(t)$$

and

$$L(0,x)=e\delta(x).$$

It is easy to see that $\Theta(t,x)=\pi(0)L(t,x)$.

We introduce the necessary notation for the Laplace transform and the Laplace-Stieltjes transform as follows:

$$A^*(s)=\int_0^{+\infty}e^{-st}A(t)\mathrm{d}t$$

and

$$A^{\sim}(s)=\int_0^{+\infty}e^{-st}\mathrm{d}A(t).$$

Now, we solve the Kolmogorov's forward equation Eq. (10.15) with the initial conditions Eq. (10.16) and Eq. (10.17). Taking the Laplace transform of Eq. (10.15) with respect to $t \to s,$ we obtain

$$sH^*(s,x) - H(0,x) + \frac{\mathrm{d}H^*(s,x)}{\mathrm{d}x}\varDelta = H^*(s,x)Q.$$

Note that $H(0,x) = \pi(0)\delta(x)$ in Eq. (10.17), we obtain

$$\frac{\mathrm{d}H^*(s,x)}{\mathrm{d}x}\varDelta = H^*(s,x)(Q - sI) + \pi(0). \tag{10.20}$$

Taking the Laplace-Stieltjes transform of Eq. (10.20) with respect to $x \to u,$ we obtain

$$H^{*\sim}(s,u)[Q - (sI + u\varDelta)] = -\pi(0),$$
$$H^{*\sim}(s,u) = -\pi(0)[Q - (sI + u\varDelta)]_{\max}^{-1},$$

which leads to

$$H^{\sim\sim}(s,u) = -s\pi(0)[Q - (sI + u\varDelta)]_{\max}^{-1}.$$

Let

$$S^{\sim\sim}(s,u) = -s[Q - (sI + u\varDelta)]_{\max}^{-1}.$$

Then

$$H^{\sim\sim}(s,u) = \pi(0)S^{\sim\sim}(s,u).$$

Note that for $s,u \geqslant 0,$ the matrix $Q - (sI + u\varDelta)$ is the infinitesimal generator of an irreducible Markov chain, we have the UL-type RG-factorization

$$Q - (sI + u\varDelta) = [I - R_U(s,u)]U_D(s,u)[I - G_L(s,u)],$$

which leads to

$$H^{\sim\sim}(s,u) = -s\pi(0)[I - G_L(s,u)]^{-1}U_D^{-1}(s,u)[I - R_U(s,u)]^{-1},$$

where

$$U_D^{-1}(s,u) = \mathrm{diag}(U_0^{-1}(s,u), U_1^{-1}(s,u), U_2^{-1}(s,u), \ldots),$$

On the other hand, we obtain the LU-type RG-factorization

$$Q - (sI + u\varDelta) = [I - \overline{R}_L(s,u)]\overline{U}_D(s,u)[I - \overline{G}_U(s,u)],$$

hence we have

$$H^{\sim\sim}(s,u) = -s\pi(0)[I - \overline{G}_U(s,u)]^{-1}\,\overline{U}_D^{-1}(s,u)[I - \overline{R}_L(s,u)]^{-1}.$$

10.3 The First Accumulated Time

In this section, we analyze the first accumulated time of the reward process to a given reward, and obtain the conditional distribution of the first accumulated time.

Let $\Gamma(x)$ be the first accumulated time to a given reward x. Then

$$\Gamma(x) = \min\{t : \Phi(t) = x\}. \tag{10.21}$$

We write

$$C(t,x) = P\{\Gamma(x) \leqslant t\}. \tag{10.22}$$

Since $\Phi(t)$ is the accumulated reward over the time interval $[0,t)$, it is clear that the event $\Gamma(x) \leqslant t$ is equivalent to the event $\Phi(t) > x$. Hence we get

$$C(t,x) = 1 - P\{\Phi(t) \leqslant x\} = 1 - \Theta(t,x). \tag{10.23}$$

Let the given reward W be a random variable with the distribution $w(x) = P\{W \leqslant x\}$. Then

$$C(t) = P\{\Gamma(W) \leqslant t\} = \int_0^{+\infty} C(t,x)\mathrm{d}w(x). \tag{10.24}$$

Now, we compute the distributions of the accumulated reward $\Phi(t)$ and the first accumulated time $\Gamma(x)$ to a given reward x. We define

$$S_{(l,i),(k,j)}(t,x) = P\{\Phi(t) \leqslant x, X(t) = (k,j) \mid X(0) = (l,i)\} \tag{10.25}$$

and

$$T_{(l,i),(k,j)}(t,x) = P\{\Gamma(x) \leqslant t, X(t) = (k,j) \mid X(0) = (l,i)\}. \tag{10.26}$$

We write

$$S_{l,k}(t,x) = (S_{(l,i),(k,j)}(t,x))_{1\leqslant i\leqslant m_l, 1\leqslant j\leqslant m_k},$$
$$S(t,x) = (S_{l,k}(t,x))_{l,k\geqslant 0};$$
$$T_{l,k}(t,x) = (T_{(l,i),(k,j)}(t,x))_{1\leqslant i\leqslant m_l, 1\leqslant j\leqslant m_k},$$
$$T(t,x) = (T_{l,k}(t,x))_{l,k\geqslant 0}.$$

It is clear that

$$\Theta(t,x) = \pi(0)S(t,x)e, \tag{10.27}$$

$$C(t,x) = \pi(0)T(t,x)e \tag{10.28}$$

and

$$[S(t,x) + T(t,x)]e = e.$$

The following theorem provides two useful relations for the two matrices $S^{\sim\sim}(s,u)$ and $T^{\sim\sim}(s,u)$, respectively.

Theorem 10.1

$$(sI + u\Delta - Q)S^{\sim\sim}(s,u) = sI \tag{10.29}$$

and

$$(sI + u\Delta - Q)T^{\sim\sim}(s,u) = u\Delta - Q. \tag{10.30}$$

Proof Consider an exponentially distributed reward requirement W with parameter u. Then it follows from Eq. (10.23) and Eq. (10.24) that

$$\begin{aligned} C(t,u) &= \int_0^{+\infty} C(t,x)\,\mathrm{d}\left[1-e^{-ux}\right] \\ &= u\int_0^{+\infty}\left[1-\Theta(t,x)\right]e^{-ux}\mathrm{d}x \\ &= 1-\Theta^{\sim}(t,u). \end{aligned}$$

Note that $C(t,u)$ is of phase type with irreducible expression $(\pi(0), Q - u\Delta)$, thus we obtain

$$1-\Theta^{\sim}(t,u) = C(t,u) = 1-\pi(0)\exp\{(Q-u\Delta)t\}e,$$

which leads to

$$\Theta^{\sim}(t,u) = \pi(0)\exp\{(Q-u\Delta)t\}e.$$

It follows from Eq. (10.27) and Eq. (10.28) that

$$S^{\sim}(t,u) = \exp\{(Q-u\Delta)t\}, \tag{10.31}$$

which leads to Eq. (10.29).

Note that

$$T^{\sim\sim}(s,u) = I - S^{\sim\sim}(s,u),$$

we have

$$(sI + u\Delta - Q)T^{\sim\sim}(s,u) = (sI + u\Delta - Q)[I - S^{\sim\sim}(s,u)] = u\Delta - Q.$$

This completes the proof.

It is easy to check that the matrix $Q-(sI+u\Delta)$ for $s,u \geqslant 0$ is the infinitesimal generator of an irreducible Markov chain. Thus, we obtain

$$S^{\sim\sim}(s,u) = -s[Q-(sI+u\Delta)]_{\max}^{-1} \tag{10.32}$$

and

$$T^{\sim\sim}(s,u) = -[Q-(sI+u\Delta)]_{\max}^{-1}(u\Delta - Q). \tag{10.33}$$

It is clear that

$$T^{\sim\sim}(s,u) = \frac{1}{s} S^{\sim\sim}(s,u)(u\Delta - Q).$$

Note that

$$[Q-(sI+u\Delta)]_{\max}^{-1} = [I-G_L(s,u)]^{-1} U_D^{-1}(s,u)[I-R_U(s,u)]^{-1}$$

or

$$[Q-(sI+u\Delta)]_{\max}^{-1} = [I-\overline{G}_U(s,u)]^{-1} \overline{U}_D^{-1}(s,u)[I-\overline{R}_L(s,u)]^{-1}.$$

Finally, we compute the first moments of the random variables $\Phi(t)$ and $\Gamma(x)$.

It follows from Eq. (10.3) and Eq. (10.11) that

$$E[\Phi(t)] = \pi(0)\int_0^t \exp\{Qx\}\mathrm{d}x\Delta e. \tag{10.34}$$

It follows from Eq. (10.22) and Eq. (10.23) that

$$P\{\Gamma(x) \leqslant t\} = 1 - P\{\Phi(t) \leqslant x\}$$

which leads to

$$E[\Gamma(x)] = \int_0^{+\infty} P\{\Phi(t) \leqslant x\}\mathrm{d}t.$$

Let $\phi(x) = E[\Gamma(x)]$. Then

$$\begin{aligned}\phi^{\sim}(u) &= \int_0^{+\infty} \Theta^{\sim}(t,u)\mathrm{d}t \\ &= \int_0^{+\infty} \pi(0)\exp\{(Q-u\Delta)t\}e\mathrm{d}t \\ &= -\pi(0)(Q-u\Delta)_{\max}^{-1}e.\end{aligned} \tag{10.35}$$

10.4 Computation of the Reward Moments

In this section, we provide a method for computing moments of the accumulated reward and the first accumulated time to a given reward.

10.4.1 The Moments of the Transient Accumulated Reward

It follows from Eq. (10.15) that

$$\frac{\partial H(t,x)}{\partial t} + \frac{\partial H(t,x)}{\partial x}\Delta = H(t,x)Q$$

with the initial conditions

$$H(t,0)=\pi(0)\delta(t)$$

and

$$H(0,x)=\pi(0)\delta(x).$$

We obtain

$$\frac{\partial H^{\sim}(t,u)}{\partial t}=H^{\sim}(t,u)(Q+u\Delta)$$

with the initial condition

$$H^{\sim}(0,u)=\pi(0).$$

Thus we have

$$H^{\sim}(t,u)=\pi(0)\exp\{(Q+u\Delta)t\},$$

which leads to

$$\Theta^{\sim}(t,u)=\pi(0)\exp\{(Q+u\Delta)t\}e.$$

It is clear that for $n\geqslant 1$,

$$E[\Phi(t)^n]=(-1)^n\frac{\partial^n}{\partial u^n}\Theta^{\sim}(t,u)_{|u=0}.$$

We write

$$\Lambda(n,i)=\frac{\partial^n}{\partial u^n}(Q-u\Delta)^i\,|_{u=0},\quad n,i\geqslant 1.$$

The following lemma provides an iterative relationship for the matrices $\Lambda(n,i)$ for $n,i\geqslant 1$. The proof is clear, and thus is omitted here.

Lemma 10.1 *For* $n,i\geqslant 0$,

$$\begin{aligned}
&\Lambda(0,0)=I, && \\
&\Lambda(n,0)=0, && n\geqslant 1,\\
&\Lambda(0,i)=Q^i, && i\geqslant 1,\\
&\Lambda(1,i)=-i\Delta Q^{i-1}, && i\geqslant 1,\\
&\Lambda(n,1)=0, && n\geqslant 2,\\
&\Lambda(n,i)=Q\Lambda(n,i-1)-n\Delta\Lambda(n-1,i-1), && n,i\geqslant 2.
\end{aligned}$$

The following theorem provides an expression for the nth moment $E[\Phi(t)^n]$.

Theorem 10.2 *For* $n\geqslant 1$, ,

$$E[\Phi(t)^n]=(-1)^n\pi(0)\sum_{i=0}^{\infty}\frac{t^i}{i!}\Lambda(n,i)e. \tag{10.36}$$

Proof It follows from Eq. (10.27) that

$$\Theta^{\sim}(t,u) = \pi(0)\exp\{(Q-u\Delta)t\}e$$
$$= \pi(0)\sum_{i=0}^{\infty}\frac{t^i}{i!}(Q-u\Delta)^i e.$$

Hence, we obtain

$$E[\Phi(t)^n] = (-1)^n \frac{\partial^n}{\partial u^n}\Theta^{\sim}(t,u)|_{u=0}$$
$$= (-1)^n \pi(0)\sum_{i=0}^{\infty}\frac{t^i}{i!}\Lambda(n,i)e.$$

This completes the proof.

10.4.2 The Moments of the First Accumulated Time

It is complicated to compute the function $E[\Gamma(x)^n]$ for $n \geqslant 1$, thus we consider the following two cases:

Case Ⅰ $f(k,i) \neq 0$ for all $k \geqslant 0$ and $1 \leqslant i \leqslant m_k$.

It follows from Eq. (10.15) that

$$\frac{\mathrm{d}H^{\sim}(s,x)}{\mathrm{d}x} = H^{\sim}(s,x)(Q-sI)\Delta^{-1}$$

with the initial condition

$$H^{\sim}(s,0) = \pi(0).$$

Therefore, we get

$$H^{\sim}(s,x) = \pi(0)\exp\{(Q-sI)\Delta^{-1}x\},$$

which leads to

$$\Theta^{\sim}(s,x) = \pi(0)\exp\{(Q-sI)\Delta^{-1}x\}e.$$

Note that

$$E[\Gamma(x)^n] = (-1)^n \frac{\partial^n}{\partial s^n}\Theta^{\sim}(s,x)_{s=0}.$$

Let

$$M(n,i) = \frac{\partial^n}{\partial s^n}[(Q-sI)\Delta^{-1}]^i_{|s=0}, \quad n,i \geqslant 0.$$

Then

$$\begin{aligned}
&M(0,0)=I, && \\
&M(n,0)=0, && n\geqslant 1,\\
&M(0,i)=(Q\varDelta^{-1})^{i}, && i\geqslant 1,\\
&M(n,i)=Q\varDelta^{-1}M(n,i-1)-nQ\varDelta^{-1}M(n-1,i-1), && n,i\geqslant 1.
\end{aligned}$$

Therefore, we obtain

$$E[\varGamma(x)^n]=(-1)^n\pi(0)\sum_{i=0}^{\infty}\frac{x^i}{i!}M(n,i)e.$$

Now, we provide another method for computing $E[\varGamma(x)^n]$.

Let $\phi^{(n)}(x)=E[\varGamma(x)^n]$ and $\widetilde{\phi^{(n)}}(u)=\int_0^{+\infty}e^{-ux}d\phi^{(n)}(x)$. Then

$$\begin{aligned}
\widetilde{\phi^{(n)}}(u)&=(-1)^n\frac{\partial^n}{\partial s^n}C^{\sim\sim}(s,u)\,|_{s=0}\\
&=(-1)^n\frac{\partial^n}{\partial s^n}\pi(0)T^{\sim\sim}(s,u)e|_{s=0}\\
&=(-1)^{n+1}\pi(0)\frac{\partial^n}{\partial s^n}[Q-(sI+u\varDelta)]_{\max}^{-1}u\varDelta e|_{s=0}\\
&=(-1)^n\pi(0)\frac{\partial^n}{\partial s^n}\sum_{i=0}^{\infty}(-1)^i(I+su^{-1}\varDelta^{-1}-u^{-1}\varDelta^{-1}Q)^ie|_{s=0}.
\end{aligned}$$

We write

$$\nabla_u(n,i)=\frac{\partial^n}{\partial s^n}(I+su^{-1}\varDelta^{-1}-u^{-1}\varDelta^{-1}Q)^i\,|_{s=0},\quad n,i\geqslant 0.$$

Thus we obtain

$$\widetilde{\phi^{(n)}}(u)=(-1)^n\pi(0)\sum_{i=0}^{\infty}(-1)^i\nabla_u(n,i)e.$$

Case Ⅱ $f(k,i)\neq 0$ for $0\leqslant k\leqslant N$ and $1\leqslant i\leqslant m_k$, while $f(k,i)=0$ for $k\geqslant N+1$ and $1\leqslant i\leqslant m_k$.

In this case, we write

$$\pi(0)=(\pi_1(0),\pi_2(0)),$$
$$\varDelta=\mathrm{diag}(\varDelta_1,0)$$

and

$$Q=\begin{pmatrix}Q_{1,1} & Q_{1,2}\\ Q_{2,1} & Q_{2,2}\end{pmatrix}$$

according to $E=\{\text{Level } k : 0 \leqslant k \leqslant N\}$ and $E^c=\{\text{Level } k : k \geqslant N+1\}$. Hence

$$[Q-(sI+u\Delta)]_{\max}^{-1}=\begin{pmatrix} Q_{1,1}-(sI+u\Delta_1) & Q_{1,2} \\ Q_{2,1} & Q_{2,2}-sI \end{pmatrix}_{\max}^{-1}.$$

Let

$$Q-(sI+u\Delta)=\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A=Q_{1,1}-(sI+u\Delta_1),$$
$$B=Q_{1,2},$$
$$C=Q_{2,1}$$

and

$$D=Q_{2,2}-sI.$$

Then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}=\begin{pmatrix} F^{-1} & -F^{-1}BD^{-1} \\ -D^{-1}CF^{-1} & D^{-1}+D^{-1}CF^{-1}BD^{-1} \end{pmatrix},$$

where

$$F=A-BD^{-1}C.$$

It follows from Eq. (10.28) and Eq. (10.33) that

$$\begin{aligned} C^{\sim\sim}(s,u) &= \pi(0)T^{\sim\sim}(s,u)e \\ &= -\pi(0)[Q-(sI+u\Delta)]_{\max}^{-1}\,u\Delta e \\ &= -(\pi_1(0),\pi_2(0))\begin{pmatrix} F^{-1} & -F^{-1}BD^{-1} \\ -D^{-1}CF^{-1} & D^{-1}+D^{-1}CF^{-1}BD^{-1} \end{pmatrix}\begin{pmatrix} u\Delta_1 e \\ 0 \end{pmatrix} \\ &= -\pi_1(0)F^{-1}u\Delta_1 e+\pi_2(0)D^{-1}CF^{-1}u\Delta_1 e. \end{aligned}$$

Let

$$C_1^{\sim\sim}(s,u)=-\pi_1(0)F^{-1}u\Delta_1 e$$

and

$$C_2^{\sim\sim}(s,u)=\pi_2(0)D^{-1}CF^{-1}u\Delta_1 e.$$

Then

$$C_1^{\sim\sim}(s,u)=\pi_1(0)\,[sI+u\Delta_1-Q_{1,1}-Q_{1,2}(sI-Q_{2,2})^{-1}Q_{2,1}]^{-1}u\Delta_1 e,$$

which leads to

$$C_1^{\sim}(s,x)=\pi_1(0)\exp\{\xi(s)x\}e,$$

where

$$\xi(s)=\varDelta_1^{-1}Q_{1,1}+\varDelta_1^{-1}Q_{1,2}(sI-Q_{2,2})^{-1}Q_{2,1}-s\varDelta_1^{-1};$$

$$\begin{aligned}C_2^{\sim\sim}(s,u)=&\pi_2(0)(sI-Q_{2,2})Q_{2,1}\\&\cdot[sI+u\varDelta_1-Q_{1,1}-Q_{1,2}(sI-Q_{2,2})^{-1}Q_{2,1}]^{-1}u\varDelta_1 e,\end{aligned}$$

which yields

$$C_2^{\sim}(s,x)=\pi_2(0)(sI-Q_{2,2})Q_{2,1}\exp\{\xi(s)x\}e.$$

We write

$$E[\varGamma(x)^n]=\phi_1^{(n)}(x)+\phi_2^{(n)}(x),$$

where

$$\phi_1^{(n)}(x)=(-1)^n\frac{\partial^n}{\partial s^n}C_1^{\sim}(s,x)\,|_{s=0}$$

and

$$\phi_2^{(n)}(x)=(-1)^n\frac{\partial^n}{\partial s^n}C_2^{\sim}(s,x)\,|_{s=0}.$$

Note that

$$C_1^{\sim}(s,x)=\pi_1(0)\sum_{i=0}^{\infty}\frac{[\xi(s)x]^i}{i!}e,$$

we obtain

$$\phi_1^{(n)}(x)=(-1)^n\sum_{i=0}^{\infty}\frac{x^i}{i!}L^{(n)}(i)e,$$

where

$$\begin{aligned}L^{(n)}(i)=&\frac{\mathrm{d}^n[\xi(s)]^i}{\mathrm{d}s^n}\,|_{s=0}\\=&\begin{cases}0, & i=0,n\geqslant 1,\\ \left(\varDelta_1^{-1}Q_{1,1}-\varDelta_1^{-1}Q_{1,2}Q_{2,2}^{-1}Q_{2,1}\right)^i, & i\geqslant 0,n=0,\\ -\varDelta_1^{-1}-\varDelta_1^{-1}Q_{1,2}Q_{2,2}^{-1}Q_{2,1}, & i=1,n=1,\\ (-1)^{n+1}n!\varDelta_1^{-1}Q_{1,2}Q_{2,2}^{-1}Q_{2,1}, & i=1,n\geqslant 2,\\ \displaystyle\sum_{l=0}^{n}\binom{n}{l}L^{(l)}(1)L^{(n-1)}(i-1), & i\geqslant 2,n\geqslant 1.\end{cases}\end{aligned}$$

Similarly, since

$$C_2^{\sim}(s,x) = \pi_2(0)(sI - Q_{2,2})Q_{2,1}\sum_{i=0}^{\infty}\frac{[\xi(s)x]^i}{i!}e$$
$$= \pi_2(0)\sum_{i=0}^{\infty}\frac{x^i}{i!}\left(sI - Q_{2,2}\right)Q_{2,1}[\xi(s)]^i e,$$

we obtain

$$\phi_2^{(n)}(x) = (-1)^n \pi_2(0)\sum_{i=0}^{\infty}\frac{x^i}{i!}H^{(n)}(i)e,$$

where

$$H^{(n)}(i) = \left.\frac{\mathrm{d}^n\{(sI - Q_{2,2})Q_{2,1}[\xi(s)]^i\}}{\mathrm{d}s^n}\right|_{s=0}$$
$$= \begin{cases} (-1)^n Q_{2,2}^{-(n+1)}Q_{2,1}, & i = 0, n \geqslant 0, \\ Q_{2,2}^{-1}Q_{2,1}(\varDelta_1^{-1}Q_{1,1} - \varDelta_1^{-1}Q_{1,2}Q_{2,2}^{-1}Q_{2,1})^i, & i \geqslant 1, n = 0, \\ \sum_{l=0}^{n}\binom{n}{l}H^{(l)}(i-1)G^{(n-l)}, & i \geqslant 1, n \geqslant 1, \end{cases}$$

and

$$G^{(n)} = \begin{cases} \varDelta_1^{-1}Q_{1,1} - \varDelta_1^{-1}Q_{1,2}Q_{2,2}^{-1}Q_{2,1}, & n = 0, \\ -\varDelta_1 - \varDelta_1 Q_{1,2}Q_{2,2}^{-1}Q_{2,1}, & n = 1, \\ (-1)^{n+1}n!\varDelta_1 Q_{1,2}Q_{2,2}^{-1}Q_{2,1}, & n \geqslant 2. \end{cases}$$

Therefore for $n \geqslant 1$,

$$E[\varGamma(x)^n] = \phi_1^{(n)}(x) = (-1)^n\sum_{i=0}^{\infty}\frac{x^i}{i!}[L^{(n)}(i) + H^{(n)}(i)]e.$$

Since the matrix $Q_{2,2}$ is of infinite size, the inverse of $Q_{2,2}$ is maximal nonpositive. We can use the *RG*-factorizations to compute the maximal nonpositive inverse $Q_{2,2}^{-1}$.

10.5 Accumulated Reward in a QBD Process

In this section, we consider the accumulated reward process of an irreducible continuous-time QBD process. We provide an iterative solution for the Laplace transforms of the conditional moments of the accumulated reward process in terms of a system of infinite-dimensional linear equations.

Let $\{x(t), t \geqslant 0\}$ be an irreducible continuous-time QBD process with infinitesimal generator Q given in Eq. (1.4). We assume that the QBD process Q is separable and Borel measurable, and its sample functions are all lower semi-continuous at the right-hand side. Therefore, the QBD process has the strong Markov property. Intuitively, if each diagonal entry of the matrix Q is finite, then the QBD process also has the strong Markov property.

We define an accumulated reward process as

$$\Phi(t) = \int_0^t f(x(u))\mathrm{d}u. \tag{10.37}$$

Suppose the initial state of the QBD process $\{x(t), t \geqslant 0\}$ is at state (k, j), i.e, $x(0) = (k, j)$. Let

$$\xi_{k,j}(t) = E\left[\int_0^t f(x(u))\mathrm{d}u \mid x(0) = (k, j)\right] \tag{10.38}$$

and

$$\xi_k(t) = (\xi_{k,1}(t), \xi_{k,2}(t), \ldots, \xi_{k,M_k}(t))^{\mathrm{T}}, \quad k \geqslant 0.$$

We denote by $\xi_k^*(s)$ the Laplace transform of the column vector $\xi_k(t)$, that is,

$$\xi_k^*(s) = \int_0^{+\infty} e^{-st}\xi_k(t)\mathrm{d}t, \quad k \geqslant 0. \tag{10.39}$$

For an arbitrary matrix $A_l^{(k)}$, we use $a_{l,(i,j)}^{(k)}$ to denote its (i, j) th entry.

The following theorem provides a system of infinite-dimensional linear equations satisfied by the vector sequence $\{\xi_k^*(s)\}$.

Theorem 10.3 *The vector sequence $\{\xi_k^*(s), k \geqslant 0\}$ satisfies the following system of infinite-dimensional linear equations,*

$$\xi_0^*(s) = \Re_0(s)\left[e + B_0\xi_0^*(s) + B_0^{(0)}\xi_1^*(s)\right], \tag{10.40}$$

and for $l \geqslant 1$,

$$\xi_l^*(s) = \Re_l(s)\left[e + B_2^{(l)}\xi_{l-1}^*(s) + B_l\xi_l^*(s) + B_0^{(l)}\xi_{l+1}^*(s)\right], \tag{10.41}$$

where for $k \geqslant 0$,

$$B_k = B_1^{(k)} - \mathrm{diag}\left(b_{1,(1,1)}^{(k)}, b_{1,(2,2)}^{(k)}, \ldots, b_{1,(M_k,M_k)}^{(k)}\right) \tag{10.42}$$

and

$$\Re_k(s) = \mathrm{diag}\left(\frac{f(k,1)}{\left[s - b_{1,(1,1)}^{(k)}\right]^2}, \frac{f(k,2)}{\left[s - b_{1,(2,2)}^{(k)}\right]^2}, \ldots, \frac{f(k,M_k)}{\left[s - b_{1,(M_k,M_k)}^{(k)}\right]^2}\right).$$

Proof We only prove Eq. (10.40), while the proof of Eq. (10.41) is similar. Let $\beta = \inf\{t : x(t) \neq (l,i)\}$. Then we have

$$\begin{aligned}
\xi_{(l,i)}(t) &= E\left[\int_0^t f(x(u))\mathrm{d}u \mid x(0)=(l,i)\right] \\
&= E\left[\int_0^t f(x(u))\mathrm{d}u, \beta > t \mid x(0)=(l,i)\right] \\
&\quad + E\left[\int_0^t f(x(u))\mathrm{d}u, \beta \leqslant t \mid x(0)=(l,i)\right]. \qquad (10.43)
\end{aligned}$$

When $x(0)=(l,i)$, it is easy to check that $P\{\beta > t\} = \exp\left\{b_{1,(i,i)}^{(l)} t\right\}$ with $b_{1,(i,i)}^{(l)} < 0$, and

$$x(\beta) = \begin{cases} (l+1, j_0), & \text{with probability} \dfrac{b_{0,(i,j_0)}^{(l)}}{-b_{1,(i,i)}^{(l)}}, \\ (l, j_1), & \text{with probability} \dfrac{b_{1,(i,j_1)}^{(l)}}{-b_{1,(i,i)}^{(l)}} \quad \text{for} \quad j_1 \neq i, \\ (l-1, j_2), & \text{with probability} \dfrac{b_{2,(i,j_2)}^{(l)}}{-b_{1,(i,i)}^{(l)}}. \end{cases}$$

Let $\mathcal{F}_\beta$ be a σ-algebra consisting of all the events of the QBD process which have occurred before the Markov time β. By the strong Markov property of the QBD process, we obtain that for $l \geqslant 1$,

$$E\left[\int_0^t f(x(u))\mathrm{d}u, \beta > t \mid x(0)=(l,i)\right] = f(l,i) t \exp\left\{b_{1,(i,i)}^{(l)} t\right\}, \qquad (10.44)$$

and

$$\begin{aligned}
& E\left[\int_0^t f(x(u))\mathrm{d}u, \beta \leqslant t \mid x(0)=(l,i)\right] \\
&= f(l,i) \sum_{j_0=1}^{M_{l+1}} b_{0,(i,j_0)}^{(l)} \int_0^t x \exp\left\{b_{1,(i,i)}^{(l)} x\right\} \xi_{(l+1,j_0)}(t-x)\mathrm{d}x \\
&\quad + f(l,i) \sum_{j_2=1}^{M_{l-1}} b_{2,(i,j_2)}^{(l)} \int_0^t x \exp\left\{b_{1,(i,i)}^{(l)} x\right\} \xi_{(l-1,j_2)}(t-x)\mathrm{d}x \\
&\quad + f(l,i) \sum_{j_1 \neq i}^{M_l} b_{1,(i,j_1)}^{(l)} \int_0^t x \exp\left\{b_{1,(i,i)}^{(l)} x\right\} \xi_{(l,j_1)}(t-x)\mathrm{d}x \qquad (10.45)
\end{aligned}$$

Taking the Laplace transforms on the both sides of Eq. (10.44) and Eq. (10.45), it follows from Eq. (10.43) that for $l \geqslant 1$,

$$\xi_l^*(s) = \Re_l(s)\left[e + B_2^{(l)} \xi_{l-1}^*(s) + B_l \xi_l^*(s) + B_0^{(l)} \xi_{l+1}^*(s)\right].$$

This completes the proof.

Let

$$\widetilde{Q}(s)=\begin{pmatrix}\Re_0(s) & & & \\ & \Re_1(s) & & \\ & & \Re_2(s) & \\ & & & \ddots\end{pmatrix}\begin{pmatrix}B_0 & B_0^{(0)} & & & \\ B_2^{(1)} & B_1 & B_0^{(1)} & & \\ & B_2^{(2)} & B_2 & B_0^{(2)} & \\ & & \ddots & \ddots & \ddots\end{pmatrix},$$

$$\Re(s)=\begin{pmatrix}\Re_0(s)e\\ \Re_1(s)e\\ \Re_2(s)e\\ \vdots\end{pmatrix} \text{ and } \xi(s)=\begin{pmatrix}\xi_0^*(s)\\ \xi_1^*(s)\\ \xi_2^*(s)\\ \vdots\end{pmatrix}.$$

Then it follows from Theorem 10.13 that

$$\xi(s)=\widetilde{Q}(s)\xi(s)+\Re(s). \tag{10.46}$$

Note that $\widetilde{Q}(s)\geqslant 0$ and $\Re(s)\geqslant 0$ for $s>0$. Let ${}_0\xi(s)=0$ and

$${}_{N+1}\xi(s)=\widetilde{Q}(s)_N\xi(s)+\Re(s), \quad N\geqslant 0.$$

Theorem 10.4 *If there exists a positive number* $K>0$ *such that* $0\leqslant f(k,j)\leqslant K$ *for all* $k\geqslant 0, 1\leqslant j\leqslant M_k$, *then for each* $s>0$, *the vector sequence* $\{_N\xi(s), N\geqslant 0\}$ *is non-decreasing and upper bounded. Hence the limit* ${}_\infty\xi(s)=\lim\limits_{N\to\infty} {}_N\xi(s)$ *exists for each* $s>0$. *Also,* ${}_\infty\xi(s)$ *is the minimal nonnegative solution to Eq.* (10.46).

Proof We first prove that for each $s>0$, the sequence $\{_N\xi(s)\}$ is non-decreasing for $N\geqslant 0$. Since $f(k,j)\geqslant 0$ for $(k,j)\in\Omega$, $\Re_k(s)\geqslant 0$ for all $k\geqslant 0$. Therefore, the matrices $\widetilde{Q}(s)$ and $\Re(s)$ are all nonnegative for each $s>0$. Therefore, ${}_0\xi(s)=0$ and

$${}_1\xi(s)=\widetilde{Q}(s)_0\xi(s)+\Re(s)=\Re(s)\geqslant {}_0\xi(s).$$

Assuming that ${}_N\xi(s)\geqslant {}_{N-1}\xi(s)$ for an arbitrarily given *N*, one can immediately show that

$${}_{N+1}\xi(s)=\widetilde{Q}(s)_N\xi(s)+\Re(s)\geqslant\widetilde{Q}(s)_{N-1}\xi(s)+\Re(s)={}_N\xi(s).$$

Thus through induction, one can show that for each $s>0$, the vector sequence $\{_N\xi(s)\}$ is non-decreasing for $N\geqslant 0$.

We now prove that for each $s>0$, the sequence $\{_N\xi(s), N\geqslant 0\}$ is upper bounded. Since there exists a positive number $K>0$ such that $0\leqslant f(k,j)\leqslant K$ for all $k\geqslant 0, 1\leqslant j\leqslant M_k$, we obtain

$$\begin{aligned}\|\xi(s)\| &= \sup_{l\geqslant 0,1\leqslant i\leqslant M_l}\left\{\xi^*_{(l,i)}(s)\right\}\\ &= \sup_{l\geqslant 0,1\leqslant i\leqslant M_l}\left\{\int_0^{+\infty} e^{-st}E\left[\int_0^t f(x(\tau))\mathrm{d}\tau \mid x(0)=(l,i)\right]\right\}\\ &\leqslant K\int_0^{+\infty} te^{-st}\mathrm{d}t = K\frac{1}{s^2}.\end{aligned}$$

A similar induction argument yields

$$0\leqslant \|_N\,\xi(s)\|\leqslant\|\xi(s)\|\leqslant K\frac{1}{s^2},$$

for every $N=0,1,2,\ldots$. Therefore, the following limit must exist for all $s>0$

$${}_{\infty}\xi(s)=\lim_{N\to\infty}\ {}_N\xi(s).$$

Finally, we prove that ${}_{\infty}\xi(s)$ is the minimal nonnegative solution to Eq. (10.46). To this end, for an arbitrary nonnegative solution $\xi(s)$ to Eq. (10.46), we have

$$0\leqslant {}_N\xi(s)\leqslant \xi(s),\quad \text{for each}\quad N\geqslant 0.$$

Hence, we obtain $0\leqslant {}_{\infty}\xi(s)\leqslant\xi(s)$, which implies that ${}_{\infty}\xi(s)$ is the minimal nonnegative solution to Eq. (10.46). This completes the proof.

Let

$$\rho(s)=\sup_{k\geqslant 0,1\leqslant j\leqslant M_k}\frac{-b^{(k)}_{1,(j,j)}f(k,j)}{\left(s-b^{(k)}_{1,(j,j)}\right)^2}.$$

Then

$$\rho(0)=\sup_{k\geqslant 0,1\leqslant j\leqslant M_k}\frac{f(k,j)}{-b^{(k)}_{1,(j,j)}}.$$

Theorem 10.5 *Suppose* $\rho(0)<1$.

(1) *The vector sequence* $\{{}_N\xi(s)\}$ *is uniformly convergent for* $s\in(0,+\infty)$ *with the convergent rate being of an exponential type.*

(2) *If D denotes the set of all bounded nonnegative solutions to Eq. (10.46), then* $D=\{{}_{\infty}\xi(s)\}$.

Proof (1) It is easy to check that

$$\|\widetilde{Q}(s)\|=\rho(s)\leqslant\sup_{k\geqslant 0,1\leqslant j\leqslant M_k}\frac{f(k,j)}{s-b^{(k)}_{1,(j,j)}}\leqslant\sup_{k\geqslant 0,1\leqslant j\leqslant M_k}\frac{f(k,j)}{-b^{(k)}_{1,(j,j)}}=\rho(0)<1.$$

Therefore, as $N\to\infty$,

$$\begin{aligned}\|{}_N\xi(s)-{}_{N-1}\xi(s)\| &\leqslant \rho(s)\|{}_{N-1}\xi(s)-{}_{N-2}\xi(s)\| \\ &\leqslant \cdots \\ &\leqslant \rho(0)^{N-1}\|{}_1\xi(s)-{}_0\xi(s)\|\to 0.\end{aligned}$$

(2) Let $\Psi(s)$ and $\xi(s)$ be any two bounded nonnegative solutions to Eq. (10.46). Then as $N\to\infty$,

$$\|\Psi(s)-\xi(s)\|\leqslant \rho(0)\|\Psi(s)-\xi(s)\|\leqslant\cdots\leqslant \rho(0)^N\|\Psi(s)-\xi(s)\|\to 0.$$

Thus, $\Psi(s)=\xi(s)$. Note that ${}_\infty\xi(s)$ is a bounded nonnegative solution to Eq. (10.46), it is clear that $\mathcal{D}=\{{}_\infty\xi(s)\}$.

Remark 10.3 *Note that the level-dependent QBD process with infinitesimal generator Q is equivalent to another QBD process with infinitesimal generator θQ, where θ is an arbitrarily given positive number, thus $\rho(0)<1$ in Theorem 10.5 can be relaxed to a weaker condition:*

$$\sup_{k\geqslant 0,1\leqslant j\leqslant M_k}\frac{f(k,j)}{-\theta b_{1,(j,j)}^{(k)}}<1.$$

In other words, the condition $\rho(0)<1$ given in Theorem 10.5 is in principle not restrictive.

In what follows we provide an iterative relationship for the conditional moments of the accumulated reward process $\Phi(t)$, which shows the usefulness of the maximal non-positive inverse for the related computations.

We write

$$m_{(k,j)}(n)=(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}s^n}\xi_{(k,j)}(s)_{|s=0},$$
$$m_k(n)=(m_{(k,1)}(n),m_{(k,2)}(n),\ldots,m_{(k,M_k)}(n))^{\mathrm{T}},$$

and

$$M(n)=(m_0(n)^{\mathrm{T}},m_1(n)^{\mathrm{T}},m_2(n)^{\mathrm{T}},\ldots)^{\mathrm{T}}.$$

It follows from Eq. (10.38) and Eq. (10.39) that for $n\geqslant 1$,

$$m_{(k,j)}(n)=E\left[\int_0^{+\infty}t^n\Phi(t)\mathrm{d}t\mid x(0)=(k,j)\right].$$

It follows from Eq. (10.46) that

$$\xi(0)=\widetilde{Q}(0)\xi(0)+\Re(0).$$

It is clear that $\widetilde{Q}(0)-I$ is the infinitesimal generator of an irreducible continuous-time QBD process. Thus, there always exists an LU-type *RG*-factorization for the matrix $\widetilde{Q}(0)-I$. Using the LU-type RG-factorization, we obtain

$$[\widetilde{Q}(0)-I]_{\max}^{-1}=(I-\overline{G}_U)^{-1}\overline{U}_D^{-1}(I-\overline{R}_L)^{-1}.$$

Therefore

$$\begin{aligned}\xi(0)&=-[\widetilde{Q}(0)-I]_{\max}^{-1}\Re(0)\\&=-(I-\overline{G}_U)^{-1}\overline{U}_D^{-1}(I-\overline{R}_L)^{-1}\Re(0).\end{aligned}$$

Taking the first n derivatives with respect to s on the both sides of Eq. (10.46) and letting $s=0$, we obtain

$$M(1)=-[\widetilde{Q}(0)-I]_{\max}^{-1}\left\{\left[\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{Q}(s)_{|s=0}\right]\xi(0)+\frac{\mathrm{d}}{\mathrm{d}s}\Re(s)_{|s=0}\right\}$$

and for $n\geqslant 2$,

$$M(n)=-[\widetilde{Q}(0)-I]_{\max}^{-1}\left\{\sum_{l=0}^{n-1}\binom{n}{l}\left[\frac{\mathrm{d}^{n-l}}{\mathrm{d}s^{n-l}}\widetilde{Q}(s)_{|s=0}\right]M(l)+\frac{\mathrm{d}^n}{\mathrm{d}s^n}\Re(s)_{|s=0}\right\}.$$

10.6 An Up-Type Reward Process in Finitely-Many Levels

In this section, we consider an up-type reward process, defined in finitely-many levels, of the QBD process given in Eq. (1.17). We explicitly express the Laplace transforms of the conditional distributions of the up-type reward process and its conditional moments.

The QBD process Q, given in Eq. (1.17), can be denoted as $\mathcal{X}=\{x_t(\omega),t\geqslant 0\}$ with state space $\Omega=\{(k,j):k\geqslant 0,1\leqslant j\leqslant M_k\}$, where $\omega\in\Omega$ is a sample path of the QBD process. It is worthwhile to note that the assumption that each diagonal element of the matrix Q is finite implies that the QBD process is a Markov jump process. Therefore, it follows from Theorem 1.1 in Chapter II of Asmussen [2] that the QBD process has the strong Markov property.

We write

$$\eta_k^{(j)}(\omega)=\begin{cases}\inf\{t\geqslant 0:x_t(\omega)=(k,j)\}, & \text{if the } t\text{-set is not empty},\\ +\infty, & \text{otherwise}.\end{cases}\tag{10.47}$$

Let $V(k,j)$ be a suitable non-zero nonnegative function defined on the set Ω. We define

$$\xi_k^{(j)}(\omega) = \int_0^{\eta_k^{(j)}(\omega)} V(x_t(\omega))\mathrm{d}t, \quad k \geqslant 0. \tag{10.48}$$

We now study the conditional distributions of the reward process $\xi_k^{(j)}(\omega)$ and its conditional moments in terms of the LU-type *RG*-factorization.

Let

$$F_{(l,i)(k,j)}(x) = P_{(l,i)}\left\{\xi_k^{(j)}(\omega) \leqslant x\right\},$$
$$\mathbf{F}_{l,k}(x) = [F_{(l,i)(k,j)}(x)]_{1\leqslant i\leqslant M_l, 1\leqslant j\leqslant M_k}$$

and

$$\mathbf{f}_{l,k}^*(s) = \int_0^{+\infty} e^{-sx}\mathrm{d}\mathbf{F}_{l,k}(x), \quad 0 \leqslant l \leqslant k-1.$$

It is clear that the (i,j) th entry of the matrix $\mathbf{f}_{l,k}^*(s)$ is given by

$$f_{(l,i)(k,j)}^*(s) = E_{(l,i)}\left[\exp\left\{-s\int_0^{\eta_k^{(j)}(\omega)} V(x_t(\omega))\mathrm{d}t\right\}\right].$$

The following theorem provides a system of matrix equations satisfied by the matrix sequence $\{\mathbf{f}_{l,k}^*(s), 0 \leqslant l \leqslant k-1\}$.

Theorem 10.6 *The matrix sequence* $\{\mathbf{f}_{l,k}^*(s), 0 \leqslant l \leqslant k-1\}$ *satisfies the system of matrix equations*

$$\mathcal{A}_1^{(0)}(s)\mathbf{f}_{0,k}^*(s) + A_0^{(0)}\mathbf{f}_{1,k}^*(s) = 0, \tag{10.49}$$

$$A_2^{(l)}\mathbf{f}_{l-1,k}^*(s) + A_1^{(l)}(s)\mathbf{f}_{l,k}^*(s) + A_0^{(l)}\mathbf{f}_{l+1,k}^*(s) = 0, \quad 1 \leqslant l \leqslant k-2, \tag{10.50}$$

$$A_2^{(k-1)}\mathbf{f}_{k-2,k}^*(s) + A_1^{(k-1)}(s)\mathbf{f}_{k-1,k}^*(s) + A_0^{(k-1)} = 0, \tag{10.51}$$

where

$$\mathcal{A}_1^{(l)}(s) = A_1^{(l)} - s \cdot \mathrm{diag}\big(V(l,1), V(l,2), \ldots, V(l,M_l)\big), \quad l \geqslant 0.$$

Proof We only prove Eq. (10.50), while Eq. (10.49) and Eq. (10.51) can be proved similarly.

Let τ be the first transition time of the QBD process from the state (l,i) to a nearby state: Either state $(l-1,i)$, state (l,i) or state $(l+1,i)$. Then

$$P_{(l,i)}\{\tau \leqslant x\} = 1 - \exp\left\{a_{1,ii}^{(l)}x\right\},$$

hence,

$$\mathbf{E}_{(l,i)}[\exp\{-sV(l,i)\tau\}] = \frac{-a_{1,ii}^{(l)}}{sV(l,i) - a_{1,ii}^{(l)}} \stackrel{\text{def}}{=} H_i^{(l)}(s).$$

It is clear that τ is a Markov time. Let $\mathcal{F}_\tau$ be a σ-algebra consisting of all the events of the QBD process which have occurred before the time τ. Applying the strong Markov property of the QBD process, we obtain

$$\begin{aligned}
f^*_{(l,i)(k,j)}(s) &= \mathbf{E}_{(l,i)}\left[\exp\left\{-s\int_0^{\eta_k^{(j)}(\omega)} V(x_t(\omega))\mathrm{d}t\right\}\right] \\
&= \mathbf{E}_{(l,i)}\left[\mathbf{E}_{(l,i)}\left[\exp\left\{-s\int_0^{\eta_k^{(j)}(\omega)} V(x_t(\omega))\mathrm{d}t\right\} \mid \mathcal{F}_\tau\right]\right] \\
&= \mathbf{E}_{(l,i)}[\exp\left\{-s\int_0^{\tau} V(x_t(\omega))\mathrm{d}t\right\} \\
&\quad \cdot \mathbf{E}_{(l,i)}\left[\exp\left\{-s\int_\tau^{\eta_k^{(j)}(\omega)} V(x_t(\omega))\mathrm{d}t\right\} \mid \mathcal{F}_\tau\right] \\
&= \mathbf{E}_{(l,i)}\left[\exp\{-sV(l,i)\tau\}\mathbf{E}_{x_{\tau(\omega)}}\left[\exp\left\{-s\int_0^{\eta_k^{(j)}(\omega)} V(x_t(\omega))\mathrm{d}t\right\}\right]\right]. \quad (10.52)
\end{aligned}$$

Note that

$$P_{(l,i)}\{x_\tau(\omega)=(l-1,j)\} = \frac{a^{(l)}_{2,ij}}{-a^{(l)}_{1,ii}},$$

$$P_{(l,i)}\{x_\tau(\omega)=(l,j)\} = \frac{a^{(l)}_{1,ij}}{-a^{(l)}_{1,ii}}, \quad i \neq j,$$

$$P_{(l,i)}\{x_\tau(\omega)=(l+1,j)\} = \frac{a^{(l)}_{0,ij}}{-a^{(l)}_{1,ii}},$$

it follows from Eq. (10.52) that

$$\begin{aligned}
f^*_{(l,i)(k,j)}(s) &= H_i^{(l)}(s)\sum_{j'=1}^{M_{l-1}} \frac{a^{(l)}_{2,ij'}}{-a^{(l)}_{1,ii}} f^*_{(l-1,j')(k,j)}(s) + H_i^{(l)}(s)\sum_{j'\neq i}^{M_l} \frac{a^{(l)}_{1,ij'}}{-a^{(l)}_{1,ii}} f^*_{(l,j')(k,j)}(s) \\
&\quad + H_i^{(l)}(s)\sum_{j'=1}^{M_{l+1}} \frac{a^{(l)}_{0,ij'}}{-a^{(l)}_{1,ii}} f^*_{(l+1,j')(k,j)}(s),
\end{aligned}$$

which leads to

$$A_2^{(l)}\mathbf{f}^*_{l-1,k}(s) + \mathcal{A}_1^{(l)}(s)\mathbf{f}^*_{l,k}(s) + A_0^{(l)}\mathbf{f}^*_{l+1,k}(s) = 0.$$

This completes the proof.

Let

$$\overline{U}_0(s) = \mathcal{A}_1^{(0)}(s),$$
$$\overline{U}_i(s) = \mathcal{A}_1^{(i)}(s) + A_2^{(i)}\left[-\overline{U}_{i-1}^{-1}(s)\right]A_0^{(i-1)}, \quad 1 \leqslant i \leqslant k-1,$$

and

$$\overline{G}_i(s)=\left[-\overline{U}_i^{-1}(s)\right]A_0^{(i)}, \quad 0\leqslant i\leqslant k-1.$$

The following theorem provides expressions for the matrices $\mathbf{f}_{l,k}^*(s)$ for $0\leqslant l\leqslant k-1$.

Theorem 10.7 *If* $s\geqslant 0$, *then*

$$\mathbf{f}_{l,k}^*(s)=\prod_{j=l}^{k-1}\overline{G}_j(s), \quad 0\leqslant l\leqslant k-1.$$

Proof Using Theorem 10.6, we have

$$\mathbf{A}_{k-1}(s)\vec{\mathbf{f}}_{k-1}^*(s)=-\vec{A}_{k-1}, \tag{10.53}$$

where

$$\vec{\mathbf{f}}_{k-1}^*(s)=\left(\mathbf{f}_{0,k}^*(s)^{\mathrm{T}},\mathbf{f}_{1,k}^*(s)^{\mathrm{T}},\ldots,\mathbf{f}_{k-2,k}^*(s)^{\mathrm{T}},\mathbf{f}_{k-1,k}^*(s)^{\mathrm{T}}\right)^{\mathrm{T}},$$

$$\vec{A}_{k-1}(s)=\left(0^{\mathrm{T}},0^{\mathrm{T}},\ldots,0^{\mathrm{T}},\left[A_0^{(k-1)}\right]^{\mathrm{T}}\right)^{\mathrm{T}}$$

and

$$\mathbf{A}_{k-1}(s)=\begin{pmatrix} \mathcal{A}_1^{(0)}(s) & A_0^{(0)} & & & \\ A_2^{(1)} & \mathcal{A}_1^{(1)}(s) & A_0^{(1)} & & \\ & \ddots & \ddots & \ddots & \\ & & A_2^{(k-2)} & \mathcal{A}_1^{(k-2)}(s) & A_0^{(k-2)} \\ & & & A_2^{(k-1)} & \mathcal{A}_1^{(k-1)}(s) \end{pmatrix}.$$

Since the QBD process given in Eq. (1.17) is irreducible, the QBD process $\mathbf{A}_{k-1}(s)$ for $s\geqslant 0$ is transient so that the matrix $\mathbf{A}_{k-1}(s)$ is invertible. Note that

$$\mathbf{A}_{k-1}(s)=[I-\overline{R}_L(s)]\overline{U}_D(s)[I-\overline{G}_U(s)],$$

it follows from Eq. (10.53) that

$$\vec{\mathbf{f}}_{k-1}^*(s)=[I-\overline{G}_U(s)]^{-1}[-\overline{U}_D(s)]^{-1}[I-\overline{R}_L(s)]^{-1}\vec{A}_{k-1}(s),$$

where

$$[I-\overline{G}_U(s)]^{-1}=\begin{pmatrix} I & Y_1^{(0)} & Y_2^{(0)} & \cdots & Y_{k-1}^{(0)} \\ & I & Y_1^{(1)} & \cdots & Y_{k-2}^{(1)} \\ & & I & \cdots & Y_{k-3}^{(2)} \\ & & & \ddots & \vdots \\ & & & & I \end{pmatrix},$$

$$[-\overline{U}_D(s)]^{-1}=\operatorname{diag}\left([-\overline{U}_0(s)]^{-1},[-\overline{U}_1(s)]^{-1},\ldots,[-\overline{U}_{k-1}(s)]^{-1}\right)$$

and

$$[I-\overline{R}_L(s)]^{-1}=\begin{pmatrix} I & & & & \\ X_1^{(1)} & I & & & \\ X_2^{(2)} & X_1^{(2)} & I & & \\ \vdots & \vdots & \vdots & \ddots & \\ X_{k-1}^{(k-1)} & X_{k-2}^{(k-1)} & X_{k-3}^{(k-1)} & \dots & I \end{pmatrix}.$$

After some matrix computations, we have

$$\mathbf{f}_{l,k}^{*}(s)=Y_{k-1-l}^{(l)}[-\overline{U}_{k-1}(s)]^{-1}A_0^{(k-1)}=\prod_{j=l}^{k-1}\overline{G}_j(s).$$

This completes the proof.

From Theorem 10.7 we easily obtain the following corollary.

Corollary 10.1 *If* $0\leqslant l\leqslant k-1$, *then*

$$\mathbf{f}_{l,k}^{*}(s)=\mathbf{f}_{l,l+1}^{*}(s)\mathbf{f}_{l+1,l+2}^{*}(s)\dots\mathbf{f}_{k-2,k-1}^{*}(s)\mathbf{f}_{k-1,k}^{*}(s).$$

Now, we compute the conditional moments of the reward process $\xi_k^{(j)}(\omega)$. Let

$$m_{(l,i)(k,j)}^{(g)}=\mathbf{E}_{(l,i)}\left[\xi_k^{(j)}(\omega)^g\right]$$

and

$$M_{l,k}^{(g)}=\left(m_{(l,i)(k,j)}^{(g)}\right)_{1\leqslant i\leqslant M_l,1\leqslant j\leqslant M_k},\quad 0\leqslant l\leqslant k-1, g\geqslant 1.$$

We write

$$\varPsi_{l,k}^{(g-1)}=\begin{cases} \operatorname{diag}(V(l,1),V(l,2),\dots,V(l,M_l))\prod_{j=l}^{k-1}\overline{G}_j(0), & \text{if } g=1,\\ \operatorname{diag}(V(l,1),V(l,2),\dots,V(l,M_l))\left(gM_{l,k}^{(g-1)}\right), & \text{if } g\geqslant 2,\end{cases}$$

for $0\leqslant l\leqslant k-1$.

Note that

$$m_{(l,i)(k,j)}^{(g)}=(-1)^g\frac{\mathrm{d}^g}{\mathrm{d}s^g}f_{(l,i)(k,j)}^{*}(s)_{|s=0},$$

it follows from Theorem 10.6 that the matrix sequence $\{M_{l,k}^{(g)}, g\geqslant 1, 0\leqslant l\leqslant k-1\}$ satisfies the system of matrix equations

$$\varPsi_{0,k}^{(g-1)}+A_1^{(0)}M_{0,k}^{(g)}+A_0^{(0)}M_{1,k}^{(g)}=0, \tag{10.54}$$

$$\varPsi_{l,k}^{(g-1)}+A_2^{(l)}M_{l-1,k}^{(g)}+A_1^{(l)}M_{l,k}^{(g)}+A_0^{(l)}M_{l+1,k}^{(g)}=0,\quad 1\leqslant l\leqslant k-2, \tag{10.55}$$

$$\Psi_{k-1,k}^{(g-1)} + A_2^{(k-1)} M_{k-2,k}^{(g)} + A_1^{(k-1)} M_{k-1,k}^{(g)} = 0. \tag{10.56}$$

To solve the system of matrix equations Eq. (10.54), Eq. (10.55) and Eq. (10.56), let the matrix sequences $\{\bar{R}_k\}$, $\{\bar{G}_l\}$ and $\{\bar{U}_i\}$ be the LU-type measures of the QBD process given in Eq. (1.17) for $N = k-2$. Based on this, the matrix sequences $\left\{X_i^{(k)}\right\}$ and $\left\{Y_j^{(l)}\right\}$ are defined by Eq. (1.32) and Eq. (1.33), respectively.

The following corollary provides an iterative relationship for the matrix sequence $\left\{M_{l,k}^{(g)}, g \geqslant 1, 0 \leqslant l \leqslant k-1\right\}$.

Corollary 10.2

$$M_{l,k}^{(g)} = \bar{G}_l M_{l+1,k}^{(g)} - \bar{U}_l^{-1} L_{l,k}^{(g)}, \quad 0 \leqslant l \leqslant k-2,$$
$$M_{k-1,k}^{(g)} = -\bar{U}_{k-1}^{-1}\left[\Psi_{k-1,k}^{(g-1)} - A_2^{(k-1)} L_{k-2,k}^{(g)}\right],$$

where for $0 \leqslant l \leqslant k-2$,

$$L_{l,k}^{(g)} = \Psi_{l,k}^{(g-1)} + \bar{R}_l \Psi_{l-1,k}^{(g-1)} + \bar{R}_l \bar{R}_{l-1} \Psi_{l-2,k}^{(g-1)} + \ldots + \bar{R}_l \bar{R}_{l-1} \ldots \bar{R}_1 \Psi_{0,k}^{(g-1)}.$$

Applying (2) of Theorem 1.2, the following corollary further expresses the matrix sequence $\left\{M_{m,k}^{(g)}, g \geqslant 1, 0 \leqslant m \leqslant k-1\right\}$.

Corollary 10.3 *For* $g \geqslant 1$, $0 \leqslant m \leqslant k-1$,

$$\begin{aligned} M_{m,k}^{(g)} = & \sum_{n=0}^{m-1}\left[\left(-\bar{U}_m^{-1}\right) X_{m-n}^{(m)} + \sum_{i=1}^{k-1-m} Y_i^{(m)}\left(-\bar{U}_{i+m}^{-1}\right) X_{i+m-n}^{(i+m)}\right] \Psi_{n,k}^{(g-1)} \\ & + \left[\left(-\bar{U}_m^{-1}\right) + \sum_{i=1}^{k-1-m} Y_i^{(m)}\left(-\bar{U}_{i+m}^{-1}\right) X_i^{(i+m)}\right] \Psi_{m,k}^{(g-1)} \\ & + \sum_{n=m+1}^{k-1}\left[Y_{n-m}^{(m)}\left(-\bar{U}_n^{-1}\right) + \sum_{i=n-m+1}^{k-1-m} Y_i^{(m)}\left(-\bar{U}_{i+m}^{-1}\right) X_{i-(n-m)}^{(i+m)}\right] \Psi_{n,k}^{(g-1)} \\ & + \left(-\bar{U}_{k-1}^{-1}\right) X_{k-1-m}^{(k-1)} \Psi_{k-1,k}^{(g-1)}. \end{aligned}$$

Proof It follows from Eq. (10.54), Eq. (10.55) and Eq. (10.56) that

$$\mathcal{A}_{k-1}^{*} \overrightarrow{\mathbf{M}}_{k-1}^{(g)} = -\overrightarrow{\Psi}_{k-1}^{(g-1)},$$

where

$$\overrightarrow{\mathbf{M}}_{k-1}^{(g)} = \left(\left[M_{0,k}^{(g)}\right]^{\mathrm{T}}, \left[M_{1,k}^{(g)}\right]^{\mathrm{T}}, \ldots, \left[M_{k-1,k}^{(g)}\right]^{\mathrm{T}}\right)^{\mathrm{T}},$$
$$\overrightarrow{\Psi}_{k-1}^{(g-1)} = \left(\left[\Psi_{0,k}^{(g-1)}\right]^{\mathrm{T}}, \left[\Psi_{1,k}^{(g-1)}\right]^{\mathrm{T}}, \ldots, \left[\Psi_{k-1,k}^{(g-1)}\right]^{\mathrm{T}}\right)^{\mathrm{T}}$$

and

$$\mathcal{A}_{k-1}^{*}=\begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & & \\ & \ddots & \ddots & \ddots & \\ & & A_2^{(k-2)} & A_1^{(k-2)} & A_0^{(k-2)} \\ & & & A_2^{(k-1)} & A_1^{(k-1)} \end{pmatrix}.$$

Since the QBD process given in Eq. (1.17) is irreducible, the matrix $\mathcal{A}_{k-1}^{*}$ is invertible for $k \geqslant 1$, using (2) in Theorem 1.2 some matrix computations yield the desired result.

10.7 An Up-Type Reward Process in Infinitely-Many Levels

In this section, we consider an up-type reward process, defined in infinitely-many levels, of the QBD process given in Eq. (1.16). We provide expressions for the Laplace transforms of the conditional distributions of the up-type reward process and its conditional moments.

We define

$$\eta_k(\omega)=\begin{cases}\inf\{t \geqslant 0 : x_t^L(\omega)=k\}, & \text{if the } t\text{-set is not empty,}\\ +\infty, & \text{otherwise,}\end{cases}$$

where $x_t^L(\omega)$ denotes the level number of the QBD process at time t. It is easy to see from Eq. (10.48) that

$$\eta_k(\omega)=\min_{1\leqslant j\leqslant M_k}\left\{\eta_k^{(j)}(\omega)\right\}. \tag{10.57}$$

If the initial state $x_0(\omega)=(l,i)$ with $0 \leqslant l<k$, then the sequence $\{\eta_k(\omega)\}$ is monotonely increasing for $k>l$. Therefore, the limit: $\lim_{k\to\infty}\eta_k(\omega)$ either $<+\infty$ or $=+\infty$, a.s.. Let

$$\eta(\omega)=\lim_{k\to\infty}\eta_k(\omega), \quad \text{a.s..} \tag{10.58}$$

Then

$$\eta(\omega)=\begin{cases}\inf\{t \geqslant 0 : x_t^L(\omega)=+\infty\}, & \text{if the } t\text{-set is not empty,}\\ +\infty, & \text{otherwise,}\end{cases}$$

where $x_t^L(\omega)=+\infty$ means that the level number of the QBD process can not be finite at time t.

We show below that $\eta(\omega)$ is also a limit of other time sequences. To do this, we define a sequence of *state jumping times* as follows:

Since the QBD process $\mathcal{X}=\{x_t(\omega), t\geqslant 0\}$ is a Markov jump process,

$$\tau_1(\omega)=\inf\{t\geqslant 0: x_t(\omega)\neq x_0(\omega)\}$$

is the first jump time of the QBD process. Define

$$\mathcal{Y}_1=\{y_t(\omega)=x_{\tau_1+t}(\omega), t\geqslant 0\}. \tag{10.59}$$

Using the strong Markov property, $\mathcal{Y}_1$ is also a Markov jump process with the same infinitesimal generator as that of the QBD process $\mathcal{X}$, and

$$\tau_2(\omega)=\inf\{t\geqslant 0: y_t(\omega)\neq y_0(\omega)\} \tag{10.60}$$

is the second jump time of the QBD process $\mathcal{X}$ (or it is also the first jump time of the Markov process $\mathcal{Y}_1$). Similar arguments to Eq. (10.59) and Eq. (10.60) indicate that $\tau_n(\omega)$ is the nth jump time of the QBD process $\mathcal{X}$ for $n\geqslant 3$. It is easy to see that

$$0=\tau_0(\omega)<\tau_1(\omega)<\tau_2(\omega)<\ldots<\tau_n(\omega)<\ldots, \quad \text{a.s..}$$

Hence, the limit: $\lim_{n\to\infty}\tau_n(\omega)$ either $<+\infty$ or $=+\infty$, a.s., and $\eta(\omega)=\lim_{n\to\infty}\tau_n(\omega)$.

Let

$$\xi_k(\omega)=\int_0^{\eta_k(\omega)} V(x_t(\omega))\mathrm{d}t, \quad k\geqslant 0,$$
$$\xi(\omega)=\int_0^{\eta(\omega)} V(x_t(\omega))\mathrm{d}t. \tag{10.61}$$

Since $\{\eta_k(\omega)\}$ is monotonely increasing for $k>l$ and the function $V(x_t(\omega))$ is non-zero nonnegative, the sequence $\{\xi_k(\omega)\}$ is also monotonely increasing for $k>l$. Hence,

$$\xi(\omega)=\lim_{k\to+\infty}\xi_k(\omega), \quad \text{a.s..}$$

Remark 10.4 (1) *If* $V(k,j)\equiv 1$ *for* $k\geqslant 0$ *and* $1\leqslant j\leqslant M_k$, *then* $\xi_k^{(i)}(\omega)=\eta_k^{(i)}(\omega)$, $\xi_k(\omega)=\eta_k(\omega)$ *and* $\xi(\omega)=\eta(\omega)$. *This implies that the reward process becomes the first passage times.*

(2) *If*

$$V(l,j)\equiv\begin{cases}1, & \text{if } 0\leqslant l\leqslant n,\\ 0, & \text{otherwise,}\end{cases}$$

then $\xi_k(\omega)$ *becomes the total sojourn time of the QBD process staying in levels 0 to n before it enters level k for the first time.*

The following lemma provides a criterion for recognizing $\xi(\omega)=+\infty$.

Lemma 10.2 *$\xi(\omega)=+\infty$, a.s., if and only if*

$$\sum_{k=0}^{\infty}\frac{V(n_k,i_k)}{-a_{1,i_k i_k}^{(n_k)}}=+\infty,\quad \text{a.s.},$$

where $\{(n_k,i_k)\}$ is a sequence of jumping states on the sample path ω of the QBD process given in Eq. (1.16).

Based on this lemma, the following theorem provides a sufficient condition under which $\xi(\omega)=+\infty$, a.s.. The sufficient condition is simple to be verified in practice.

Theorem 10.8 *Suppose that the function $V(k,j)$ for $k\geqslant 0, 1\leqslant j\leqslant M_k$ is non-zero nonnegative. If the QBD process is recurrent, then*

$$\xi(\omega)=+\infty,\quad \text{a.s.}.$$

Proof If the function $V(k,j)$ for $k\geqslant 0, 1\leqslant j\leqslant M_k$ is non-zero nonnegative, then there always exists a state (k^*,j^*) such that $V(k^*,j^*)>0$. Since the QBD process is recurrent, it follows from Proposition 1.2 in Asmussen [2] that there must exist a subsequence $\{(n_{k_j},i_{k_j})\}$ of the state sequence $\{(n_k,i_k)\}$ on the sample path ω such that

$$n_{k_j}=k^*,\quad i_{k_j}=j^*,\quad j\geqslant 0.$$

Hence,

$$\sum_{k=0}^{\infty}\frac{V(n_k,i_k)}{-a_{1,i_k i_k}^{(n_k)}}\geqslant\sum_{j=0}^{\infty}\frac{V(n_{k_j},i_{k_j})}{-a_{1,i_{k_j} i_{k_j}}^{(n_{k_j})}}=\sum_{j=0}^{\infty}\frac{V(k^*,j^*)}{-a_{1,j^* j^*}^{(k^*)}}=+\infty,\quad \text{a.s.}.$$

Using Lemma 10.2 leads to $\xi(\omega)=+\infty$, a.s.. This completes the proof.

We now express the Laplace transforms of the conditional distributions of the reward process $\xi(\omega)<+\infty$ a.s. and its conditional moments in terms of the UL-type RG-factorization.

We write

$$\phi_{(l,i)}^*(s)=E_{(l,i))}[\exp\{-s\xi(\omega)\}],$$
$$\varPhi_l^*(s)=\left(\phi_{(l,1)}^*(s),\phi_{(l,2)}^*(s),\ldots,\phi_{(l,M_l)}^*(s)\right)^{\mathrm{T}},\quad l\geqslant 0.$$

The following theorem provides a system of vector equations satisfied by the vector sequence $\left\{\varPhi_l^*(s)\right\}$.

Theorem 10.9 *The vector sequence $\left\{\varPhi_l^*(s)\right\}$ satisfies the system of vector equations*

$$\mathcal{A}_1^{(0)}(s)\Phi_0^*(s) + A_0^{(0)}\Phi_1^*(s) = 0 \tag{10.62}$$

and

$$A_2^{(l)}\Phi_{l-1}^*(s) + \mathcal{A}_1^{(l)}(s)\Phi_l^*(s) + A_0^{(l)}\Phi_{l+1}^*(s) = 0, \quad l \geqslant 1. \tag{10.63}$$

Proof We only prove Eq. (10.63), while Eq. (10.62) can be proved similarly. Using a similar method to Theorem 10.6 leads to

$$\begin{aligned}
\phi_{(l,i)}^*(s) &= E_{(l,i)}\left[\exp\left\{-s\int_0^{\eta(\omega)} V(x_t(\omega))\mathrm{d}t\right\}\right] \\
&= E_{(l,i)}\left[\exp\{-sV(l,i)\tau\}E_{x_{\tau(\omega)}}\left[\exp\left\{-s\int_0^{\eta(\omega)} V(x_t(\omega))\mathrm{d}t\right\}\right]\right] \\
&= H_i^{(l)}(s)\sum_{j'=1}^{M_{l-1}} \frac{a_{2,ij'}^{(l)}}{-a_{1,ii}^{(l)}}\phi_{(l-1,j')}^*(s) + H_i^{(l)}(s)\sum_{j'\neq i}^{M_l} \frac{a_{1,ij'}^{(l)}}{-a_{1,ii}^{(l)}}\phi_{(l,j')}^*(s) \\
&\quad + H_i^{(l)}(s)\sum_{j'=1}^{M_{l+1}} \frac{a_{0,ij'}^{(l)}}{-a_{1,ii}^{(l)}}\phi_{(l+1,j')}^*(s),
\end{aligned}$$

which can be written in matrix form

$$A_2^{(l)}\Phi_{l-1}^*(s) + \mathcal{A}_1^{(l)}(s)\Phi_l^*(s) + A_0^{(l)}\Phi_{l+1}^*(s) = 0.$$

This completes the proof.

We now use the UL-type *RG*-factorization Eq. (1.21) to solve the system of vector Eq. (10.62) and Eq. (10.63). Based on this, we provide an expression for the vector sequence $\left\{\Phi_l^*(s)\right\}$. We use the notation $\{R_l(s)\}$, $\{G_k(s)\}$ and $\{U_l(s)\}$ to denote the UL-type measures of the QBD process with infinitesimal generator, given by

$$\mathcal{Q}(s) = \begin{pmatrix}
\mathcal{A}_1^{(0)}(s) & A_0^{(0)} & & & \\
A_2^{(1)} & \mathcal{A}_1^{(1)}(s) & A_0^{(1)} & & \\
 & A_2^{(2)} & \mathcal{A}_1^{(2)}(s) & A_0^{(2)} & \\
 & & \ddots & \ddots & \ddots
\end{pmatrix}.$$

For the matrix sequence $\{R_l(s)\}$ for $s > 0$, following identically Theorem 3.1 in Latouche, Pearce and Taylor [27] or Lemma 7 in Bean, Pollett and Taylor [3], we give the following lemma, which will be used later in the proof of Theorem 10.10.

Lemma 10.3 *For $s > 0$, there exists a sequence $\{\rho_k(s), k \geqslant 0\}$ of positive scalars and a sequence $\{z_k(s), k \geqslant 0\}$ of non-zero nonnegative column vectors such that*

$$z_k(s) = \rho_k(s)R_k(s)z_{k+1}(s), \quad k \geqslant 0.$$

Using Lemma 10.3 and the *RG*-factorization Eq. (1.21), the following theorem describes the vector sequence $\left\{\Phi_k^*(s)\right\}$.

Theorem 10.10 *For* $s \geqslant 0$, ,

$$\Phi_0^*(s) = [-U_0(s)]^{-1} R_0(s) z_1(s),$$
$$\Phi_1^*(s) = G_1(s)[-U_0(s)]^{-1} R_0(s) z_1(s) + [-U_1(s)]^{-1} z_1(s)$$

and for $k \geqslant 2$,

$$\Phi_k^*(s) = \prod_{j=k}^{1} G_j(s)[-U_0(s)]^{-1} R_0(s) z_1(s)$$
$$+\sum_{i=1}^{k} \prod_{j=k}^{i+1} G_j(s)[-U_i(s)]^{-1} \prod_{l=1}^{i-1} \rho_l(s) z_i(s).$$

Proof According to Eq. (1.21), the UL-type *RG*-factorization of the QBD process $Q(s)$ for $s \geqslant 0$ is written as

$$Q(s) = [I - R_U(s)] U_D(s) [I - G_L(s)].$$

It follows from Theorem 10.9 that $Q(s)\vec{\Phi}(s) = 0$, where

$$\vec{\Phi}(s) = \left(\Phi_0^*(s)^{\mathrm{T}}, \Phi_1^*(s)^{\mathrm{T}}, \Phi_2^*(s)^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}.$$

Hence,

$$[I - R_U(s)][-U_D(s)][I - G_L(s)]\vec{\Phi}(s) = 0. \tag{10.64}$$

We now solve the equation Eq. (10.64) by two steps. In the first step, let

$$v(s) = [-U_D(s)][I - G_L(s)]\vec{\Phi}(s)$$

partitioned according to the levels as

$$v(s) = (v_0(s)^{\mathrm{T}}, v_1(s)^{\mathrm{T}}, v_2(s)^{\mathrm{T}}, \ldots)^{\mathrm{T}}.$$

This is equivalent to

$$\Phi_0^*(s) = [-U_0(s)]^{-1} v_0(s), \tag{10.65}$$

$$\Phi_k^*(s) = G_k(s)\Phi_{k-1}^*(s) + [-U_k(s)]^{-1} v_k(s), \quad k \geqslant 1. \tag{10.66}$$

To determine the sequence $\{v_k(s)\}$, in the second step we solve

$$[I - R_U(s)] v(s) = 0,$$

which is equivalent to

$$v_k(s) = R_k(s) v_{k+1}(s), \quad k \geqslant 0. \tag{10.67}$$

Using Lemma 10.3 and Eq. (10.67) leads to

$$v_0(s) = R_0(s) z_1(s), \tag{10.68}$$

$$v_1(s) = z_1(s), \tag{10.69}$$

$$v_k(s) = \prod_{l=1}^{k-1} \rho_l(s) z_k(s), \quad k \geqslant 2. \tag{10.70}$$

Substituting Eq. (10.68), Eq. (10.68) and Eq. (10.70) into Eq. (10.65) and Eq. (10.66) leads to

$$\varPhi_0^*(s) = [-U_0(s)]^{-1} R_0(s) z_1(s),$$
$$\varPhi_1^*(s) = G_1(s)[-U_0(s)]^{-1} R_0(s) z_1(s) + [-U_1(s)]^{-1} z_1(s)$$

and for $k \geqslant 2$,

$$\begin{aligned}
\varPhi_k^*(s) &= G_k(s)\varPhi_{k-1}^*(s) + [-U_k(s)]^{-1} \prod_{l=1}^{k-1} \rho_l(s) z_k(s) \\
&= G_k(s) G_{k-1}(s) \varPhi_{k-2}^*(s) + G_k(s)[-U_{k-1}(s)]^{-1} \prod_{l=1}^{k-2} \rho_l(s) z_{k-1}(s) \\
&\quad + [-U_k(s)]^{-1} \prod_{l=1}^{k-1} \rho_l(s) z_k(s) \\
&= \cdots \\
&= \prod_{j=k}^{1} G_j(s)[-U_0(s)]^{-1} R_0(s) z_1(s) \\
&\quad + \sum_{i=1}^{k} \prod_{j=k}^{i+1} G_j(s)[-U_i(s)]^{-1} \prod_{l=1}^{i-1} \rho_l(s) z_i(s).
\end{aligned}$$

This completes the proof.

We now compute the conditional moments of the reward process.

For $g \geqslant 1$, $l \geqslant 0$, we write

$$m_{(l,i)}^{(g)} = E_{(l,i))}[\xi(\omega)^g],$$
$$M_l^{(g)} = \left(m_{(l,1)}^{(g)}, m_{(l,2)}^{(g)}, \ldots, m_{(l,M_l)}^{(g)} \right)^{\mathrm{T}}$$

and

$$\varPsi_l^{(g-1)} = \begin{cases} \operatorname{diag}\left(V(l,1), V(l,2), \ldots, V(l,M_l)\right) \varPhi_l^*(0), & \text{if } g = 1, \\ \operatorname{diag}\left(V(l,1), V(l,2), \ldots, V(l,M_l)\right)\left(g M_l^{(g-1)}\right), & \text{if } g \geqslant 2. \end{cases}$$

Based on Theorem 10.9, some computations yield that the vector sequence $\{M_l^{(g)}\}$ satisfies the system of vector equations

$$\Psi_0^{(g-1)} + A_1^{(0)} M_0^{(g)} + A_0^{(0)} M_1^{(g)} = 0 \tag{10.71}$$

and

$$\Psi_l^{(g-1)} + A_2^{(l)} M_{l-1}^{(g)} + A_1^{(l)} M_l^{(g)} + A_0^{(l)} M_{l+1}^{(g)} = 0, \quad g \geqslant 1, l \geqslant 1. \tag{10.72}$$

Using (1) of Theorem 1.2 to solve the system of vector Eq. (10.71) and Eq. (10.72), we have the following corollary.

Corollary 10.4 *The vector sequence* $\{M_m^{(g)}, g \geqslant 1, m \geqslant 0\}$ *is given by*

$$\begin{aligned} M_m^{(g)} = &\sum_{n=0}^{m-1} \left[\left(-\bar{U}_m^{-1}\right) X_{m-n}^{(m)} + \sum_{i=1}^{\infty} Y_i^{(m)} \left(-\bar{U}_{i+m}^{-1}\right) X_{i+m-n}^{(i+m)} \right] \Psi_n^{(g-1)} \\ &+ \left[\left(-\bar{U}_m^{-1}\right) + \sum_{i=1}^{\infty} Y_i^{(m)} \left(-\bar{U}_{i+m}^{-1}\right) X_i^{(i+m)} \right] \Psi_m^{(g-1)} \\ &+ \sum_{n=m+1}^{\infty} \left[Y_{n-m}^{(m)} \left(-\bar{U}_n^{-1}\right) + \sum_{i=n-m+1}^{\infty} Y_i^{(m)} \left(-\bar{U}_{i+m}^{-1}\right) X_{i-(n-m)}^{(i+m)} \right] \Psi_n^{(g-1)}. \end{aligned}$$

It is easy to see from Corollary 10.4 that $\xi(\omega) = +\infty$, a.s., if and only if $M_m^{(1)}$ are infinite for some $m \geqslant 0$. For example, $M_m^{(1)}$ has a simple expression for a birth-death process. In this case, $M_m^{(1)} = +\infty$ is a useful condition under which $\xi(\omega) = +\infty$, a.s..

10.8 A Down-Type Peward Process

In this section, we consider a down-type reward process and a return-type reward process for the QBD process given in Eq. (1.16). By means of the *RG*-factorizations, we first express the Laplace transforms of the conditional distributions of the down-type reward process and its conditional moments. Based on the up- and down-type reward processes, we then provide a way to study the return-type reward process.

Let

$$\xi_0^{(j)}(\omega) = \int_0^{\eta_0^{(j)}(\omega)} V(x_t(\omega)) \mathrm{d}t,$$

which is called a down-type reward process.

Given that a level-independent QBD process starts at level 1, the down-type reward process $\xi_0^{(j)}(\omega)$ is a reward-type generalization of the busy period or the fundamental period, which is referred to Subsection 2.2 in Neuts [40] for more details.

We write

$$G_{k,(i,j)}(x) = P_{(k,i)}\left\{\xi_0^{(j)}(\omega) \leqslant x\right\},$$
$$G_k(x) = [G_{k,(i,j)}(x)]_{1\leqslant i\leqslant M_k, 1\leqslant j\leqslant M_0}$$

and

$$\mathbf{g}_k^*(s) = \int_0^{+\infty} e^{-sx} \mathrm{d}G_k(x).$$

A similar argument to the proof of Theorem 10.6 yields the following lemma.

Lemma 10.4 *The matrix sequence* $\left\{\mathbf{g}_k^*(s)\right\}$ *satisfies the system of matrix equations*

$$A_2^{(1)} + \mathcal{A}_1^{(1)}(s)\mathbf{g}_1^*(s) + A_0^{(1)}\mathbf{g}_2^*(s) = 0 \tag{10.73}$$

and

$$A_2^{(k)}\mathbf{g}_{k-1}^*(s) + \mathcal{A}_1^{(k)}(s)\mathbf{g}_k^*(s) + A_0^{(k)}\mathbf{g}_{k+1}^*(s) = 0, \quad k \geqslant 2. \tag{10.74}$$

Based on Lemma 10.4, we explicitly express the sequence $\left\{\mathbf{g}_k^*(s)\right\}$ in the following theorem.

Theorem 10.11 *For* $s \geqslant 0$,

$$\mathbf{g}_k^*(s) = G_k(s)G_{k-1}(s)G_{k-2}(s)\dots G_1(s), \quad k \geqslant 1,$$

where the sequence $\left\{G_k(s)\right\}$ *is the UL-type G-measure of the QBD process whose infinitesimal generator is given by*

$$Q(s) = \begin{pmatrix} \mathcal{A}_1^{(1)}(s) & A_0^{(1)} & & & \\ A_2^{(2)} & \mathcal{A}_1^{(2)}(s) & A_0^{(2)} & & \\ & A_2^{(3)} & \mathcal{A}_1^{(3)}(s) & A_0^{(3)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Proof Let $\vec{g}(s) = \left(\mathbf{g}_1^*(s)^{\mathrm{T}}, \mathbf{g}_2^*(s)^{\mathrm{T}}, \mathbf{g}_3^*(s)^{\mathrm{T}}, \dots\right)^{\mathrm{T}}$. Then it follows from Lemma 10.4 that $Q(s)\vec{g}(s) = \left(-\left[A_2^{(1)}\right]^{\mathrm{T}}, 0^{\mathrm{T}}, 0^{\mathrm{T}}, \dots\right)^{\mathrm{T}}$. Using the UL-type RG-factorization leads to

$$\begin{aligned} \vec{g}(s) &= [I - G_L(s)]^{-1}[U_D(s)]^{-1}[I - R_U(s)]^{-1}\left(-\left[A_2^{(1)}\right]^{\mathrm{T}}, 0^{\mathrm{T}}, 0^{\mathrm{T}}, \dots\right)^{\mathrm{T}} \\ &= \left(\left\{[-U_1(s)]^{-1}A_2^{(1)}\right\}^{\mathrm{T}}, \left\{Z_1^{(1)}[-U_1(s)]^{-1}A_2^{(1)}\right\}^{\mathrm{T}}, \dots\right)^{\mathrm{T}}, \end{aligned} \tag{10.75}$$

where

$$Z_k^{(k)} = G_k(s)G_{k-1}(s)G_{k-2}(s)\ldots G_2(s), \quad k \geqslant 1.$$

Notice that $\left[-U_1^{-1}(s)\right]A_2^{(1)} = G_1(s)$, and so Eq. (10.75) gives the desired result.

We write

$$w^*_{k,(i,j)}(s) = E_{(k,i)}\left[\exp\left\{-s\int_0^{\eta_{k-1}^{(j)}} V(x_t)\mathrm{d}t\right\}\right]$$

and the matrix

$$w^*_k(s) = \left[w^*_{k,(i,j)}(s)\right]_{1\leqslant i\leqslant M_k, 1\leqslant j\leqslant M_{k-1}}.$$

The following theorem provides expression for the sequence $\{\mathbf{g}^*_k(s)\}$ in terms of the skip-free property of the QBD process.

Theorem 10.12

$$\mathbf{g}^*_k(s) = w^*_k(s)w^*_{k-1}(s)\ldots w^*_1(s), \tag{10.76}$$

where the matrix sequence $\{w^*_l(s)\}$ *satisfies the system of matrix equations*

$$A_2^{(k)} + \mathcal{A}_1^{(k)}(s)w^*_k(s) + A_0^{(k)}w^*_{k+1}(s)w^*_k(s) = 0, \quad k \geqslant 1. \tag{10.77}$$

Proof Similar argument to the proof of Theorem 10.6 leads to

$$\begin{aligned}\mathbf{w}^*_{k,(i,j)}(s) = {} & H_i^{(k)}(s)\frac{a_{2,ij}^{(k)}}{-a_{1,ii}^{(k)}} + H_i^{(k)}(s)\sum_{j'\neq i}^{M_k}\frac{a_{1,ij'}^{(k)}}{-a_{1,ii}^{(k)}}\mathbf{w}^*_{k,(j',j)}(s) \\ & + H_i^{(k)}(s)\sum_{j'=1}^{M_{k+1}}\frac{a_{0,ij'}^{(k)}}{-a_{1,ii}^{(k)}}\sum_{m=1}^{M_k}\mathbf{w}^*_{k+1,(j',m)}(s)\mathbf{w}^*_{k,(m,j)}(s),\end{aligned}$$

which is equivalent to Eq. (10.77). Note that when the QBD process arrives at level 0 from the initial level k, it certainly needs to pass through level $k-1$, level $k-2,\ldots,$ level 2 and level 1. Therefore, we get

$$\mathbf{g}^*_{k,(i,j)}(s) = \sum_{i_1=1}^{M_1}\sum_{i_2=1}^{M_2}\cdots\sum_{i_{k-1}=1}^{M_{k-1}}\mathbf{w}^*_{k,(i,i_{k-1})}(s)\mathbf{w}^*_{k-1,(i_{k-1},i_{k-2})}(s)\ldots\mathbf{w}^*_{1,(i_1,j)}(s),$$

which is equivalent to Eq. (10.76).

For a level-independent QBD process, according to Theorem 10.12 we have the following corollary.

Corollary 10.5 *If* $A_2^{(l)} = A_2$ *for* $l \geqslant 2$, $A_1^{(k)} = A_1$ *and* $A_0^{(k)} = A_0$ *for* $k \geqslant 1$, *then*

$$w^*_l(s) = w^*(s), \quad l \geqslant 2,$$
$$\mathbf{g}^*_k(s) = w^*(s)^{k-1}w^*_1(s).$$

In particular,

$$\mathbf{g}_k^*(0) = G^{k-1}G_1, \quad k \geqslant 1,$$

where $G_1 = w_1^*(0)$ *and* G *is the minimal nonnegative solution to the matrix equation*

$$A_2 + A_1 G + A_0 G^2 = 0.$$

Remark 10.5 (1) *Let* $G_k = w_k^*(0)$ *for* $k \geqslant 1$. *Then the matrix sequence* $\{G_k\}$ *is the minimal nonnegative solution to the system of matrix equations*

$$A_2^{(k)} + A_1^{(k)} G_k + A_0^{(k)} G_{k+1} G_k = 0, \quad k \geqslant 1.$$

(2) *If* $V(k, j) = 1$ *for all* $k \geqslant 0, 1 \leqslant j \leqslant M_k$, *then the matrix sequence* $\{G_k, k \geqslant 1\}$ *is the same as* (43) *in Ramaswami and Taylor [46].*

We now compute the conditional moments of the random variable $\xi_0^{(j)}(\omega)$. Let

$$m_{k,(i,j)}^{(g)} = E_{(k,i)}\left[\xi_0^{(j)}(\omega)^g\right]$$

and the matrix

$$\mathcal{M}_k^{(g)} = \left(m_{k,(i,j)}^{(g)}\right)_{1 \leqslant i \leqslant M_k, 1 \leqslant j \leqslant M_0}, \quad g \geqslant 1.$$

We write

$$\varPsi_k^{(g)} = \begin{cases} \operatorname{diag}(V(k,1), V(k,2), \dots, V(k, M_k)) \prod_{j=k}^{1} G_j, & \text{if } g = 0, \\ \operatorname{diag}(V(k,1), V(k,2), \dots, V(k, M_k)) \left(g \mathcal{M}_k^{(g-1)}\right), & \text{if } g \geqslant 1, \end{cases}$$

where the matrix sequence $\{G_k\}$ is given in (1) of Remark 10.5.

Note that

$$m_{k,(i,j)}^{(g)} = (-1)^g \frac{\mathrm{d}^g}{\mathrm{d}s^g} g_{k,(i,j)}^*(s)_{|s=0},$$

it follows from Eq. (10.73) and Eq. (10.74) that the matrix sequence $\{\mathcal{M}_m^{(g)}, g \geqslant 1, m \geqslant 1\}$. satisfies the system of matrix equations

$$\varPsi_1^{(g-1)} + A_1^{(1)} \mathcal{M}_1^{(g)} + A_0^{(1)} \mathcal{M}_2^{(g)} = 0 \tag{10.78}$$

and

$$\varPsi_k^{(g-1)} + A_2^{(k)} \mathcal{M}_{k-1}^{(g)} + A_1^{(k)} \mathcal{M}_k^{(g)} + A_0^{(k)} \mathcal{M}_{k+1}^{(g)} = 0, \quad k \geqslant 2. \tag{10.79}$$

The following corollary gives an expression for the matrix sequence $\{\mathcal{M}_m^{(g)}, g \geqslant 1, m \geqslant 1\}$.

Corollary 10.6 *For* $g \geqslant 1,\ m \geqslant 1,$

$$\begin{aligned}\mathcal{M}_m^{(g)} = &\sum_{n=0}^{m-1}\left[\left(-\overline{U}_m^{-1}\right)X_{m-n}^{(m)} + \sum_{i=1}^{\infty} Y_i^{(m)}\left(-\overline{U}_{i+m}^{-1}\right)X_{i+m-n}^{(i+m)}\right]\Psi_n^{(g-1)} \\ &+\left[\left(-\overline{U}_m^{-1}\right) + \sum_{i=1}^{\infty} Y_i^{(m)}\left(-\overline{U}_{i+m}^{-1}\right)X_i^{(i+m)}\right]\Psi_m^{(g-1)} \\ &+\sum_{n=m+1}^{\infty}\left[Y_{n-m}^{(m)}\left(-\overline{U}_n^{-1}\right) + \sum_{i=n-m+1}^{\infty} Y_i^{(m)}\left(-\overline{U}_{i+m}^{-1}\right)X_{i-(n-m)}^{(i+m)}\right]\Psi_n^{(g-1)},\end{aligned}$$

where $\{\overline{R}_k\}$, $\{\overline{G}_l\}$ *and* $\{\overline{U}_l\}$ *are the LU-type measures of the QBD process whose infinitesimal generator is given by*

$$\mathcal{A} = \begin{pmatrix} A_1^{(1)} & A_0^{(1)} & & & \\ A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & & \\ & A_2^{(3)} & A_1^{(3)} & A_0^{(3)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{10.80}$$

and $\left\{X_k^{(l)}, 1 \leqslant k \leqslant l\right\}$ *and* $\left\{Y_k^{(l)}, k \geqslant 1, l \geqslant 0\right\}$ *are respectively expressed by* $\{\overline{R}_k\}$ *and* $\{\overline{G}_l\}$ *according to Eq.* (1.32) *and Eq.* (1.33).

Proof It follows from Eq. (10.78) and Eq. (10.79) that $\mathcal{A}\overrightarrow{\mathbf{M}}^{(g)} = -\overrightarrow{\Psi}^{(g-1)}$, where

$$\overrightarrow{\mathbf{M}}^{(g)} = \left(\left[\mathcal{M}_1^{(g)}\right]^{\mathrm{T}}, \left[\mathcal{M}_2^{(g)}\right]^{\mathrm{T}}, \left[\mathcal{M}_3^{(g)}\right]^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}$$

and

$$\overrightarrow{\Psi}^{(g-1)} = \left(\left[\Psi_1^{(g-1)}\right]^{\mathrm{T}}, \left[\Psi_2^{(g-1)}\right]^{\mathrm{T}}, \left[\Psi_3^{(g-1)}\right]^{\mathrm{T}}, \ldots\right)^{\mathrm{T}}.$$

Using (1) of Theorem 1.2 to solve the equation $\mathcal{A}\overrightarrow{\mathbf{M}}^{(g)} = -\overrightarrow{\Psi}^{(g-1)}$, we obtain the desired result.

Finally, we study a return-type reward process and express the Laplace transforms of its conditional distributions in terms of the up- and down-type reward process discussed above.

If the QBD process starts at state (k, i) at time 0, then $\eta_k^{(j)}(\omega)$ is called the first return time to state (k, i).

We write

$$h_{k,(i,j)}^{*}(s) = E_{(k,i)}\left[e^{-s\xi_k^{(j)}(\omega)}\right]$$

and the matrix

$$h_k^{*}(s) = (h_{k,(i,j)}(s))_{1 \leqslant i,j \leqslant M_k}.$$

Using the law of total probability, we obatin

$$h_0^*(s) = -[\mathcal{A}_1^{(0)}(s)]^{-1} A_0^{(0)} w_1^*(s) \tag{10.81}$$

and

$$h_k^*(s) = -\left[\mathcal{A}_1^{(k)}(s)\right]^{-1}\left[A_2^{(k)} f_{k-1,k}^*(s) + A_0^{(k)} w_{k+1}^*(s)\right], \quad k \geqslant 1. \tag{10.82}$$

Clearly, it is not difficult to further derive the conditional moments of the return-type reward process according to Eq. (10.81) and Eq. (10.82).

10.9 Discrete-Time Markov Reward Processes

In this section, we provide a simple introdution to reward process of an irreducible discrete-time block-structured Markov chain. By using the UL- and LU-types of *RG*-factorizations, we provide expressions for conditional distributions and conditional moments of the reward process.

Consider an irreducible discrete-time block-structured Markov chain $\{X_n, n \geqslant 0\}$ whose transition probability matrix is given by Eq. (2.1), or

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & \cdots \\ P_{2,0} & P_{2,1} & P_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Let $\pi(0) = (\pi_0(0), \pi_1(0), \pi_2(0), \dots)$ be the initial probability vector of the Markov chain $\{X_n, n \geqslant 0\}$, that is, the (k,i)th entry of $\pi(0)$ is given by

$$\pi_{k,i}(0) = P\{X_0 = (k,i)\}.$$

Obviously, the transient probability vector of the Markov chain at time n is given by $\pi(n) = \pi(0)P^n$, $n \geqslant 0$.

Let $f(x)$ be a reward rate with repect to state x of the Markov chain $\{X_n, n \geqslant 0\}$, i.e., the reward rate is $f(k,i)$ if the Markov chain $\{X_n, n \geqslant 0\}$ is at state (k,i). Then the random variable $f(X_n)$ is the instantaneous reward rate at time n. Hence

$$E[f(X_n)] = \sum_{(k,i)\in\Omega} \pi_{k,i}(n) f(k,i) = \sum_{k=0}^{\infty} \pi_k(n) f_k = \pi(n) f \tag{10.83}$$

and

$$E[f(X_n)^r] = \sum_{(k,i)\in\Omega} \pi_{k,i}(n) f(k,i)^r, \quad r \geqslant 2.$$

At the same time, the probability distribution of the random variable $f(X_n)$ is given by

$$P\{f(X_n)\leqslant x\}=\sum_{\substack{f(k,i)\leqslant x\\(k,i)\in\Omega}}\pi_{k,i}(n).$$

Also, the limit distribution of the random variable $f(X_n)$ is given by

$$\lim_{n\to\infty}P\{f(X_n)\leqslant x\}=\sum_{\substack{f(k,i)\leqslant x\\(k,i)\in\Omega}}\pi_{k,i}.$$

If the Markov chain $\{X_n, n\geqslant 0\}$ is positive recurrent and π is the stationary probability vector of the Markov chain, then $\lim_{n\to\infty}P^n=e\pi$ and $\lim_{n\to\infty}\pi(n)=\pi$. We write $\eta=\lim_{n\to\infty}E[f(X_n)]$. It is easy to see that

$$\eta=\sum_{(k,i)\in\Omega}\pi_{k,i}f(k,i)=\sum_{k=0}^{\infty}\pi_k f_k=\pi f. \tag{10.84}$$

Now, we analyze the accumulated reward over these times $0,1,\ldots,N$ as follows:

$$\varPsi(N)=\sum_{n=0}^{N}f(X_n). \tag{10.85}$$

Let $L(N)=\pi(0)\sum_{n=0}^{N}P^n$ and $\xi(N)=E[\varPsi(N)]$. It is easy to check that

$$\xi(N)=\sum_{(k,i)\in\Omega}L_{k,i}(N)f(k,i)=\sum_{k=0}^{\infty}L_k(N)f_k=L(N)f \tag{10.86}$$

and

$$E[\varPsi(N)^r]=\sum_{(k,i)\in\Omega}L_{k,i}(N)f(k,i)^r,\quad r\geqslant 2.$$

It follows from Eq. (10.86) that

$$\xi(0)=\pi(0)f \tag{10.87}$$

$$\xi(N)=\xi(N-1)+\pi(0)P^N f,\quad N\geqslant 1. \tag{10.88}$$

Using the iterative relations Eq. (10.87) and Eq. (10.88), we can compute the reward value $\xi(N)$ for each $N\geqslant 0$.

The following proposition provides a useful property for the limit of the sequence $\{\xi(N)\}$.

Proposition 10.1 *If the Markov chain P is positive recurrent,* $\pi(0)e > 0$ *and* $f > 0$, *then* $\lim_{N\to\infty} \xi(N) = +\infty$.

Proof Since the reward rate vector $f > 0$, it is easy to see that the sequence $\{\xi(N)\}$ is non-decreasing. Thus the limit $\lim_{N\to\infty} \xi(N)$ is either finite or infinite.

We assume that $\xi = \lim_{N\to\infty} \xi(N) < +\infty$. Taking the limit $N \to \infty$ for the Eq. (10.88), and using $P^N \to e\pi$, we obtain

$$\xi = \xi + \pi(0)e \cdot \pi f,$$

which leads to

$$\pi(0)e \cdot \pi f = 0.$$

This yields a contradiction due to the fact that $\pi(0)e > 0$ and $\pi f > 0$. Therefore, $\xi = \lim_{N\to\infty} \xi(N) = +\infty$. This completes the proof.

Remark 10.6 *If the Markov chain P is transient, then*

$$L(\infty) = \pi(0)(I - P)^{-1}_{\min}.$$

Thus we obtain

$$\xi = \pi(0)(I - P)^{-1}_{\min} f.$$

Proposition 10.1 provides a sufficient condition under which $\xi = +\infty$. However, this condition can have many forms once guaranteing that $\lim_{N\to\infty} \pi(0)P^N f \neq 0$. For this, we should introduce a discounted reward function as follows:

$$\Phi(\beta, N) = \sum_{n=0}^{N} \beta^n f(X_n),$$

where $\beta \in (0,1]$ is a discounted factor. It is clear that $\Phi(1, N) = \Psi(N)$, a.s.. Let $\varphi(\beta, N) = E[\Phi(\beta, N)]$. Then $\varphi(1, N) = \xi(N)$. In this case, we can compute the sequence $\{\varphi(\beta, N)\}$ according to the following iterative relations:

$$\begin{aligned}
&\varphi(\beta, 0) = \pi(0) f \\
&\varphi(\beta, N) = \varphi(\beta, N-1) + \beta^N \pi(0) P^N f, \quad N \geqslant 1.
\end{aligned}$$

If $\beta^N P^N \to 0$ as $N \to \infty$, then

$$\varphi(\beta) = \varphi(\beta, \infty) = \pi(0)(I - \beta P)^{-1}_{\min} f. \tag{10.89}$$

Using the UL-type *RG*-factorization, we have

$$\varphi(\beta) = \pi(0)[I - G_L(\beta)]^{-1}[I - \Psi_D(\beta)]^{-1}[I - R_U(\beta)]^{-1} f$$

or applying the LU-type RG-factorization, we obtain

$$\varphi(\beta) = \pi(0)[I - \overline{G}_U(\beta)]^{-1}[I - \Phi_D(\beta)]^{-1}[I - \overline{R}_L(\beta)]^{-1} f.$$

10.10 Notes in the Literature

Markov reward processes can accurately model practical systems that evolve stochastically over time. A Markov reward process consists of two elements: A Markov environment and an associated reward structure. Important examples include Naddor [38] for inventory systems, Daley [13] and Daley and Jacobs [14] for queues, Puri [41] for biological models, Kulkarni, Nicola and Trivedi [26], Masuda and Sumita [33], Donatiello and Grassi [17], Nabli and Sericola [37], and Vaidyanathan, Harper, Hunter and Trivedi [58] for communication networks.

Karlin and McGregor [24], Wang [59], Puri [41,42], and McNeil [35] studied Markov reward processes for a birth-death process. Darling and Kac [15], Kesten [25], Mclean and Neuts [34], Puri [43], Howard [23], Meyer [36], Masuda and Sumita [33], and Stenberg, Manca and Silvestrov [51] analyzed Markov reward processes for Markov or semi-Markov processes. Glasserman [18,19,20], Ho and Cao [22], Cassandras [9], and Cao [7,8] studied Markov reward processes by means of infinitesimal perturbation analysis. Bobbio and Trivedi [5] provided a method for computing the first completion time distribution of a continuous-time Markov reward chain. Reibman, Smith and Trivedi [47] gave an overview for transient numerical approach of Markov reward models. Readers may further refer to Trivedi and Wagner [57], Reibman and Trivedi [48], Ciardo, Marie, Sericola and Trivedi [11], Ciardo, Blakemore, Chimento, Muppala and Trivedi [10], Ciardo and Trivedi [12], Rubino and Sericola [49], Masuda [32], Qureshi and Sanders [44], Mallubhatla, Pattipati and Viswanadham [31], Telek, Pfening and Fodor [54,55], Brenner and Kumar [6], Abdallah and Hamza [1], de Souza e Silva and Gail [16], Telek and Rácz [56], Bladt, Meini, Neuts and Sericola [4], Rácz [45], Telek, Horváth and Horváth [52,53], Grassmann and Luo [21], Stefanov [50], and Lisnianski [30].

In this chapter, we mainly refer to Li [28,29], Reibman, Smith and Trivedi [47], Telek, Pfening and Fodor [54,55], and Rácz [45]. At the same time, we have also added some new results without publication for a more systematical organization.

Problems

10.1 For a continuous-time level-dependent QBD process $\{X_t, t \geqslant 0\}$ with finitely-many levels, discuss the two reward processes $\{f(X_t)\}$ and $\{\Phi(t)\}$.

10.2 For a discrete-time level-dependent QBD process $\{X_n, n \geqslant 0\}$ with finitely-many levels, discuss the two reward processes $\{f(X_n)\}$ and $\{\Phi(n)\}$, where $\Phi(n) = \sum_{k=0}^{n} f(X_k)$.

10.3 For a continuous-time Markov chain of $GI/M/1$ type, study the reward process $\{\Phi(t)\}$.

10.4 For a continuous-time level-dependent QBD process $\{X_t, t \geqslant 0\}$ with infinitely-many levels, $\Gamma(x) = \min\{t : \Phi(t) \leqslant x\}$, $C(t,x) = P\{\Gamma(x) \leqslant t\}$, discuss the probability distribution $C(t,x)$.

10.5 For a continuous-time Markov chain of $M/G/1$ type, study the probability distribution $C(t,x)$. Further, extend the results to a Markov chain of $GI/G/1$ type.

10.6 Using the Markov reward process to study the multivariate PH distribution.

10.7 Consider an $M/G/1$ queue with server breakdowns and repairs, where the distributions of the life time and repair time are exponential and general, respectively. A reward rate of 1 is assigned to all the system operational states and a reward rate of 0 is assigned to all the system failure states. Compute $E[f(X(t))]$, $E[\Phi(t)]$, $E[\Phi(t)]/t$ and $\lim_{t \to +\infty} E[\Phi(t)]/t$.

10.8 Consider a $PH/PH/1$ queue. Let $N(t)$ be the number of customers in the system at time t. We define the reward rate

$$f(N(t)) = \begin{cases} 1, & 0 \leqslant N(t) \leqslant M, \\ 0, & N(t) \geqslant M+1. \end{cases}$$

Let $\Phi)(t) = \int_0^t f(N(x))\mathrm{d}x$. Compute $E[\Phi(t)]$, $E[\Phi(t)]/t$ and $\lim_{t \to +\infty} E[\Phi(t)]/t$.

10.9 Consider a $BMAP/G/1$ queue. Let $N(t)$ be the number of customers in the system at time t. We define the reward rate

$$f(N(t)) = \begin{cases} 1, & 0 \leqslant N(t) \leqslant M, \\ 0, & N(t) \geqslant M+1. \end{cases}$$

Let $\Phi(t) = \int_0^t f(N(x))\mathrm{d}x$. Compute $E[\Phi(t)], E[\Phi(t)]/t$ and $\lim_{t \to +\infty} E[\Phi(t)]/t$.

10.10 Consider a $M/M/c$ retrial queue, where the retrial time is exponential. Let $N(t)$ be the number of customers in the system at time t. We define the reward rate

$$f(N(t)) = \begin{cases} 1, & 1 \leqslant N(t) \leqslant M, \\ 0, & N(t) = 0 \text{ or } N(t) \geqslant M+1. \end{cases}$$

Let $\Phi(t) = \int_0^t f(N(x))\mathrm{d}x$. Compute $E[\Phi(t)], E[\Phi(t)]/t$ and $\lim_{t \to +\infty} E[\Phi(t)]/t$.

10.11 Consider an accumulated reward in an irreducible continuous-tine Markov chain which is either finite-state or infinite-state, provide some sufficient conditions for iteratively computing the distribution of the accumulated reward.

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11 Sensitivity Analysis and Evolutionary Games

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Abstract In this chapter, we present sensitivity analysis for performance measures of an irreducible perturbed Markov chain which is either discrete-time or continuous-time. By using the UL- and LU-types of *RG*-factorizations, we can express the *n*th derivatives of the performance measures, including the stationary, transient and discounted cases. Furthermore, we apply the sensitivity analysis to study symmetric evolutionary games by perturbed birth death processes and asymmetric evolutionary games by perturbed QBD processes.

Keywords stochastic models, *RG*-factorization, sensitivity analysis, evolutionary game, perturbed Markov chain, perturbed birth death process, perturbed QBD process.

In this chapter, we present sensitivity analysis for performance measures of an irreducible perturbed Markov chain which is either discrete-time or continuous-time. By using the UL- and LU-types of *RG*-factorizations, we can express the *n*th derivatives of the performance measures, including the stationary, transient and discounted cases. Further, we apply the sensitivity analysis to study symmetric evolutionary games by perturbed birth death processes, and asymmetric evolutionary games by perturbed QBD processes.

This chapter is organized as follows. Section 11.1 provides sensitivity analysis for the stationary performance measures of a perturbed discrete-time Markov chain with either finitely-many levels or infinitely-many levels. Also, it gives the censored structure for sensitivity analysis of a perturbed discrete-time Markov chain on either a large state space or an infinite state space. Section 11.2 obtains sensitivity analysis for a perturbed Markov chain of $GI/M/1$ type and a perturbed Markov chain of $M/G/1$ type by means of the matrix-geometric solution and the matrix-iterative solution, respectively. Section 11.3 discusses a perturbed continuous-time Markov chain. Section 11.4 derives the *n*th derivative of the conditional moments of the accumulated reward process for a QBD process. Section 11.5

discusses a perturbed $MAP/PH/1$ queue, including sensitivity analysis for a perturbed PH distribution and also for a perturbed MAP. Sections 11.6 and 11.7 use perturbed birth death processes to analyze the evolutionary stable strategy (ESS) of symmetric evolutionary games. Section 11.8 discusses the ESS of asymmetric evolutionary games by terms of perturbed QBD processes. Finally, Section 11.9 summarizes notes for the references related to the results of this chapter.

11.1 Perturbed Discrete-Time Markov Chains

In this section, we provide sensitivity analysis for performance measures of a perturbed discrete-time Markov chain with either finitely-many levels or infinitely-many levels. Applying the *RG*-factorizations, we can express the *n*th derivative for the performance measures for $n \geqslant 1$.

11.1.1 Markov Chains with Finitely-Many Levels

Consider an irreducible discrete-time block-structured Markov chain with N levels whose transition probability matrix is given by

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,N} \\ P_{1,0} & P_{1,1} & \cdots & P_{1,N} \\ \vdots & \vdots & & \vdots \\ P_{N,0} & P_{N,1} & \cdots & P_{N,N} \end{pmatrix},$$

and introduce a perturbed real matrix or a perturbed-directional matrix as follows:

$$V = \begin{pmatrix} V_{0,0} & V_{0,1} & \cdots & V_{0,N} \\ V_{1,0} & V_{1,1} & \cdots & V_{1,N} \\ \vdots & \vdots & & \vdots \\ V_{N,0} & V_{N,1} & \cdots & V_{N,N} \end{pmatrix}.$$

Note that the block structure of the matrix V is the same as that of the matrix P. We assume that there exists a sufficiently small number $\varepsilon > 0$ such that

$$\begin{aligned} \widetilde{P}_{\varepsilon} &= P + \varepsilon V \\ &= \begin{pmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,N} \\ P_{1,0} & P_{1,1} & \cdots & P_{1,N} \\ \vdots & \vdots & & \vdots \\ P_{N,0} & P_{N,1} & \cdots & P_{N,N} \end{pmatrix} + \varepsilon \begin{pmatrix} V_{0,0} & V_{0,1} & \cdots & V_{0,N} \\ V_{1,0} & V_{1,1} & \cdots & V_{1,N} \\ \vdots & \vdots & & \vdots \\ V_{N,0} & V_{N,1} & \cdots & V_{N,N} \end{pmatrix} \end{aligned} \tag{11.1}$$

is still a transition probability matrix whose irreducibility and state classification are the same as those of the matrix P.

We assume that the Markov chain P and the perturbed Markov chain $\widetilde{P}_\varepsilon$ are both positive recurrent. In this case, it is clear that $Ve=0$, where e is a column vector of ones with suitable size. Let $\tilde{\pi}_\varepsilon$ and π be the stationary probability vectors of the perturbed Markov chain $\widetilde{P}_\varepsilon$ and the Markov chain P, respectively. We denote by f the reward rate vector with respect to the Markov chain P. In what follows we discuss the perturbed stationary performance measure $\tilde{\eta}_\varepsilon$, where $\tilde{\eta}_\varepsilon = \tilde{\pi}_\varepsilon f$. Note that the stationary performance measure of the Markov chain P is given by $\eta = \pi f = \lim_{\varepsilon\to 0} \tilde{\eta}_\varepsilon$.

It is clear that $\tilde{\pi}_\varepsilon \tilde{P}_\varepsilon = \tilde{\pi}_\varepsilon$ and $\tilde{\pi}_\varepsilon e = 1$, thus we have

$$\tilde{\pi}_\varepsilon (P + \varepsilon V) = \tilde{\pi}_\varepsilon \tag{11.2}$$

and

$$\tilde{\pi}_\varepsilon e = 1. \tag{11.3}$$

Taking derivatives of the both sides of Eq. (11.2) for the variable ε, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon \cdot (P + \varepsilon V) + \tilde{\pi}_\varepsilon V, \tag{11.4}$$

let $\varepsilon = 0$ in Eq. (11.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon |_{\varepsilon=0} \cdot (I - P) = \pi V. \tag{11.5}$$

It follows from Eq. (11.3) that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon |_{\varepsilon=0} \cdot e = 0,$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon |_{\varepsilon=0} \cdot e\pi = 0.$$

Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon |_{\varepsilon=0} \cdot (I - P + e\pi) = \pi V.$$

Note that $I - P + e\pi$ is the fundamental matrix of the Markov chain P with finitely-many levels, and $I - P + e\pi$ is always invertible, hence we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon |_{\varepsilon=0} = \pi V (I - P + e\pi)^{-1}. \tag{11.6}$$

Continuously taking derivatives of both sides of Eq. (11.2) in the variable ε for $n \geqslant 2$, we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_\varepsilon \cdot [I-(P+\varepsilon V)] = n\frac{\mathrm{d}^{n-1}}{\mathrm{d}\varepsilon^{n-1}}\tilde{\pi}_\varepsilon \cdot V,$$

let $\varepsilon = 0$ we have

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_\varepsilon|_{\varepsilon=0} \cdot (I-P) = n\frac{\mathrm{d}^{n-1}}{\mathrm{d}\varepsilon^{n-1}}\tilde{\pi}_\varepsilon|_{\varepsilon=0} \cdot V. \tag{11.7}$$

Note that

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_\varepsilon|_{\varepsilon=0} \cdot e\pi = 0, \tag{11.8}$$

it follows from Eq. (11.7) and Eq. (11.8) that for $n \geqslant 2$,

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_\varepsilon|_{\varepsilon=0} \cdot (I-P+e\pi) = n\frac{\mathrm{d}^{n-1}}{\mathrm{d}\varepsilon^{n-1}}\tilde{\pi}_\varepsilon|_{\varepsilon=0} \cdot V.$$

Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_\varepsilon|_{\varepsilon=0} &= \frac{\mathrm{d}^{n-1}}{\mathrm{d}\varepsilon^{n-1}}\tilde{\pi}_\varepsilon|_{\varepsilon=0} \cdot nV(I-P+e\pi)^{-1} \\ &= n!\pi[V(I-P+e\pi)^{-1}]^n. \end{aligned} \tag{11.9}$$

Further, for the perturbed stationary performance measure we can obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_\varepsilon|_{\varepsilon=0} = n!\pi[V(I-P+e\pi)^{-1}]^n f, \quad n \geqslant 1.$$

Now, we provide the generalized inverse expression for the nth derivative $\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_\varepsilon|_{\varepsilon=0}$. Note that the matrix $I-P$ is singular, we write the generalized inverse as

$$(I-P)^{\#} = (I-P+e\pi)^{-1} - e\pi.$$

Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon|_{\varepsilon=0} \cdot (I-P)(I-P)^{\#} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon|_{\varepsilon=0} \cdot (I-P)[(I-P+e\pi)^{-1} - e\pi] \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon|_{\varepsilon=0} \cdot (I-P)(I-P+e\pi)^{-1} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon|_{\varepsilon=0} \cdot (I-P+e\pi)(I-P+e\pi)^{-1} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon|_{\varepsilon=0}\end{aligned}$$

and

$$\pi V(I-P)^{\#} = \pi V[(I-P+e\pi)^{-1} - e\pi]$$
$$= \pi V(I-P+e\pi)^{-1},$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_\varepsilon|_{\varepsilon=0} = \pi V(I-P)^{\#}.$$

Further, we can obtain that for $n \geqslant 2$,

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_\varepsilon|_{\varepsilon=0} = n!\pi[V(I-P)^{\#}]^n.$$

Therefore, we obtain that for $n \geqslant 1$,

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_\varepsilon|_{\varepsilon=0} = n!\pi[V(I-P)^{\#}]^n f.$$

Now, we provide a mathematical interpretation for the derivative $\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_\varepsilon|_{\varepsilon=0}$ for $n \geqslant 1$ as follows:

(1) If for $n = 1, 2, ..., 2k-1$,

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_\varepsilon|_{\varepsilon=0} = 0,$$

and

$$\frac{\mathrm{d}^{2k}}{\mathrm{d}\varepsilon^{2k}}\tilde{\eta}_\varepsilon|_{\varepsilon=0} < 0 \text{ (or } > 0),$$

then the perturbed stationary performance measure $\tilde{\eta}_\varepsilon$ can reach its maximal (or minimal) value at $\varepsilon = 0$. In this case, system performance is not sensitive to the designed parameters of the system, thus such designed parameters are satisfied for system operations.

(2) If for $n = 1, 2, ..., 2k$,

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_\varepsilon|_{\varepsilon=0} = 0,$$

and

$$\frac{\mathrm{d}^{2k+1}}{\mathrm{d}\varepsilon^{2k+1}}\tilde{\eta}_\varepsilon|_{\varepsilon=0} \neq 0,$$

then the perturbed stationary performance measure $\tilde{\eta}_\varepsilon$ does not exist the maximal (or minimal) value at $\varepsilon = 0$. In this case, system performance is sensitive to the designed parameters of the system, it is necessary to redesign (or readjust) system parameters.

11.1.2 Markov Chains with Infinitely-Many Levels

Consider an irreducible discrete-time block-structured Markov chain whose transition probability matrix is given by

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & \cdots \\ P_{2,0} & P_{2,1} & P_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

and the perturbed-directional matrix is given by

$$V = \begin{pmatrix} V_{0,0} & V_{0,1} & V_{0,2} & \cdots \\ V_{1,0} & V_{1,1} & V_{1,2} & \cdots \\ V_{2,0} & V_{2,1} & V_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Note that the block structure of V is the same as that of P, we assume that there exists a sufficiently small number $\varepsilon > 0$ such that

$$\begin{aligned} \tilde{P}_\varepsilon &= P + \varepsilon V \\ &= \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & \cdots \\ P_{2,0} & P_{2,1} & P_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} + \varepsilon \begin{pmatrix} V_{0,0} & V_{0,1} & V_{0,2} & \cdots \\ V_{1,0} & V_{1,1} & V_{1,2} & \cdots \\ V_{2,0} & V_{2,1} & V_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \end{aligned} \tag{11.10}$$

is still a transition probability matrix whose irreducibility and state classification are the same as those of the matrix P.

We now discuss the perturbed stationary performance measure $\tilde{\eta}_\varepsilon$ with respect to the perturbed Markov chain $\tilde{P}_\varepsilon$ and the reward rate vector f. We assume that the two Markov chains P and $\tilde{P}_\varepsilon$ are both irreducible and positive recurrent. Then $\tilde{\pi}_\varepsilon \tilde{P}_\varepsilon = \tilde{\pi}_\varepsilon$, which leads to

$$\tilde{\pi}_\varepsilon (P + \varepsilon V) = \tilde{\pi}_\varepsilon. \tag{11.11}$$

Taking derivatives of the both sides of Eq. (11.11) for the variable ε, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_{\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_{\varepsilon} \cdot (P + \varepsilon V) + \tilde{\pi}_{\varepsilon} V, \tag{11.12}$$

let $\varepsilon = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_{\varepsilon}|_{\varepsilon=0} \cdot (I - P) = \pi V, \tag{11.13}$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_{\varepsilon}|_{\varepsilon=0} = \pi V (I - P)^{-1}_{\min}, \tag{11.14}$$

where

$$(I - P)^{-1}_{\min} = \sum_{n=0}^{\infty} P^n.$$

In order to compute the minimal nonnegative inverse $(I - P)^{-1}_{\min}$, we apply the RG-factorizations. Using the UL-type RG-factorization, we obtain

$$(I - P)^{-1}_{\min} = (I - G_L)^{-1}(I - \Psi_D)^{-1}(I - R_U)^{-1}. \tag{11.15}$$

Note that $(I - G_L)^{-1}$ and $(I - R_U)^{-1}$ are ordinary matrix inverses, while $(I - \Psi_D)^{-1}$ contains the group inverse of a matrix of finite size which is indicated by

$$(I - \Psi_D)^{-1} = \mathrm{diag}((I - \Psi_0)^{\#}, (I - \Psi_1)^{-1}, (I - \Psi_2)^{-1}, (I - \Psi_3)^{-1}, \ldots), \tag{11.16}$$

where the group inverse is given by

$$(I - \Psi_0)^{\#} = (I - \Psi_0 + e x_0)^{-1} - e x_0, \tag{11.17}$$

and x_0 is the stationary probability vector of the finite-state Markov chain Ψ_0.

Applying the LU-type RG-factorization, we obtain

$$(I - P)^{-1}_{\min} = (I - \overline{G}_U)^{-1}(I - \Phi_D)^{-1}(I - \overline{R}_L)^{-1}. \tag{11.18}$$

Continuously taking derivatives of both sides of Eq. (11.11) for the variable ε, we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_{\varepsilon} \cdot [I - (P + \varepsilon V)] = n \frac{\mathrm{d}^{n-1}}{\mathrm{d}\varepsilon^{n-1}}\tilde{\pi}_{\varepsilon} \cdot V,$$

let $\varepsilon = 0$ we have

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_{\varepsilon}|_{\varepsilon=0} \cdot (I - P) = n \frac{\mathrm{d}^{n-1}}{\mathrm{d}\varepsilon^{n-1}}\tilde{\pi}_{\varepsilon}|_{\varepsilon=0} \cdot V,$$

which leads to

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_{\varepsilon}|_{\varepsilon=0}=n\frac{\mathrm{d}^{n-1}}{\mathrm{d}\varepsilon^{n-1}}\tilde{\pi}_{\varepsilon}|_{\varepsilon=0}\cdot V(I-P)_{\min}^{-1}$$
$$=n!\pi\left[V(I-P)_{\min}^{-1}\right]^n.$$

By means of Eq. (11.15) or using Eq. (11.18), we have

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_{\varepsilon}|_{\varepsilon=0}=n!\pi[V(I-G_L)^{-1}(I-\Psi_D)^{-1}(I-R_U)^{-1}]^n \tag{11.19}$$

or

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\pi}_{\varepsilon}|_{\varepsilon=0}=n!\pi[V(I-\overline{G}_U)^{-1}(I-\Phi_D)^{-1}(I-\overline{R}_L)^{-1}]^n. \tag{11.20}$$

Note that $\tilde{\eta}_{\varepsilon}=\tilde{\pi}_{\varepsilon}f$, it follows from Eq. (11.19) or Eq. (11.20) that for $n\geqslant 1$,

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_{\varepsilon}|_{\varepsilon=0}=n!\pi[V(I-G_L)^{-1}(I-\Psi_D)^{-1}(I-R_U)^{-1}]^n f \tag{11.21}$$

or

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_{\varepsilon}|_{\varepsilon=0}=n!\pi[V(I-\overline{G}_U)^{-1}(I-\Phi_D)^{-1}(I-\overline{R}_L)^{-1}]^n f. \tag{11.22}$$

11.1.3 The Realization Matrix and Potential Vector

Now, we discuss computations for the realization matrix D of the Markov chain P with either finitely-many levels or infinitely-many levels. Note that $D=eg^{\mathrm{T}}-ge^{\mathrm{T}}$, where g is a potential vector. Note that for an arbitrary constant c, $g+ce$ is also a potential vector. The realization matrix plays an important role in sensitivity analysis of stochastic models based on the following relationship

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\pi}_{\varepsilon}|_{\varepsilon=0}=\pi VD^{\mathrm{T}}\pi^{\mathrm{T}}=\pi Vg.$$

For the Markov chain P, the potential vector g satisfies the Poisson equation

$$(I-P)g=f-\pi fe. \tag{11.23}$$

Thus, using the UL-type RG-factorization, it follows from Eq. (11.23) that

$$g=(I-P)_{\min}^{-1}(f-\pi fe)$$
$$=(I-G_L)^{-1}(I-\Psi_D)^{-1}(I-R_U)^{-1}(f-\pi fe) \tag{11.24}$$

or using the LU-type RG-factorization, we have

$$\begin{aligned} g &= (I-P)_{\min}^{-1}(f-\pi fe) \\ &= (I-\overline{G}_U)^{-1}(I-\Phi_D)^{-1}(I-\overline{R}_L)^{-1}(f-\pi fe). \end{aligned} \tag{11.25}$$

It follows from Eq. (11.24) and Eq. (11.25) that the realization matrix D is given by

$$\begin{aligned} D &= eg^{\mathrm{T}} - ge^{\mathrm{T}} \\ &= e(f-\pi fe)^{\mathrm{T}}[(I-R_U)^{-1}]^{\mathrm{T}}[(I-\Psi_D)^{-1}]^{\mathrm{T}}[(I-G_L)^{-1}]^{\mathrm{T}} \\ &\quad -(I-G_L)^{-1}(I-\Psi_D)^{-1}(I-R_U)^{-1}(f-\pi fe)e^{\mathrm{T}} \end{aligned} \tag{11.26}$$

or

$$\begin{aligned} D &= eg^{\mathrm{T}} - ge^{\mathrm{T}} \\ &= e(f-\pi fe)^{\mathrm{T}}[(I-\overline{R}_L)^{-1}]^{\mathrm{T}}[(I-\Phi_D)^{-1}]^{\mathrm{T}}[(I-\overline{G}_U)^{-1}]^{\mathrm{T}} \\ &\quad -(I-\overline{G}_U)^{-1}(I-\Phi_D)^{-1}(I-\overline{R}_L)^{-1}(f-\pi fe)e^{\mathrm{T}}. \end{aligned} \tag{11.27}$$

11.1.4 The Censored Structure in Sensitivity Analysis

When the perturbed discrete-time Markov chain has a large state space or an infinite state space, the censored approximation is a useful method. Let the state space $\Omega = E \bigcup E^c$ where $E = \{0,1,2,\dots,N\}$, we write

$$P = \begin{pmatrix} P_A & P_B \\ P_C & P_D \end{pmatrix}, \quad V = \begin{pmatrix} V_A & V_B \\ V_C & V_D \end{pmatrix},$$

based on the two subsets E and E^c. Hence we have

$$\tilde{P}_\varepsilon = \begin{pmatrix} P_A + \varepsilon V_A & P_B + \varepsilon V_B \\ P_C + \varepsilon V_C & P_D + \varepsilon V_D \end{pmatrix}.$$

Obviously, the censored chain of the perturbed Markov chain $\tilde{P}_\varepsilon$ to the set E is given by the transition probability matrix

$$\tilde{P}_\varepsilon^E = (P_A + \varepsilon V_A) + (P_B + \varepsilon V_B)\sum_{n=0}^{\infty}(P_D + \varepsilon V_D)^n (P_C + \varepsilon V_C).$$

Lemma 11.1 *Let*

$$\tilde{P}_\varepsilon^E \approx K + \varepsilon L.$$

Then

$$K = P^E = P_A + P_B \sum_{n=0}^{\infty} P_D^n P_C$$

and

$$L = V_A + P_B \sum_{n=0}^{\infty} P_D^n V_C + V_B \sum_{n=0}^{\infty} P_D^n P_C + P_B \left(\sum_{n=0}^{\infty} P_D^n \right)^2 V_D P_C$$
$$= V_A + P_B \sum_{n=0}^{\infty} P_D^n V_C + V_B \sum_{n=0}^{\infty} P_D^n P_C + P_B V_D \left(\sum_{n=0}^{\infty} P_D^n \right)^2 P_C.$$

Proof We write the following two general forms

$$(I - P_D - \varepsilon V_D)_{\min}^{-1} = \sum_{n=0}^{\infty} (P_D + \varepsilon V_D)^n$$

and

$$(I - P_D)_{\min}^{-1} = \sum_{n=0}^{\infty} (P_D)^n.$$

Note that

$$\begin{aligned}
\tilde{P}_\varepsilon^E &= (P_A + \varepsilon V_A) + (P_B + \varepsilon V_B) \sum_{n=0}^{\infty} (P_D + \varepsilon V_D)^n (P_C + \varepsilon V_C) \\
&= (P_A + \varepsilon V_A) + (P_B + \varepsilon V_B)(I - P_D - \varepsilon V_D)_{\min}^{-1} (P_C + \varepsilon V_C) \\
&\approx P_A + \varepsilon V_A + (P_B + \varepsilon V_B)(I - P_D)_{\min}^{-1} \left[I + \varepsilon (I - P_D)_{\min}^{-1} V_D \right] (P_C + \varepsilon V_C) \\
&\approx P_A + P_B (I - P_D)_{\min}^{-1} P_C + \varepsilon [V_A + P_B (I - P_D)_{\min}^{-1} V_C \\
&\quad + V_B (I - P_D)_{\min}^{-1} P_C + P_B (I - P_D)_{\min}^{-2} V_D P_C],
\end{aligned}$$

which leads to the stated results. This completes the proof.

Let

$$f^E = (f_0, f_1, f_2, \dots, f_N)^{\mathrm{T}} + P_B \sum_{n=0}^{\infty} P_D^n (f_{N+1}, f_{N+2}, f_{N+3}, \dots)^{\mathrm{T}},$$

and $\tilde{\pi}_\varepsilon^E$ and π^E the stationary probability vectors of the censored Markov chains $\tilde{P}_\varepsilon^E$ and P^E, respectively. Then $\tilde{\eta}_\varepsilon^E = \tilde{\pi}_\varepsilon^E f^E$. Therefore, for the perturbed stationary performance measure we can obtain that for $n \geqslant 1$,

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n} \tilde{\eta}_\varepsilon^E |_{\varepsilon=0} = n! \pi^E [L(I - K + e\pi^E)^{-1}]^n f^E.$$

11.1.5 The Transient Performance Measure

We provide sensitivity analysis for the transient performance measures. Let

$$\tilde{\xi}_\varepsilon(N) = \pi(0)\sum_{k=0}^{N}(\tilde{P}_\varepsilon)^k f, \quad N \geqslant 0. \tag{11.28}$$

Then

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\xi}_\varepsilon(N) = \pi(0)\sum_{k=n}^{N}k(k-1)\ldots(k-n+1)(\tilde{P}_\varepsilon)^{k-n}V^n f, \quad 1 \leqslant n \leqslant N. \tag{11.29}$$

Hence, it is easy to see from Eq. (11.29) that

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\xi}_\varepsilon(N)|_{\varepsilon=0} = \pi(0)\sum_{k=n}^{N}k(k-1)\ldots(k-n+1)\,P^{k-n}V^n f, \quad 1 \leqslant n \leqslant N. \tag{11.30}$$

11.1.6 The Discounted Performance Measure

We provide sensitivity analysis for the β-discounted performance measures. Let

$$\tilde{\varphi}_\varepsilon(\beta) = \pi(0)(I - \beta\tilde{P}_\varepsilon)^{-1}_{\min} f. \tag{11.31}$$

Then

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\varphi}_\varepsilon(\beta) = n!\beta^n\pi(0)(I - \beta\tilde{P}_\varepsilon)^{-1-n}_{\min}V^n f,$$

which leads to

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\varphi}_\varepsilon(\beta)|_{\varepsilon=0} = n!\beta^n\pi(0)(I - \beta P)^{-1-n}_{\min}V^n f. \tag{11.32}$$

Note that

$$(I-\beta P)^{-n}_{\min} = \{[I - G_L(\beta)]^{-1}[I - \Psi_D(\beta)]^{-1}[I - R_U(\beta)]^{-1}\}^n \tag{11.33}$$

or

$$(I-\beta P)^{-n}_{\min} = \{[I - \overline{G}_U(\beta)]^{-1}[I - \Phi_D(\beta)]^{-1}[I - \overline{R}_L(\beta)]^{-1}\}^n. \tag{11.34}$$

11.2 Two Important Markov Chains

In this section, we provide sensitivity analysis for the stationary performance measures of Markov chains of $GI/M/1$ type and Markov chains of $M/G/1$ type in terms of the matrix-geometric solution and the matrix-iterative solution, respectively. Using the matrix R or G, we express the nth derivative for the stationary performance measures.

11.2.1 Perturbed Markov Chains of *GI/M/*1 Type

Consider an irreducible discrete-time Markov chain of $GI/M/1$ type whose transition matrix is given by

$$P = \begin{pmatrix} B_1 & B_0 & & & & \\ B_2 & A_1 & A_0 & & & \\ B_3 & A_2 & A_1 & A_0 & & \\ B_4 & A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{11.35}$$

where B_1 is a square matrix of finite size m_0, all A_i are square matrices of finite size m, the sizes of the other block-entries are determined accordingly and all empty entries are zero.

We define a perturbed matrix of $GI/M/1$ type as

$$\tilde{P}_\varepsilon = P + \varepsilon V, \tag{11.36}$$

where $\varepsilon > 0$ is sufficiently small, and the perturbed-directional matrix V is of $GI/M/1$ type as follows:

$$V = \begin{pmatrix} D_1 & D_0 & & & & \\ D_2 & C_1 & C_0 & & & \\ D_3 & C_2 & C_1 & C_0 & & \\ D_4 & C_3 & C_2 & C_1 & C_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We assume that the irreduciblity and state classification of $\tilde{P}_\varepsilon$ are the same as those of P.

We provide sensitivity analysis for the stationary performance measure

$$\tilde{\eta}_\varepsilon = \tilde{\pi}_\varepsilon f = \sum_{k=0}^{\infty} \tilde{\pi}_\varepsilon(k) f_k, \tag{11.37}$$

where

$$\tilde{\pi}_\varepsilon(0) = \kappa(\varepsilon) x_0(\varepsilon) \tag{11.38}$$

and

$$\tilde{\pi}_{\varepsilon}(k) = \kappa(\varepsilon) x_0(\varepsilon) R_{0,1}(\varepsilon) R(\varepsilon)^{k-1}, \quad k \geqslant 1, \tag{11.39}$$

the matrix $R(\varepsilon)$ is the minimal nonnegative solution to the nonlinear matrix equation

$$R(\varepsilon) = \sum_{k=0}^{\infty} R(\varepsilon)^k (A_k + \varepsilon C_k), \tag{11.40}$$

the matrix $R_{0,1}(\varepsilon)$ is given by

$$R_{0,1}(\varepsilon) = (B_0 + \varepsilon D_0) \left[I - \sum_{k=1}^{\infty} R(\varepsilon)^{k-1} (A_k + \varepsilon C_k) \right]^{-1}, \tag{11.41}$$

$x_0(\varepsilon)$ is the stationary probability vector of the censored chain to level 0 whose transition probability matrix is given by

$$\Psi_0(\varepsilon) = (B_1 + \varepsilon D_1) + \sum_{k=1}^{\infty} R(\varepsilon)^k (B_{k+1} + \varepsilon D_{k+1}), \tag{11.42}$$

and the positive constant $\kappa(\varepsilon)$ is given by

$$\kappa(\varepsilon) = \frac{1}{1 + x_0(\varepsilon) R_{0,1}(\varepsilon) e + x_0(\varepsilon) R_{0,1}(\varepsilon) [I - R(\varepsilon)]^{-1} e}. \tag{11.43}$$

To give the sensitivity analysis, let the matrix R be the minimal nonnegative solution to the nonlinear matrix equation $R = \sum_{k=0}^{\infty} R^k A_k$. It is seen later that each expression in the sensitivity analysis will be given by the minimal nonnegative solution R and the matrix sequences $\{A_k\}$, $\{B_k\}$, $\{C_k\}$ and $\{D_k\}$. It is worthwhile to note that $R_{0,1} = B_0 \left(I - \sum_{k=1}^{\infty} R^{k-1} A_k \right)^{-1}$, $\Psi_0 = B_1 + \sum_{k=1}^{\infty} R^k B_{k+1}$, x_0 is the stationary probability vector of the censored chain Ψ_0, and $\kappa = [1 + x_0 R_{0,1} e + x_0 R_{0,1} (I - R)^{-1} e]^{-1}$.

Our computations for the sensitivity analysis are listed in the following steps:

(1) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R(\varepsilon)|_{\varepsilon=0}$

It follows from (11.40) that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R(\varepsilon)|_{\varepsilon=0} = \sum_{k=0}^{\infty} R^k C_k \cdot \left(I - \sum_{k=1}^{\infty} k R^{k-1} A_k \right)^{-1}. \tag{11.44}$$

(2) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} x_0(\varepsilon)|_{\varepsilon=0}$

Since $x_0(\varepsilon) = x_0(\varepsilon)\Psi_0(\varepsilon)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} x_0(\varepsilon)|_{\varepsilon=0} = x_0 \left[D_1 + \sum_{k=1}^{\infty} R^k D_{k+1} \right.$$
$$\left. + \frac{\mathrm{d}}{\mathrm{d}\varepsilon} R(\varepsilon)|_{\varepsilon=0} \cdot \sum_{k=1}^{\infty} k R^{k-1} B_{k+1} \right] (I - \Psi_0)^{\#}. \tag{11.45}$$

(3) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_{0,1}(\varepsilon)|_{\varepsilon=0}$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_{0,1}(\varepsilon)|_{\varepsilon=0} = D_0 \left(I - \sum_{k=1}^{\infty} R^{k-1} A_k \right)^{-1} + B_0 \left(I - \sum_{k=1}^{\infty} R^{k-1} A_k \right)^{-2}$$
$$\cdot \left[\sum_{k=1}^{\infty} R^{k-1} C_k + \frac{\mathrm{d}}{\mathrm{d}\varepsilon} R(\varepsilon)|_{\varepsilon=0} \cdot \sum_{k=2}^{\infty} (k-1) R^{k-2} A_k \right]. \tag{11.46}$$

(4) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \kappa(\varepsilon)|_{\varepsilon=0}$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \kappa(\varepsilon)|_{\varepsilon=0} = -\left[1 + x_0 R_{0,1} e + x_0 R_{0,1} (I - R)^{-1} e \right]^{-2}$$
$$\cdot \left\{ \frac{\mathrm{d}}{\mathrm{d}\varepsilon} x_0(\varepsilon)|_{\varepsilon=0} \cdot R_{0,1} \left[I + (I - R)^{-1} \right] e \right.$$
$$+ x_0 \frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_{0,1}(\varepsilon)|_{\varepsilon=0} \cdot \left[I + (I - R)^{-1} \right] e$$
$$\left. + x_0 R_{0,1} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} R(\varepsilon)|_{\varepsilon=0} \cdot (I - R)^{-2} e \right\}.$$

(5) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \tilde{\eta}_\varepsilon|_{\varepsilon=0}$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \tilde{\eta}_\varepsilon|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left\{ \kappa(\varepsilon) \left[x_0(\varepsilon) f_0 + \sum_{k=1}^{\infty} x_0(\varepsilon) R_{0,1}(\varepsilon) R(\varepsilon)^{k-1} f_k \right] \right\} |_{\varepsilon=0}$$
$$= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \kappa(\varepsilon)|_{\varepsilon=0} \cdot \left(x_0 f_0 + x_0 R_{0,1} f_1 + \sum_{k=2}^{\infty} x_0 R_{0,1} R^{k-1} f_k \right)$$
$$+ \kappa \frac{\mathrm{d}}{\mathrm{d}\varepsilon} x_0(\varepsilon)|_{\varepsilon=0} \cdot f + \kappa \frac{\mathrm{d}}{\mathrm{d}\varepsilon} x_0(\varepsilon)|_{\varepsilon=0} \cdot R_{0,1} f + \kappa x_0 \frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_{0,1}(\varepsilon)|_{\varepsilon=0} \cdot f$$
$$+ \kappa \sum_{k=2}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} x_0(\varepsilon)|_{\varepsilon=0} \cdot R_{0,1} R^{k-1} f_k + \kappa \sum_{k=2}^{\infty} x_0 \frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_{0,1}(\varepsilon)|_{\varepsilon=0} \cdot R^{k-1} f_k$$
$$+ \kappa \sum_{k=2}^{\infty} (k-1) x_0 R_{0,1} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} R(\varepsilon)|_{\varepsilon=0} \cdot R^{k-2} f_k.$$

Therefore, the derivative $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\eta}_\varepsilon|_{\varepsilon=0}$ can be given by Eq. (11.44), Eq. (11.45) and Eq. (11.46).

11.2.2 Perturbed Markov Chains of *M*/*G*/1 Type

Consider an irreducible discrete-time Markov chain of $M/G/1$ type whose transition matrix is given by

$$P=\begin{pmatrix} B_1 & B_2 & B_3 & B_4 & B_5 & \cdots \\ B_0 & A_1 & A_2 & A_3 & A_4 & \cdots \\ & A_0 & A_1 & A_2 & A_3 & \cdots \\ & & A_0 & A_1 & A_2 & \cdots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{11.47}$$

where B_1 is a square matrix of finite size m_0, all A_i are square matrices of finite size m, the sizes of the other block-entries are determined accordingly and all empty entries are zero.

We define a perturbed matrix of $M/G/1$ type as

$$\tilde{P}_\varepsilon = P+\varepsilon V, \tag{11.48}$$

where $\varepsilon>0$ is sufficiently small, and the perturbed-directional matrix V is given by

$$V=\begin{pmatrix} D_1 & D_2 & D_3 & D_4 & D_5 & \cdots \\ D_0 & C_1 & C_2 & C_3 & C_4 & \cdots \\ & C_0 & C_1 & C_2 & C_3 & \cdots \\ & & C_0 & C_1 & C_2 & \cdots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We assume that the irreducibility and state classification of $\tilde{P}_\varepsilon$ are the same as those of P.

We provide sensitivity analysis for the stationary performance measure

$$\tilde{\eta}_\varepsilon=\tilde{\pi}_\varepsilon f=\sum_{k=0}^{\infty}\tilde{\pi}_\varepsilon(k)f_k, \tag{11.49}$$

where

$$\tilde{\pi}_\varepsilon(0)=\kappa(\varepsilon)x_0(\varepsilon) \tag{11.50}$$

and

$$\tilde{\pi}_\varepsilon(k)=\tilde{\pi}_\varepsilon(0)R_{0,k}(\varepsilon)+\sum_{i=1}^{k-1}\tilde{\pi}_\varepsilon(i)R_{k-i}(\varepsilon), \tag{11.51}$$

the matrix $G(\varepsilon)$ is the minimal nonnegative solution to the nonlinear matrix equation

$$G(\varepsilon)=\sum_{k=0}^{\infty}(A_k+\varepsilon C_k)G(\varepsilon)^k, \tag{11.52}$$

the matrix $R_{0,k}(\varepsilon)$ is given by

$$R_{0,k}(\varepsilon)=\sum_{i=1}^{\infty}(B_{k+i}+\varepsilon D_{k+i})G(\varepsilon)^{i-1} \cdot\left[I-\sum_{k=1}^{\infty}(A_k+\varepsilon C_k)G(\varepsilon)^{k-1}\right]^{-1}, \tag{11.53}$$

the matrix $R_k(\varepsilon)$ is given by

$$R_k(\varepsilon)=\sum_{i=1}^{\infty}(A_{k+i}+\varepsilon C_{k+i})G(\varepsilon)^{i-1} \cdot\left[I-\sum_{k=1}^{\infty}(A_k+\varepsilon C_k)G(\varepsilon)^{k-1}\right]^{-1}, \tag{11.54}$$

and $x_0(\varepsilon)$ is the stationary probability vector of the censored chain to level 0 whose transirion probability matrix is given by

$$\Psi_0(\varepsilon)=(B_1+\varepsilon D_1)+\sum_{k=1}^{\infty}(B_{k+1}+\varepsilon D_{k+1})G(\varepsilon)^{k-1} \cdot\left[I-\sum_{k=1}^{\infty}(A_k+\varepsilon C_k)G(\varepsilon)^{k-1}\right]^{-1}(B_0+\varepsilon D_0). \tag{11.55}$$

To give the sensitivity analysis, let the matrix G be the minimal nonnegative solution to the nonlinear matrix equation $G=\sum_{k=0}^{\infty}A_kG^k$. It is seen later that each expression in the sensitivity analysis will be given by the minimal nonnegative solution G and the matrix sequences $\{A_k\}$, $\{B_k\}$, $\{C_k\}$ and $\{D_k\}$. It is worthwhile to note that

$$R_{0,k}=\sum_{i=1}^{\infty}B_{k+i}G^{i-1}\left(I-\sum_{k=1}^{\infty}A_kG^{k-1}\right)^{-1},$$

$$R_k=\sum_{i=1}^{\infty}A_{k+i}G^{i-1}\left(I-\sum_{k=1}^{\infty}A_kG^{k-1}\right)^{-1},$$

$$\Psi_0=B_1+\sum_{k=1}^{\infty}B_{k+1}G^{k-1}\left(I-\sum_{k=1}^{\infty}A_kG^{k-1}\right)^{-1}B_0$$

and x_0 is the stationary probability vector of the censored chain Ψ_0.

Our computations for the sensitivity analysis are listed in the following steps:

(1) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}G(\varepsilon)|_{\varepsilon=0}$

It follows from Eq. (11.52) that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}G(\varepsilon)|_{\varepsilon=0}=\left(I-\sum_{k=1}^{\infty}kA_kG^{k-1}\right)^{-1}\sum_{k=0}^{\infty}C_kG^k. \tag{11.56}$$

(2) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}R_{0,k}(\varepsilon)|_{\varepsilon=0}$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}R_{0,k}(\varepsilon)|_{\varepsilon=0}=&\sum_{i=1}^{\infty}D_{k+i}G^{i-1}\left(I-\sum_{k=1}^{\infty}A_kG^{k-1}\right)^{-1}\\&+\sum_{i=1}^{\infty}(i-1)B_{k+i}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}G(\varepsilon)|_{\varepsilon=0}\cdot G^{i-2}\left(I-\sum_{k=1}^{\infty}A_kG^{k-1}\right)^{-1}\\&+\sum_{i=1}^{\infty}B_{k+i}G^{i-1}\left(I-\sum_{k=1}^{\infty}A_kG^{k-1}\right)^{-2}\\&\cdot\left[\sum_{k=1}^{\infty}C_kG^{k-1}+\sum_{k=1}^{\infty}(k-1)A_kG^{k-2}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}G(\varepsilon)|_{\varepsilon=0}\right].\end{aligned} \tag{11.57}$$

(3) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}R_k(\varepsilon)|_{\varepsilon=0}$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}R_k(\varepsilon)|_{\varepsilon=0}=&\sum_{i=1}^{\infty}C_{k+i}G^{i-1}\left(I-\sum_{k=1}^{\infty}A_kG^{k-1}\right)^{-1}\\&+\sum_{i=2}^{\infty}(i-1)A_{k+i}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}G(\varepsilon)|_{\varepsilon=0}\cdot G^{i-2}\left(I-\sum_{k=1}^{\infty}A_kG^{k-1}\right)^{-1}\\&+\sum_{i=1}^{\infty}A_{k+i}G^{i-1}\left(I-\sum_{k=1}^{\infty}A_kG^{k-1}\right)^{-2}\\&\cdot\left[\sum_{k=1}^{\infty}C_kG^{k-1}+\sum_{k=2}^{\infty}(k-1)A_kG^{k-2}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}G(\varepsilon)|_{\varepsilon=0}\right].\end{aligned} \tag{11.58}$$

(4) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}x_0(\varepsilon)|_{\varepsilon=0}$

Since $x_0(\varepsilon)=x_0(\varepsilon)\Psi_0(\varepsilon),$ we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon} x_0(\varepsilon)|_{\varepsilon=0} = x_0 \Bigg\{ & D_1 + \sum_{k=1}^{\infty} (B_{k+1} + D_{k+1}) G^{k-1} \left(I - \sum_{k=1}^{\infty} A_k G^{k-1} \right)^{-1} B_0 \\
& + \sum_{k=2}^{\infty} (k-1) B_{k+1} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} G(\varepsilon)|_{\varepsilon=0} \cdot G^{k-2} \left(I - \sum_{k=1}^{\infty} A_k G^{k-1} \right)^{-1} B_0 \\
& + \sum_{k=1}^{\infty} B_{k+1} G^{k-1} \left(I - \sum_{k=1}^{\infty} A_k G^{k-1} \right)^{-2} \\
& \cdot \left[\sum_{k=1}^{\infty} C_k G^{k-1} + \sum_{k=2}^{\infty} (k-1) A_k \frac{\mathrm{d}}{\mathrm{d}\varepsilon} G(\varepsilon)|_{\varepsilon=0} \cdot G^{k-2} B_0 \right] \Bigg\} (I - \Psi_0)^{\#}.
\end{aligned} \tag{11.59}$$

(5) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\kappa(\varepsilon)|_{\varepsilon=0}$

It follows from Eq. (11.50) and Eq. (11.51) that

$$\tilde{\pi}_\varepsilon(0) = \kappa(\varepsilon) x_0(\varepsilon)$$
$$\tilde{\pi}_\varepsilon(1) = \kappa(\varepsilon) x_0(\varepsilon) R_{0,1}(\varepsilon)$$

and

$$\begin{aligned}
\tilde{\pi}_\varepsilon(k) = & \kappa(\varepsilon) x_0(\varepsilon) R_{0,k}(\varepsilon) \\
& + \kappa(\varepsilon) x_0(\varepsilon) \sum_{m=2}^{k} \sum_{\substack{i_1+i_2+\ldots+i_m=k \\ 1\leqslant i_1, i_2, \ldots, i_m \leqslant k}} R_{0,i_1}(\varepsilon) R_{i_2}(\varepsilon) \ldots R_{i_m}(\varepsilon),
\end{aligned}$$

thus, we get

$$\kappa(\varepsilon) = \frac{1}{1 + x_0(\varepsilon) \sum_{k=1}^{\infty} \left[R_{0,k}(\varepsilon) + \sum_{m=2}^{k} \sum_{\substack{i_1+i_2+\ldots+i_m=k \\ 1\leqslant i_1, i_2, \ldots, i_m \leqslant k}} R_{0,i_1}(\varepsilon) R_{i_2}(\varepsilon) \ldots R_{i_m}(\varepsilon) \right] e},$$

which easily leads to the expression of $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\kappa(\varepsilon)|_{\varepsilon=0}$.

(6) Compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\eta}_\varepsilon|_{\varepsilon=0}$

Note that

$$\begin{aligned}
\tilde{\eta}_\varepsilon = & \kappa(\varepsilon) x_0(\varepsilon) f_0 + \kappa(\varepsilon) x_0(\varepsilon) R_{0,1}(\varepsilon) f_1 + \kappa(\varepsilon) x_0(\varepsilon) \sum_{k=2}^{\infty} R_{0,k}(\varepsilon) f_k \\
& + \kappa(\varepsilon) x_0(\varepsilon) \sum_{k=2}^{\infty} \sum_{m=2}^{k} \sum_{\substack{i_1+i_2+\ldots+i_m=k \\ 1\leqslant i_1, i_2, \ldots, i_m \leqslant k}} R_{0,i_1}(\varepsilon) R_{i_2}(\varepsilon) \ldots R_{i_m}(\varepsilon) f_k.
\end{aligned}$$

Therefore, the derivative $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\eta}_\varepsilon|_{\varepsilon=0}$ can be given by means of the above steps (1) to (5).

11.3 Perturbed Continuous-Time Markov Chains

In this section, we provide sensitivity analysis for the stationary performance measure of a perturbed continuous-time Markov chain which is irreducible and positive recurrent. By means of the UL- and LU-types of *RG*-factorizations, we express the *n*th derivative for the stationary performance measure.

We consider an irreducible perturbed continuous-time Markov chain $\{x_\varepsilon(t), t \geqslant 0\}$ whose infinitesimal generator is given by

$$\widetilde{Q}_\varepsilon = Q + \varepsilon V, \tag{11.60}$$

where

$$Q = \begin{pmatrix} B_{0,0} & B_{0,1} & B_{0,2} & B_{0,3} & \cdots \\ B_{1,0} & B_{1,1} & B_{1,2} & B_{1,3} & \cdots \\ B_{2,0} & B_{2,1} & B_{2,2} & B_{2,3} & \cdots \\ B_{3,0} & B_{3,1} & B_{3,2} & B_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

is the infinitesimal generator of an irreducible continuous-time Markov chain with block structure, the perturbed-directional matrix V is given by

$$V = \begin{pmatrix} C_{0,0} & C_{0,1} & C_{0,2} & C_{0,3} & \cdots \\ C_{1,0} & C_{1,1} & C_{1,2} & C_{1,3} & \cdots \\ C_{2,0} & C_{2,1} & C_{2,2} & C_{2,3} & \cdots \\ C_{3,0} & C_{3,1} & C_{3,2} & C_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

is a non-zero matrix with the size of each block $C_{i,j}$ equal to that of $B_{i,j}$ for $i, j \geqslant 0$, and ε is a sufficiently small positive number such that $\widetilde{Q}_\varepsilon$ is still the infinitesimal generator of a continuous-time Markov chain whose irreducibility and state classification are the same as those of the Markov chain Q. We assume that if the Markov chain Q is positive recurrent, then the perturbed Markov chain $\widetilde{Q}_\varepsilon$ can also be positive recurrent for each sufficiently small $\varepsilon > 0$. In this case, $Ve = 0$.

We assume that the two Markov chains Q and $\widetilde{Q}_\varepsilon$ are both irreducible and positive recurrent. Let π and $\tilde{\pi}_\varepsilon$ be the stationary probability vectors of the Markov chains Q and $\widetilde{Q}_\varepsilon$, respectively. We write $\eta = \pi f$ and $\tilde{\eta}_\varepsilon = \tilde{\pi}_\varepsilon f$. In what follows we provide sensitivity analysis for the stationary performance measures $\tilde{\eta}_\varepsilon$.

Note that $\tilde{\pi}_\varepsilon$ is the unique positive solution to the system of equations $\tilde{\pi}_\varepsilon \widetilde{Q}_\varepsilon = 0$ and $\tilde{\pi}_\varepsilon e = 1$, we obtain

$$\frac{\mathrm{d}\tilde{\pi}_\varepsilon}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \cdot Q = -\pi V, \tag{11.61}$$

$$\frac{\mathrm{d}^n\tilde{\pi}_\varepsilon}{\mathrm{d}\varepsilon^n}\Big|_{\varepsilon=0} \cdot Q = -\frac{\mathrm{d}^{n-1}\tilde{\pi}_\varepsilon}{\mathrm{d}\varepsilon^{n-1}}\Big|_{\varepsilon=0} \cdot nV, \quad n \geqslant 2, \tag{11.62}$$

and

$$\frac{\mathrm{d}^n\tilde{\pi}_\varepsilon}{\mathrm{d}\varepsilon^n}\Big|_{\varepsilon=0} \cdot e = 0, \quad n \geqslant 1. \tag{11.63}$$

Since Q always has the maximal nonpositive inverse $Q_{\max}^{-1}$, it follows from Eq. (11.61) and Eq. (11.62) that

$$\frac{\mathrm{d}^n\tilde{\pi}_\varepsilon}{\mathrm{d}\varepsilon^n}\Big|_{\varepsilon=0} = n!\pi\left[\left(-Q_{\max}^{-1}\right)V\right]^n, \quad n \geqslant 1. \tag{11.64}$$

Using Eq. (11.63) and Eq. (11.64), we have

$$\pi\left[\left(-Q_{\max}^{-1}\right)V\right]^n e = 0, \quad n \geqslant 1, \tag{11.65}$$

For the stationary performance measure $\tilde{\eta}_\varepsilon = \tilde{\pi}_\varepsilon f$,

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_\varepsilon\Big|_{\varepsilon=0} = n!\pi\left[\left(-Q_{\max}^{-1}\right)V\right]^n f. \tag{11.66}$$

Therefore, it follows from the UL-type *RG*-factorization that for $n \geqslant 1$,

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_\varepsilon\Big|_{\varepsilon=0} = n!\pi\left[(I - G_L)^{-1}\left(-U_D^{-1}\right)(I - R_U)^{-1}V\right]^n f. \tag{11.67}$$

or from the LU-type *RG*-factorization that for $n \geqslant 1$,

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\eta}_\varepsilon\Big|_{\varepsilon=0} = n!\pi\left[(I - \overline{G}_U)^{-1}\left(-\overline{U}_D^{-1}\right)(I - \overline{R}_L)^{-1}V\right]^n f. \tag{11.68}$$

In what follows we simply discuss the realization matrix of the continuous-time Markov chain. The realization matrix D is expressed in terms of a potential vector g which satisfies the Poisson equation

$$Qg = -f + \pi f e. \tag{11.69}$$

Thus, using the UL-type RG-factorization, it follows from Eq. (11.69) that

$$g=-Q_{\max}^{-1}(f-\pi fe)=(I-G_L)^{-1}\left(-U_D^{-1}\right)(I-R_U)^{-1}(f-\pi fe) \tag{11.70}$$

or using the LU-type RG-factorization, we have

$$g=-Q_{\max}^{-1}(f-\pi fe)=(I-\overline{G}_U)^{-1}\left(-\overline{U}_D^{-1}\right)(I-\overline{R}_L)^{-1}(f-\pi fe). \tag{11.71}$$

The realization matrix is given by

$$\begin{aligned} D &= eg^{\mathrm{T}}-ge^{\mathrm{T}} \\ &= e(f-\pi fe)^{\mathrm{T}}\left[(I-R_U)^{-1}\right]^{\mathrm{T}}\left(-U_D^{-1}\right)^{\mathrm{T}}\left[(I-G_L)^{-1}\right]^{\mathrm{T}} \\ &\quad -(I-G_L)^{-1}\left(-U_D^{-1}\right)(I-R_U)^{-1}(f-\pi fe)e^{\mathrm{T}}. \end{aligned} \tag{11.72}$$

or

$$\begin{aligned} D &= eg^{\mathrm{T}}-ge^{\mathrm{T}} \\ &= e(f-\pi fe)^{\mathrm{T}}\left[(I-\overline{R}_L)^{-1}\right]^{\mathrm{T}}\left(-\overline{U}_D^{-1}\right)^{\mathrm{T}}\left[(I-\overline{G}_U)^{-1}\right]^{\mathrm{T}} \\ &\quad -(I-\overline{G}_U)^{-1}\left(-\overline{U}_D^{-1}\right)(I-\overline{R}_L)^{-1}(f-\pi fe)e^{\mathrm{T}}. \end{aligned} \tag{11.73}$$

As an important example, we consider a perturbed continuous-time level-dependent QBD process with infinitely-many levels whose infinitesimal generator is given by

$$Q(\varepsilon)=Q+\varepsilon V, \tag{11.74}$$

where ε is a sufficiently small positive number,

$$Q=\begin{pmatrix} B_1^{(0)} & B_0^{(0)} & & & \\ B_2^{(1)} & B_1^{(1)} & B_0^{(1)} & & \\ & B_2^{(2)} & B_1^{(2)} & B_0^{(2)} & \\ & & B_2^{(3)} & B_1^{(3)} & B_0^{(3)} \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{11.75}$$

and the perturbed-directional matrix

$$V=\begin{pmatrix} C_1^{(0)} & C_0^{(0)} & & & \\ C_2^{(1)} & C_1^{(1)} & C_0^{(1)} & & \\ & C_2^{(2)} & C_1^{(2)} & C_0^{(2)} & \\ & & C_2^{(3)} & C_1^{(3)} & C_0^{(3)} \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ be the stationary probability vector of the QBD process Q, partitioned according to the levels. We denote by the matrix sequence $\{R_k\}$ the minimal nonnegative solution to the system of matrix equations

$$B_0^{(k)} + R_k B_1^{(k+1)} + R_k R_{k+1} B_2^{(k+2)} = 0, \quad k \geqslant 0.$$

Then

$$\pi_0 = \kappa x_0$$

and

$$\pi_k = \kappa x_0 R_0 R_1 \ldots R_{k-1}, \quad k \geqslant 1,$$

where x_0 is the stationary probability vector of the censored chain to level 0 with infinitesimal generator $U_0 = B_1^{(0)} + R_0 B_2^{(1)}$, κ is a normalization constant such that $\sum_{k=0}^{\infty} \pi_k e = 1$, i.e.,

$$\kappa = \frac{1}{1 + \sum_{k=0}^{\infty} x_0 R_0 R_1 \ldots R_k e}.$$

In principle, Eq. (11.68), using the LU-type RG-factorization, provides expression for the nth derivative of the stationary performance measure $\tilde{\eta}_\varepsilon$ at $\varepsilon = 0$. It is easy to see that for a small $n \geqslant 1$, $\eta^{(n)}$ can be further simplified using the LU-type R-, U- and G-measures. The following proposition provides expression for $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \tilde{\eta}_\varepsilon |_{\varepsilon=0}$ by means of the LU-type R-, U- and G-measures.

Proposition 11.1

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \tilde{\eta}_\varepsilon |_{\varepsilon=0} = & \left[\pi_0 C_1^{(0)} + \pi_1 C_1^{(1)}\right] \sum_{k=0}^{\infty} Y_k^{(0)} \left(-\bar{U}_k^{-1}\right) \sum_{i=0}^{k} X_i^{(k)} f_{k-i} \\ & + \sum_{m=1}^{\infty} \left[\pi_{m-1} C_0^{(m-1)} + \pi_m C_1^{(m)} + \pi_{m+1} C_2^{(m+1)}\right] \\ & \cdot \sum_{k=m}^{\infty} Y_{k-m}^{(m)} \left(-\bar{U}_k^{-1}\right) \sum_{i=0}^{k} X_i^{(k)} f_{k-i},\end{aligned}$$

where

$$X_0^{(l)} = I, \quad l \geqslant 0,$$
$$X_k^{(l)} = \bar{R}_l \bar{R}_{l-1} \bar{R}_{l-2} \ldots \bar{R}_{l-k+1}, \quad l \geqslant k \geqslant 1,$$

and

$$Y_0^{(l)} = I, \quad l \geqslant 0,$$
$$Y_k^{(l)} = \bar{G}_l \bar{G}_{l+1} \bar{G}_{l+2} \ldots \bar{G}_{l+k-1}, \quad k \geqslant 1, l \geqslant 0.$$

Proof Setting $n=1$ in Eq. (11.68), we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\eta}_\varepsilon|_{\varepsilon=0}=\pi V(I-\overline{G}_U)^{-1}\left(-\overline{U}_D^{-1}\right)(I-\overline{R}_L)^{-1}f.$$

Thus, we obtain

$$\pi V=(\pi_0 C_1^{(0)}+\pi_1 C_1^{(1)},\pi_0 C_0^{(0)}+\pi_1 C_1^{(1)}+\pi_2 C_2^{(2)},\pi_1 C_0^{(1)}+\pi_2 C_1^{(2)}+\pi_3 C_2^{(3)},\ldots),$$

and

$$(I-\overline{G}_U)^{-1}\left(-\overline{U}_D^{-1}\right)(I-\overline{R}_L)^{-1}f=\begin{pmatrix}\sum\limits_{k=0}^{\infty}Y_k^{(0)}\left(-\overline{U}_k^{-1}\right)\sum\limits_{i=0}^{k}X_i^{(k)}f_{k-i}\\ \sum\limits_{k=1}^{\infty}Y_{k-1}^{(1)}\left(-\overline{U}_k^{-1}\right)\sum\limits_{i=0}^{k}X_i^{(k)}f_{k-i}\\ \sum\limits_{k=2}^{\infty}Y_{k-2}^{(2)}\left(-\overline{U}_k^{-1}\right)\sum\limits_{i=0}^{k}X_i^{(k)}f_{k-i}\\ \vdots\end{pmatrix}.$$

Simple computations yield the stated result. This completes the proof.

Now, we provide sensitivity analysis for the transient performance measure $\gamma(t)$ for $t\geqslant 0$, where

$$\gamma(t)=E[\Phi(t)]=E\left[\int_0^t f(X_u)\mathrm{d}u\right].$$

It is easy to see from Eq. (10.3) that

$$\gamma(t)=\pi(0)\int_0^t\exp\{Qx\}\mathrm{d}xf. \tag{11.76}$$

Based on Eq. (11.76), for the perturbed Markov chain $\widetilde{Q}_\varepsilon$ we can write the perturbed transient performance measure as

$$\tilde{\gamma}_\varepsilon(t)=\pi(0)\int_0^t\exp\{\widetilde{Q}_\varepsilon x\}\mathrm{d}xf.$$

Therefore, we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\gamma}_\varepsilon(t)=\pi(0)\int_0^t x^n\exp\{\widetilde{Q}_\varepsilon x\}\mathrm{d}xV^nf,\quad n\geqslant 1,$$

which leads to

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\tilde{\gamma}_\varepsilon(t)|_{\varepsilon=0}=\pi(0)\int_0^t x^n\exp\{Qx\}\mathrm{d}xV^nf,\quad n\geqslant 1. \tag{11.77}$$

Finally, we provide sensitivity analysis for the β-discounted performance measure $\phi(\beta)$, where

$$\phi(\beta) = E\left[\int_0^{+\infty} e^{-\beta u} f(X_u)\mathrm{d}u\right].$$

It is easy to see from Eq. (10.11) that

$$\phi(\beta) = -\pi(0)(Q-\beta I)_{\max}^{-1} f. \tag{11.78}$$

Based on Eq. (11.78), for the perturbed Markov chain $\widetilde{Q}_\varepsilon$ we can write the perturbed discounted performance measure as

$$\widetilde{\phi}_\varepsilon(\beta) = -\pi(0)\left(\widetilde{Q}_\varepsilon - \beta I\right)_{\max}^{-1} f.$$

Therefore, we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\widetilde{\phi}_\varepsilon(\beta) = (-1)^{n+1} n!\pi(0)\left(\widetilde{Q}_\varepsilon - \beta I\right)_{\max}^{-1-n} V^n f, \quad n \geqslant 1,$$

which leads to

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\widetilde{\phi}_\varepsilon(\beta)\,|_{\varepsilon=0} = (-1)^{n+1} n!\pi(0)(Q-\beta I)_{\max}^{-1-n} V^n f, \quad n \geqslant 1. \tag{11.79}$$

11.4 Perturbed Accumulated Reward Processes

In this section, we derive the nth derivative for the conditional moments of the accumulated reward process of an irreducible continuous-time perturbed QBD process.

Consider the accumulated reward process of an irreducible continuous-time perturbed QBD process. We assume that the perturbed QBD process $\{x_\varepsilon(t), t \geqslant 0\}$, given in Eq. (11.74), is separable and Borel measurable, and its sample functions are almost all lower semi-continuous at the right-hand side. Therefore, the perturbed QBD process has the strong Markov property.

Similar to (10.38), we define

$$\psi_{k,j}(t,\varepsilon) = E\left[\int_0^t f(x_\varepsilon(u))\mathrm{d}u \mid x_\varepsilon(0) = (k,j)\right].$$

Let

$$\Psi_k(t,\varepsilon) = (\psi_{k,1}(t,\varepsilon), \psi_{k,2}(t,\varepsilon), \ldots, \psi_{k,M_k}(t,\varepsilon))^{\mathrm{T}}$$

and

$$\Psi_k^*(s,\varepsilon)=\int_0^{+\infty}e^{-st}\Psi_k(t,\varepsilon)\mathrm{d}t,\quad k\geqslant 0.$$

The sequence $\{\Psi_k^*(s,\varepsilon)\}$ then satisfies the following system of infinite-dimensional linear equations, as in Theorem 10.3,

$$\Psi_0^*(s,\varepsilon)=\Re_0(s,\varepsilon)\left\{e+\mathcal{A}_0(\varepsilon)\Psi_0^*(s,\varepsilon)+\left[B_0^{(0)}+\varepsilon C_0^{(0)}\right]\Psi_1^*(s,\varepsilon)\right\},\quad (11.80)$$

and for $l\geqslant 1$,

$$\Psi_l^*(s,\varepsilon)=\Re_l(s,\varepsilon)\left\{e+\left[B_2^{(l)}+\varepsilon C_2^{(l)}\right]\Psi_{l-1}^*(s,\varepsilon)\right.$$
$$\left.+\mathcal{A}_l(\varepsilon)\Psi_l^*(s,\varepsilon)+\left[B_0^{(l)}+\varepsilon C_0^{(l)}\right]\Psi_{l+1}^*(s,\varepsilon)\right\},\quad (11.81)$$

where for $k\geqslant 0$,

$$\Re_k(s,\varepsilon)=\mathrm{diag}\left(\frac{f(k,1)}{\left[s-b_{1,(1,1)}^{(k)}-\varepsilon c_{1,(1,1)}^{(k)}\right]^2},\ldots,\frac{f(k,M_k)}{\left[s-b_{1,(M_k,M_k)}^{(k)}-\varepsilon c_{1,(M_k,M_k)}^{(k)}\right]^2}\right)$$

and

$$\mathcal{A}_k(\varepsilon)=B_1^{(k)}+\varepsilon C_1^{(k)}-\mathrm{diag}\left(b_{1,(1,1)}^{(k)}+\varepsilon c_{1,(1,1)}^{(k)},\ldots,b_{1,(M_k,M_k)}^{(k)}+\varepsilon c_{1,(M_k,M_k)}^{(k)}\right).$$

Note that $\mathcal{A}_k(\varepsilon)=\mathcal{B}_k+\varepsilon\mathcal{C}_k$, where $\mathcal{B}_k$ is given in Eq. (10.42) and

$$\mathcal{C}_k=C_1^{(k)}-\mathrm{diag}\left(c_{1,(1,1)}^{(k)},c_{1,(2,2)}^{(k)},\ldots,c_{1,(M_k,M_k)}^{(k)}\right).$$

Let

$$\widetilde{Q}(s,\varepsilon)=\begin{pmatrix}\Re_0(s,\varepsilon)&&&\\&\Re_1(s,\varepsilon)&&\\&&\Re_2(s,\varepsilon)&\\&&&\ddots\end{pmatrix}$$
$$\cdot\begin{pmatrix}\mathcal{A}_0(\varepsilon)&B_0^{(0)}+\varepsilon C_0^{(0)}&&&\\B_2^{(1)}+\varepsilon C_2^{(1)}&\mathcal{A}_1(\varepsilon)&B_0^{(1)}+\varepsilon C_0^{(1)}&&\\&B_2^{(2)}+\varepsilon C_2^{(2)}&\mathcal{A}_2(\varepsilon)&B_0^{(2)}+\varepsilon C_0^{(2)}&\\&&\ddots&\ddots&\ddots\end{pmatrix},$$

$$\Re(s,\varepsilon)=\begin{pmatrix}\Re_0(s,\varepsilon)e\\\Re_1(s,\varepsilon)e\\\Re_2(s,\varepsilon)e\\\vdots\end{pmatrix},\quad \Psi(s,\varepsilon)=\begin{pmatrix}\Psi_0^*(s,\varepsilon)\\\Psi_1^*(s,\varepsilon)\\\Psi_2^*(s,\varepsilon)\\\vdots\end{pmatrix}.$$

Then it follows from Eq. (11.80) and Eq. (11.81) that

$$\Psi(s,\varepsilon)=\widetilde{Q}(s,\varepsilon)\Psi(s,\varepsilon)+\Re(s,\varepsilon). \tag{11.82}$$

Let

$$\theta_k(s,\varepsilon)=\operatorname{diag}\left(\left[s-b_{1,(1,1)}^{(k)}-\varepsilon c_{1,(1,1)}^{(k)}\right]^2,\dots,\left[s-b_{1,(M_k,M_k)}^{(k)}-\varepsilon c_{1,(M_k,M_k)}^{(k)}\right]^2\right),$$

$$\Theta(s,\varepsilon)=\operatorname{diag}(\theta_0(s,\varepsilon),\theta_1(s,\varepsilon),\theta_2(s,\varepsilon),\dots),$$

$$f_k=(f(k,1),f(k,2),\dots,f(k,M_k))^{\mathrm{T}},$$

and $f=\left(f_0^{\mathrm{T}},f_1^{\mathrm{T}},f_2^{\mathrm{T}},\dots\right)^{\mathrm{T}}$. Then, using Eq. (11.82), we yield

$$\Theta(s,\varepsilon)\Psi(s,\varepsilon)=\left(\widetilde{Q}_1+\varepsilon\widetilde{Q}_2\right)\Psi(s,\varepsilon)+f, \tag{11.83}$$

where

$$\widetilde{Q}_1=\begin{pmatrix}\mathcal{B}_0 & B_0^{(0)} & & & \\ B_2^{(1)} & \mathcal{B}_1 & B_0^{(1)} & & \\ & B_2^{(2)} & \mathcal{B}_2 & B_0^{(2)} & \\ & & \ddots & \ddots & \ddots\end{pmatrix}$$

and

$$\widetilde{Q}_2=\begin{pmatrix}\mathcal{C}_0 & C_0^{(0)} & & & \\ C_2^{(1)} & \mathcal{C}_1 & C_0^{(1)} & & \\ & C_2^{(2)} & \mathcal{C}_2 & C_0^{(2)} & \\ & & \ddots & \ddots & \ddots\end{pmatrix}.$$

Let

$$\xi^{(n)}(s)=\frac{\partial^n}{\partial\varepsilon^n}\Psi(s,\varepsilon)_{|\varepsilon=0},\quad \Theta^{(n)}(s)=\frac{\partial^n}{\partial\varepsilon^n}\Theta(s,\varepsilon)_{|\varepsilon=0},\quad n\geqslant 1.$$

Taking the derivatives with respect to ε on the both sides of Eq. (11.83) and letting $\varepsilon=0$, we have

$$\left[\widetilde{Q}_1-\Theta(s)\right]\xi^{(1)}(s)=-\left[\widetilde{Q}_2-\Theta^{(1)}(s)\right]\xi(s),$$

where $\Theta(s)=\Theta(s,0)$ and $\xi(s)={}_{\infty}\xi(s)$ are given in Theorem 10.4. Hence, we obtain

$$\xi^{(1)}(s)=-\left[\widetilde{Q}_1-\Theta(s)\right]_{\max}^{-1}\left[\widetilde{Q}_2-\Theta^{(1)}(s)\right]\xi(s). \tag{11.84}$$

Further, for the nth derivative we can obtain

$$\left[\widetilde{Q}_1-\Theta(s)\right]\xi^{(n)}(s)=-n\left[\widetilde{Q}_2-\Theta^{(1)}(s)\right]\xi^{(n-1)}(s)$$
$$+\sum_{l=0}^{n-1}\binom{n}{l}\Theta^{(n-l)}(s)\xi^{(l)}(s).$$

Therefore,

$$\xi^{(n)}(s)=-n\left[\widetilde{Q}_1-\Theta(s)\right]_{\max}^{-1}\left[\widetilde{Q}_2-\Theta^{(1)}(s)\right]\xi^{(n-1)}(s)$$
$$+\left[\widetilde{Q}_1-\Theta(s)\right]_{\max}^{-1}\sum_{l=0}^{n-1}\binom{n}{l}\Theta^{(n-l)}(s)\xi^{(l)}(s), \tag{11.85}$$

where

$$\left(\widetilde{Q}_1-\Theta(s)\right)_{\max}^{-1}=[I-G_L(s)]^{-1}U_D(s)^{-1}[I-R_U(s)]^{-1}$$

or

$$\left(\widetilde{Q}_1-\Theta(s)\right)_{\max}^{-1}=[I-\overline{G}_U(s)]^{-1}\overline{U}_D(s)^{-1}[I-\overline{R}_L(s)]^{-1}.$$

11.5 A Perturbed *MAP*/*PH*/1 Queue

This section provides sensitivity analysis for the stationary queue length of a perturbed *MAP*/*PH*/1 queue. To achieve this, we must first provide a sensitivity analysis for the PH distribution as well as the MAP.

11.5.1 A Perturbed PH Distribution

Consider a PH distribution $F(x)$ for $x \geqslant 0$ with irreducible representation (α, T) of order γ_1, where $\alpha e=1$. Thus we have

$$F(x)=P\{X \leqslant x\}=1-\alpha\exp\{Tx\}e,$$

where X is the corresponding PH random variable. We denote by S a matrix of order γ_1, which satisfies that the perturbed matrix $T+\varepsilon S$ has negative diagonal elements and nonnegative off-diagonal elements with $(T+\varepsilon S)e \lneqq 0$ for each safficiently small $\varepsilon > 0$. Let $T^0=-Te$ and $S^0=-Se$. We assume that the irreducibility of the matrix $T+T^0\alpha$ is the same as that for the matrix $(T+\varepsilon S)+(T^0+\varepsilon S^0)\alpha$. Obviously, $(\alpha, T+\varepsilon S)$ is the irreducible representation of order γ_1 of a perturbed PH distribution $F_\varepsilon(x)$ with

$$F_\varepsilon(x)=P\{X_\varepsilon \leqslant x\}=1-\alpha\exp\{(T+\varepsilon S)x\}e.$$

Let $f_\varepsilon^*(s)$ be the Laplace-Stieltjes transform of $F_\varepsilon(x)$. Then

$$f_\varepsilon^*(s) = \alpha(sI - T - \varepsilon S)^{-1}(T^0 + \varepsilon S^0).$$

We obtain

$$\frac{\partial^n}{\partial \varepsilon^n} f_\varepsilon^*(s)_{|\varepsilon=0} = n!\alpha(sI - T)^{-n}[(sI - T)^{-1}S^n T^0 - S^n e], \quad n \geqslant 1.$$

In addition, from $E\left[X_\varepsilon^k\right] = (-1)^k k![\alpha(T + \varepsilon S)^{-k} e]$, we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n} E\left[X_\varepsilon^k\right]_{|\varepsilon=0} = (-1)^{k+n}(k+n-1)!k\alpha T^{-(k+n)} S^n e, \quad k \geqslant 1, n \geqslant 1.$$

For the PH distribution, the maximal eigenvalue of the matrix T is an important quantity. In what follows we provide sensitivity analysis for the maximal eigenvalue. Let c and d, c_ε and d_ε be the left and right Perron-Frobenius eigenvectors of the matrices T and $T + \varepsilon S$, respectively. Assume $cd = 1$. We denote by λ and λ_ε the maximal eigenvalues of T and $T + \varepsilon S$, respectively. It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} c_{\varepsilon|\varepsilon=0}(\lambda I - T) = cS - \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda_{\varepsilon|\varepsilon=0} c.$$

Note that $(\lambda I - T)d = 0$ and $cd = 1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda_{\varepsilon|\varepsilon=0} = cSd.$$

11.5.2 A Perturbed MAP

Consider a MAP with irreducible matrix descriptor (C_1, D_1) of order γ_2, where the diagonal elements of the matrix C_1 are all negative, the off-diagonal elements of the matrix C_1 are all nonnegative, each element of the matrix D_1 is nonnegative and the Markov chain with infinitesimal generator $C_1 + D_1$ is irreducible and positive recurrent. Let θ be the stationary probability vector of the Markov chain with infinitesimal generator $C_1 + D_1$. Then $r = \theta D_1 e$ is the stationary arrival rate. Let C_2 and D_2 be two matrices of order γ_2 such that for each sufficiently small $\varepsilon > 0$, the diagonal elements of the matrix $C_1 + \varepsilon C_2$ are all negative, the off-diagonal elements of the matrix $C_1 + \varepsilon C_2$ are all nonnegative, each element of the matrix $D_1 + \varepsilon D_2$ is nonnegative and $(C_2 + D_2)e = 0$. It is clear that $(C_1 + \varepsilon C_2, D_1 + \varepsilon D_2)$ is the irreducible matrix descriptor of a perturbed MAP.

Let θ_ε be the stationary probability vector of the perturbed Markov chain with infinitesimal generator $(C_1 + D_1) + \varepsilon(C_2 + D_2)$. Then

$$\theta_\varepsilon\left[(C_1 + D_1) + \varepsilon(C_2 + D_2)\right] = 0$$

and $\theta_\varepsilon e = 1$. Thus, we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\theta_\varepsilon e = 0, \quad \text{for} \quad n \geqslant 1, \varepsilon \geqslant 0, \tag{11.86}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\theta_{\varepsilon|\varepsilon=0}(C_1 + D_1) = -\theta(C_2 + D_2). \tag{11.87}$$

It follows from Eq. (11.86) and Eq. (11.87) that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\theta_{\varepsilon|\varepsilon=0}(C_1 + D_1 + \kappa e\theta) = -\theta(C_2 + D_2),$$

where $\kappa = \min_{1 \leqslant i \leqslant \gamma_2}\{c_{1,(i,i)} + d_{1,(i,i)}\}$. Note that the matrix $C_1 + D_1 + \kappa e\theta$ is invertible, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\theta_{\varepsilon|\varepsilon=0} = (-1)^n n\theta\left[(C_2 + D_2)(C_1 + D_1 + \kappa e\theta)^{-1}\right]^n, \quad n \geqslant 1.$$

Therefore, the nth derivative of the stationary arrival rate r_ε of the perturbed MAP is given by

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n} r_{\varepsilon|\varepsilon=0} = (-1)^{n-1} n!\theta\left[(C_2 + D_2)(C_1 + D_1 + \kappa e\theta)^{-1}\right]^{n-1} \\ \cdot\left\{D_2 e - \left[(C_2 + D_2)(C_1 + D_1 + \kappa e\theta)^{-1}\right] D_1 e\right\}, \quad n \geqslant 1.\end{aligned}$$

11.5.3 A Perturbed *MAP*/*PH*/1 Queue

Consider a *MAP*/*PH*/1 queue. We denote by $q(t)$, $I(t)$ and $J(t)$ the number of customers in the system, the phase numbers of the MAP input and the PH service time at time t, respectively. It is obvious that $\{q(t), I(t), J(t), t \geqslant 0\}$ is a level-independent QBD process whose infinitesimal generator is given by

$$\mathcal{Q}_1 = \begin{pmatrix} C_1 & D_1 \otimes \alpha & & & \\ I \otimes T^0 & C_1 \oplus T & D_1 \otimes I & & \\ & I \otimes T^0\alpha & C_1 \oplus T & D_1 \otimes I & \\ & & I \otimes T^0\alpha & C_1 \oplus T & D_1 \otimes I \\ & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{11.88}$$

Let θ and ω be the stationary probability vectors of the Markov chains $C_1 + D_1$ and $T + T^0\alpha$, respectively. Then the QBD process with infinitesimal generator given in Eq. (11.88) is positive recurrent, null recurrent or transient according to $\theta D_1 e < \omega T^0$, $\theta D_1 e = \omega T^0$ or $\theta D_1 e > \omega T^0$, respectively.

Now, we impose an infinitesimal perturbation on both the input stream and the service times, i.e., $(C_1 + \varepsilon C_2, D_1 + \varepsilon D_2)$ and $(\alpha, T + \varepsilon S)$, respectively. The infinitesimal generator of the perturbed QBD process $\{q(t), I(t), J(t), t \geqslant 0\}$ is given by

$$Q_\varepsilon = Q_1 + \varepsilon Q_2, \tag{11.89}$$

where Q_1 is given in Eq. (11.88) and

$$Q_2 = \begin{pmatrix} C_2 & D_2 \otimes \alpha & & & \\ I \otimes S^0 & C_2 \oplus S & D_2 \otimes I & & \\ & I \otimes S^0\alpha & C_2 \oplus S & D_2 \otimes I & \\ & & I \otimes S^0\alpha & C_2 \oplus S & D_2 \otimes I \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let R and R_ε be the minimal nonnegative solutions to the matrix equations

$$(D_1 \otimes I) + R(C_1 \oplus T) + R^2(I \otimes T^0\alpha) = 0$$

and

$$[(D_1 + \varepsilon D_2) \otimes I] + R_\varepsilon[(C_1 + \varepsilon C_2) \oplus (T + \varepsilon S)] + R_\varepsilon^2[I \otimes (T^0 + S^0)\alpha] = 0,$$

respectively. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_{\varepsilon|\varepsilon=0} = &-[(D_2 \otimes I) + R(C_2 \oplus S) + R^2(I \otimes S^0\alpha)] \\ &\cdot [(C_1 \oplus T) + 2R(I \otimes T^0\alpha)]^{-1}. \end{aligned}$$

Since $\varepsilon > 0$ is sufficiently small, the perturbed $MAP/PH/1$ queue will keep the properties of state classification for the original $MAP/PH/1$ queue. We now provide sensitivity analysis for the stationary queue length in the perturbed $MAP/PH/1$ queue as an example.

If $\theta D_1 e < \omega T^0$, then the perturbed $MAP/PH/1$ queue is positive recurrent. To analyze the sensitivity of the stationary queue length, we need the derivatives of the stationary probability vector of the perturbed QBD process, since $P\{L_\varepsilon = k\} = \pi_k(\varepsilon)e$ for $k \geqslant 0$, where $L_\varepsilon = \lim_{t \to +\infty} q_\varepsilon(t)$ almost everywhere.

Let $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ and $\pi(\varepsilon) = (\pi_0(\varepsilon), \pi_1(\varepsilon), \pi_2(\varepsilon), \ldots)$ be the stationary probability vectors of the two level-independent QBD processes with infinitesimal generators Q_1 given in Eq. (11.88) and Q_ε given in Eq. (11.89), respectively. Then

$$\pi_k(\varepsilon) = \pi_1(\varepsilon) R_\varepsilon^{k-1}, \quad k \geqslant 2, \tag{11.90}$$

and $(\pi_0(\varepsilon), \pi_1(\varepsilon))$ is the unique positive solution to the system of equations

$$\pi_0(\varepsilon)(C_1 + \varepsilon C_2) + \pi_1(\varepsilon)\left[I \otimes (T^0 + \varepsilon S^0) \right] = 0, \tag{11.91}$$

$$\begin{aligned} \pi_0(\varepsilon)[(D_1 + \varepsilon D_2) \otimes \alpha] + \pi_1(\varepsilon)\{(C_1 + \varepsilon C_2) \oplus (T + \varepsilon S) \\ + R_\varepsilon[I \otimes (T^0 + \varepsilon S^0)]\alpha\} = 0, \end{aligned} \tag{11.92}$$

and $\pi_0(\varepsilon)e + \pi_1(\varepsilon)(I - R_\varepsilon)^{-1} e = 1$. It is obvious that $\pi_l = \pi_l(0)$ for all $l \geqslant 0$.

It follows from Eq. (11.91) and Eq. (11.92) that

$$\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \pi_0(\varepsilon)_{|\varepsilon=0}, \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \pi_1(\varepsilon)_{|\varepsilon=0} \right) \begin{pmatrix} C_1 & D_1 \otimes \alpha \\ I \otimes T^0 & C_1 \oplus T + R(I \otimes T^0 \alpha) \end{pmatrix} = \pi_0 C_2, A,$$

where

$$A = \pi_0(D_2 \otimes \alpha) + \pi_1 \left[C_2 \oplus S + R(I \otimes S^0 \alpha) + \frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_{\varepsilon|\varepsilon=0}(I \otimes T^0 \alpha) \right].$$

Note that

$$\begin{pmatrix} C_1 & D_1 \otimes \alpha \\ I \otimes T^0 & C_1 \oplus T + R(I \otimes T^0 \alpha) \end{pmatrix}^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$\begin{aligned} A_{11} &= C_1^{-1} + C_1^{-1}(D_1 \otimes \alpha) B^{-1} (I \otimes T^0) C_1^{-1}, \\ A_{12} &= C_1^{-1}(D_1 \otimes \alpha) B^{-1}, \\ A_{21} &= B^{-1}(I \otimes T^0) C_1^{-1}, \\ A_{22} &= B^{-1}, \\ B &= I - \left[C_1 \oplus T + R(I \otimes T^0 \alpha) \right] - (I \otimes T^0) C_1^{-1} (D_1 \otimes \alpha), \end{aligned}$$

we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \pi_0(\varepsilon)_{|\varepsilon=0} &= \pi_0 C_2 A_{11} + A A_{21}, \\ \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \pi_1(\varepsilon)_{|\varepsilon=0} &= \pi_0 C_2 A_{12} + A A_{22}. \end{aligned} \tag{11.93}$$

It follows from Eq. (11.90) and Eq. (11.93) that for $k \geqslant 2$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \pi_k(\varepsilon)_{|\varepsilon=0} = (\pi_0 C_2 A_{12} + A A_{22}) R^{k-1} - (k-1) \pi_1 R^{k-2} [(D_2 \otimes I) + R(C_2 \oplus S) \\ + R^2 (I \otimes S^0 \alpha)][(C_1 \oplus T) + 2R(I \otimes T^0 \alpha)]^{-1}. \end{aligned} \tag{11.94}$$

Note that $E[L_\varepsilon] = \pi_1(\varepsilon)(I - R_\varepsilon)^{-2} e,$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} E[L_\varepsilon]_{|\varepsilon=0} = (\pi_0 C_2 A_{12} + A A_{22})(I-R)^{-2} e + 2\pi_1 (I-R)^{-3} [(D_2 \otimes I) + R(C_2 \oplus S) + R^2 (I \otimes S^0 \alpha)][(C_1 \oplus T) + 2R(I \otimes T^0 \alpha)]^{-1} e. \tag{11.95}$$

The derivatives in Eq. (11.93), Eq. (11.94) and Eq. (11.95) can be obtained once matrix R is calculated numerically.

As an illustration, we now consider a perturbed $M/M/1$ queue with perturbed arrival rate $\lambda + a\varepsilon$ and perturbed service rate $\mu + b\varepsilon$. Let $\rho = \lambda/\mu < 1$ and $\rho(\varepsilon) = (\lambda + a\varepsilon)/(\mu + b\varepsilon) < 1$. It is clear that the perturbed $M/M/1$ queue is stable if $\rho < 1$. Simple calculations show that $\pi_k(\varepsilon) = [1 - \rho(\varepsilon)]\rho(\varepsilon)^k$ for $k \geqslant 0$. Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \pi_k(\varepsilon)_{|\varepsilon=0} = \rho^{k-1}[k - (k+1)\rho]\frac{a - b\rho}{\mu}, \tag{11.96}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \pi_k(\varepsilon)_{|\varepsilon=0} = \rho^{k-2}\frac{a - b\rho}{\mu^2}\{[a(k-1) - b\rho(k+1)][k - (k+1)\rho] - \rho(k+1)(\alpha - b\rho)\}. \tag{11.97}$$

Since the mean of the stationary queue length in the perturbed $M/M/1$ queue is $E[L_\varepsilon] = \rho(\varepsilon)/[1 - \rho(\varepsilon)],$ we have

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n} E[L_\varepsilon]_{|\varepsilon=0} = (-1)^{n+1}\frac{n!(a\mu - b\lambda)}{(\mu - \lambda)^{n+1}}, \quad n \geqslant 1. \tag{11.98}$$

11.6 Symmetric Evolutionary Games

In this section, we apply the perturbed birth death processes to study a 2×2 symmetric evolutionary game, and obtain the evolutionary stable strategy (ESS) of the symmetric evolutionary game by means of the stationary probability distribution of the perturbed birth-death process. Note that a 2×2 evolutionary game contains 2 types of players, and each player has 2 strategies.

The evolutionary game is a useful mathematical tool developed by biologists for predicting population dynamics in the context of interactions. The Evolutionary Stable Strategy (ESS) is an important concept which is characterized by a property of robustness against invaders (or mutations). More specifically,

(1) if an ESS is reached, then the proportions of each population do not change

in time; and

(2) at the ESS, the populations are immune from being invaded by other small populations.

Obviously, the ESS is stronger than Nash equilibrium in which it is only requested that a single user would not benefit by a change (mutation) of its behavior.

Now, it is necessary to provide a mathematical description for the ESS, which will be useful in the rest of this section later.

Consider a large population of players. Each tagged individual needs to take some strategies with the strategy set K. We denote by $J(p, q)$ the expected payoff for our tagged individual if it uses a strategy p when meeting another individual who adopts the strategy q. In general, the payoff may be regarded as a fitness, and strategies with larger fitness are expected to propagate faster in a population. Let p and q belong to a set K of available strategies. In the standard framework for evolutionary games, there are a finite number of pure strategies, and a general strategy of an individual is a probability distribution over the pure strategies. Note that $J(p, q)$ is linear in p and q.

Suppose that the whole population uses a strategy q, and a small fraction $\varepsilon > 0$ (called mutations) adopts another strategy p. Evolutionary forces are expected to select against p if

$$J(q, \varepsilon p + (1-\varepsilon)q) > J(p, \varepsilon p + (1-\varepsilon)q).$$

A strategy q is said to be ESS if for every $p \neq q$ there exists some $\overline{\varepsilon} > 0$ such that the above inequality holds for all $\varepsilon \in (0, \overline{\varepsilon})$.

In fact, we expect that if for all $p \neq q$,

$$J(q,q) > J(q,p).$$

This indicates that the mutation fraction in the population will tend to decrease, since it has a lower reward which leads to a lower growth rate. Thus the strategy q is immune to the mutations. On the other hand, if for all $p \neq q$, $J(q,q) = J(p,q)$ and $J(q,p) > J(p,p)$, then a population using strategy q are weakly immune against a mutation using strategy p. In this case, if the mutant's population grows, then we shall frequently have individuals with strategy q competing with the mutations, since the condition $J(q,p) > J(p,p)$ ensures that the growth rate of the original population exceeds that of the mutations.

Based on the above analysis, a strategy q is said to be ESS if for all $p \neq q$,

$$J(q,q) > J(q,p);$$

or

$$J(q,q) = J(p,q) \text{ and } J(q,p) > J(p,p).$$

In the remainder of this section, we use the perturbed Markov chains to study the ESS for some symmetric evolutionary games and asymmetric evolutionary games.

We consider a 2×2 symmetric evolutionary game whose payoff matrix is given by

$$A = \begin{array}{cc} & \begin{array}{cc} \sigma_1 & \sigma_2 \end{array} \\ \begin{array}{c} \sigma_1 \\ \sigma_2 \end{array} & \begin{pmatrix} (a,a) & (b,c) \\ (c,b) & (d,d) \end{pmatrix} \end{array}, \tag{11.99}$$

where σ_1 and σ_2 denote the two strategies for each player. We assume that the populations of players of types 1 and 2 are both N. For simplicity of description, we may only study the players of type 1 with the payoff matrix, given by

$$A = \begin{array}{cc} & \begin{array}{cc} \sigma_1 & \sigma_2 \end{array} \\ \begin{array}{c} \sigma_1 \\ \sigma_2 \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}. \tag{11.100}$$

In fact, it is seen from Eq. (11.99) that the payoff matrix for players of type 2 is given by

$$B = \begin{array}{cc} & \begin{array}{cc} \sigma_1 & \sigma_2 \end{array} \\ \begin{array}{c} \sigma_1 \\ \sigma_2 \end{array} & \begin{pmatrix} a & c \\ b & d \end{pmatrix} \end{array}.$$

We denote by $z(t)$ the number of players of type 1 who are playing strategy σ_1 at time t. Then $z(t)\in S=\{0,1,2,\ldots,N-1,N\}$. Using the total probability law, the profitability of choosing strategies σ_1 and σ_2 can be defined as

$$f_{\sigma_1}(z)=\frac{z}{N}a+\frac{N-z}{N}b \tag{11.101}$$

and

$$f_{\sigma_2}(z)=\frac{z}{N}c+\frac{N-z}{N}d, \tag{11.102}$$

respectively. Note that the players switch from one strategy to another according to the bounded rationality, using the two functions $f_{\sigma_1}(z)$ and $f_{\sigma 2}(z)$, the state transition rates are defined as

$$\lambda_i(\varepsilon)=\varepsilon+\kappa\max\{f_{\sigma_1}(i)-f_{\sigma_2}(i),0\},\quad 0\leqslant i\leqslant N-1, \tag{11.103}$$

and

$$\mu_i(\varepsilon)=\varepsilon+\kappa\max\{f_{\sigma_2}(i)-f_{\sigma_1}(i),0\},\quad 1\leqslant i\leqslant N, \tag{11.104}$$

where ε is the exogenous mutation rate, κ estimates in some sense the speed at which the boundedly rational players react to their environment, thus κ is called the learning ability rate of the players.

Based on Eq. (11.103) and Eq. (11.104), it is seen from the average meaning that $\{z(t), t \geqslant 0\}$ is a continuous-time birth death process on state space S whose infinitesimal generator is given by

$$Q_{\varepsilon}=\begin{pmatrix} -\lambda_0(\varepsilon) & \lambda_0(\varepsilon) & & & & \\ \mu_1(\varepsilon) & -\gamma_1(\varepsilon) & \lambda_1(\varepsilon) & & & \\ & \mu_2(\varepsilon) & -\gamma_2(\varepsilon) & \lambda_2(\varepsilon) & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{N-1}(\varepsilon) & -\gamma_{N-1}(\varepsilon) & \lambda_{N-1}(\varepsilon) \\ & & & & \mu_N(\varepsilon) & -\mu_N(\varepsilon) \end{pmatrix}, \tag{11.105}$$

where

$$\gamma_i(\varepsilon)=\lambda_i(\varepsilon)+\mu_i(\varepsilon), \quad 1 \leqslant i \leqslant N-1.$$

We assume that the perturbation parameter $\varepsilon > 0$ is small enough, it is clear that $\lambda_i(\varepsilon) > 0$ for $0 \leqslant i \leqslant N-1$ and $\mu_j(\varepsilon) > 0$ for $1 \leqslant j \leqslant N$, so that the QBD process Q_ε is irreducible. Since the state space is finite and $Q_\varepsilon e = 0$, the QBD process Q_ε is postive recurrent. Let

$$\pi^{\varepsilon}=\left(\pi_0^{\varepsilon}, \pi_1^{\varepsilon}, \pi_2^{\varepsilon}, \ldots, \pi_{N-1}^{\varepsilon}, \pi_N^{\varepsilon}\right)$$

be the stationary probability vector of the QBD process Q_ε. We write

$$\xi_0^{\varepsilon}=1$$

and

$$\xi_k^{\varepsilon}=\frac{\lambda_0(\varepsilon)\lambda_1(\varepsilon)\ldots\lambda_{k-1}(\varepsilon)}{\mu_1(\varepsilon)\mu_2(\varepsilon)\ldots\mu_k(\varepsilon)}. \tag{11.106}$$

Then

$$\pi_n^{\varepsilon}=\frac{\xi_n^{\varepsilon}}{\sum\limits_{k=0}^{N}\xi_k^{\varepsilon}}, \quad 0 \leqslant n \leqslant N. \tag{11.107}$$

It follows from Eq. (11.101) and Eq. (11.102) that

$$f_{\sigma_1}(z)=\frac{a-b}{N}z+b \tag{11.108}$$

and

$$f_{\sigma_2}(z)=\frac{c-d}{N}z+d, \tag{11.109}$$

respectively. Thus, $f_{\sigma_1}(z)$ and $f_{\sigma_2}(z)$ are both linear functions to z. Based on the two linear functions, we can distinguish three cases as follows.

Case Ⅰ The two lines cross in the interval $[0, N]$.

In this case, there exists a unique $z^* \in [0, N]$ such that

$$f_{\sigma_1}(z^*) = f_{\sigma_2}(z^*),$$

which leads to

$$z^* = \frac{d-b}{(a-c)+(d-b)}N.$$

Let i^* be the largest integer part of z^*. Then $i^* \in \{0,1,2,\dots,N-1,N\}$ and $i^* \leqslant z^*$. We write

$$a_i = \begin{cases} \kappa\left[f_{\sigma_2}(i) - f_{\sigma_1}(i)\right], & \text{if } i \leqslant \dfrac{N-1}{2}, \\ \kappa\left[f_{\sigma_1}(i) - f_{\sigma_2}(i)\right], & \text{if } i > \dfrac{N-1}{2}, \end{cases}$$

$$B_N = \frac{a_{\frac{N-1}{2}+1} a_{\frac{N-1}{2}+2} \dots a_{N-1}}{a_1 a_2 \dots a_{\frac{N-1}{2}}}$$

and

$$\alpha = \frac{a_0 a_1 \dots a_{i^*-1}}{a_0 a_1 \dots a_{i^*-1}\left(1 + \dfrac{a_{i^*}}{a_{i^*+1}}\right)}.$$

The following theorem describes the stationary probability vector π^ε in Case Ⅰ.

Theorem 11.1 (1) *If* $b \leqslant d$, *then the stationary probability vector* π^ε *puts probability* 1 *on state* N *when* $i^* < (N-2)/2$, *on state 0 when* $i^* > (N-2)/2$. *It puts probability* $1/(1+B_N)$ *on state* 0 *and* $B_N/(1+B_N)$ *on state* N *when* $i^* = (N-2)/2$.

(2) *If* $b > d$, *then the stationary probability vector* π^ε *puts probability* α *on state* i^* *and probability* $1-\alpha$ *on state* i^*+1.

Proof (1) If $b \leqslant d$, then it follows from Eq. (11.101) and Eq. (11.102) that $f_{\sigma_1}(0) \leqslant f_{\sigma_2}(0)$. Note that $f_{\sigma_1}(z)$ and $f_{\sigma_2}(z)$ are linear functions, we obtain that $f_{\sigma_1}(i) \leqslant f_{\sigma_2}(i)$ when $i \leqslant i^*$ and $f_{\sigma_1}(j) > f_{\sigma_2}(j)$ when $j > i^*$. Let

$$a_i = \begin{cases} \kappa\left[f_{\sigma_2}(i) - f_{\sigma_1}(i)\right], & \text{if } i \leqslant i^*, \\ \kappa\left[f_{\sigma_1}(i) - f_{\sigma_2}(i)\right], & \text{if } i > i^*. \end{cases}$$

Then $\lambda_i(\varepsilon)=\varepsilon, \mu_i(\varepsilon)=\varepsilon+a_i$ for $i \leqslant i^*$, and $\lambda_j(\varepsilon)=\varepsilon+a_j, \mu_j(\varepsilon)=\varepsilon$ for $j>i^*$.
We write

$$A_j(\varepsilon)=\frac{1}{(\varepsilon+a_1)(\varepsilon+a_2)\ldots(\varepsilon+a_j)}$$

and

$$B_j(\varepsilon)=\frac{(\varepsilon+a_{i^*+1})(\varepsilon+a_{i^*+2})\ldots(\varepsilon+a_{j-1})}{(\varepsilon+a_1)(\varepsilon+a_2)\ldots(\varepsilon+a_{i^*})}.$$

Therefore, we have

$$\xi_0^\varepsilon=1$$

and

$$\xi_j^\varepsilon=\begin{cases}\varepsilon^j A_j(\varepsilon), & \text{for } 1\leqslant j\leqslant i^*-1,\\ \varepsilon^{i^*}A_{i^*}(\varepsilon), & \text{for } j=i^*, i^*+1,\\ \varepsilon^{2i^*-j+1}B_j(\varepsilon), & \text{for } j\geqslant i^*+2.\end{cases} \tag{11.110}$$

For simplicity of description, we suppress the dependence of $A_j(\varepsilon)$ and $B_j(\varepsilon)$ on ε, and write $A_j(\varepsilon)=A_j$ and $B_j(\varepsilon)=B_j$. In these notation, it follows from Eq. (11.107) that

$$\pi_j^\varepsilon=\begin{cases}\dfrac{1}{1+A_1\varepsilon+\ldots+A_{i^*}\varepsilon^{i^*}+B_{i^*+2}\varepsilon^{i^*-1}+\ldots+B_N\varepsilon^{2i^*-N+1}}, & \text{for } j=0,\\ \dfrac{\varepsilon^j A_j(\varepsilon)}{1+A_1\varepsilon+\ldots+A_{i^*}\varepsilon^{i^*}+B_{i^*+2}\varepsilon^{i^*-1}+\ldots+B_N\varepsilon^{2i^*-N+1}}, & \text{for } 1\leqslant j\leqslant i^*+1,\\ \dfrac{\varepsilon^{2i^*-j+1}B_j(\varepsilon)}{1+A_1\varepsilon+\ldots+A_{i^*}\varepsilon^{i^*}+B_{i^*+2}\varepsilon^{i^*-1}+\ldots+B_N\varepsilon^{2i^*-N+1}}, & \text{for } i^*+2\leqslant j\leqslant N.\end{cases} \tag{11.111}$$

If $i^*>(N-2)/2$, it is clear from Eq. (11.111) that

$$\lim_{\varepsilon\to 0}\pi_j^\varepsilon=\begin{cases}1, & \text{for } j=0,\\ 0, & \text{for } 1\leqslant j\leqslant N.\end{cases}$$

If $i^*<(N-2)/2$, it is seen from Eq. (11.111) that

$$\lim_{\varepsilon\to 0}\pi_j^{\varepsilon}=\begin{cases}0, & \text{for } 0\leqslant j\leqslant N-1,\\ 1, & \text{for } j=N.\end{cases}$$

If $i^*=(N-2)/2$, it is clear that N is odd and

$$\xi_N^{\varepsilon}=\varepsilon^{2i^*-N+1}B_N(\varepsilon)=B_N(\varepsilon).$$

Based on Eq. (11.111), we obtain

$$\pi_0^{\varepsilon}=\frac{1}{1+A_1\varepsilon+\ldots+A_{i^*}\varepsilon^{i^*}+B_{i^*+2}\varepsilon^{i^*-1}+\ldots+B_{N-1}\varepsilon+B_N},$$

which leads to

$$\lim_{\varepsilon\to 0}\pi_0^{\varepsilon}=\frac{1}{1+B_N(0)}.$$

It is obvious from Eq. (11.111) that

$$\lim_{\varepsilon\to 0}\pi_j^{\varepsilon}=0,\quad 1\leqslant j\leqslant N-1.$$

Note that

$$\pi_N^{\varepsilon}=\frac{B_N}{1+A_1\varepsilon+\ldots+A_{i^*}\varepsilon^{i^*}+B_{i^*+2}\varepsilon^{i^*-1}+\ldots+B_{N-1}\varepsilon+B_N},$$

we obtain

$$\lim_{\varepsilon\to 0}\pi_N^{\varepsilon}=\frac{B_N(0)}{1+B_N(0)}.$$

(2) If $b>d$, then it follows from Eq. (11.101) and Eq. (11.102) that $f_{\sigma_1}(0)>f_{\sigma_2}(0)$, we obtain that $f_{\sigma_1}(i)>f_{\sigma_2}(i)$ when $i\leqslant i^*$ and $f_{\sigma_1}(j)<f_{\sigma_2}(j)$ when $j>i^*$. Let

$$a_i=\begin{cases}\kappa\left[f_{\sigma_1}(i)-f_{\sigma_2}(i)\right], & \text{if } i\leqslant i^*,\\ \kappa\left[f_{\sigma_2}(i)-f_{\sigma_1}(i)\right], & \text{if } i\geqslant i^*+1.\end{cases}$$

Then $\lambda_i(\varepsilon)=\varepsilon+a_i$, $\mu_i(\varepsilon)=\varepsilon$, for $i\leqslant i^*$, and $\lambda_i(\varepsilon)=\varepsilon$, $\mu_j(\varepsilon)=\varepsilon+a_j$ for $j\geqslant i^*+1$. We write

$$\xi_0^{\varepsilon}=1$$

and

$$\xi_j^\varepsilon = \begin{cases} \dfrac{(\varepsilon+a_0)(\varepsilon+a_1)\ldots(\varepsilon+a_{j-1})}{\varepsilon^j}, & \text{for } 1 \leqslant j \leqslant i^*, \\ \dfrac{(\varepsilon+a_0)(\varepsilon+a_1)\ldots(\varepsilon+a_{i^*})}{\varepsilon^{i^*}(\varepsilon+a_{i^*+1})}, & \text{for } j = i^*+1, \\ \dfrac{(\varepsilon+a_0)(\varepsilon+a_1)\ldots(\varepsilon+a_{i^*})}{\varepsilon^{2i^*-j+1}(\varepsilon+a_{i^*+1})\ldots(\varepsilon+a_j)} & \text{for } j \geqslant i^*+2. \end{cases} \tag{11.112}$$

Using Eq. (11.110), we easily obtain

$$\lim_{\varepsilon\to 0}\pi_j^\varepsilon = \begin{cases} \alpha, & \text{for } j = i^*, \\ 1-\alpha, & \text{for } j = i^*+1, \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof.

The following theorem further provides some useful properties for the stationary probability vector π^ε in Case Ⅰ.

Theorem 11.2 *For any $\varepsilon > 0$, the stationary probability vector π^ε satisfies the following properties:*

(1) *If $b \leqslant d$, then*

$$\pi_0^\varepsilon > \pi_1^\varepsilon > \ldots > \pi_{i^*}^\varepsilon$$

and

$$\pi_{i^*+1}^\varepsilon < \pi_{i^*+2}^\varepsilon < \ldots < \pi_N^\varepsilon.$$

At the same time, π_j^ε attains its minimum at $j = i^*$ and $j = i^*+1$, and $\pi_{i^*}^\varepsilon = \pi_{i^*+1}^\varepsilon$.

(2) *If $b > d$, then*

$$\pi_0^\varepsilon < \pi_1^\varepsilon < \ldots < \pi_{i^*}^\varepsilon,$$

$$\pi_{i^*+1}^\varepsilon > \pi_{i^*+2}^\varepsilon > \ldots > \pi_N^\varepsilon,$$

and $\pi_{i^*}^\varepsilon \leqslant \pi_{i^*+1}^\varepsilon$ if and only if $z^* \geqslant (2i^*+1)/2$. At the same time, π_j^ε attains its maximum at $j = i^*$ if $z^* < (2i^*+1)/2$ and at $j = i^*+1$ if $z^* > (2i^*+1)/2$.

Proof We first prove (1)

For $i \leqslant i^*-2$, it follows from Eq. (11.110) that

$$\xi_{i+1}^\varepsilon = \frac{\varepsilon^{i+1}}{(\varepsilon+a_1)(\varepsilon+a_2)\ldots(\varepsilon+a_i)(\varepsilon+a_{i+1})} = \xi_i^\varepsilon \frac{\varepsilon}{\varepsilon+a_{i+1}}.$$

Note that $\varepsilon/(\varepsilon+a_{i+1}) < 1$ when $a_{i+1} > 0$, we obtain $\xi_{i+1}^\varepsilon < \xi_i^\varepsilon$, which leads to

$$\pi_0^\varepsilon > \pi_1^\varepsilon > \ldots > \pi_{i^*}^\varepsilon.$$

For $i \geqslant i^* + 2,$ it follows from Eq. (11.110) that

$$\xi_{i+1}^\varepsilon = \frac{\varepsilon^{i^*+1}(\varepsilon + a_{i^*+1})(\varepsilon + a_{i^*+2})\ldots(\varepsilon + a_i)}{\varepsilon^{i+1-i^*}(\varepsilon + a_1)(\varepsilon + a_2)\ldots(\varepsilon + a_{i^*})} = \xi_i^\varepsilon \frac{\varepsilon + a_i}{\varepsilon}.$$

Note that $(\varepsilon + a_i)/\varepsilon > 1$ when $a_{i+1} > 0,$ we obtain $\xi_{i+1}^\varepsilon > \xi_i^\varepsilon,$ which leads to

$$\pi_{i^*+1}^\varepsilon < \pi_{i^*+2}^\varepsilon < \ldots < \pi_N^\varepsilon.$$

For $i = i^*, i^* + 1,$ it follows from Eq. (11.110) that

$$\xi_i^\varepsilon = \frac{\varepsilon^{i^*}}{(\varepsilon + a_1)(\varepsilon + a_2)\ldots(\varepsilon + a_{i^*})},$$

which shows that $\xi_{i^*}^\varepsilon = \xi_{i^*+1}^\varepsilon,$ which yields that $\pi_{i^*}^\varepsilon = \pi_{i^*+1}^\varepsilon.$

Now, We prove (2)

For $i \leqslant i^* - 1,$ it follows from Eq. (11.110) that

$$\begin{aligned}\xi_{i+1}^\varepsilon &= \frac{(\varepsilon + a_0)(\varepsilon + a_1)\ldots(\varepsilon + a_{i-1})(\varepsilon + a_i)}{\varepsilon^{i+1}} \\ &= \xi_i^\varepsilon \frac{\varepsilon + a_i}{\varepsilon} > \xi_i^\varepsilon,\end{aligned}$$

thus we obtain

$$\pi_0^\varepsilon < \pi_1^\varepsilon < \ldots < \pi_{i^*}^\varepsilon.$$

For $i \geqslant i^* + 1,$ it follows from Eq. (11.112) that

$$\begin{aligned}\xi_{i+1}^\varepsilon &= \frac{(\varepsilon + a_0)(\varepsilon + a_1)\ldots(\varepsilon + a_{i^*})}{\varepsilon^{2i^*-i}(\varepsilon + a_{i^*+1})\ldots(\varepsilon + a_{i+1})} \\ &= \xi_i^\varepsilon \frac{\varepsilon}{\varepsilon + a_{i+1}} < \xi_i^\varepsilon,\end{aligned}$$

we get

$$\pi_{i^*+1}^\varepsilon > \pi_{i^*+2}^\varepsilon > \ldots > \pi_N^\varepsilon.$$

It follows from Eq. (11.112) that

$$\xi_{i^*+1}^\varepsilon = \xi_{i^*}^\varepsilon \frac{\varepsilon + a_{i^*}}{\varepsilon + a_{i^*+1}}.$$

Since $a_{i^*} \leqslant a_{i^*+1}$ if and only if $z^* \geqslant (2i^*+1)/2$, $\pi_{i^*}^{\varepsilon} \leqslant \pi_{i^*+1}^{\varepsilon}$ if and only if $z^* \geqslant (2i^*+1)/2$. Hence, it is clear that π_j^{ε} attains its maximum at $j = i^*$ if $z^* < (2i^*+1)/2$ and at $j = i^*+1$ if $z^* > (2i^*+1)/2$. This completes the proof.

Case Ⅱ The two lines do not cross in the interval $[0, N]$, that is, there exists no $z^* \in [0, N]$ such that $f_{\sigma_1}(z^*) = f_{\sigma_2}(z^*)$.

In this case, note that $f_{\sigma_1}(z)$ and $f_{\sigma_2}(z)$ are both linear functions to *z*, we have two different classes: $f_{\sigma_1}(z) > f_{\sigma_2}(z)$ and $f_{\sigma_1}(z) < f_{\sigma_2}(z)$.

The following theorem provides some useful properties for the stationary probability vector π^{ε} in Case Ⅱ.

Theorem 11.3 (1) *If* $b \leqslant d$, *then for any* $\varepsilon > 0$,

$$\pi_0^{\varepsilon} < \pi_1^{\varepsilon} < \ldots < \pi_N^{\varepsilon},$$

and for $1 \leqslant i \leqslant N-1$,

$$\begin{cases} \pi_{i+1}^{\varepsilon} - \pi_i^{\varepsilon} > \pi_i^{\varepsilon} - \pi_{i-1}^{\varepsilon}, & \text{if } i^* < 0, \\ \pi_{i+1}^{\varepsilon} - \pi_i^{\varepsilon} < \pi_i^{\varepsilon} - \pi_{i-1}^{\varepsilon}, & \text{if } i^* > N. \end{cases}$$

At the same time, we have

$$\lim_{\varepsilon \to 0} \pi_i^{\varepsilon} = \begin{cases} 0, & \text{for } 0 \leqslant i \leqslant N-1, \\ 1, & \text{for } i = N. \end{cases}$$

(2) *If* $b > d$, *then for any* $\varepsilon > 0$,

$$\pi_0^{\varepsilon} > \pi_1^{\varepsilon} > \ldots > \pi_N^{\varepsilon},$$

and for $1 \leqslant i \leqslant N-1$,

$$\begin{cases} \pi_{i+1}^{\varepsilon} - \pi_i^{\varepsilon} < \pi_i^{\varepsilon} - \pi_{i-1}^{\varepsilon}, & \text{if } i^* < 0, \\ \pi_{i+1}^{\varepsilon} - \pi_i^{\varepsilon} > \pi_i^{\varepsilon} - \pi_{i-1}^{\varepsilon}, & \text{if } i^* > N. \end{cases}$$

At the same time, we have

$$\lim_{\varepsilon \to 0} \pi_i^{\varepsilon} = \begin{cases} 1, & \text{for } i = 0, \\ 0, & \text{for } 1 \leqslant i \leqslant N. \end{cases}$$

Proof We only need to prove (1), while (2) can be proved similarly.

If $b \leqslant d$, then for any $\varepsilon > 0$, $f_{\sigma_1}(z) \leqslant f_{\sigma_2}(z)$ for each $z \in [0, N]$. Since there exists no $z^* \in [0, N]$ such that $f_{\sigma_1}(z^*) = f_{\sigma_2}(z^*)$, we obtain $f_{\sigma_1}(z) < f_{\sigma_2}(z)$ for each $z \in [0, N]$. In this case,

$$a_i = \kappa[f_{\sigma_2}(i) - f_{\sigma_1}(i)], \quad \text{for } 0 \leqslant i \leqslant N,$$

which leads to that $\lambda_i(\varepsilon)=\varepsilon$, $\mu_i(\varepsilon)=\varepsilon+a_i$ for $0\leqslant i\leqslant N$. It follows from Eq. (11.110) that for $0\leqslant i\leqslant N$,

$$\begin{aligned}\xi_{i+1}^{\varepsilon}&=\frac{(\varepsilon+a_0)(\varepsilon+a_1)\ldots(\varepsilon+a_{i-1})(\varepsilon+a_i)}{\varepsilon^{i+1}}\\&=\xi_i^{\varepsilon}\frac{(\varepsilon+a_i)}{\varepsilon}>\xi_i^{\varepsilon},\end{aligned}$$

thus we obtain

$$\pi_0^{\varepsilon}<\pi_1^{\varepsilon}<\ldots<\pi_N^{\varepsilon}.$$

We consider the case with $i^*<0$. Note that for any $\varepsilon>0$, $f_{\sigma_1}(z)$ and $f_{\sigma_2}(z)$ are linear functions, and $f_{\sigma_1}(z)<f_{\sigma_2}(z)$ for each $z\in[0,N]$, it is easy to see that $a_i=f_{\sigma_2}(i)-f_{\sigma_1}(i)$ is increasing in $i=0,1,2,\ldots,N$. Since

$$\xi_{i+1}^{\varepsilon}-\xi_i^{\varepsilon}=\xi_i^{\varepsilon}\frac{a_i}{\varepsilon},$$

it is easy to see that $\xi_{i+1}^{\varepsilon}-\xi_i^{\varepsilon}$ is increasing in $i=0,1,2,\ldots,N$. Therefore, $\pi_{i+1}^{\varepsilon}-\xi_{i+1}^{\varepsilon}-\xi_i^{\varepsilon}$ is increasing in $i=0,1,2,\ldots,N$.

Similarly, we can analyze the case with $i^*>N$. The details of this proof are omitted here.

Note that when $\lambda_i(\varepsilon)=\varepsilon$, $\mu_i(\varepsilon)=\varepsilon+a_i$ for $0\leqslant i\leqslant N$, it follows from Eq. (11.111) for $0\leqslant i\leqslant N$,

$$\pi_j^{\varepsilon}=\begin{cases}\dfrac{1}{1+A_1\varepsilon+\ldots+A_{i^*}\varepsilon^{i^*}+B_{i^*+2}\varepsilon^{i^*-1}+\ldots+B_N\varepsilon^{2i^*-N+1}}, & \text{for } j=0,\\[2ex] \dfrac{\varepsilon^j A_j(\varepsilon)}{1+A_1\varepsilon+\ldots+A_{i^*}\varepsilon^{i^*}+B_{i^*+2}\varepsilon^{i^*-1}+\ldots+B_N\varepsilon^{2i^*-N+1}}, & \text{for } 1\leqslant j\leqslant N.\end{cases}$$

it is clear that

$$\lim_{\varepsilon\to0}\pi_i^{\varepsilon}=\begin{cases}1, & \text{for } i=0,\\0, & \text{for } 1\leqslant i\leqslant N.\end{cases}$$

This completes the proof.

Case Ⅲ The two lines are identical in the interval $[0,N]$, that is, the two functions $f_{\sigma_1}(z)$ and $f_{\sigma_2}(z)$ are equal for all $z=0,1,2,\ldots,N$.

The following theorem provides some useful properties for the stationary probability vector π^{ε} in Case Ⅲ.

Theorem 11.4 *For any* $\varepsilon > 0$ *and each* $i = 0,1,2,\ldots,N$,

$$\pi_i^{\varepsilon} = \frac{1}{N}$$

and

$$\lim_{\varepsilon \to 0} \pi_i^{\varepsilon} = \frac{1}{N}.$$

Proof Note that the two functions $f_{\sigma_1}(z)$ and $f_{\sigma_2}(z)$ are equal for all $z = 0,1,2,\ldots,N$, thus it is seen that $\lambda_i(\varepsilon) = \varepsilon$, $\mu_i(\varepsilon) = \varepsilon$ for $0 \leqslant i \leqslant N$. It follows from Eq. (11.106) that for each $i = 0,1,2,\ldots,N$,

$$\xi_i^{\varepsilon} = 1,$$

which, together with Eq. (11.107), leads to that for any $\varepsilon > 0$,

$$\pi_i^{\varepsilon} = \frac{1}{N}.$$

Therefore,

$$\lim_{\varepsilon \to 0} \pi_i^{\varepsilon} = \frac{1}{N}.$$

This completes the proof.

For the 2×2 symmetric evolutionary game under the above three cases, we can provide further analysis from the payoff matrix given in Eq. (11.100). We summarize the following three classes.

Class 1 The class of a strictly dominant strategy:

(1) If $a > c$ and $b > d$, then strategy σ_1 strictly dominates strategy σ_2.

(2) If $a < c$ and $b < d$, then strategy σ_2 strictly dominates strategy σ_1.

Class 2 The coordination class (when $a > c$ and $b < d$) with its two pure Nash equilibria (a,a) and (d,d) and one mixed-strategy equilibrium $p^* = (d-b)/[(a-c)+(d-b)]$.

Class 3 When $a < c$ and $b > d$, there is no pure-strategy symmetric Nash equilibrium. The game has a unique equilibrium in mixed-strategy equilibrium $p^* = (d-b)/[(a-c)+(d-b)]$.

It is easy to see the risk dominance of strategy σ_1 (resp. strategy σ_2), that is, $a-c > d-b$ (resp. $a-c < d-b$), is equivalent to $z^* < N/2$ (resp. $z^* > N/2$).

The following definition provides a slightly more risk restrictive concept.

Definition 11.1 *Strategy* σ_1 *is called strictly more dominant than strategy* σ_2 *if* $z^* < (N-1)/2$. *On the other hand, Strategy* σ_2 *is called strictly more risk dominant than strategy* σ_1 *if* $z^* > (N+1)/2$.

Theorem 11.5 (1) *In class 1, the stationary distribution* π^{ε} *puts probability 1 on the dominant strategy.*

(2) *In class 2, the stationary distribution* π^{ε} *puts probability 1 on the pure Nash equilibrium that is strictly more risk dominant.*

(3) *In class 3, the expected value of* $\lim_{\varepsilon\to 0}\pi^{\varepsilon}$ *is equal to* p^*, *that is,*

$$\sum_{k=0}^{N} k \lim_{\varepsilon\to 0}\pi_k^{\varepsilon} = (d-b)/[(a-c)+(d-b)].$$

Proof (1) In class 1, when $a > c$ and $b > d$, it follows from Eq. (11.101) and Eq. (11.102) that for each $z \in [0, N]$,

$$f_{\sigma_1}(z) - f_{\sigma_2}(z) = \frac{z}{N}(a-c) + \frac{N-z}{N}(b-d) > 0.$$

Hence, using (2) in Theorem 11.3 we can obtain the desired result.

(2) In class 2, when $b < d$, it follows from Eq. (11.108) and Eq. (11.109) that $f_{\sigma_1}(0) < f_{\sigma_2}(0)$, and

$$z^* = \frac{(d-b)}{(a-c)+(d-b)}N.$$

Note that $a > c$, we obtain

$$\frac{(d-b)}{(a-c)+(d-b)} < 1$$

which leads to $z^* \in [0, N]$. As seen above, the inequality $z^* < N/2$ is equivalent to saying that the pure Nash equilibrium (a, a) is risk dominant. But, in this case it might happen that $i^* = (N-1)/2$, which does not imply by Theorem 11.1 that the stationary distribution π^{ε} puts probability 1 on state N. On the contrary, if we assume that strategy σ_1 is strictly more dominant than strategy σ_2, then the inequality $i^* < (N-1)/2$ follows, which implies the desired result. Likewise, we can analyze the case with strategy σ_2 which is strictly more dominant than strategy σ_1.

(3) When $a < c$ and $b > d$, we can obatin

$$\frac{(i^*+1) - z^*}{z^* - i^*} = \frac{\alpha}{1-\alpha},$$

which implies the desired result. This completes the proof.

It is necessary to consider the following two borderline cases: (1) When $a = c$ and $b = d$, the two functions $f_{\sigma_1}(z)$ and $f_{\sigma_2}(z)$ are equal for all $z = 0, 1, 2, \ldots, N$, the two strategies are indistinguishable, thus Theorem 11.4 indicates that the stationary distribution π^{ε} is uniform on the state space $S=\{0,1,2,\ldots,N\}$. (2) When $a-c=b-d$, then $z^{*}=1/2$, using Theorem 11.1 we obtain that the stationary distribution π^{ε} puts probability $1/2$ on state 0 and probability $1/2$ on state N.

11.7 Constructively Perturbed Birth Death Process

In this section, we further construct the perturbed birth death process to study the ESS of the 2×2 symmetric evolutionary game.

11.7.1 An Embedded Chain

For the 2×2 symmetric evolutionary game, based on Eq. (11.100) we write

$$f_{\sigma_1}(z)=\frac{z-1}{N-1}a+\frac{N-z}{N-1}b \tag{11.113}$$

and

$$f_{\sigma_2}(z)=\frac{z}{N-1}c+\frac{N-1-z}{N-1}d. \tag{11.114}$$

It is easy to see from Eq. (11.113) that at least one player can play strategy σ_1, and from Eq. (11.114) that at least one player can play strategy σ_2. Under the two conditions, the best-response decision rule can be defined as follows:

(1) when playing strategy σ_1, switch to strategy σ_2 if $f_{\sigma_1}(z)<f_{\sigma_2}(z-1)$; and

(2) when playing strategy σ_2, switch to strategy σ_1 if $f_{\sigma_1}(z+1)>f_{\sigma_2}(z)$.

From $f_{\sigma_1}(z)<f_{\sigma_2}(z-1)$, it follows from Eq. (11.113) and Eq. (11.114) that

$$z<n_{\sigma_1}=(N-1)\alpha^{*}+1,$$

while using $f_{\sigma_1}(z+1)>f_{\sigma_2}(z)$ we have

$$z>n_{\sigma_2}=(N-1)\alpha^{*},$$

where

$$\alpha^{*}=\frac{d-b}{a-c+d-b}.$$

Therefore, we obtain that

(1) when playing strategy σ_1, switch to strategy σ_2 if $z<n_{\sigma_1}$; and

(2) when playing strategy σ_2, switch to strategy σ_1 if $z>n_{\sigma_2}$.

It is clear that $n_{\sigma_1} - n_{\sigma_2} = 1$. Hence the two thresholds n_{σ_1} and n_{σ_2} are closely linked to the mixed equilibrium strategy α^* with $n_{\sigma_2}/N \leqslant \alpha^* \leqslant n_{\sigma_1}/N$.

In what follows, we shall consider two different cases based on whether or nor n_{σ_1} (or n_{σ_2}) is an integer.

Case I n_{σ_2} (or $n_{\sigma_1} = n_{\sigma_2} + 1$) is not an integer.

In this case, we denote by n^* the unique integer between n_{σ_1} and n_{σ_2}. Obviously, n^* can be taken to be an integer approximation to the mixed equilibrium strategy α^*. Therefore, when playing strategy σ_1, switch to strategy σ_2 if $z \leqslant n^*$; and when playing strategy σ_2, switch to strategy σ_1 if $z > n^*$.

Now, we consider the dynamics. In each period, one player is sampled at uniformly random from the population and receives the opportunity with probabilities z/N (for σ_1) and $(N-z)/N$ (for σ_2) to revise his strategy. With probability $1-\varepsilon$, the selected player takes the action prescribed by the myopic best reply. With probability ε for $0 < \varepsilon < 1/2$, he mutates and takes the opposite action.

Let $p_{i,j}(\varepsilon)$ be the probability of transition from a state i to another state j. Clearly, if $p_{i,j}(\varepsilon) = 0$ whenever $|i-j| > 1$, then the Markov chain $P(\varepsilon) = (p_{i,j}(\varepsilon))$ is a birth death process. We take

$$p_{n,n-1}(\varepsilon) = \begin{cases} (1-\varepsilon)\dfrac{n}{N}, & 1 \leqslant n \leqslant n^*, \\ \varepsilon\dfrac{n}{N}, & n^*+1 \leqslant n \leqslant N, \end{cases}$$

$$p_{n,n+1}(\varepsilon) = \begin{cases} (1-\varepsilon)\dfrac{N-n}{N}, & 0 \leqslant n \leqslant n^*-1, \\ \varepsilon\dfrac{N-n}{N}, & n^* \leqslant n \leqslant N-1, \end{cases}$$

and

$$p_{n,n}(\varepsilon) = 1 - p_{n,n-1}(\varepsilon) - p_{n,n+1}(\varepsilon), \quad 0 \leqslant n \leqslant N.$$

Let $\pi(\varepsilon) = (\pi_0(\varepsilon), \pi_1(\varepsilon), \pi_2(\varepsilon), \ldots, \pi_{N-1}(\varepsilon), \pi_N(\varepsilon))$ be the stationary probability vector of the perturbed birth death process $P(\varepsilon)$. Then $\pi(\varepsilon)$ is given by

$$\pi_n(\varepsilon) = \begin{cases} C, & n = 0, \\ C\beta^n \dbinom{N}{n}, & 1 \leqslant n \leqslant n^*, \\ C\beta^{2n^*-n} \dbinom{N}{n}, & n^*+1 \leqslant n \leqslant N, \end{cases}$$

where

$$\beta = \frac{\varepsilon}{1-\varepsilon}$$

and

$$C = \left[1 + \sum_{n=1}^{n^*} \beta^n \binom{N}{n} + \sum_{n=n^*+1}^{N} \beta^{2n^*-n} \binom{N}{n} \right]^{-1}.$$

Theorem 11.6 (1) *If* $2n^* > N$, *then*

$$\lim_{\varepsilon \to 0} \pi_n(\varepsilon) = \begin{cases} 1, & n = 0, \\ 0, & 1 \leqslant n \leqslant N. \end{cases}$$

(2) *If* $2n^* = N$, *then*

$$\lim_{\varepsilon \to 0} \pi_n(\varepsilon) = \begin{cases} \frac{1}{2}, & n = 0, \\ \frac{1}{2}, & n = N, \\ 0, & 1 \leqslant n \leqslant N-1. \end{cases}$$

(3) *If* $2n^* < N$, *then*

$$\lim_{\varepsilon \to 0} \pi_n(\varepsilon) = \begin{cases} 1, & n = 2n^*, \\ 0, & \text{others.} \end{cases}$$

Case Ⅱ n_{σ_2} (or $n_{\sigma_1} = n_{\sigma_2} + 1$) is an integer.

In this case, the payoff ties occur at state n_{σ_1} and n_{σ_2}. We denote by η a fixed probability of switching from the current strategy if a tie occurs, reflecting, e.g., un-modeled switching cost. It is easy to see that when playing strategy σ_1, player 1 switches to σ_2 if $0 \leqslant z \leqslant n_{\sigma_1} - 1$, does not change if $z \geqslant n_{\sigma_1} + 1$, and switches randomly if $z = n_{\sigma_1}$. On the other hand, when playing strategy σ_2, player 1 switches to σ_1 if $z \geqslant n_{\sigma_2} + 1$, does not change if $0 \leqslant z \leqslant n_{\sigma_2} - 1$, and switches randomly if $z = n_{\sigma_2}$.

Based on the switching probability, the birth death process $P(\varepsilon) = (p_{i,j}(\varepsilon))$ is described as

$$p_{n,n-1}(\varepsilon) = \begin{cases} (1-\varepsilon)\dfrac{n}{N}, & 1 \leqslant n \leqslant n_{\sigma_2}, \\ [(1-\varepsilon)\eta + (1-\eta)\varepsilon]\dfrac{n}{N}, & n = n_{\sigma_1}, \\ \varepsilon \dfrac{n}{N}, & n_{\sigma_1} + 1 \leqslant n \leqslant N, \end{cases}$$

$$p_{n,n+1}(\varepsilon)=\begin{cases}(1-\varepsilon)\dfrac{N-n}{N}, & 0\leqslant n\leqslant n_{\sigma_2}-1,\\ [(1-\varepsilon)\eta+(1-\eta)\varepsilon]\dfrac{N-n}{N}, & n=n_{\sigma_2},\\ \varepsilon\dfrac{N-n}{N}, & n_{\sigma_1}\leqslant n\leqslant N-1,\end{cases}$$

and

$$p_{n,n}(\varepsilon)=1-p_{n,n-1}(\varepsilon)-p_{n,n+1}(\varepsilon),\quad 0\leqslant n\leqslant N.$$

Thus, the stationary probability $\pi(\varepsilon)$ is given by

$$\pi_n(\varepsilon)=\begin{cases}C, & n=0,\\ C\beta^n\dbinom{N}{n}, & 1\leqslant n\leqslant n_{\sigma_2},\\ C\beta^{n_{\sigma_2}}\dbinom{N}{n}, & n=n_{\sigma_1},\\ C\beta^{2n_{\sigma_1}-n-1}\dbinom{N}{n}, & n_{\sigma_1}+1\leqslant n\leqslant N,\end{cases}$$

where

$$C=\left[1+\sum_{n=1}^{n_{\sigma_2}}\beta^n\binom{N}{n}+\beta^{n_{\sigma_1}-1}\binom{N}{n_{\sigma_1}}+\sum_{n=n_{\sigma_1}+1}^{N}\beta^{2n_{\sigma_1}-n-1}\binom{N}{n}\right]^{-1}.$$

Theorem 11.7 (1) *If* $2n_{\sigma_1}>N+1,$ *then*

$$\lim_{\varepsilon\to 0}\pi_n(\varepsilon)=\begin{cases}1, & n=0,\\ 0, & 1\leqslant n\leqslant N.\end{cases}$$

(2) *If* $2n_{\sigma_1}=N+1,$ *then*

$$\lim_{\varepsilon\to 0}\pi_n(\varepsilon)=\begin{cases}\dfrac{1}{2}, & n=0,\\ \dfrac{1}{2}, & n=N,\\ 0, & 1\leqslant n\leqslant N-1.\end{cases}$$

(3) *If* $2n_{\sigma_1}<N+1,$ *then*

$$\lim_{\varepsilon\to 0}\pi_n(\varepsilon)=\begin{cases}1, & n=2n_{\sigma_1}-1,\\ 0, & \text{others}.\end{cases}$$

11.7.2 A Probabilistic Construction

For the 2×2 symmetric evolutionary game, based on Eq. (11.100) we write

$$f_i = \frac{(i-1)a + (N-i)b}{N-1} \tag{11.115}$$

and

$$g_i = \frac{ic + (N-1-i)d}{N-1}. \tag{11.116}$$

At every step, one individual is chosen to reproduce. The probability that an individual using strategy σ_1 is chosen is given by

$$\frac{if_i}{if_i + (N-i)g_i}.$$

We assume that with probability $\varepsilon > 0$, a σ_1-offspring is mutant which plays strategy σ_2 instead of strategy σ_1, and with probability $\kappa\varepsilon > 0$, a σ_2-offspring is mutant which plays strategy σ_1. After reproduction, the offspring replaces a randomly chosen member of the population, so that the population size is constant. The process that describes the number of individuals playing strategy σ_1 is a birth death process whose transition probability matrix $P(\varepsilon) = (p_{i,j}(\varepsilon))$ with $p_{i,j}(\varepsilon) = 0$ for $|i-j|>1$ is given by

$$p_{0,1}(\varepsilon) = 1 - p_{0,0}(\varepsilon) = \kappa\varepsilon,$$
$$p_{N,N-1}(\varepsilon) = 1 - p_{N,N}(\varepsilon) = \varepsilon,$$

and for $1 \leqslant i \leqslant N-1$,

$$p_{i,i+1}(\varepsilon) = \frac{if_i(1-\varepsilon) + (N-i)g_i\kappa\varepsilon}{if_i + (N-i)g_i}\,\frac{N-i}{N},$$

$$p_{i,i-1}(\varepsilon) = \frac{if_i\varepsilon + (N-i)g_i(1-\kappa\varepsilon)}{if_i + (N-i)g_i}\,\frac{i}{N}$$

and

$$p_{i,i}(\varepsilon) = 1 - p_{i,i-1}(\varepsilon) - p_{i,i+1}(\varepsilon).$$

Because of the presence of mutations, the birth death process is irreducible and positive recurrent, and thus the stationary probability vector $\pi^\varepsilon = (\pi_0^\varepsilon, \pi_1^\varepsilon, \pi_2^\varepsilon, \ldots, \pi_{N-1}^\varepsilon, \pi_N^\varepsilon)$. Let $\pi^* = \lim_{\varepsilon \to 0} \pi^\varepsilon$ with $\pi_i^* = \lim_{\varepsilon \to 0} \pi_i^\varepsilon$ for $0 \leqslant i \leqslant N$.

We take a new birth death process with the transition probability matrix $\hat{P}=(\hat{p}_{i,j})$. We write

$$\hat{p}_{0,0}=\hat{p}_{N,N}=1,$$

and for $1\leqslant i\leqslant N-1$,

$$\hat{p}_{i,i+1}=\frac{if_i}{if_i+(N-i)g_i}\frac{N-i}{N},$$
$$\hat{p}_{i,i-1}=\frac{(N-i)g_i}{if_i+(N-i)g_i}\frac{i}{N}$$

and

$$\hat{p}_{i,i}=1-\hat{p}_{i,i-1}-\hat{p}_{i,i+1}.$$

It is clear that $\hat{p}_{i,j}=\lim\limits_{\varepsilon\to 0}p_{i,j}(\varepsilon)$ for $0\leqslant i,j\leqslant N$, which leads to $\hat{P}=\lim\limits_{\varepsilon\to 0}P(\varepsilon)$. Thus, π^* is the stationary probability vector of the birth death process $\hat{P}$. Let

$$\gamma=\prod_{i=1}^{N-1}\frac{f_i}{g_i}. \tag{11.117}$$

For very small mutation rates, the moran process spends nearly all the time at one of the two states 0 and N. It is easy to see that π_0^* and π_N^* are the limits of the fractions of time that the moran process spends at the states "all σ_1" and "all σ_2", respectively. Therefore, we obtain

$$\pi_0^*=\frac{\kappa\gamma}{1+\kappa\gamma} \tag{11.118}$$

and

$$\pi_N^*=\frac{1}{1+\kappa\gamma}. \tag{11.119}$$

The following lemma provides a useful bound for π_0^* and π_N^*. The proof is clear and is omitted here.

Lemma 11.2 (1) $\pi_0^*>1/2$ (*or* $\pi_N^*<1/2$) *if and only if* $\gamma>1/\kappa$,

(2) $\pi_0^*<1/2$ (*or* $\pi_N^*>1/2$) *if and only if* $\gamma<1/\kappa$, *and*

(3) $\pi_0^*=1/2$ (*or* $\pi_N^*=1/2$) *if and only if* $\gamma=1/\kappa$.

Now, we construct a two-state Markov chain whose stationary probability vector is $\left(\pi_0^*,\pi_N^*\right)$. Such a constructed Markov chain can be used to study the underlying game with more than two pure strategies. For the birth death process $\hat{P}$, the states 0 and N are absorbing and the other states are all transient. Let $\rho_{1,2}$

be the probability of absorption at state 0, if the birth death process is initially at state $N-1$. Thus, $\rho_{1,2}$ is the probability that a single individual that plays strategy σ_2 takes over a population where everyone else plays strategy σ_1. Define $\rho_{2,1}$ analogously. It is easy to check that

$$\rho_{1,2} = \frac{\prod_{j=1}^{N-1} \frac{g_j}{f_j}}{1+\sum_{i=1}^{N-1}\prod_{j=1}^{i}\frac{g_j}{f_j}}$$

and

$$\rho_{2,1} = \frac{1}{1+\sum_{i=1}^{N-1}\prod_{j=1}^{i}\frac{g_j}{f_j}}.$$

It is clear that $\gamma = \rho_{2,1}/\rho_{1,2}$. We take a two-state Markov chain whose transition probability matrix is given by

$$P = \begin{pmatrix} 1-\kappa\rho_{2,1} & \kappa\rho_{2,1} \\ \rho_{1,2} & 1-\rho_{1,2} \end{pmatrix},$$

Simple computation shows that two-state Markov chain P has stationary probability vector $\left(\pi_0^*, \pi_N^*\right)$.

In what follows, we discuss how the limiting distribution $\left(\pi_0^*, \pi_N^*\right)$ depends on the size N of the population and the payoff matrix. In this situation, we rewrite π^* as $\pi^*(N) = \left(\pi_0^*(N), \pi_1^*(N), \pi_2^*(N), \ldots, \pi_N^*(N)\right)$ in order to indicate the dependence on the number N.

Definition 11.2 *Strategy σ is called to be favored by the moran process if $\pi_i^*(N) > 1/2$, and Strategy σ is called to be selected by the moran process if $\lim_{N\to\infty} \pi_i^*(N) = 1$.*

To determine favored and selected strategies, we rewrite Eq. (11.117) as

$$\gamma(N) = \frac{\prod_{i=1}^{N-1}[ib+(N-1-i)a]}{\prod_{i=1}^{N-1}[ic+(N-1-i)d]}.$$

Following Lemma 11.2, the following theorem indicates how the favored and selected strategies depend on the payoff matrix. The proof is easy and is omitted here.

Theorem 11.8 *Suppose* $\kappa = 1$.

(1) *If* $b > c$ *and* $a > d$, *then* $\pi_0^*(N) > 1/2$ *for each N.*

(2) *If* $b > c$ *and* $a < d$, *then* $\pi_0^*(N) > 1/2$ *may depend on the population size N. A sufficient condition for* $\pi_0^*(N) > 1/2$ *is given by*

$$N < 2 + \frac{b-c}{d-a}.$$

Recall that in a 2×2 game, a strategy is risk dominant if it is the unique best response to the distribution (1/2,1/2) for the two strategies σ_1 and σ_2. Thus, strategy σ_1 is risk dominant if $a + b > c + d$; while strategy σ_2 is risk dominant if $a + b < c + d$. Strategy σ_1 is Pareto-dominant if $a > d$; while strategy σ_2 is Pareto-dominant if $a < d$. The following theorem provides sufficient conditions for the favored and selected strategies.

Theorem 11.9 (1) *If* $b > c$ *and* $a > d$, *then* $\lim_{N\to\infty} \pi_0^*(N) = 1$.

(2) *If* $b > d > a > c$, *then* $\lim_{N\to\infty} \pi_0^*(N) = 1$.

(3) *If* $d > b > c > a$, *then* $\lim_{N\to\infty} \pi_0^*(N) = 0$.

(4) *If* $d > b > a > c$ *or* $d > a > b > c$, *then there are two pure-strategy Nash equlibibria, and* $\lim_{N\to\infty} \pi_0^*(N)$ *is either* 1 *or* 0 *as* $\int_0^1 \ln\left[b + (a-b)x\right]\mathrm{d}x$ *is greater or less than* $\int_0^1 \ln[d + (c-d)x]\mathrm{d}x$. *The risk dominant equilibrium need not be selected, even if it is also Pareto-dominant.*

(5) *If* $b > c > d > a$ *or* $b > d > c > a$, *then there are two pure-strategy Nash equlibibria, and* $\lim_{N\to\infty} \pi_0^*(N)$ *is either* 1 *or 0 as* $\int_0^1 \ln[b + (a-b)x]\mathrm{d}x$ *is greater or less than* $\int_0^1 \ln[d + (c-d)x]\mathrm{d}x$.

Proof (1) The ratio of each pair of terms in $\gamma(N)$ is bounded away from 1, hence $\lim_{N\to\infty} \gamma(N) = +\infty$, which leads to $\lim_{N\to\infty} \pi_0^*(N) = 1$.

(2) If $b > d > a > c$, then it is easy to check that

$$\gamma(N) = \frac{[(N-1)b][(N-2)b+a]\ldots[2b+(N-3)a][b+(N-2)a]}{[c+(N-2)d][2c+(N-3)d]\ldots[(N-2)c+d][(N-1)c]} > 1,$$

which leads to $\lim_{N\to\infty} \pi_0^*(N) = 1$.

(3) If $d > b > c > a$, then when $N > 2 + \max\{(b-c)/(d-b), (b-c)/(c-a)\}$, it is easy to see that $\gamma(N) < 1$ which leads to $\lim_{N\to\infty} \pi_0^*(N) = 0$.

(4) and (5) To prove the two results, we need to approximate $\gamma(N)$ as $N \to \infty$. Note that

$$\gamma(N) = \frac{d}{a} \prod_{j=0}^{N-1} \frac{aj + b(N-1-j)}{cj + d(N-1-j)},$$

$$d\prod_{j=0}^{N-1} [aj + b(N-1-j)] \approx d \exp\left\{N \int_0^1 \ln[a + (b-a)x]dx\right\} + O(1)$$

and

$$a\prod_{j=0}^{N-1} [cj + d(N-1-j)] \approx a \exp\left\{N \int_0^1 \ln[c + (d-c)x]dx\right\} + O(1).$$

Therefore, $\lim_{N \to \infty} \gamma(N)$ is equal to 0 or $+\infty$ by means of the comparison for the two above integrals. Based on this, (4) and (5) can be proved. This completes the proof.

11.8 Asymmetric Evolutionary Games

In this section, we apply the perturbed QBD processes to study the ESS for asymmetric evolutionary games under a unified algorithmic framework. For a 2×2 asymmetric evolutionary game, we discuss three cases: Independent structure, dependent structure and information interaction, and provide numerical solution for the ESS of each case by means of the *RG*-factorizations. Further, we simply discuss the ESS for a 3×2 asymmetric evolutionary game.

11.8.1 A 2×2 Game with Independent Structure

Consider a 2×2 asymmetric evolutionary game with independent structure, where each player has two strategies σ_1 and σ_2, the populations of players of type 1 and 2 are *M* and *N*, and their payoff matrices are given by

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

respectively.

Based on the populations and the payoff matrices, we can setup a transition rate of players from strategy σ_i to σ_j under an average setting. Thus, for the first type of players we take

$$f_{\sigma_1}^1(m) = \frac{a_1 m + b_1(M-m)}{M}$$

and

$$f_{\sigma_2}^1(m)=\frac{c_1 m+d_1(M-m)}{M}.$$

Similarly, for the second type of players we write

$$f_{\sigma_1}^2(n)=\frac{a_2 n+b_2(N-n)}{N}$$

and

$$f_{\sigma_2}^2(n)=\frac{c_2 n+d_2(N-n)}{N}.$$

Further, for the first type of players, the transition rate switching strategy from σ_2 to σ_1 is defined as

$$\lambda_\varepsilon^1(i)=\varepsilon+\kappa_1\max\left\{f_{\sigma_1}^1(i)-f_{\sigma_2}^1(i),0\right\},\quad i=0,1,\ldots,M-1,$$

and the transition rate switching strategy from σ_1 to σ_2 is defined as

$$\mu_\varepsilon^1(i)=\varepsilon+\kappa_1\max\left\{f_{\sigma_2}^1(i)-f_{\sigma_1}^1(i),0\right\},\quad i=1,2,\ldots,M.$$

Similarly, for the second type of players, the transition rate switching strategy from σ_2 to σ_1 is defined as

$$\lambda_\varepsilon^2(j)=\varepsilon+\kappa_2\max\left\{f_{\sigma_1}^2(j)-f_{\sigma_2}^2(j),0\right\},\quad j=0,1,\ldots,N-1,$$

and the transition rate switching strategy from σ_1 to σ_2 is defined as

$$\mu_\varepsilon^2(j)=\varepsilon+\kappa_2\max\left\{f_{\sigma_2}^2(j)-f_{\sigma_1}^2(j),0\right\},\quad j=1,2,\ldots,N.$$

Note that ε and κ_i are called an exogenous mutation rate and a learning ability rate.

Let $z(t)=(z_1(t),z_2(t))$, where $z_i(t)$ is the number of the players of type i who are playing strategy σ_1 at time t for $i=1,2$. Note that each player switches from one strategy to another according to the bounded rationality with myopia, inertia and mutation for a small probability $\varepsilon>0$. Based on this, we can model this game as a continuous-time level-dependent QBD process $\{z(t),t\geqslant 0\}$ on the state space

$$\begin{aligned}S=\{&(0,0),(0,1),\ldots,(0,N);(1,0),(1,1),\ldots,(1,N);\\&\ldots;(M,0),(M,1),\ldots,(M,N)\},\end{aligned}$$

where the first type of players are regarded as "level" while the second type of players are regarded as "phase". Therefore, the infinitesimal generator of this QBD process is given by

$$Q(\varepsilon)=\begin{pmatrix} A_1^0(\varepsilon) & A_0^0(\varepsilon) & & & & & \\ A_2^1(\varepsilon) & A_1^1(\varepsilon) & A_0^1(\varepsilon) & & & & \\ & A_2^2(\varepsilon) & A_1^2(\varepsilon) & A_0^2(\varepsilon) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_2^{M-2}(\varepsilon) & A_1^{M-2}(\varepsilon) & A_0^{M-2}(\varepsilon) & \\ & & & & A_2^{M-1}(\varepsilon) & A_1^{M-1}(\varepsilon) & A_0^{M-1}(\varepsilon) \\ & & & & & A_2^{M}(\varepsilon) & A_1^{M}(\varepsilon) \end{pmatrix},$$

where for $0 \leqslant k \leqslant M$,

$$A_1^k(\varepsilon)=\begin{pmatrix} -a_\varepsilon(k,0) & \lambda_\varepsilon^2(0) & & & & \\ \mu_\varepsilon^2(1) & -a_\varepsilon(k,1) & \lambda_\varepsilon^2(1) & & & \\ & \mu_\varepsilon^2(2) & -a_\varepsilon(k,2) & \lambda_\varepsilon^2(2) & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_\varepsilon^2(N-1) & -a_\varepsilon(k,N-1) & \lambda_\varepsilon^2(N-1) \\ & & & & \mu_\varepsilon^2(N) & -a_\varepsilon(k,N) \end{pmatrix}$$

with for $k=0$,

$$a_\varepsilon(0,l)=\begin{cases} \lambda_\varepsilon^1(0)+\lambda_\varepsilon^2(0), & l=0, \\ \lambda_\varepsilon^1(0)+\lambda_\varepsilon^2(l)+\mu_\varepsilon^2(l), & 1 \leqslant l \leqslant N-1, \\ \lambda_\varepsilon^1(0)+\mu_\varepsilon^2(N), & l=N, \end{cases}$$

for $k=M$,

$$a_\varepsilon(M,l)=\begin{cases} \mu_\varepsilon^1(M)+\lambda_\varepsilon^2(0), & l=0, \\ \mu_\varepsilon^1(M)+\lambda_\varepsilon^2(l)+\mu_\varepsilon^2(l), & 1 \leqslant l \leqslant N-1, \\ \mu_\varepsilon^1(M)+\mu_\varepsilon^2(N), & l=N, \end{cases}$$

and for $1 \leqslant k \leqslant M-1$,

$$a_\varepsilon(k,l)=\begin{cases} \lambda_\varepsilon^1(k)+\mu_\varepsilon^1(k)+\lambda_\varepsilon^2(0), & l=0, \\ \lambda_\varepsilon^1(k)+\mu_\varepsilon^1(k)+\lambda_\varepsilon^2(l)+\mu_\varepsilon^2(l), & 1 \leqslant l \leqslant N-1, \\ \lambda_\varepsilon^1(k)+\mu_\varepsilon^1(k)+\mu_\varepsilon^2(N), & l=N. \end{cases}$$

At the same time, for $0 \leqslant m \leqslant M-1$,

$$A_0^m(\varepsilon) = \lambda_\varepsilon^1(m)\mathrm{diag}(1,1,1,\ldots,1,1),$$

and for $1 \leqslant n \leqslant M$,

$$A_2^n(\varepsilon) = \mu_\varepsilon^1(n)\mathrm{diag}(1,1,1,\ldots,1,1).$$

For the QBD process, we write

$$U_M(\varepsilon) = A_1^M(\varepsilon),$$

$$U_k(\varepsilon) = A_1^k(\varepsilon) + A_0^k(\varepsilon)[-U_{k+1}(\varepsilon)]^{-1}A_2^{k+1}(\varepsilon), \quad 0 \leqslant k \leqslant M-1,$$

and

$$R_k(\varepsilon) = A_0^k(\varepsilon)[-U_{k+1}(\varepsilon)]^{-1}, \quad 0 \leqslant k \leqslant M-1.$$

Let

$$\pi(\varepsilon) = (\pi_0(\varepsilon), \pi_1(\varepsilon), \pi_2(\varepsilon), \ldots, \pi_{M-1}(\varepsilon), \pi_M(\varepsilon))$$

with

$$\pi_k(\varepsilon) = (\pi_k^0(\varepsilon), \pi_k^1(\varepsilon), \pi_k^2(\varepsilon), \ldots, \pi_k^{N-1}(\varepsilon), \pi_k^N(\varepsilon))$$

be the stationary probability vector of the QBD process $Q(\varepsilon)$. Then

$$\pi_k(\varepsilon) = \tau(\varepsilon)v_0(\varepsilon)R_0(\varepsilon)R_1(\varepsilon)R_2(\varepsilon)\ldots R_{k-1}(\varepsilon), \quad 0 \leqslant k \leqslant M,$$

where $v_0(\varepsilon)$ is the stationary probability vector of the censored Markov chain $U_0(\varepsilon)$ to level 0, and the constant $\tau(\varepsilon)$ is given by

$$\tau(\varepsilon) = \frac{1}{1 + v_0(\varepsilon)\sum_{k=1}^{M} R_0(\varepsilon)R_1(\varepsilon)R_2(\varepsilon)\ldots R_{k-1}(\varepsilon)e}.$$

To compute the ESS of this game numerically, we need to study the limiting distribution

$$\pi^* = \lim_{\varepsilon \to 0} \pi(\varepsilon)$$

where

$$\tau^* = \lim_{\varepsilon \to 0} \tau(\varepsilon), \quad v_0^* = \lim_{\varepsilon \to 0} v_0(\varepsilon), \quad R_k^* = \lim_{\varepsilon \to 0} R_k(\varepsilon).$$

In what follows we provide two numerical examples to indicate the structure of the two probabilities $p_i(\varepsilon)$ and $q_j(\varepsilon)$ with respect to the two learning ability rates κ_1 and κ_2, and the exogenous mutation rate ε. For simplification of description, we write

$$p_i(\varepsilon) = P\{z_1(+\infty) = i\} = \sum_{j=0}^{N} \pi_i^j(\varepsilon),$$

$$q_j(\varepsilon) = P\{z_2(+\infty) = j\} = \sum_{i=0}^{M} \pi_i^j(\varepsilon).$$

Example 11.1 (*The learning ability rate*) *In the* 2×2 *asymmetric evolutionary game, the parameters are given by*

$$A_1 = \begin{pmatrix} 20 & 0 \\ 2 & 8 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 40 & 1 \\ 0 & 10 \end{pmatrix}$$

$$M = 6, \quad N = 10, \quad \varepsilon = 0.5,$$

$$\kappa_1 = \kappa_2 = \kappa = 0.1, 0.2, 0.4, 0.6, 0.8, 1.0.$$

Table 11.1 indicates how the two probabilities $p_i(\varepsilon)$ *and* $q_j(\varepsilon)$ *depend on the learning ability rate* κ.

Table 11.1 The role of the learning ability rate

For the first type of players

$z_1(\infty)=$	0	1	2	3	4	5	6
$\kappa = 0.1$	0.0457	0.0264	0.0264	0.0299	0.0598	0.1715	0.6402
$\kappa = 0.2$	0.0171	0.0070	0.0070	0.0088	0.0264	0.1251	0.8087
$\kappa = 0.4$	0.0046	0.0012	0.0012	0.0018	0.0090	0.0760	0.9063
$\kappa = 0.6$	0.0019	0.0004	0.0004	0.0006	0.0044	0.0539	0.9384
$\kappa = 0.8$	0.0010	0.0001	0.0001	0.0003	0.0026	0.0417	0.9541
$\kappa = 1.0$	0.0006	0.0001	0.0001	0.0002	0.0017	0.0340	0.9634

For the second type of players

$z_2(\infty)=0$	$z_2(\infty)=1$	$z_2(\infty)=2$	$z_2(\infty)=3$	$z_2(\infty)=4$	$z_2(\infty)=5$	$z_2(\infty)=6$	$z_2(\infty)=7$	$z_2(\infty)=8$	$z_2(\infty)=9$	$z_2(\infty)=10$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0025	0.0155	0.1088	0.8725
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0047	0.0620	0.9328
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0013	0.0332	0.9654
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	0.0227	0.9767
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0172	0.9825
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0138	0.9859

Example 11.2 (*The exogenous mutation rate*) *In the* 2×2 *asymmetric evolutionary game, the parameters are given by*

$$A_1 = \begin{pmatrix} 20 & 0 \\ 2 & 8 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 40 & 1 \\ 0 & 10 \end{pmatrix}$$

$$M = 6, \quad N = 10, \quad \kappa_1 = \kappa_2 = \kappa = 0.6,$$

$$\varepsilon = 0.2, 0.6, 1.0, 1.5, 2.0, 4.0.$$

Table 11.2 indicates how the two probabilities $p_i(\varepsilon)$ *and* $q_j(\varepsilon)$ *depend on the exogenous mutation rate* ε.

Table 11.2 The role of the exogenous mutation rate

For the first type of players

$z_1(\infty) =$	0	1	2	3	4	5	6
$\varepsilon = 0.2$	0.0002	0.0000	0.0000	0.0001	0.0008	0.0232	0.9757
$\varepsilon = 0.6$	0.0029	0.0006	0.0006	0.0010	0.0061	0.0631	0.9257
$\varepsilon = 1.0$	0.0082	0.0026	0.0026	0.0036	0.0144	0.0950	0.8737
$\varepsilon = 1.5$	0.0171	0.0070	0.0070	0.0088	0.0264	0.1251	0.8087
$\varepsilon = 2.0$	0.0270	0.0128	0.0128	0.0154	0.0385	0.1465	0.7469
$\varepsilon = 4.0$	0.0613	0.0395	0.0395	0.0435	0.0761	0.1827	0.5573

For the second type of players

$z_2(\infty)$=0	$z_2(\infty)$=1	$z_2(\infty)$=2	$z_2(\infty)$=3	$z_2(\infty)$=4	$z_2(\infty)$=5	$z_2(\infty)$=6	$z_2(\infty)$=7	$z_2(\infty)$=8	$z_2(\infty)$=9	$z_2(\infty)$=10
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0093	0.9906
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0009	0.0269	0.9722
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0023	0.0433	0.9543
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0047	0.0620	0.9328
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0009	0.0079	0.0791	0.9120
0.0001	0.0001	0.0001	0.0001	0.0001	0.0004	0.0012	0.0050	0.0241	0.1333	0.8354

11.8.2 A 2×2 Game with Dependent Structure

Consider a 2×2 asymmetric evolutionary game with dependent structure, where each player has two strategies σ_1 and σ_2, the populations of players of types 1 and 2 are *M* and *N*, their payoff matrices are given by

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

respectively. We assume that the dependent structure is expressed as four factors $D_1^\lambda(i,j)$, $D_1^\mu(i,j)$, $D_2^\lambda(i,j)$ and $D_2^\mu(i,j)$ for $0 \leqslant i \leqslant M$ and $0 \leqslant j \leqslant N$. Note that when there are i players of type 1 playing strategy σ_1 and j players of type 2 playing strategy σ_1, $D_1^{\lambda}(i,j)$ and $D_1^{\mu}(i,j)$ are two dependent factors for the players of type 1 switching strategy σ_2 to strategy σ_1 and switching strategy σ_1 to strategy σ_2, respectively; while $D_2^{\lambda}(i,j)$ and $D_2^{\mu}(i,j)$ can be similarly explained for the players of type 2.

Let

$$f_{\sigma_1}^1(m) = \frac{a_1 m + b_1(M-m)}{M},$$
$$f_{\sigma_2}^1(m) = \frac{c_1 m + d_1(M-m)}{M};$$
$$f_{\sigma_1}^2(n) = \frac{a_2 n + b_2(N-n)}{N}$$

and

$$f_{\sigma_2}^2(n) = \frac{c_2 n + d_2(N-n)}{N}.$$

Applying the dependently structured factors, we can further define the transition rate either switching strategy σ_2 to strategy σ_1 or switching strategy σ_1 to strategy σ_2 as follows. For the first type of players, the transition rate switching strategy from σ_2 to σ_1 is defined as that for $i = 0, 1, \ldots, M-1$,

$$a_{\varepsilon}^1(i,j) = D_1^{\lambda}(i,j)\left[\varepsilon + \kappa_1 \max\left\{f_{\sigma_1}^1(i) - f_{\sigma_2}^1(i), 0\right\}\right] = \lambda_{\varepsilon}^1(i) D_1^{\lambda}(i,j),$$

and the transition rate switching strategy from σ_1 to σ_2 is defined as that for $i = 1, 2, \ldots, M$,

$$b_{\varepsilon}^1(i,j) = D_1^{\mu}(i,j)\left[\varepsilon + \kappa_1 \max\left\{f_{\sigma_2}^1(i) - f_{\sigma_1}^1(i), 0\right\}\right] = \mu_{\varepsilon}^1(i) D_1^{\mu}(i,j).$$

Similarly, for the second type of players, the transition rate switching strategy from σ_2 to σ_1 is defined as that for $j = 0, 1, \ldots, N-1$,

$$a_{\varepsilon}^2(i,j) = D_2^{\lambda}(i,j)\left[\varepsilon + \kappa_2 \max\left\{f_{\sigma_1}^2(j) - f_{\sigma_2}^2(j), 0\right\}\right] = D_2^{\lambda}(i,j)\lambda_{\varepsilon}^2(j),$$

and the transition rate switching strategy from σ_1 to σ_2 is defined as that for $j = 1, 2, \ldots, N$,

$$b_{\varepsilon}^2(i,j) = D_2^{\mu}(i,j)\left[\varepsilon + \kappa_2 \max\left\{f_{\sigma_2}^2(j) - f_{\sigma_1}^2(j), 0\right\}\right] = D_2^{\mu}(i,j)\mu_{\varepsilon}^2(j).$$

Remark 11.1 *The four dependently structured factors* $D_1^{\lambda}(i,j)$, $D_1^{\mu}(i,j)$, $D_2^{\lambda}(i,j)$ *and* $D_2^{\mu}(i,j)$ *for* $0 \leqslant i \leqslant M$ *and* $0 \leqslant j \leqslant N$ *can be chosen extensively, for example,*

$$D_1^{\lambda}(i,j) = c_1 \alpha^i \beta^j, c_1 i^2 j^3, c_1 \ln(i)\ln(j), c_1\left(\alpha^i + \beta^j\right), c_1 \frac{\alpha^i}{\ln(j)}.$$

Specifically, if the four dependently structured factors are all equal to one, then this game becomes the case with independent structure, as studied above.

Let $z(t) = (z_1(t), z_2(t))$, where $z_i(t)$ is the number of the players of type i who are playing strategy σ_1 at time t for $i = 1, 2$. Then we can model this game as a continuous-time level-dependent QBD process $\{z(t), t \geqslant 0\}$ on the state space

$$\begin{aligned} S = \{&(0,0),(0,1),\ldots,(0,N);(1,0),(1,1),\ldots,(1,N);\\ &\ldots;(M,0),(M,1),\ldots,(M,N)\},\end{aligned}$$

and its state transition relation is given in Fig. 11.1.

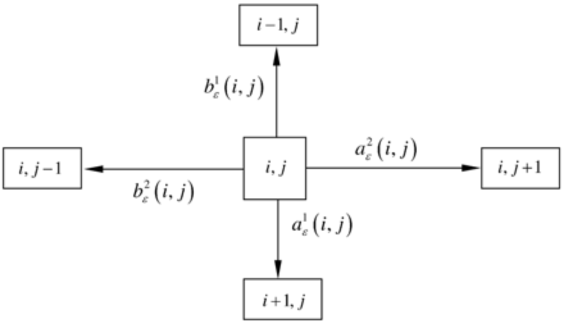

Figure 11.1 The state transition relation

Based on Fig. 11.1, the infinitesimal generator of this QBD process is given by

$$Q(\varepsilon) = \begin{pmatrix} A_1^0(\varepsilon) & A_0^0(\varepsilon) & & & & & \\ A_2^1(\varepsilon) & A_1^1(\varepsilon) & A_0^1(\varepsilon) & & & & \\ & A_2^2(\varepsilon) & A_1^2(\varepsilon) & A_0^2(\varepsilon) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_2^{M-2}(\varepsilon) & A_1^{M-2}(\varepsilon) & A_0^{M-2}(\varepsilon) & \\ & & & & A_2^{M-1}(\varepsilon) & A_1^{M-1}(\varepsilon) & A_0^{M-1}(\varepsilon) \\ & & & & & A_2^{M}(\varepsilon) & A_1^{M}(\varepsilon) \end{pmatrix},$$

where for $0 \leqslant k \leqslant M$,

$$A_1^k(\varepsilon)=\begin{pmatrix} -\xi_\varepsilon(k,0) & a_\varepsilon^2(k,0) & & & & \\ b_\varepsilon^2(k,1) & -\xi_\varepsilon(k,1) & a_\varepsilon^2(k,1) & & & \\ & b_\varepsilon^2(k,2) & -\xi_\varepsilon(k,2) & a_\varepsilon^2(k,2) & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_\varepsilon^2(k,N-1) & -\xi_\varepsilon(k,N-1) & a_\varepsilon^2(k,N-1) \\ & & & & b_\varepsilon^2(k,N) & -\xi_\varepsilon(k,N) \end{pmatrix}$$

with for $k=0$,

$$\xi_\varepsilon(0,l)=\begin{cases} a_\varepsilon^1(0,0)+a_\varepsilon^2(0,0), & l=0, \\ a_\varepsilon^1(0,l)+a_\varepsilon^2(0,l)+b_\varepsilon^2(0,l), & 1\leqslant l\leqslant N-1, \\ a_\varepsilon^1(0,N)+b_\varepsilon^2(0,N) & l=N, \end{cases}$$

for $k=M$,

$$\xi_\varepsilon(M,l)=\begin{cases} b_\varepsilon^1(M,0)+a_\varepsilon^2(M,0), & l=0, \\ b_\varepsilon^1(M,l)+a_\varepsilon^2(M,l)+b_\varepsilon^2(M,l), & 1\leqslant l\leqslant N-1, \\ b_\varepsilon^1(M,N)+b_\varepsilon^2(M,N), & l=N, \end{cases}$$

and for $1\leqslant k\leqslant M-1$,

$$\xi_\varepsilon(k,l)=\begin{cases} \lambda_\varepsilon^1(k,0)+\mu_\varepsilon^1(k,0)+\lambda_\varepsilon^2(k,0), & l=0, \\ \lambda_\varepsilon^1(k,l)+\mu_\varepsilon^1(k,l)+\lambda_\varepsilon^2(k,l)+\mu_\varepsilon^2(k,l), & 1\leqslant l\leqslant N-1, \\ \lambda_\varepsilon^1(k,N)+\mu_\varepsilon^1(k,N)+\mu_\varepsilon^2(k,N), & l=N. \end{cases}$$

At the same time, for $0\leqslant m\leqslant M-1$,

$$A_0^m(\varepsilon)=\operatorname{diag}\left(a_\varepsilon^1(m,0),a_\varepsilon^1(m,1),a_\varepsilon^1(m,2),\ldots,a_\varepsilon^1(m,N-1),a_\varepsilon^1(m,N)\right),$$

for $1\leqslant n\leqslant M$,

$$A_2^n(\varepsilon)=\operatorname{diag}\left(b_\varepsilon^1(n,0),b_\varepsilon^1(n,1),b_\varepsilon^1(n,2),\ldots,b_\varepsilon^1(n,N-1),b_\varepsilon^1(n,N)\right).$$

For the QBD process, we write

$$U_M(\varepsilon)=A_1^M(\varepsilon)$$

and for $0\leqslant k\leqslant M-1$,

$$U_k(\varepsilon)=A_1^k(\varepsilon)+A_0^k(\varepsilon)\left[-U_{k+1}(\varepsilon)\right]^{-1}A_2^{k+1}(\varepsilon),$$

and

$$R_k(\varepsilon)=A_0^k(\varepsilon)\left[-U_{k+1}(\varepsilon)\right]^{-1},\quad 0\leqslant k\leqslant M-1.$$

Let

$$\pi(\varepsilon) = (\pi_0(\varepsilon), \pi_1(\varepsilon), \pi_2(\varepsilon), \ldots, \pi_{M-1}(\varepsilon), \pi_M(\varepsilon))$$

with

$$\pi_k(\varepsilon) = (\pi_k^0(\varepsilon), \pi_k^1(\varepsilon), \pi_k^2(\varepsilon), \ldots, \pi_k^{N-1}(\varepsilon), \pi_k^N(\varepsilon))$$

be the stationary probability vector of the QBD process $Q(\varepsilon)$. Then

$$\pi_k(\varepsilon) = \tau(\varepsilon) v_0(\varepsilon) R_0(\varepsilon) R_1(\varepsilon) R_2(\varepsilon) \cdots R_{k-1}(\varepsilon), \quad 0 \leqslant k \leqslant M,$$

where $v_0(\varepsilon)$ is the stationary probability vector of the Markov chain $U_0(\varepsilon)$ and the constant $\tau(\varepsilon)$ is given by

$$\tau(\varepsilon) = \frac{1}{1 + v_0(\varepsilon) \sum_{k=1}^{M} R_0(\varepsilon) R_1(\varepsilon) R_2(\varepsilon) \ldots R_{k-1}(\varepsilon) e}.$$

To compute the ESS numerically, we need to study the limiting distribution

$$\pi^* = \lim_{\varepsilon \to 0} \pi(\varepsilon)$$

where

$$\tau^* = \lim_{\varepsilon \to 0} \tau(\varepsilon), \quad v_0^* = \lim_{\varepsilon \to 0} v_0(\varepsilon), \quad R_k^* = \lim_{\varepsilon \to 0} R_k(\varepsilon), \quad 0 \leqslant k \leqslant M-1.$$

In what follows we provide a comparable numerical example to indicate how the limiting distribution π^* depends on the dependently structured factors $D_1^\lambda(i,j)$, $D_1^\mu(i,j)$, $D_2^\lambda(i,j)$ and $D_2^\mu(i,j)$ for $0 \leqslant i \leqslant M$ and $0 \leqslant j \leqslant N$.

Example 11.3 (*The dependently structured factors*) *In the* 2×2 *asymmetric evolutionary game, the payoff matrix is given by*

$$A_1 = A_2 = \begin{pmatrix} 20 & 0 \\ 2 & 8 \end{pmatrix},$$

$$M = N = 8, \quad \kappa_1 = \kappa_2 = 0.8, \quad \varepsilon = 0.1,$$

$$D_1^\lambda(i,j) = (1+\alpha)^i (1-\beta)^j, \quad D_1^\mu(i,j) = (1-\alpha)^i (1+\beta)^j,$$

$$D_2^\lambda(i,j) = (1-\xi)^i (1+\eta)^j, \quad D_2^\mu(i,j) = (1+\xi)^i (1-\eta)^j,$$

where in the original model $\alpha = \beta = \xi = \eta = 0$; *while in a comparable model* $\alpha = \beta = 0.2, \xi = 0.6, \eta = 0.4$. *Figure 11.2 indicates how the two probabilities* $p_i(\varepsilon)$ *and* $q_j(\varepsilon)$ *depend on the dependently structured factors. It is seen that when this game has an independent structure, two types of players can choose* *their dominant strategies* $P\{z_i(+\infty)=8\}=1$ *for* $i=1,2$. *However, when introducing the dependent structure to this game, the dominant strategies become that*

$$P\{z_1(+\infty)=8\}=1, \quad P\{z_2(+\infty)=0\}=1.$$

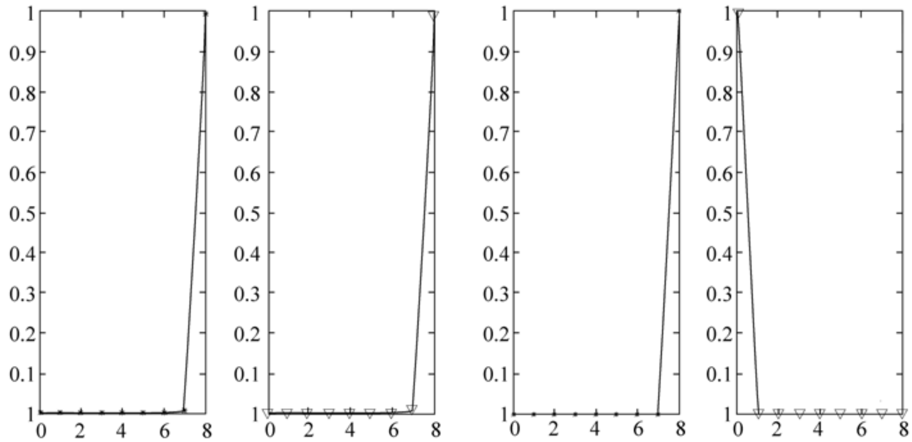

Figure 11.2 The role of the dependent structure

11.8.3 A 2×2 Game with Information Interaction

Consider a 2×2 asymmetric evolutionary game with information interaction, where each player has two strategies σ_1 and σ_2, the populations of the players of types 1 and 2 are *M* and *N*, their payoff matrices are given by

$$A_1=\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad A_2=\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

respectively.

We define

$$f_{\sigma_1}^1(n)=\frac{a_1 n+b_1(N-n)}{N},$$

$$f_{\sigma_2}^1(n)=\frac{c_1 n+d_1(N-n)}{N};$$

$$f_{\sigma_1}^2(m)=\frac{a_2 m+b_2(M-m)}{M}$$

and

$$f_{\sigma_2}^2(m) = \frac{c_2 m + d_2(M-m)}{M}.$$

For the first type of players, the transition rate switching strategy from σ_2 to σ_1 is defined as

$$\lambda_\varepsilon^1(i) = \varepsilon + \kappa_1 \max\left\{f_{\sigma_1}^1(i) - f_{\sigma_2}^1(i), 0\right\}, \quad i = 0,1,\ldots,N-1,$$

and the transition rate switching strategy from σ_1 to σ_2 is defined as

$$\mu_\varepsilon^1(i) = \varepsilon + \kappa_1 \max\left\{f_{\sigma_2}^1(i) - f_{\sigma_1}^1(i), 0\right\}, \quad i = 1,2,\ldots,N.$$

Similarly, for the second type of players, the transition rate switching strategy from σ_2 to σ_1 is defined as

$$\lambda_\varepsilon^2(i) = \varepsilon + \kappa_2 \max\left\{f_{\sigma_1}^2(i) - f_{\sigma_2}^2(i), 0\right\}, \quad i = 0,1,\ldots M-1,$$

and the transition rate switching strategy from σ_1 to σ_2 is defined as

$$\mu_\varepsilon^2(i) = \varepsilon + \kappa_2 \max\left\{f_{\sigma_2}^2(i) - f_{\sigma_1}^2(i), 0\right\}, \quad i = 1,2,\ldots,M.$$

Let $z(t) = (z_1(t), z_2(t))$, where $z_i(t)$ is the number of the players of type i who are playing strategy σ_1 at time t for $i = 1,2$. We can model this game as a continuous-time level-dependent QBD process $\{z(t), t \geqslant 0\}$ on the state space

$$\begin{aligned} S = \{&(0,0),(0,1),\ldots,(0,M);(1,0),(1,1),\ldots,(1,M); \\ &\ldots;(N,0),(N,1),\ldots,(N,M)\}, \end{aligned}$$

and the infinitesimal generator of this QBD process is given by

$$Q(\varepsilon) = \begin{pmatrix} A_1^0(\varepsilon) & A_0^0(\varepsilon) & & & & & \\ A_2^1(\varepsilon) & A_1^1(\varepsilon) & A_0^1(\varepsilon) & & & & \\ & A_2^2(\varepsilon) & A_1^2(\varepsilon) & A_0^2(\varepsilon) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_2^{N-2}(\varepsilon) & A_1^{N-2}(\varepsilon) & A_0^{N-2}(\varepsilon) & \\ & & & & A_2^{N-1}(\varepsilon) & A_1^{N-1}(\varepsilon) & A_0^{N-1}(\varepsilon) \\ & & & & & A_2^N(\varepsilon) & A_1^N(\varepsilon) \end{pmatrix},$$

where for $0 \leqslant k \leqslant N$,

$$A_1^k(\varepsilon)=\begin{pmatrix} -a_\varepsilon(k,0) & \lambda_\varepsilon^2(0) & & & & \\ \mu_\varepsilon^2(1) & -a_\varepsilon(k,1) & \lambda_\varepsilon^2(1) & & & \\ & \mu_\varepsilon^2(2) & -a_\varepsilon(k,2) & \lambda_\varepsilon^2(2) & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_\varepsilon^2(M-1) & -a_\varepsilon(k,M-1) & \lambda_\varepsilon^2(M-1) \\ & & & & \mu_\varepsilon^2(M) & -a_\varepsilon(k,M) \end{pmatrix},$$

for $k=0$,

$$a_\varepsilon(0,l)=\begin{cases} \lambda_\varepsilon^1(0)+\lambda_\varepsilon^2(0), & l=0, \\ \lambda_\varepsilon^1(0)+\lambda_\varepsilon^2(l)+\mu_\varepsilon^2(l), & 1\leqslant l\leqslant M-1, \\ \lambda_\varepsilon^1(0)+\mu_\varepsilon^2(M), & l=M, \end{cases}$$

for $k=N$,

$$a_\varepsilon(N,l)=\begin{cases} \mu_\varepsilon^1(N)+\lambda_\varepsilon^2(0), & l=0, \\ \mu_\varepsilon^1(N)+\lambda_\varepsilon^2(l)+\mu_\varepsilon^2(l), & 1\leqslant l\leqslant M-1, \\ \mu_\varepsilon^1(N)+\mu_\varepsilon^2(M), & l=M. \end{cases}$$

and for $1\leqslant k\leqslant N-1$,

$$a_\varepsilon(k,l)=\begin{cases} \lambda_\varepsilon^1(k)+\mu_\varepsilon^1(k)+\lambda_\varepsilon^2(0), & l=0, \\ \lambda_\varepsilon^1(k)+\mu_\varepsilon^1(k)+\lambda_\varepsilon^2(l)+\mu_\varepsilon^2(l), & 1\leqslant l\leqslant M-1, \\ \lambda_\varepsilon^1(k)+\mu_\varepsilon^1(k)+\mu_\varepsilon^2(M), & l=M, \end{cases}$$

At the same time, for $0\leqslant m\leqslant N-1$,

$$A_0^m(\varepsilon)=\operatorname{diag}\left(\lambda_\varepsilon^1(0),\lambda_\varepsilon^1(1),\lambda_\varepsilon^1(2),\ldots,\lambda_\varepsilon^1(M-1),\lambda_\varepsilon^1(M)\right),$$

for $1\leqslant n\leqslant N$,

$$A_2^n(\varepsilon)=\operatorname{diag}\left(\mu_\varepsilon^1(0),\mu_\varepsilon^1(1),\mu_\varepsilon^1(2),\ldots,\mu_\varepsilon^1(M-1),\mu_\varepsilon^1(M)\right).$$

For the QBD process, we write

$$U_N(\varepsilon)=A_1^N(\varepsilon),$$
$$U_k(\varepsilon)=A_1^k(\varepsilon)+A_0^k(\varepsilon)[-U_{k+1}(\varepsilon)]^{-1}A_2^{k+1}(\varepsilon),\quad \text{for } 0\leqslant k\leqslant N-1,$$

and

$$R_k(\varepsilon)=A_0^k(\varepsilon)[-U_{k+1}(\varepsilon)]^{-1},\quad 0\leqslant k\leqslant N-1.$$

Let

$$\pi(\varepsilon) = (\pi_0(\varepsilon), \pi_1(\varepsilon), \pi_2(\varepsilon), \ldots, \pi_{N-1}(\varepsilon), \pi_N(\varepsilon))$$

with

$$\pi_k(\varepsilon) = \left(\pi_k^0(\varepsilon), \pi_k^1(\varepsilon), \pi_k^2(\varepsilon), \ldots, \pi_k^{M-1}(\varepsilon), \pi_k^M(\varepsilon)\right)$$

be the stationary probability vector of the QBD process $Q(\varepsilon)$. Then

$$\pi_k(\varepsilon) = \tau(\varepsilon) v_0(\varepsilon) R_0(\varepsilon) R_1(\varepsilon) R_2(\varepsilon) \ldots R_{k-1}(\varepsilon), \quad 0 \leqslant k \leqslant N,$$

where $v_0(\varepsilon)$ is the stationary probability vector of the censored Markov chain $U_0(\varepsilon)$ to level 0, and the constant $\tau(\varepsilon)$ is given by

$$\tau(\varepsilon) = \frac{1}{1 + v_0(\varepsilon) \sum_{k=1}^{N} R_0(\varepsilon) R_1(\varepsilon) R_2(\varepsilon) \ldots R_{k-1}(\varepsilon) e}.$$

To compute the ESS numerically, we need to study the limiting distribution

$$\pi^* = \lim_{\varepsilon \to 0} \pi(\varepsilon)$$

where

$$\tau^* = \lim_{\varepsilon \to 0} \tau(\varepsilon), \quad v_0^* = \lim_{\varepsilon \to 0} v_0(\varepsilon), \quad R_k^* = \lim_{\varepsilon \to 0} R_k(\varepsilon), \quad 0 \leqslant k \leqslant N-1.$$

In what follows we provide two numerical examples to indicate how the limiting distribution π^* depends on the information interaction.

Example 11.4 (*The information interaction*) *In the* 2×2 *asymmetric evolutionary game, the payoff matrices are given by*

$$A_1 = \begin{pmatrix} 20 & 10 \\ 2 & 8 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 10 & 1 \\ 0 & 40 \end{pmatrix}$$

$$M = 8, \quad N = 10, \quad \varepsilon = 0.1, \quad \kappa_1 = \kappa_2 = \kappa = 0.8.$$

Figure 11.3 indicates how the two probabilities $p_i(\varepsilon)$ *and* $q_j(\varepsilon)$ *depend on the information interaction.*

Example 11.5 (*The information interaction*) *In the* 2×2 *asymmetric evolutionary game, the payoff matrices are given by*

$$A_1 = \begin{pmatrix} 2 & 20 \\ 10 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 12 & 1 \\ 0 & 8 \end{pmatrix}$$

$$M = 8, \quad N = 10, \quad \varepsilon = 0.1, \quad \kappa_1 = \kappa_2 = \kappa = 0.8.$$

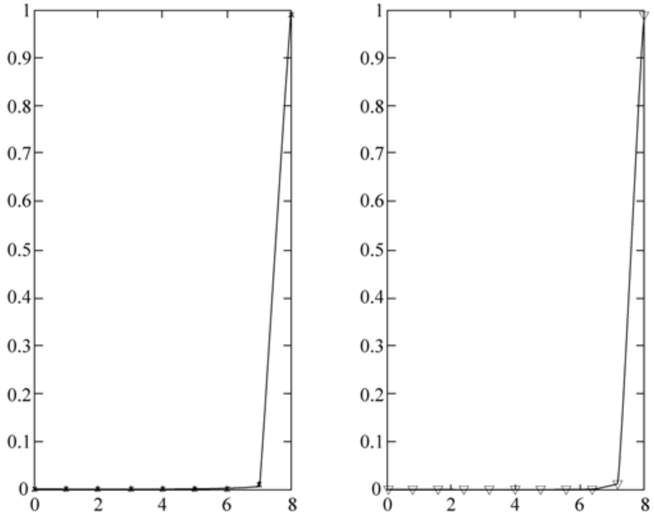

Figure 11.3 The role of the information interaction

Figure 11.4 indicates how the two probabilities $p_i(\varepsilon)$ *and* $q_j(\varepsilon)$ *depend on the information interaction.*

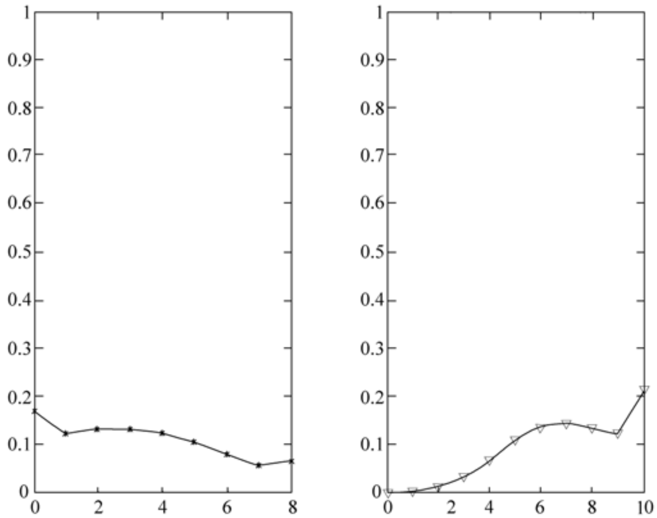

Figure 11.4 The role of the information interaction

11.8.4 A 3×2 Asymmetric Evolutionary Game

Consider a 3×2 asymmetric evolutionary game with information interaction, where there are three types of players and each player has two strategies σ_1 and

σ_2, the populations of the players of type k are N_k for $k=1,2,3,$ their payoff matrices are given by

$$A_1=\begin{pmatrix} c^1_{1,1,1} & c^1_{1,1,2} & c^1_{1,2,1} & c^1_{1,2,2} \\ c^1_{2,1,1} & c^1_{2,1,2} & c^1_{2,2,1} & c^1_{2,2,2} \end{pmatrix},$$

$$A_2=\begin{pmatrix} c^2_{1,1,1} & c^2_{1,1,2} & c^2_{2,1,1} & c^2_{2,1,2} \\ c^2_{1,2,1} & c^2_{1,2,2} & c^2_{2,2,1} & c^2_{2,2,2} \end{pmatrix}$$

and

$$A_3=\begin{pmatrix} c^3_{1,1,1} & c^3_{1,2,1} & c^3_{2,1,1} & c^3_{2,2,1} \\ c^3_{1,1,2} & c^3_{1,2,2} & c^3_{2,1,2} & c^3_{2,2,2} \end{pmatrix},$$

respectively. For example, $c^2_{1,1,2}$ denotes the payoff for the players of type 2 when the players of types 1, 2 and 3 play strategies σ_1, σ_2 and σ_3, respectively.

For $0\leqslant k\leqslant N_1, 0\leqslant i\leqslant N_2$ and $0\leqslant j\leqslant N_3$, we define

$$\begin{aligned} f^1_{\sigma_1}(\cdot,i,j)=&\,c^1_{1,1,1}\frac{i}{N_2}\frac{j}{N_3}+c^1_{1,1,2}\frac{i}{N_2}\left(1-\frac{j}{N_3}\right)+c^1_{1,2,1}\left(1-\frac{i}{N_2}\right)\frac{j}{N_3}\\ &+c^1_{1,2,2}\left(1-\frac{i}{N_2}\right)\left(1-\frac{j}{N_3}\right), \end{aligned}$$

$$\begin{aligned} f^1_{\sigma_2}(\cdot,i,j)=&\,c^1_{2,1,1}\frac{i}{N_2}\frac{j}{N_3}+c^1_{2,1,2}\frac{i}{N_2}\left(1-\frac{j}{N_3}\right)+c^1_{2,2,1}\left(1-\frac{i}{N_2}\right)\frac{j}{N_3}\\ &+c^1_{2,2,2}\left(1-\frac{i}{N_2}\right)\left(1-\frac{j}{N_3}\right); \end{aligned}$$

$$\begin{aligned} f^2_{\sigma_1}(k,\cdot,j)=&\,c^2_{1,1,1}\frac{k}{N_1}\frac{j}{N_3}+c^2_{1,1,2}\frac{k}{N_1}\left(1-\frac{j}{N_3}\right)+c^2_{2,1,1}\left(1-\frac{k}{N_1}\right)\frac{j}{N_3}\\ &+c^2_{2,1,2}\left(1-\frac{k}{N_1}\right)\left(1-\frac{j}{N_3}\right), \end{aligned}$$

$$\begin{aligned} f^2_{\sigma_2}(k,\cdot,j)=&\,c^2_{1,2,1}\frac{k}{N_1}\frac{j}{N_3}+c^2_{1,2,2}\frac{k}{N_1}\left(1-\frac{j}{N_3}\right)+c^2_{2,2,1}\left(1-\frac{k}{N_1}\right)\frac{j}{N_3}\\ &+c^2_{2,2,2}\left(1-\frac{k}{N_1}\right)\left(1-\frac{j}{N_3}\right), \end{aligned}$$

$$f_{\sigma_1}^3(k,i,\cdot) = c_{1,1,1}^3 \frac{k}{N_1}\frac{i}{N_2} + c_{1,2,1}^3 \frac{k}{N_1}\left(1-\frac{i}{N_2}\right) + c_{2,1,1}^3\left(1-\frac{k}{N_1}\right)\frac{i}{N_2} + c_{2,2,1}^3\left(1-\frac{k}{N_1}\right)\left(1-\frac{i}{N_2}\right),$$

$$f_{\sigma_2}^3(k,i,\cdot) = c_{1,1,2}^3 \frac{k}{N_1}\frac{i}{N_2} + c_{1,2,2}^3 \frac{k}{N_1}\left(1-\frac{i}{N_2}\right) + c_{2,1,2}^3\left(1-\frac{k}{N_1}\right)\frac{i}{N_2} + c_{2,2,2}^3\left(1-\frac{k}{N_1}\right)\left(1-\frac{i}{N_2}\right),$$

For the first type of players, the transition rate switching strategy from σ_2 to σ_1 is defined as that for $i = 0,1,\ldots,N_2-1$ and $j = 0,1,\ldots,N_3-1$,

$$\lambda_\varepsilon^1(\cdot,i,j) = \varepsilon + \kappa_1 \max\left\{f_{\sigma_1}^1(\cdot,i,j) - f_{\sigma_2}^1(\cdot,i,j), 0\right\},$$

and the transition rate switching strategy from σ_1 to σ_2 is defined as that for $i = 1,2,\ldots,N_2$ and $j = 1,2,\ldots,N_3$,

$$\mu_\varepsilon^1(\cdot,i,j) = \varepsilon + \kappa_1 \max\left\{f_{\sigma_2}^1(\cdot,i,j) - f_{\sigma_1}^1(\cdot,i,j), 0\right\}.$$

For the second type of players, the transition rate switching strategy from σ_2 to σ_1 is defined as that for $k = 0,1,\ldots,N_1-1$ and $j = 0,1,\ldots,N_3-1$,

$$\lambda_\varepsilon^2(k,\cdot,j) = \varepsilon + \kappa_2 \max\left\{f_{\sigma_1}^2(k,\cdot,j) - f_{\sigma_2}^2(k,\cdot,j), 0\right\},$$

and the transition rate switching strategy from σ_1 to σ_2 is defined as that for $k = 1,2,\ldots,N_1$ and $j = 1,2,\ldots,N_3$,

$$\mu_\varepsilon^2(k,\cdot,j) = \varepsilon + \kappa_2 \max\left\{f_{\sigma_2}^2(k,\cdot,j) - f_{\sigma_1}^2(k,\cdot,j), 0\right\}.$$

For the third type of players, the transition rate switching strategy from σ_2 to σ_1 is defined as that for $k = 0,1,\ldots,N_1-1$ and $i = 0,1,\ldots,N_2-1$,

$$\lambda_\varepsilon^3(k,i,\cdot) = \varepsilon + \kappa_3 \max\left\{f_{\sigma_1}^3(k,i,\cdot) - f_{\sigma_2}^3(k,i,\cdot), 0\right\},$$

and the transition rate switching strategy from σ_1 to σ_2 is defined as that for $k = 1,2,\ldots,N_1$ and $i = 1,2,\ldots,N_2$,

$$\mu_\varepsilon^3(k,i,\cdot) = \varepsilon + \kappa_3 \max\left\{f_{\sigma_2}^3(k,i,\cdot) - f_{\sigma_1}^3(k,i,\cdot), 0\right\}.$$

Let $z(t)=(z_1(t),z_2(t),z_3(t))$, where $z_i(t)$ is the number of the players of type i who are playing strategy σ_1 at time t for $i=1,2,3$. We can model this game as a continuous-time level-dependent QBD process $\{z(t),t\geqslant 0\}$ on the state space

$$\begin{aligned}S=\{&(0,0,0),(0,0,1),\ldots,(0,0,N_1);(0,1,0),(0,1,1),\ldots,(0,1,N_1);\\&\ldots;(0,N_2,0),(0,N_2,1),\ldots,(0,N_2,N_1);(1,0,0),(1,0,1),\ldots,(1,0,N_1);\\&\ldots;(0,N_2,N_1),(1,N_2,N_1),\ldots,(N_3,N_2,N_1)\}\end{aligned}$$

with the state transition relation shown in Fig. 11.5.

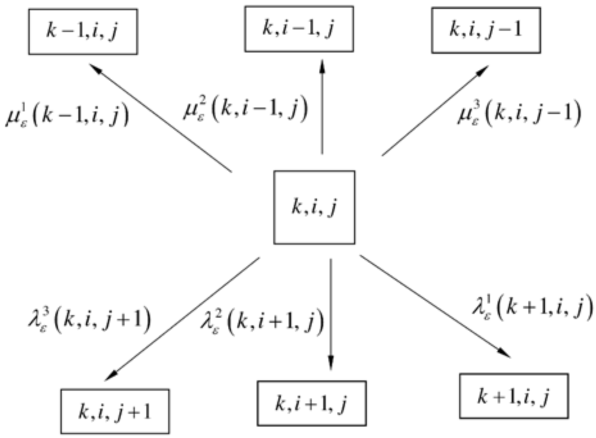

Figure 11.5 The state transition relation

For a given integer $j=0,1,\ldots,N_3$, we have the transition rates $\lambda_\varepsilon^1(\cdot,i,j)$ and $\mu_\varepsilon^1(\cdot,i,j)$ for the players of type 1, and $\lambda_\varepsilon^2(k,\cdot,j)$ and $\mu_\varepsilon^2(k,\cdot,j)$ for the players of type 2, thus we can construct a QBD process with the infinitesimal generator $Q_\varepsilon^{1,2}(j)$ with respect to the players of types 1 and 2, as indicated above. Applying the state transition relation, the infinitesimal generator of the QBD process describing the 3×2 asymmetric evolutionary game is given by

$$Q^{1,2,3}(\varepsilon)=\begin{pmatrix}\Lambda^{1,2}(0) & \varPsi_\varepsilon & & & \\ \varPhi_\varepsilon & \Lambda^{1,2}(1) & \varPsi_\varepsilon & & \\ & \ddots & \ddots & \ddots & \\ & & \varPhi_\varepsilon & \Lambda^{1,2}(N_3-1) & \varPsi_\varepsilon \\ & & & \varPhi_\varepsilon & \Lambda^{1,2}(N_3)\end{pmatrix},$$

where

$$\Lambda^{1,2}(l)=\begin{cases} Q_\varepsilon^{1,2}(0)-\Psi_\varepsilon, & l=0, \\ Q_\varepsilon^{1,2}(l)-\Psi_\varepsilon-\Phi_\varepsilon, & 1\leqslant l\leqslant N_3-1, \\ Q_\varepsilon^{1,2}(N_3)-\Phi_\varepsilon, & l=N_3, \end{cases}$$

$$\Psi_\varepsilon=\mathrm{diag}(\lambda_\varepsilon^3(0,0,\cdot),\lambda_\varepsilon^3(0,1,\cdot),\ldots,\lambda_\varepsilon^3(0,N_2,\cdot);\lambda_\varepsilon^3(1,0,\cdot),\lambda_\varepsilon^3(1,1,\cdot),\\ \ldots,\lambda_\varepsilon^3(1,N_2,\cdot);\ldots,\lambda_\varepsilon^3(N_1,0,\cdot),\lambda_\varepsilon^3(N_1,1,\cdot),\ldots,\lambda_\varepsilon^3(N_1,N_2,\cdot))$$

and

$$\Phi_\varepsilon=\mathrm{diag}(\mu_\varepsilon^3(0,0,\cdot),\mu_\varepsilon^3(0,1,\cdot),\ldots,\mu_\varepsilon^3(0,N_2,\cdot);\mu_\varepsilon^3(1,0,\cdot),\mu_\varepsilon^3(1,1,\cdot),\\ \ldots,\mu_\varepsilon^3(1,N_2,\cdot);\ldots,\mu_\varepsilon^3(N_1,0,\cdot),\mu_\varepsilon^3(N_1,1,\cdot),\ldots,\mu_\varepsilon^3(N_1,N_2,\cdot)).$$

For the QBD process $Q^{1,2,3}(\varepsilon)$, we write

$$U_{N_3}(\varepsilon)=Q_\varepsilon^{1,2}(N_3)-\Phi_\varepsilon,$$

for $1\leqslant k\leqslant N-1$,

$$U_k(\varepsilon)=Q_\varepsilon^{1,2}(k)-\Psi_\varepsilon-\Phi_\varepsilon+\Psi_\varepsilon[-U_{k+1}(\varepsilon)]^{-1}\Phi_\varepsilon,$$

and

$$U_0(\varepsilon)=Q_\varepsilon^{1,2}(0)-\Psi_\varepsilon+\Psi_\varepsilon[-U_1(\varepsilon)]^{-1}\Phi_\varepsilon.$$

We write

$$R_k(\varepsilon)=\Psi_\varepsilon[-U_{k+1}(\varepsilon)]^{-1},\quad 0\leqslant k\leqslant N-1.$$

Let

$$\pi(\varepsilon)=\left(\pi_0(\varepsilon),\pi_1(\varepsilon),\pi_2(\varepsilon),\ldots,\pi_{N_3-1}(\varepsilon),\pi_{N_3}(\varepsilon)\right)$$

be the stationary probability vector of the QBD process $Q^{1,2,3}(\varepsilon)$. Then

$$\pi_k(\varepsilon)=\tau(\varepsilon)v_0(\varepsilon)R_0(\varepsilon)R_1(\varepsilon)R_2(\varepsilon)\ldots R_{k-1}(\varepsilon),\quad 0\leqslant k\leqslant N_3,$$

where $v_0(\varepsilon)$ is the stationary probability vector of the censored Markov chain $U_0(\varepsilon)$ to level 0, and the constant $\tau(\varepsilon)$ is given by

$$\tau(\varepsilon)=\frac{1}{1+v_0(\varepsilon)\sum_{k=1}^{N_3}R_0(\varepsilon)R_1(\varepsilon)R_2(\varepsilon)\ldots R_{k-1}(\varepsilon)e}.$$

To compute the ESS numerically, we need to study the limiting distribution

$$\pi^*=\lim_{\varepsilon\to 0}\pi(\varepsilon)$$

where

$$\tau^* = \lim_{\varepsilon \to 0} \tau(\varepsilon), \quad v_0^* = \lim_{\varepsilon \to 0} v_0(\varepsilon), \quad R_k^* = \lim_{\varepsilon \to 0} R_k(\varepsilon), \quad 0 \leqslant k \leqslant N_3 - 1.$$

11.9 Notes in the Literature

Infinitesimal perturbation analysis is a sensitivity-analytic technique for discrete-event dynamic systems using single sample path performance measures. The basic approach is to obtain unbiased or strongly consistent estimates for derivatives of the stationary performance measures. Representative early works include Suri and Zazanis [71], Glasserman [32], Ho and Cao [44], Cassandras [18], Cao [9, 14] and Bao, Cassandras and Zananis [4].

For infinitesimal perturbation analysis of Markov chains, Cao, Yuan and Qiu [17] proposed a novel approach in terms of the realization factor. Cao [10, 11, 12, 13] developed two fundamental concepts: the realization matrix and the performance potential vector, to express derivatives of the stationary performance measures. The subsequent works may be referred to Cao and Chen [15], Cao and Wan [16] and the book by Cao [14].

Perturbation analysis of Markov chains is quite extensive, e.g., see Delebecque [23], Hunter [49, 50, 51], Korolyuk and Turbin [54], Cao and Chen [14], Roberts, Rosenthal and Schwartz [65], Bielecki and Stettner [5], Altman, Avrachenkov and Núñez-Queija [2], Solan and Vieille [70], Li and Liu [57] and Gambin, Krzyanowski and Pokarowski [31]. For Perturbation analysis of queueing systems, readers may refer to Latouche [55], Gong, Pan and Cassandras [33], Chang and Nelson [19], He [40], Grassmann and Chen [35], Grassmann [34] and He and Neuts [41].

The study of evolutionary game theory was originated in mathematical biology. Maynard Smith and Price [62] first introduced an important equilibrium concept: evolutionarily stable strategy (ESS), which is used to capture the possible stable outcomes of a dynamic evolutionary process, readers may also refer to Maynard Smith [60, 61] for more details. To derive the ESS, Taylor and Jonker [77] provided a basic method: replicator dynamics, which can be described as the ordinary differential equations based on the natural selection and the associated fitness. Based on this, the stability of the ordinary differential equations for the replicator dynamics can be used to express the ESS. Important examples of the replicator dynamics include the study of animal behavior, population ecology, population genetics, road traffic, computer networks, industrial evolutionary structure, and complex networks etc. Since 1980s, significant advances on the replicator dynamics have been made by the efforts of many researchers. Readers may refer to, such as, Vincent [79], Hines [42], Harsanyi and Selten [36], Robson [66], Cressman [22], Björnerstedt and Weibull [7], Vega-Redondo [78], Samuelson [68], Schlag [69], Hofbauer and Sigmund [47], Hofbauer [45], Hofbauer and Sandholm [46], Hart and Mas-Colell [38, 39], Hofbauer and Sigmund [48] and Suri [72].

For a stochastic evolutionary game, the replicator dynamics is not effective no longer when considering the strategic behavior over a long time horizon. To deal with the stochastic evolutionary games, perturbed Markov chains can be applied to analyze the ESS, that is, the perturbed stationary probability vector directly expresses the ESS based on its perturbed limit structure, which is the time proportion spent by this process in each population state. Foster and Young [27] gave stochastic evolutionary dynamics by means of the perturbed Markov chains whose stationary probability vector and the associated perturbed limit indicates the ESS. The subsequent works may refer to, for example, Takada and Kigami [74], Young [80, 81], Ellison [24], Kandori, Mailath and Rob [52], Nöldeke and Samuelson [63], Kandori and Rob [53], Binmore, Samuelson and Vaughan [6], Amir and Berninghaus [3], Robson and Vega-Redondo [67], Rhode and Stegeman [64], Fudenberg and Levine [28], Ellison [25], Corradi and Sarin [21], Tanaka [75], Hart [37], Blume [8], Tadj and Touzene [73], Taylor, Fudenberg, Sasaki and Nowak [76], Alos-Ferrer and Neustadt [1], Fudenberg and Imhof [29], Fudenberg, Nowakb, Taylorb and Imhof [30], Chen and Chow [20] and Fishman [26].

In this chapter, we mainly refer to Cao, Yuan and Qiu [17], Cao and Chen [15], Li and Zhao [58, 59], Li and Cao [56] and Li and Liu [57] for perturbation analysis of Markov chains; and Amir and Berninghaus [3], Tadj and Touzene [73], Alos-Ferrer and Neustadt [1], Fudenberg and Imhof [29] and Fudenberg, Nowakb, Taylorb and Imhof [30] for evolutionary games. At the same time, we also add some new results without publication for a more systematical organization of this chapter.

Problems

11.1 For a perturbed discrete-time level-independent QBD process,

(1) use the UL-type *RG*-factorization to compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\eta}_\varepsilon|_{\varepsilon=0}$;

(2) use the LU-type *RG*-factorization to compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\eta}_\varepsilon|_{\varepsilon=0}$;

(3) provide numerical examples to compare the results given in (1) and (2).

11.2 For a perturbed continuous-time level-independent QBD process,

(1) use the UL-type *RG*-factorization to compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\xi}_\varepsilon(N)|_{\varepsilon=0}$;

(2) use the UL-type *RG*-factorization to compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\varphi}_\varepsilon(\beta)|_{\varepsilon=0}$ for $0<\beta<1$.

11.3 For a perturbed PH distribution and a perturbed MAP which form a *MAP*/*PH*/1 queue,

(1) Provide sensitivity analysis for the mean of stationary waiting time;

(2) Provide sensitivity analysis for the mean of busy period.

11.4 Applying the matrix-geometric solution, provide numerical experiments for sensitivity analysis of a perturbed Markov chain of $GI/M/1$ type.

11.5 Applying the matrix iterative solution, provide numerical experiments for sensitivity analysis of a perturbed Markov chain of $M/G/1$ type.

11.6 For the perturbed $MAP/G/1$ queue, apply the UL-type *RG*-factorization to provide sensitivity analysis for the mean of the stationary queue length, and give some numerical experimentations.

11.7 For the perturbed $SM/PH/1$ queue, apply the LU-type *RG*-factorization to provide sensitivity analysis for the mean of the stationary queue length, and give some numerical experimentations.

11.8 Apply the *RG*-factorizations to provide sensitivity analysis for Markov chains of $GI/G/1$ type.

11.9 Provide a sensitivity analysis for the BMAP with irreducible matrix descriptor $\{D_k, k \geqslant 0\}$.

11.10 Provide a sensitivity analysis for the MMAP[K] with irreducible matrix descriptor $\{D_n, \boldsymbol{n} \in \mathbf{N}^k\}$.

11.11 Consider the 3×2, 2×3, $N\times 2$, $2\times M$ symmetric evolutionary games, and provide their ESS numerically.

11.12 Consider the 3×2, 2×3, $N\times 2$, $2\times M$ asymmetric evolutionary games, and provide their ESS numerically.

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Appendix

Appendix A Matrix Notation and Computation

In this appendix, we provide some notation for Kronecker product of matrices (e.g., see Graham [1]), Perron-Probenius theory of nonnegative matrices (e.g., see Seneta [2]), and inverses of matrices of infinite size.

A.1 Kronecker Product

Let A and B be two matrices, where the (i, j) th element of A is $a_{i,j}$. The Kronecker product of the two matrices A and B is defined as

$$A \otimes B = (a_{i,j} B),$$

and the Kronecker sum of the two matrices A and B is defined as

$$A \oplus B = I \otimes B + A \otimes I.$$

The useful properties of the Kronecker product are listed as follows:

(1) $(A + B) \otimes C = A \otimes C + B \otimes C$; $A \otimes (B + C) = A \otimes B + A \otimes C$.

(2) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

(3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

(4) $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$.

(5) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

(6) $\exp\{(A \oplus B)x\} = \exp\{Ax\} \otimes \exp\{Bx\}$.

(7) $\dfrac{\mathrm{d}}{\mathrm{d}x}[\exp\{Ax\} \otimes \exp\{Bx\}] = [\exp\{Ax\} \otimes \exp\{Bx\}](A \oplus B)$.

(8) If $A\alpha = \lambda\alpha$ and $B\beta = \mu\beta$, then

$$(A \otimes B)(\alpha \otimes \beta) = \lambda\mu(\alpha \otimes \beta)$$

and

$$(A \oplus B)(\alpha \otimes \beta) = (\lambda + \mu)(\alpha \otimes \beta).$$

A.2 Perron-Frobenius Theory

Let $\eta_1, \eta_2, \ldots \eta_{m-1}, \eta_m$ be all the eigenvalues of the matrix A of size m. The spectral radius of the matrix A is defined as

$$sp(A) = \max\{\eta_1, \eta_2, \ldots, \eta_{m-1}, \eta_m\}.$$

For simplicity of description, we also write $\rho(A) = sp(A)$.

Consider a square matrix $A = (a_{i,j})_{1 \leqslant i,j \leqslant m}$. If $a_{i,j} \geqslant 0$ for each pair (i, j) with $1 \leqslant i, j \leqslant m$, then A is called to be nonnegative; if $a_{i,j} > 0$ for each pair (i, j) with $1 \leqslant i, j \leqslant m$, then A is called to be positive. The nonnegative matrix A is said to be primitive if there exists a positive integer k such that $A^k > 0$. Obviously, A is primitive if $A > 0$. The nonnegative matrix A is said to be irreducible if $(I + A)^{m-1} > 0$.

For the nonnegative matrices, we list some useful properties as follows:

(1) If $0 \leqslant A \leqslant B$, then $0 \leqslant \rho(A) \leqslant \rho(B)$. If $0 < A < B$, then $0 < \rho(A) < \rho(B)$.

(2) If $A \geqslant 0$, then $\rho(I + A) = 1 + \rho(A)$.

(3) If $A \geqslant 0$, then $\rho(A)$ is an eigenvalue of the matrix A, and there exists a vector $x \gneq 0$ such that $Ax = \rho(A)x$ or $xA = \rho(A)x$.

(4) If $A \geqslant 0$ and there exists a vector $x > 0$ such that $Ax = \lambda x$, then $\lambda = \rho(A)$. If $A > 0$ and there exists a vector $x \gneq 0$ such that $Ax = \lambda x$, then $\lambda = \rho(A)$.

(5) If $A \geqslant 0$, $x > 0$ and $\alpha x \leqslant Ax \leqslant \beta x$, then $\alpha \leqslant \rho(A) \leqslant \beta$. If $A \geqslant 0$, $x > 0$ and $\alpha x < Ax < \beta x$, then $\alpha < \rho(A) < \beta$.

(6) If $A > 0$ or A is primitive, then there exists a unique positive vector x with $xe = 1$ such that $Ax = \rho(A)x$ or $xA = \rho(A)x$.

The following two theorems are always useful in the study of nonnegative matrices and stochastic models.

Theorem A.1 *Suppose the nonnegative matrix A is primitive.*

(1) $\rho(A) > 0$ *and* $\rho(A)$ *is an eigenvalue of the matrix A.*

(2) *There exist two vectors* $x > 0$ *and* $y > 0$ *such that* $Ax = \rho(A)x$ *and* $yA = \rho(A)y$.

(3) $\rho(A) > |\lambda|$ *for any eigenvalue* $\lambda \neq \rho(A)$.

(4) $\rho(A)$ *is a simple root of the characteristic equation* $\det(\lambda I - A) = 0$.

(5) $\lim_{k \to \infty} \left[\dfrac{A}{\rho(A)}\right]^k = yx.$

Theorem A.2 *Suppose the nonnegative matrix A is irreducible.*

(1) $\rho(A) > 0$ *and* $\rho(A)$ *is an eigenvalue of the matrix A.*

(2) *There exist two vectors* $x>0$ *and* $y>0$ *such that* $Ax=\rho(A)x$ *and* $yA=\rho(A)y$.

(3) $\rho(A)$ *is a simple root of the characteristic equation* $\det(\lambda I-A)=0$.

(4) *If* $\rho(A)=|\lambda|$ *for an eigenvalue* $\lambda\neq\rho(A)$, *then* $\lambda=\rho(A)\exp\left\{2\pi i\dfrac{k}{n}\right\}$ *with* $i^2=-1$. *At the same time,* $\rho(A)\exp\left\{2\pi i\dfrac{k}{n}\right\}$ *is a simple root of the characteristic equation* $\det(\lambda I-A)=0$.

A.3 Inverses of Matrices of Infinite Size

We now compute the inverses of the two matrices $I-R_U$ and $I-G_L$, which are useful in the study of stochastic models. The following two cases are listed.

Case Ⅰ The inverse of the matrix $I-R_U$

Let

$$R_U=\begin{pmatrix} 0 & R_{0,1} & R_{0,2} & R_{0,3} & \cdots \\ & 0 & R_{1,2} & R_{1,3} & \cdots \\ & & 0 & R_{2,3} & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Then

$$(I-R_U)^{-1}=\begin{pmatrix} I & Z_{0,1} & Z_{0,2} & Z_{0,3} & \cdots \\ & I & Z_{1,2} & Z_{1,3} & \cdots \\ & & I & Z_{2,3} & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix},$$

where

$$Z_{a,b}=\begin{cases}\displaystyle\sum_{k=0}^{b-a-1} X_k^{(a,b)}, & b\geqslant a+2,\\ R_{a,b}, & b=a+1,\end{cases}$$

$$X_k^{(a,b)}=\begin{cases} R_{a,a+1}R_{a+1,a+2}\cdots R_{b-2,b-1}R_{b-1,b}, & k=0,\\ \displaystyle\sum_{a<m_1<m_2<\cdots<m_k<b} Y^{(a,b)}_{m_1,m_2,\ldots,m_k}, & 1\leqslant k\leqslant b-a-1,\end{cases}$$

$$
\begin{aligned}
Y^{(a,b)}_{m_1,m_2,\ldots,m_k} &= R_{a,a+1}R_{a+1,a+2}\ldots R_{m_1-1,N_{(m_1)}}R_{N_{(m_1)},N_{(N_{(m_1)})}} \\
&\quad \ldots R_{m_k-1,N_{(m_k)}}R_{N_{(m_k)},N_{(N_{(m_k)})}}\ldots R_{b-2,b-1}R_{b-1,b},
\end{aligned}
$$

$$
N_{(m_k)} = \begin{cases} m_k+1, & m_k+1 \neq m_{k+1}, \\ N_{(m_{k+1})}, & m_k+1 = m_{k+1}. \end{cases}
$$

To understand the sequence $\left\{Y^{(a,b)}_{m_1,m_2,\ldots,m_k}\right\}$, we provide four examples as follows:

$$
\begin{gathered}
Y^{(0,5)}_{2,4} = R_{0,1}R_{1,3}R_{3,5}, \quad Y^{(0,5)}_{2,3,4} = R_{0,5}, \\
Y^{(2,6)}_{3,4} = R_{2,5}R_{5,6}, \quad Y^{(2,6)}_{3} = R_{2,4}R_{4,5}R_{5,6}.
\end{gathered}
$$

It is necessary to consider the following two special cases: Markov chains of $GI/G/1$ type and level-dependent QBD processes.

For a Markov chain of $GI/G/1$ type, let

$$
\widetilde{R}_U = \begin{pmatrix} 0 & R_1 & R_2 & R_3 & \ldots \\ & 0 & R_1 & R_2 & \ldots \\ & & 0 & R_1 & \ldots \\ & & & 0 & \ldots \\ & & & & \ddots \end{pmatrix}.
$$

Then

$$
(I-\widetilde{R}_U)^{-1} = \begin{pmatrix} I & Z_1 & Z_2 & Z_3 & \ldots \\ & I & Z_1 & Z_2 & \ldots \\ & & I & Z_1 & \ldots \\ & & & I & \ldots \\ & & & & \ddots \end{pmatrix},
$$

where

$$
Z_l = \sum_{i=1}^{\infty} \sum_{\substack{n_1+n_2+\ldots+n_i=l \\ n_j \geqslant 1,\, 1\leqslant j\leqslant i}} R_{n_1}R_{n_2}\ldots R_{n_i}.
$$

For a level-dependent QBD process, we write

$$
R_U = \begin{pmatrix} 0 & R_0 & & & \\ & 0 & R_1 & & \\ & & 0 & R_2 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.
$$

Then

$$(I-R_U)^{-1}=\begin{pmatrix} I & X_1^{(0)} & X_2^{(0)} & X_3^{(0)} & \cdots \\ & I & X_1^{(1)} & X_2^{(1)} & \cdots \\ & & I & X_1^{(2)} & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{pmatrix},$$

where

$$X_k^{(l)}=R_lR_{l+1}R_{l+2}\dots R_{l+k-1},\quad k\geqslant 1, l\geqslant 0.$$

Case Ⅱ The inverses of the matrix $I-G_L$

Let

$$G_L=\begin{pmatrix} 0 & & & & \\ G_{1,0} & 0 & & & \\ G_{2,0} & G_{2,1} & 0 & & \\ G_{3,0} & G_{3,1} & G_{3,2} & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$(I-G_L)^{-1}=\begin{pmatrix} I & & & & \\ T_{1,0} & I & & & \\ T_{2,0} & T_{2,1} & I & & \\ T_{3,0} & T_{3,1} & T_{3,2} & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$T_{a,b}=\begin{cases} G_{a,b}, & a=b+1, \\ \sum_{k=0}^{a-b-1} H_k^{(a,b)}, & a\geqslant b+2, \end{cases}$$

$$H_k^{(a,b)}=\begin{cases} G_{a,a-1}G_{a-1,a-2}\dots G_{b+2,b+1}G_{b+1,b}, & k=0, \\ \sum_{a>m_1>m_2>\dots>m_k>b} F_{m_1,m_2,\dots,m_k}^{(a,b)}, & 1\leqslant k\leqslant a-b-1, \end{cases}$$

$$F_{m_1,m_2,\dots,m_k}^{(a,b)}=G_{a,a-1}G_{a-1,a-2}\dots G_{m_1+1,L_{(m_1)}}G_{L_{(m_1)},L_{(L_{(m_1)})}} \dots G_{m_k+1,L_{(m_k)}}G_{L_{(m_k)},L_{(L_{(mk)})}}\dots G_{b+2,b+1}G_{b+1,b},$$

$$L_{(m_k)}=\begin{cases} m_k-1, & m_k-1\neq m_{k+1}, \\ L_{(m_{k+1})}, & m_k-1=m_{k+1}. \end{cases}$$

To understand the sequence $\left\{F_{m_1,m_2,\ldots,m_k}^{(a,b)}\right\}$, we provide two examples as follows:

$$F_{4,2}^{(5,0)} = G_{5,3}G_{3,1}G_{1,0}, \quad F_{4,3,2}^{(5,0)} = G_{5,1}G_{1,0}.$$

It is necessary to consider the following two special cases: Markov chains of $GI/G/1$ type and level-dependent QBD processes.

For a Markov chain of $GI/G/1$ type, let

$$\widetilde{G}_L = \begin{pmatrix} 0 & & & & \\ G_1 & 0 & & & \\ G_2 & G_1 & 0 & & \\ G_3 & G_2 & G_1 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$(I-\widetilde{G}_L)^{-1} = \begin{pmatrix} I & & & & \\ W_1 & I & & & \\ W_2 & W_1 & I & & \\ W_3 & W_2 & W_1 & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$W_l = \sum_{i=1}^{\infty} \sum_{\substack{n_1+n_2+\ldots+n_i=l \\ n_j \geqslant 1,\, 1\leqslant j\leqslant i}} G_{n_1}G_{n_2}\ldots G_{n_i}, \quad l \geqslant 1.$$

For a level-dependent QBD process, we write

$$G_L = \begin{pmatrix} 0 & & & & \\ G_1 & 0 & & & \\ & G_2 & 0 & & \\ & & G_3 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Then

$$(I-G_L)^{-1} = \begin{pmatrix} I & & & & \\ Y_1^{(1)} & I & & & \\ Y_2^{(2)} & Y_1^{(2)} & I & & \\ Y_3^{(3)} & Y_2^{(3)} & Y_1^{(3)} & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$Y_k^{(l)} = G_l G_{l-1} G_{l-2} \dots G_{l-k+1}, \quad l \geqslant k \geqslant 1.$$

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Appendix B Heavy-Tailed Distributions

In this appendix, we provide definitions and preliminary properties for light tail, heavy tail, long tail, subexponentiality and regular variation for sequences of nonnegative matrices. These preliminaries are useful in the study of block-structured stochastic models.

Following the standard definition of the light tail for a sequence of non-negative scalars, the light tail of a sequence of nonnegative matrices is defined as follows.

Definition B.1 *For a sequence $\{C_k\}$ of nonnegative scalars, it is called light-tailed if*

$$\sum_{k=1}^{\infty} C_k \exp\{\varepsilon k\} < +\infty, \quad \text{for some } \varepsilon > 0.$$

For a sequence $\{C_k\}$ of nonnegative matrices of size $m \times n$, it is called light-tailed if for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, the sequences $\{C_k(i, j)\}$ of nonnegative scalars are light-tailed, where $C_k(i, j)$ is the (i, j)th entry of C_k.

For the study of the light tail, readers may refer to Wilf [13] and Abate and Whitt [1] for more details, some of which are used in Sections 4.3 and 4.4.

Based on the light-tailed definition, we now define the heavy tail as follows.

Definition B.2 *For a sequence $\{C_k\}$ of nonnegative scalars, it is called heavy-tailed if*

$$\sum_{k=1}^{\infty} C_k \exp\{\varepsilon k\} = +\infty, \quad \text{for all } \varepsilon > 0.$$

For a sequence $\{C_k\}$ of nonnegative matrices of size $m \times n$, it is called heavy-tailed if there exists at least one pair (i_0, j_0) such that the sequences $\{C_k(i_0, j_0)\}$ of nonnegative scalars are heavy-tailed.

According to Subsection 1.3 and Subsection 1.4 in Embrechts, Klüppelberg and Mikosch [6], we provide the following definitions for a sequence of nonnegative scalars to be heavy-tailed, long-tailed, subexponential and regularly varying.

Definition B.3 (1) *A sequence* $\{c_n\}$ *of nonnegative scalars with* $\sum_{n=0}^{\infty} c_n < +\infty$ *is called heavy-tailed if for all* $\varepsilon > 0$

$$\sum_{n=0}^{\infty} c_n \exp\{\varepsilon n\} = +\infty. \tag{A}$$

Otherwise, $\{c_n\}$ *is called light-tailed. Denote by* $\mathcal{H}$ *the class of the heavy-tailed sequences.*

(2) *A sequence* $\{c_n\}$ *of nonnegative scalars with* $\sum_{n=0}^{\infty} c_n < +\infty$ *is called long-tailed if* $\overline{c_{\leqslant n}} > 0$ *for all* $n > N$, *where N is a large enough positive integer, and*

$$\lim_{n\to\infty} \frac{\overline{c_{\leqslant n+m}}}{\overline{c_{\leqslant n}}} = 1, \quad \text{for any integer } m \geqslant 0. \tag{B}$$

Denote $\mathcal{L}$ *as the class of the long-tailed sequences.*

(3) *A probability sequence* $\{c_n\}$ *is called subexponential if*

$$\lim_{n\to\infty} \frac{\overline{c_{\leqslant n}^{2*}}}{\overline{c_{\leqslant n}}} = 2. \tag{C}$$

Denote $\mathcal{S}$ *as the class of the subexponential sequences.*

(4) (i) *A sequence* $\{l_n\}$ *of nonnegative scalars with* $\sum_{n=0}^{\infty} l_n < +\infty$ *is called slowly varying if* $\overline{l_{\leqslant n}} > 0$ *for* $n > N$, *where N is a large enough positive integer, and* $\lim_{n\to\infty} \frac{\overline{l_{\leqslant [\lambda n]}}}{\overline{l_{\leqslant n}}} = 1$ *for any* $\lambda > 0$. *Denote* $\Re_0$ *as the class of the slowly varying sequences.* (ii) *A sequence* $\{c_n\}$ *of nonnegative scalars with* $\sum_{n=0}^{\infty} c_n < +\infty$ *is called regularly varying with index* $\alpha \in (-\infty, +\infty)$ *if* $\overline{c_{\leqslant n}} = n^{\alpha}\, \overline{l_{\leqslant n}}$ *for all* $n \geqslant N$. *Denote* $\Re_\alpha$ *as the class of the regularly varying sequences with index* α.

Let $\{c_k\}$ be a sequence of nonnegative scalars with $\sum_{k=0}^{\infty} c_k = c < +\infty$, then $\overline{c_{\leqslant n}^{2*}} = c^2 - c_{\leqslant n}^{2*}$, and

$$\lim_{n\to\infty}\frac{\overline{c^{2*}_{\leqslant n}}}{\overline{\overline{c_{\leqslant n}}}}=2c \tag{D}$$

if and only if $\left\{\frac{1}{c}c_k\right\}$ is subexponential. According to Teugels [15], properties of a subexponential sequence also hold for a sequence of nonnegative scalars satisfying (D).

To characterize subexponential asymptotics, we need to introduce the class $\mathcal{S}^*$, where $\mathcal{S}^*\subset\mathcal{S}$. For a sequence $\{c_k\}$ of nonnegative scalars with $\mu_c=\sum_{k=0}^{\infty}kc_k<+\infty$, we define $c_k^{(I)}=\frac{1}{\mu_c}\sum_{l=0}^{k}\overline{c_{\leqslant l}}$. Clearly, $\left\{c_k^{(I)}\right\}$ is a probability sequence. Following Klüppelberg [11], the integral tail of the sequence $\{c_k\}$ is defined as $\overline{c^{(I)}_{\leqslant k}}$ for $k\geqslant 1$. Klüppelberg [11] illustrated that for $\{c_k\}\in\mathcal{S}$, it is possible that $\left\{c_k^{(I)}\right\}\notin\mathcal{S}$. Klüppelberg [11] provided a useful sufficient condition under which $\left\{c_k^{(I)}\right\}\in\mathcal{S}$.

Definition B.4 *A sequence $\{c_k\}$ of nonnegative scalars is in $\mathcal{S}^*$ if $\mu_c<+\infty$ and* $\lim_{k\to\infty}\frac{\overline{c_{\leqslant k}}\circledast\overline{c_{\leqslant k}}}{\overline{\overline{c_{\leqslant k}}}}=2\mu_c$.

Proposition B.1 (1) *If $\{c_k\}\in\mathcal{S}^*$, then $\left\{c_k^{(I)}\right\}\in\mathcal{S}$.*

(2) *If $\{c_k\}\in\Re_{-\alpha}$ for $\alpha>1$, then $\left\{c_k^{(I)}\right\}\in\Re_{-(\alpha-1)}$.*

In (c) of Theorem 5.1 in Goldie and Klüppelberg [10], a condition on the closeness of convolution associated with two subexponential sequences was provided, which is restated in the following proposition.

Proposition B.2 *If $\{p_k\}$, $\{q_k\}\in S$, then $\{p_k\circledast q_k\}\in S$ if and only if $\{\lambda p_k+(1-\lambda)q_k\}\in S$ for all $\lambda\in(0,1)$.*

Proof The proof is clear by noting the following two relationships: (1) The sequence $\{p_k\}\in\mathcal{S}$ if and only if the function $p_{\leqslant k}\in\mathcal{S}$; and (2) the sequence $\{p_k\circledast q_k\}\in\mathcal{S}$ if and only if the function of convolution $p_{\leqslant k}*q_{\leqslant k}\in\mathcal{S}$, since $p_{\leqslant k}*q_{\leqslant k}=\sum_{l=0}^{k}p_l\circledast q_l$.

Definition B.5 (1) *(Tail-equivalent) Two sequences $\{c_k\}$ and $\{d_k\}$ of nonnegative scalars are called tail-equivalent, denoted as $\overline{c_{\leqslant k}}\sim\xi\overline{d_{\leqslant k}}$, if* $\lim_{k\to\infty}\frac{\overline{c_{\leqslant k}}}{d_{\leqslant k}}=\xi\in(0,+\infty)$.

(2) *(Tail-lighter) A sequence* $\{c_k\}$ *of nonnegative scalars is tail-lighter than a sequence* $\{d_k\}$ *of nonnegative scalars, or* $\{d_k\}$ *is tail-heavier than* $\{c_k\}$, *denoted as* $\overline{c_{\leqslant k}} = o(\overline{d_{\leqslant k}})$, *if* $\lim_{k\to\infty} \frac{\overline{c_{\leqslant k}}}{\overline{d_{\leqslant k}}} = 0.$

Remark B.1 (1) *It is easy to check that* $\mathcal{H}, \mathcal{L}$ *and* $\Re_{-\alpha}$ *are all closed with respect to tail-equivalence.*

(2) *Teugels [15] proved that* $\mathcal{S}$ *is closed with respect to tail-equivalence.*

(3) *Goldie and Klüppelberg [10]* (*p. 445*) *illustrated that* $\mathcal{S}^*$ *is closed with respect to tail-equivalence.*

Now, we extend the above notion for sequences of nonnegative scalars to that for sequences of nonnegative matrices. We will abuse the notation, without any confusion, by using the same $\mathcal{H}, \mathcal{L}, \mathcal{S}, \mathcal{S}^*$ and $\Re_\alpha$ for the classes of heavy-tailed, long-tailed, subexponential and regularly varying matrix sequences, respectively.

Definition B.6 *We assume that the nonnegative matrices* B_n *for* $n \geqslant 1$ *have the same size and* $\sum_{n=0}^{\infty} B_n$ *is finite.*

(1) *The sequence* $\{B_n\}$ *of nonnegative matrices is called heavy-tailed if there exists at least one entry sequence of* $\{B_n\}$ *which is heavy-tailed. Otherwise,* $\{B_n\}$ *is called light-tailed. Denote* $\mathcal{H}$ *as the class of the heavy-tailed matrix sequences of all sizes.*

(2) *The sequence* $\{B_n\}$ *of nonnegative matrices is called long-tailed if there exists at least one entry sequence of* $\{B_n\}$ *which is long-tailed and all the other entry sequences are either long-tailed or tail-lighter than a long-tailed entry sequence of* $\{B_n\}$. *Denote* $\mathcal{L}$ *as the class of the long-tailed matrix sequences of all sizes.*

(3) *The sequence* $\{B_n\}$ *of nonnegative matrices is called subexponential if there exists at least one entry sequence of* $\{B_n\}$ *which is subexponential and all the other entry sequences are either subexponential or tail-lighter than a subexponential entry sequence of* $\{B_n\}$. *Denote* $\mathcal{S}$ *as the class of subexponential matrix sequences of all sizes.*

(4) *The sequence* $\{B_n\}$ *of nonnegative matrices is called regularly varying with index* $\alpha \in (-\infty, +\infty)$ *if there exists at least one entry sequence of* $\{B_n\}$ *which is regularly varying with index* α *and all the other entry sequences are either regularly varying with index* $\beta \in (-\infty, \alpha]$ *or tail-lighter than an entry sequence in* $\Re_\alpha$ *of* $\{B_n\}$. *Denote* $\Re_\alpha$ *as the class of the regularly varying matrix sequences with index* α *of all sizes.*

(5) *The sequence* $\{B_n\}$ *of nonnegative matrices is in* $\mathcal{S}^*$ *if there exists at least one entry sequence of* $\{B_n\}$ *which is in* $\mathcal{S}^*$ *and all the other entry sequences are*

either in $\mathcal{S}^*$ or tail-lighter than a entry sequence in $\mathcal{S}^*$ of $\{B_n\}$.

We denote $b(i,j)$ as the (i,j) th entry of the matrix B. For a sequence $\{B_k\}$ of matrices, $B_{\leqslant k}$ and $\overline{B_{\leqslant k}}$ are defined elementwise as $B_{\leqslant k} = (b_{\leqslant k}(i,j))$ and $\overline{B_{\leqslant k}} = (\overline{b_{\leqslant k}}(i,j))$, respectively.

We denote Ω as the class of heavy-tailed matrix sequences satisfying the property that for each sequence $\{B_k\}$ in Ω there exists a heavy-tailed scalar sequence $\{\beta_k\}$ and a finite, non-zero nonnegative matrix W such that $\lim\limits_{k\to\infty} \dfrac{\overline{B_{\leqslant k}}}{\overline{\beta_{\leqslant k}}} = W$. The sequence $\{\beta_k\}$ of nonnegative scalars and the matrix W are called a uniformly dominant sequence of the matrix sequence $\{B_k\}$, and the associated ratio matrix, respectively.

Proposition B.3 *A heavy-tailed matrix sequence $\{B_k\}$ is in Ω if and only if there exists at least a pair (i_0, j_0) such that the sequence $\{b_k(i_0, j_0)\}$ is heavy-tailed and the limit $\lim\limits_{k\to\infty} \dfrac{\overline{b_{\leqslant k}}(i,j)}{\overline{b_{\leqslant k}}(i_0,j_0)}$ is either zero or a positive number for all i and j.*

Proof For the sufficiency, if there exists at least a pair (i_0, j_0) such that the sequence $\{b_k(i_0, j_0)\}$ is heavy-tailed and the limit $\lim\limits_{k\to\infty} \dfrac{\overline{b_{\leqslant k}}(i,j)}{\overline{b_{\leqslant k}}(i_0,j_0)}$ is either zero or a positive number for all i and j, then the matrix $W = \lim\limits_{k\to\infty} \dfrac{\overline{B_{\leqslant k}}}{\overline{b_{\leqslant k}}(i_0,j_0)}$ is finite, non-zero, and nonnegative. We take $\beta_k = b_k(i_0, j_0)$ for $k \geqslant 1$. This implies $\{B_k\} \in \Omega$.

For the necessity, if $\{B_k\} \in \Omega$, then there exists a heavy-tailed scalar sequence $\{\beta_k\}$ and a finite, non-zero, nonnegative matrix W such that $\lim\limits_{k\to\infty} \dfrac{\overline{B_{\leqslant k}}}{\overline{\beta_{\leqslant k}}} = W$. We assume that the (i_0, j_0) th entry $w(i_0, j_0)$ of the matrix W is not zero. Then

$$\lim_{k\to\infty} \frac{\overline{b_{\leqslant k}}(i,j)}{\overline{b_{\leqslant k}}(i_0,j_0)} = \lim_{k\to\infty} \frac{\dfrac{\overline{b_{\leqslant k}}(i,j)}{\overline{\beta_{\leqslant k}}}}{\dfrac{\overline{b_{\leqslant k}}(i_0,j_0)}{\overline{\beta_{\leqslant k}}}} = \frac{w(i,j)}{w(i_0,j_0)}$$

for all i and j, which is either zero or a positive number. Since $\overline{b_{\leqslant k}}(i_0,j_0) \backsim w(i_0,j_0)\overline{\beta_{\leqslant k}}$, $\{\beta_k\}$ is heavy-tailed and $w(i_0,j_0) > 0$, $\{b_k(i_0,j_0)\}$ is obviously heavy-tailed. This completes the proof.

Remark B.2 *It is possible that a heavy-tailed matrix sequence is not in Ω.*

To illustrate this, we consider the vector sequence

$$\overline{B_{\leqslant k}} = \left(\frac{1}{k}, \frac{1}{k}[1 + a\sin(2\pi \log k)] \right)$$

for $k \geqslant 1$. *Using* $\overline{G}_4$ *in Embrechts and Omey [7] (p. 81 – 82) yields that* $\{b_k(1,2)\} \in \mathcal{S}$ *while* $\{b_k(1,1)\} \in \Re_0$, *hence* $\{B_k\} \in \mathcal{S}$. *It is clear that* $\dfrac{\overline{b_{\leqslant k}}(1,2)}{\overline{b_{\leqslant k}}(1,1)} = 1 + a\sin(2\pi \log k)$ *has neither a finite nor an infinite limit.*

The following proposition provides a way of using a sequence of nonnegative scalars to characterize the tail of a sequence of nonnegative matrices. The proof is clear according to Definition B.6 and Remark B.1.

Proposition B.4 *Assume a heavy-tailed matrix sequence* $\{B_k\} \in \Omega$ *with a uniformly dominant sequence* $\{\beta_k\}$ *and the associated ratio matrix W.*

(1) $\{B_k\}$ *is long-tailed if and only if* $\{\beta_k\}$ *is long-tailed.*

(2) $\{B_k\}$ *is subexponential if and only if* $\{\beta_k\}$ *is subexponential.*

(3) $\{B_k\}$ *is regularly varying with index* $\alpha \in (-\infty, +\infty)$ *if and only if* $\{\beta_k\}$ *is regularly varying with index* α.

(4) $\{B_k\} \in \mathcal{S}^*$ *if and only if* $\{\beta_k\} \in \mathcal{S}^*$.

We provide some basic properties for heavy-tailed matrix sequences which are useful for characterizing the tail behavior of block-structured stochastic models. For simplicity, we assume that all the nonnegative matrices involved are square matrices with a common size *m*.

Proposition B.5 *For two sequences* $\{B_k\}$ *and* $\{C_k\}$ *of nonnegative matrices, if* (1) *there exists a nonnegative invertible matrix W such that* $B_k \geqslant WC_k$ *for all* $k > N$, *where N is a large enough positive integer, and* (2) $\{C_k\}$ *is heavy-tailed, then* $\{B_k\}$ *is heavy-tailed.*

Proof If $\{C_k\}$ is heavy-tailed, then there exists at least a pair (i_0, j_0) such that the (i_0, j_0) th entry sequence $\{C_k(i_0, j_0)\}$ is heavy-tailed. Since *W* is invertible, each column of *W* is not zero. For the i_0 th column of *W*, we assume that the (i_1, i_0) th entry $w(i_1, i_0) > 0$. Then we obtain

$$\sum_{l=1}^{m} w(i_1, l) C_k(l, j_0) \geqslant w(i_1, i_0) C_k(i_0, j_0).$$

Since $B_k \geqslant WC_k$, $b_k(i_1, j_0) \geqslant w(i_1, i_0)\, C_k(i_0, j_0)$. Notice that $w(i_1, i_0) > 0$ and $\{C_k(i_0, j_0)\}$ is heavy-tailed. It follows from (1) in Definition B.3 that $\{B_k(i_1, j_0)\}$ is heavy-tailed. Therefore, $\{B_k\}$ is heavy-tailed according to (1) in Definition B.6.

Remark B.3 *In Proposition B.5, condition* (1) *can be replaced, for example, by* $W > 0$, *or by* $WB_k \geqslant C_k$ *for all* $k > N$, *where* N *is a large enough positive integer.*

Proposition B.6 *For two sequences* $\{B_k\}$ *and* $\{C_k\}$ *of nonnegative matrices, suppose that* (1) *there exists a nonnegative invertible matrices* V *and a matrix* $W \geqslant V$ *such that* $VC_k \leqslant B_k \leqslant WC_k$ *for all* $k > N$, *where* N *is a large enough positive integer,* (2) $\{C_k\} \in \Omega$ *and* (3) $\{B_k\} \in \mathcal{L}$.

(1) *If* $\{C_k\} \in \mathcal{S}$, *then* $\{B_k\} \in \mathcal{S}$.

(2) *If* $\{C_k\} \in \mathcal{S}^*$, *then* $\{B_k\} \in \mathcal{S}^*$.

(3) *If* $\{C_k\} \in \Re_\alpha$ *for* $\alpha \in (-\infty, +\infty)$, *then* $\{B_k\} \in \Re_\alpha$.

Proof We only prove (1); (2) and (3) can be similarly proved.

Under assumptions that $\{C_k\} \in \mathcal{S}$ and $\{C_k\} \in \Omega$, it follows from Propositions B.3 and B.4 that there exists a pair (i_0, j_0) such that the (i_0, j_0)th entry sequence $\{C_k(i_0, j_0)\} \in \mathcal{S}$ and the limit $\lim\limits_{k\to\infty} \dfrac{\overline{C_{\leqslant k}}(i,j)}{\overline{\overline{C_{\leqslant k}}}(i_0, j_0)}$ is equal to either zero or a positive constant. Let $\lim\limits_{k\to\infty} \dfrac{\overline{C_{\leqslant k}}}{\overline{\overline{C_{\leqslant k}}}(i_0, j_0)} = \Gamma$ with the (i,j)th entry being $\tau(i,j)$. Then Γ is finite, non-zero, and nonnegative. Hence, for an arbitrarily small number $\varepsilon > 0$ there always exists a large enough positive integer N_0 such that

$$\Gamma - \varepsilon ee^{\mathrm{T}} < \frac{\overline{C_{\leqslant k}}}{\overline{\overline{C_{\leqslant k}}}(i_0, j_0)} < \Gamma + \varepsilon ee^{\mathrm{T}} \tag{E}$$

for all $k > N_0$. This, together with $VC_k \leqslant B_k \leqslant WC_k$ for all $k > N$ means that for all $k > \max\{N, N_0\}$,

$$V(\Gamma - \varepsilon ee^{\mathrm{T}})\overline{C_{\leqslant k}}(i_0, j_0) \leqslant \overline{B_{\leqslant k}} \leqslant W(\Gamma + \varepsilon ee^{\mathrm{T}})\overline{C_{\leqslant k}}(i_0, j_0). \tag{F}$$

We denote $h(i,j)$ and $f(i,j)$ as the (i,j)th entries of the matrices $H = V(\Gamma - \varepsilon ee^{\mathrm{T}})$ and $F = W(\Gamma + \varepsilon ee^{\mathrm{T}})$, respectively. Let $\Theta = \{(i,j) : h(i,j) > 0\}$ and $\gamma = \max\limits_{1\leqslant i,j\leqslant m} \{\tau(i,j)\} = \tau(i^*, j^*)$. Then $\gamma > 0$ due to $\Gamma \gneqq 0$. We take $0 < \varepsilon < \gamma$. Then $V(\Gamma - \varepsilon ee^{\mathrm{T}}) \neq 0$, since $\tau(i^*, j^*) - \varepsilon > 0$ and V is invertible. Since V is invertible and nonnegative, each row of V is not zero. In the i_1th row of V, we assume that the (i_1, i^*)th entry $v(i_1, i^*) > 0$. Therefore, for a small enough $\varepsilon > 0$,

$$h(i_1, j^*) = \sum_{l=1}^{m} v(i_1, l)[\tau(l, j^*) - \varepsilon] \geqslant v(i_1, i^*)\gamma - \varepsilon \sum_{l=1}^{m} v(i_1, l) > 0,$$

which implies that the set Θ is not empty. Now, we assume the pair $(i_1, j_1) \in \Theta$, then it follows from (F) that for all $k > \max\{N, N_0\}$,

$$h(i_1, j_1)\overline{c_{\leqslant k}}(i_0, j_0) \leqslant \overline{b_{\leqslant k}}(i_1, j_1) \leqslant f(i_1, j_1)\overline{c_{\leqslant k}}(i_0, j_0), \tag{G}$$

which illustrates that the sequence $\{b_k(i_1, j_1)\}$ is heavy tailed due to the two facts that $h(i_1, j_1) > 0$ and $\{c_k(i_0, j_0)\} \in \mathcal{S}$. For an arbitrary pair $(i_2, j_2) \neq (i_1, j_1)$, it is clear that for all $k > \max\{N, N_0\}$,

$$0 \leqslant \overline{b_{\leqslant k}}(i_2, j_2) \leqslant f(i_2, j_2)\overline{c_{\leqslant k}}(i_0, j_0). \tag{H}$$

It follows from (H) and the right-hand side of (G) that for all $k > \max\{N, N_0\}$,

$$0 \leqslant \overline{b_{\leqslant k}}(i_2, j_2) \leqslant \frac{f(i_2, j_2)}{h(i_1, j_1)}\overline{b_{\leqslant k}}(i_1, j_1), \tag{I}$$

which shows that the sequence $\{b_k(i_1, j_1)\}$ is not tail-lighter than the sequence $\{b_k(i_2, j_2)\}$ for each pair $(i_2, j_2) \neq (i_1, j_1)$. Therefore, the assumption that $\{B_k\} \in \mathcal{L}$ implies that $\{b_k(i_1, j_1)\} \in \mathcal{L}$. Notice that $\{c_k(i_0, j_0)\} \in \mathcal{S}$, $\{b_k(i_1, j_1)\} \in \mathcal{L}$, $h(i_1, j_1) > 0$ and $f(i_1, j_1) > 0$, it follows from (F) and (a) of Theorem 2.1 in Klüppelberg [11] that $\{b_k(i_1, j_1)\} \in \mathcal{S}$. Similarly, we can check that for an arbitrary pair (i, j) with $1 \leqslant i,\ j \leqslant m$, the entry sequence $\{b_k(i, j)\}$ is either subexponential or tail-lighter than the subexponential entry sequence $\{b_k(i_1, j_1)\}$ according to (I). It follows from (3) in Definition B.6 that $\{B_k\} \in \mathcal{S}$.

Remark B.4 (1) *In Proposition B.6, conditions* (2) *and* (3) *are necessary. Refer to (a) of Theorem 2.1 in Klüppelberg [11] for details. Condition* (1) *can be replaced, for example, by* $V > 0$ *and* $W > 0$.

For two sequences $\{B_k\}$ and $\{C_k\}$ of matrices, $B_{\leqslant k} * C_{\leqslant k}$ is defined elementwise as

$$B_{\leqslant k} * C_{\leqslant k} = \left(\sum_r b_{\leqslant k}(i, r) * c_{\leqslant k}(r, j) \right).$$

The following three propositions characterize tail behavior of convolutions for sequences of nonnegative matrices.

Proposition B.7 *If* (1) $\{p_k\} \in \mathcal{S}$, $\{q_k\}$ *is any probability sequence and* $\overline{q_{\leqslant k}} = o(\overline{p_{\leqslant k}})$, *and* (2) $\overline{B_{\leqslant k}} \backsim W\overline{p_{\leqslant k}}$, $\overline{C_{\leqslant k}} \backsim V\overline{q_{\leqslant k}}$, *then*

$$\overline{B_{\leqslant k} * C_{\leqslant k}} \backsim WV\overline{p_{\leqslant k}}.$$

Proof It is easy to check that

$$\overline{B_{\leqslant k} * C_{\leqslant k}} = \left(\sum_{r=1}^{m} \overline{b_{\leqslant k}(i,r) * c_{\leqslant k}(r,j)} \right).$$

Since $\overline{B_{\leqslant k}} \backsim W \overline{p_{\leqslant k}}$ and $\overline{C_{\leqslant k}} \backsim V \overline{q_{\leqslant k}}$, we obtain

$$\overline{b_{\leqslant k}(i,r)} \backsim w(i,r)\overline{p_{\leqslant k}} \text{ and } \overline{c_{\leqslant k}(r,j)} \backsim v(r,j)\overline{c_{\leqslant k}(r,j)}.$$

If $w(i,r)=0$ or $v(r,j)=0$, then we take $\overline{b_{\leqslant k}(i,r) * c_{\leqslant k}(r,j)} \backsim 0$. If $w(i,r) \neq 0$ and $v(r,j) \neq 0$, then

$$\overline{b_{\leqslant k}(i,r) * c_{\leqslant k}(r,j)} = w(i,r)v(r,j) \cdot \overline{\frac{b_{\leqslant k}(i,r)}{w(i,r)} * \frac{c_{\leqslant k}(r,j)}{v(r,j)}}.$$

Since

$$\overline{\frac{b_{\leqslant k}(i,r)}{w(i,r)}} \backsim \overline{p_{\leqslant k}}, \quad \overline{\frac{c_{\leqslant k}(r,j)}{v(r,j)}} \backsim \overline{q_{\leqslant k}},$$

$\{p_k\} \in \mathcal{S}$ and $\overline{q_{\leqslant k}} = o(\overline{p_{\leqslant k}})$, it follows from Proposition 2.7 in Sigman [14] that $\overline{p_{\leqslant k} * q_{\leqslant k}} \backsim \overline{p_{\leqslant k}}$, and so

$$\overline{b_{\leqslant k}(i,r) * c_{\leqslant k}(r,j)} = w(i,r)v(r,j)\overline{p_{\leqslant k}}.$$

Therefore, we obtain

$$\overline{B_{\leqslant k} * C_{\leqslant k}} \backsim \left(\sum_{r=1}^{m} w(i,r)v(r,j)\overline{p_{\leqslant k}} \right) = WV\overline{p_{\leqslant k}}.$$

This completes the proof.

Proposition B.8 *If $\{p_k\} \in \mathcal{S}$, and two sequences $\{C_k^{(1)}\}$ and $\{C_k^{(2)}\}$ of nonnegative matrices satisfy $\overline{C_{\leqslant k}^{(l)}} \backsim H_l \overline{p_{\leqslant k}}$ for $l=1,2$, where H_1 and H_2 are two finite, non-zero nonnegative matrices, then*

$$\overline{C_{\leqslant k}^{(1)} * C_{\leqslant k}^{(2)}} \backsim (H_1 e e^{\mathrm{T}} + e e^{\mathrm{T}} H_2)\overline{p_{\leqslant k}}.$$

Proof The condition that $\overline{C_{\leqslant k}^{(l)}} \backsim H_l \overline{p_{\leqslant k}}$ *for* $l=1,2$ implies

$$\overline{c_{\leqslant k}^{(1)}}(i,r) \backsim h_1(i,r)\overline{p_{\leqslant k}}, \quad \overline{c_{\leqslant k}^{(2)}}(r,j) \backsim h_2(r,j)\overline{p_{\leqslant k}}.$$

Note that

$$\overline{C_{\leqslant k}^{(1)} * C_{\leqslant k}^{(2)}} = \left(\sum_{r=1}^{m} \overline{c_{\leqslant k}^{(1)}(i,r) * c_{\leqslant k}^{(2)}(r,j)} \right),$$

using Theorem 5.1 in Goldie and Klüppelberg [10] leads to

$$\overline{c_{\leqslant k}^{(1)}(i,r) * c_{\leqslant k}^{(2)}(r,j)} = \left[h_1(i,r) + h_2(r,j)\right] \overline{p_{\leqslant k}}.$$

Simple computations lead to

$$\overline{C_{\leqslant k}^{(1)} * C_{\leqslant k}^{(2)}} \backsim \left(H_1 ee^{\mathrm{T}} + ee^{\mathrm{T}} H_2\right) \overline{p_{\leqslant k}}.$$

This completes the proof.

Proposition B.9 *If* $\overline{B_{\leqslant k}} \backsim Wk^{-\alpha} \overline{l_{\leqslant k}^{(1)}}$ *and* $\overline{C_{\leqslant k}} \backsim Vk^{-\beta} \overline{l_{\leqslant k}^{(2)}}$, *where* $\alpha,\ \beta > 0$, *the scalar sequences* $\{l_k^{(1)}\}$, $\{l_k^{(2)}\} \in \Re_0$, *and* W *and* V *are two finite, non-zero, nonnegative matrices, then*

$$\overline{B_{\leqslant k} * C_{\leqslant k}} = \begin{cases} WVk^{-\alpha} \overline{l_{\leqslant k}^{(1)}}, & \text{if } \alpha < \beta, \\ WVk^{-\beta} \overline{l_{\leqslant k}^{(2)}}, & \text{if } \alpha > \beta, \\ k^{-\alpha} \left[Wee^{\mathrm{T}} \overline{l_{\leqslant k}^{(1)}} + ee^{\mathrm{T}} V \overline{l_{\leqslant k}^{(2)}} \right], & \text{if } \alpha = \beta. \end{cases}$$

Proof The first two equalities are obtained by Proposition B.7. The last one follows from a proposition in Feller [8] (p. 278).

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