Abstract

In this paper, we consider Lur’e type differential-algebraic systems (LDS) and introduce the concept of strongly absolute stability. Such a notion is a generalization of absolute stability for Lur’e type standard state-space systems (LSS). By a Lur’e type Lyapunov function, we derive an LMI based stability criterion for LDS to be strongly absolutely stable. Using extended strictly positive realness (ESPR), we present the frequency-domain interpretation of the obtained criterion, by which we simplify the criterion and show that the criterion is a generalization of the well-known Popov criterion. Finally, we illustrate the effectiveness of the main results by a numerical example.

Keywords: Lur’e type systems; Differential-algebraic systems; Strongly absolute stability; Popov criterion; Linear matrix inequality (LMI)

1. Introduction

In the last two decades, differential-algebraic systems have attracted much attention due to their comprehensive applications in the Leontief dynamic model [1], electrical and mechanical models [2,3], etc. Depending on the applicable areas, these models are also called singular systems, semi-state systems, descriptor systems, or generalized state-space systems. As to linear time-invariant differential-algebraic systems, many fundamental theory have been reported...
(e.g., [4–6]). However, nonlinear differential-algebraic systems have not been thoroughly investigated. In [7–9], the researchers investigated the stability of nonlinear differential-algebraic systems under the assumption that the set of consistent initial conditions is given. And [10] considered stability of differential-algebraic systems consisting of a constant coefficient linear part and a small nonlinearity. [11,12] investigated optimal control problem of nonlinear differential-algebraic systems. In [13], the authors presented a sufficient condition for the system to be locally asymptotically stable.

In 1944, Lur’e and Postnikov introduced a novel method to deal with stability problem of nonlinear systems, which is called “nonlinearities isolation method” later and has been developed as the absolute stability theory. For many practical systems, by using this method, the nonlinear characteristic can be separated, which results in a feedback system called Lur’e type system whose forward path is a linear time-invariant system and the feedback path is a nonlinearity with sector constraints [14]. Lur’e type standard state-space systems (LSS) have been widely investigated and the most celebrated ones are Popov criterion (PC) and circle criterion (CC) [15,16]. PC is less conservative than CC because the Lyapunov function used by PC is a Lur’e type Lyapunov function which explicitly depends on the nonlinearity, while CC is related to a fixed quadratic Lyapunov function. And CC can deal with more diverse nonlinearities including time-varying ones. However, investigation on Lur’e type differential-algebraic systems (LDS) is very few. In [17], an LMI based strictly positive real (SPR) lemma is given for discrete-time differential-algebraic systems. Under the admissibility and SPR assumption of the involved linear time-invariant differential-algebraic system, it shows that the globally asymptotic stability of the feedback connection is guaranteed for the whole class of memoryless time-varying nonlinearities with dynamics constrained in the first and third quadrants. But it does not consider the impulse behavior of the overall system.

In this paper, we investigate the stability of LDS. First, the notion of index one is revised and strongly absolute stability of LDS is defined to be globally asymptotically stable and index one. Such a concept is a generalization of the absolute stability of LSS as well as the admissibility of linear time-invariant differential-algebraic systems. Then, an LMI based stability criterion is derived by constructing a Lur’e-type Lyapunov function. Furthermore, we present the frequency-domain interpretation of the LMI based stability criterion, which simplifies the criterion and shows that the criterion is a generalization of the well-known Popov criterion. Finally, we convert the nonstrict LMI conditions involved in the stability criterion into strict LMI conditions and illustrate the effectiveness of the obtained method by a numerical example.

The notations used here are standard in most respects. We use $\mathbb{R}$ to denote the set of real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n_1 \times n_2}$ are the obvious extensions to vectors and matrices of the specified dimensions. Let $I$ or $I_r$ denote the identity matrix with appropriate dimension. $M$ is a matrix with appropriate dimension, $M^T$ and $M^H$ stand for the transpose and complex conjugate transpose of $M$, respectively. $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ denote the real part and the image part of a complex number, respectively.

2. Preliminaries

Consider a linear time-invariant differential-algebraic system

$$ E \dot{x} = Ax $$

where $x \in \mathbb{R}^n$ is the state variable, matrices $A, E \in \mathbb{R}^{n \times n}$, $\text{rank}(E) = r \leq n$. 

We state here some basic definitions which will be used in the sequel and can be found in [2] and [5]. If \( \det(sE - A) \neq 0 \) for some complex number \( s \), then the pair \((E, A)\) is said to be regular. A regular pair \((E, A)\) is called impulsive-free if \( \deg \det(sE - A) = \rank(E) \). If all roots of \( \det(sE - A) = 0 \) lie in \( \Re(s) < 0 \), \((E, A)\) is called stable. And the pair \((E, A)\) is called admissible if it is regular, impulsive-free and stable. It is proved in [4] that \((E, A)\) is regular if and only if there exist two nonsingular matrices \( M \) and \( N \) such that \((E, A)\) can be transformed to the Weierstrass canonical form

\[
MEN = \begin{bmatrix} I_r & 0 \\ 0 & J \end{bmatrix}, \quad MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}
\]

where \( J \in \mathbb{R}^{(n-r) \times (n-r)} \) is a nilpotent matrix, \( A_1 \in \mathbb{R}^{r \times r} \). And system \((E, A)\) is impulsive-free if and only if \( J = 0 \).

The notion of index plays a key role in the classification and behavior of nonlinear differential-algebraic systems and can be thought of as the generalization of the nilpotent index of a linear time-invariant differential-algebraic system. [18] defined the concept of uniform index one which is revised here to perform our study.

Consider the nonlinear differential-algebraic system

\[
E \dot{x} = F(x)
\]

where \( F : \mathbb{R}^n \to \mathbb{R}^n \) is smooth enough and \( F(0) \equiv 0 \).

**Definition 1.** System (3) is said to be index one if the constant coefficient system

\[
E \dot{w} - F_x(\hat{x})w = g(t)
\]

is regular and impulsive-free for all \( \hat{x} \) in a neighborhood of the equilibrium point \( x = 0 \), where \( F_x \) is the Jacobian matrix \( \partial F / \partial x \).

**Remark 2.** By Definition 1, system (1) is regular and impulsive-free if and only if it is index one. Therefore, an index one system (3) has no impulse behavior around the equilibrium \( x = 0 \). By implicit function theorem, system (3) is solvable if it is index one [13,18].

Now, consider the following Lur’e type differential-algebraic system

\[
E \dot{x} = Ax + Bu,
\sigma = Cx,
\]

\[
u = -\phi(\sigma)
\]

where \( x \in \mathbb{R}^n \) is state variable, matrices \( A, E \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, \rank(E) = r \leq n \), \( \phi(\sigma) = [\phi_1(\sigma_1), \phi_2(\sigma_2), \ldots, \phi_m(\sigma_m)]^T \), \( \phi_i(\cdot) \) is assumed to be a time-invariant and smooth enough nonlinearity.

We call \( \phi(\sigma) \in F[0, K] \) if \( \phi_i(0) \equiv 0 \) and

\[
0 \leq \phi_i(\sigma_i)/\sigma_i \leq k_i, \quad i = 1, 2, \ldots, m,
\]

where \( K = \text{diag}[k_1, k_2, \ldots, k_m] \).

**Definition 3.** LDS (5) is said to be strongly absolutely stable with respect to \( F[0, K] \), if for \( \forall \phi \in F[0, K] \), LDS (5) is globally asymptotically stable and index one.
Remark 4. Just like on study of linear time-invariant descriptor systems, authors usually concern admissibility rather than only stability (see, e.g., [6]). Strongly absolute stability defined in Definition 3 concerns not only stability but also impulse behavior and solvability of LDS (5). So it is different from the notion of absolute stability given in [17] which only considers the global stability.

Remark 5. If $E = I$, LDS (5) reduces to an LSS that has been widely studied. It is easy to see that Definition 3 is a generalization of absolute stability of LSS as well as admissibility of linear time-invariant differential-algebraic systems.

The following lemmas and definitions will be used in the derivation of our main results.

Lemma 6. (See [19].) The pair $(E, A)$ is admissible if and only if there exists $X \in \mathbb{R}^{n \times n}$ such that
\[
E^T X = X^T E \geq 0,
\]
\[
A^T X + X^T A < 0.
\]

Lemma 7. The pair $(E, A)$ is regular and impulsive-free if there exists $X \in \mathbb{R}^{n \times n}$ such that
\[
E^T X = X^T E,
\]
\[
A^T X + X^T A < 0.
\] (7)

See Appendix A for the proof.

Lemma 8. (See [5].) Assume that $(E, A)$ is in the form of
\[
E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\]
then $(E, A)$ is regular and impulsive-free if and only if $A_{22}$ is nonsingular.

The transfer matrix of system
\[
E \dot{x} = Ax + Bu,
\]
\[
y = Cx + Du
\] (8)
is defined by $G(s) = C(sE - A)^{-1}B + D$.

Definition 9. (See [22,23].)

1. $G(s)$ is said to be positive real (PR) if $G(s)$ is analytic in $\text{Re}(s) > 0$ and satisfies $G(s) + G^*(s) \geq 0$ for $\text{Re}(s) > 0$;
2. $G(s)$ is said to be strictly positive real (SPR) if $G(s)$ is analytic in $\text{Re}(s) \geq 0$ and satisfies $G(j\omega) + G^*(j\omega) > 0$ for $\omega \in [0, +\infty)$;
3. $G(s)$ is said to be extended strictly positive real (ESPR) if it is SPR and satisfies $G(j\infty) + G^*(j\infty) > 0$. 
Lemma 10. (See [23].) The following statements are equivalent.

1. \((E, A)\) is admissible, \(D + DT > 0\) and \(G(s)\) is ESPR.
2. The following LMIs are feasible
   \[
   \begin{bmatrix}
   A^T X + X^T A & C^T - X^T B \\
   C - B^T X & -(D + DT)
   \end{bmatrix} < 0,
   \]
   \[E^T X = X^T E \geq 0.\]

3. Time-domain criterion

In this section, we will derive a time-domain criterion for LDS (5) to be strongly absolutely stable by a Lur’e type Lyapunov function.

Theorem 11. LDS (5) is strongly absolutely stable with respect to \(F[0, K]\) if there exist

- \(\tilde{A} = \text{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m) \geq 0, \quad A = \text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_m) \geq 0\)
- matrices \(X \in \mathbb{R}^{n \times n}, \overline{Q} \in \mathbb{R}^{n \times m}, Q \in \mathbb{R}^{n \times m}\), such that

\[
\begin{align*}
E^T X &= X^T E \geq 0, \\
C^T \tilde{A} &= E^T \overline{Q}, \\
C^T A &= E^T Q, \\
\begin{bmatrix}
A^T (X + \overline{Q} KC) + (X^T + C^T K Q^T)A & (1, 2) \\
(1, 2)^T & (2, 2)
\end{bmatrix} &< 0
\end{align*}
\]

where block matrix \((1, 2) = C^T + A^T (\overline{Q} - Q) - (X + \overline{Q} KC)^T B,\) and \((2, 2) = -2K^{-1} - (\overline{Q} - Q)^T B - B^T (\overline{Q} - Q)\).

Proof. Assume that all conditions of this theorem hold. By Lemma 6, it is easy to show that \((E, A)\) is admissible. Then, there exist two nonsingular matrices \(M, N \in \mathbb{R}^{n \times n}\), such that

\[
M EN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \quad MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}
\]

where \(A_1 \in \mathbb{R}^{r \times r}\). Compatible with (14), partition \(MB\) and \(CN\) as follows

\[
MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \quad CN = \begin{bmatrix} C_1 & C_2 \end{bmatrix}
\]

and let

\[
N^{-1} x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]

Thus LDS (5) is decomposed as

\[
\begin{align*}
\dot{x}_1 &= A_1 x_1 + B_1 u, \\
x_2 &= -B_2 u, \\
\sigma &= C_1 x_1 - C_2 B_2 u, \\
u &= -\phi(\sigma).
\end{align*}
\]
It is easy to see that LDS (5) is strongly absolutely stable with respect to $F[0, K]$ if and only if system (16) is strongly absolutely stable with respect to $F[0, K]$. Without loss of generality, we assume that LDS (5) is in the form of (16).

Next, the proof is mainly divided into two parts.

(i) We will prove that system (16) is index one for any $\phi \in F[0, K]$.

For any diagonal matrix $K_\Delta = \text{diag}(k_{\Delta_1}, \ldots, k_{\Delta_m})$ with $0 \leq K_\Delta \leq K$, pre- and post-multiplying (13) by \[
\begin{bmatrix}
I & C^T K_\Delta \\
0 & I
\end{bmatrix}
\]
and its transpose, respectively, we have
\[
\begin{bmatrix}
(1, 1) & (1, 2) \\
(1, 2)^T & -2K^{-1} - (\overline{Q} - Q)^T B - B^T (\overline{Q} - Q)
\end{bmatrix} < 0
\]
where block matrix
\[
(1, 1) = (A - BK_\Delta C)^T (X + QKC + (\overline{Q} - Q)K_\Delta C)
+ (X + QKC + (\overline{Q} - Q)K_\Delta C)^T (A - BK_\Delta C)
+ 2C^T (K_\Delta - K_\Delta K^{-1} K_\Delta) C
\]
and
\[
(1, 2) = C^T (I - 2K\Delta K^{-1}) + (A - BK_\Delta C)^T (\overline{Q} - Q)
+ (\hat{X} + (\overline{Q} - Q)K_\Delta C)^T B.
\]
Thus
\[
(A - BK_\Delta C)^T (X + QKC + (\overline{Q} - Q)K_\Delta C)
+ (X + QKC + (\overline{Q} - Q)K_\Delta C)^T (A - BK_\Delta C) < 0.
\]
(17)
At the same time, it is easy to check that
\[
E^T (X + QKC + (\overline{Q} - Q)K_\Delta C) = (X + QKC + (\overline{Q} - Q)K_\Delta C)^T E
\]
then, from Lemma 7, $(E, A - BK_\Delta C)$ is regular and impulsive-free, which implies, by Lemma 8,
\[
det(I - B_2 K_\Delta C_2) \neq 0
\]
in view of
\[
E = \begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix}, \quad A - BK_\Delta C = \begin{bmatrix}
A_1 - B_1 K_\Delta C_1 & -B_1 K_\Delta C_2 \\
-B_2 K_\Delta C_1 & I - B_2 K_\Delta C_2
\end{bmatrix}.
\]
From (16), we have
\[
\sigma = C_1 x_1 + C_2 B_2 \phi(\sigma).
\]
We claim that $x_1 = 0$ implies $\sigma = 0$. If it is not the case, there exists $\sigma_0 \neq 0$ satisfying $\sigma_0 = C_2 B_2 \phi(\sigma_0)$. 

Since $\phi \in F[0, K]$, there exists a diagonal matrix $K_{\sigma_0}$ with $0 \leq K_{\sigma_0} \leq K$ such that $\phi(\sigma_0) = K_{\sigma_0}\sigma_0$. Then

$$\sigma_0 = C_2 B_2 K_{\sigma_0}\sigma_0$$

which indicates $I - C_2 B_2 K_{\Delta_0}$ is singular, which contradicts with (18).

Then $x_1 = 0$ implies $\sigma = 0$, $\phi(\sigma) = 0$ and $x_2 = 0$.

Rewrite system (16) in the following form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix}$$

where $F_1(x_1, x_2) = A_1 x_1 - B_1 \phi(\sigma)$, $F_2(x_1, x_2) = x_2 - B_2 \phi(\sigma)$.

Then,

$$\frac{\partial F_2}{\partial x_2}|_{x_1=0, x_2=0} = I - B_2 \frac{\partial \phi}{\partial \sigma} (\frac{\partial \sigma}{\partial x_2})|_{x_1=0, x_2=0} C_2$$

$$= I - B_2 (\frac{\partial \phi}{\partial \sigma}|_{\sigma=0}) C_2$$

Since $\phi \in F[0, K]$, there exists a diagonal matrix $K_{\Delta}$ with $0 \leq K_{\Delta} \leq K$ such that

$$\frac{\partial \phi}{\partial \sigma}|_{\sigma=0} = K_{\Delta}$$

then inequality (18) implies that $\frac{\partial F_2}{\partial x_2}|_{x_1=0, x_2=0}$ is nonsingular, so is $\frac{\partial F_2}{\partial x_2}$ around the point $x_1 = 0, x_2 = 0$ by the continuity of $\frac{\partial \phi}{\partial \sigma}$. Hence, by Lemma 8, system (16) is index one for $\forall \phi \in F[0, K]$.

(ii) We will prove that LDS (16) is globally asymptotically stable for any $\phi \in F[0, K]$.

Consider the LSS

$$\begin{bmatrix} \dot{x}_1 \\ \sigma \end{bmatrix} = \begin{bmatrix} A_1 x_1 + B_1 u \\ C_1 x_1 - C_2 B_2 u \end{bmatrix}, \quad u = -\phi(\sigma)$$

which is obtained from (16) by removing the second equation.

Partition

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

conformably to (16), then, by (10)–(12), we have

$$X_{11} = X_{11}^T \geq 0, \quad X_{12} = 0, \quad C_1^T \bar{A} = \bar{Q}_1, \quad C_2^T \bar{A} = 0, \quad C_1^T \Delta = \bar{Q}_1, \quad C_2^T \Delta = 0.$$

Then (13) can be written as

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \Pi_{12}^T & \Pi_{22} & \Pi_{23} \\ \Pi_{13}^T & \Pi_{23}^T & \Pi_{33} \end{bmatrix} < 0$$

where
\[
\Pi_{11} = A_1^T (X_{11} + C_1^T \Delta K C_1) + (X_{11} + C_1^T \Delta K C_1)^T A_1,
\]
\[
\Pi_{12} = X_{21} + C_1^T K Q_2^T + A_1^T C_1^T \Delta K C_2,
\]
\[
\Pi_{13} = C_1^T + A_1^T C_1^T (\bar{\Lambda} - \Delta) - (X_{11} + C_1^T \Delta K C_1)^T B_1 - (X_{21} + Q_2 K C_1)^T B_2,
\]
\[
\Pi_{22} = X_{22} + X_{22}^T + Q_2 K C_2 + C_2^T K Q_2^T,
\]
\[
\Pi_{23} = C_2^T + (\bar{Q}_2 - Q_2) - C_2^T K \Delta C_1 B_1 - (X_{22} + Q_2 K C_2)^T B_2,
\]
\[
\Pi_{33} = -2K^{-1} - (\bar{\Lambda} - \Delta) C_1 B_1 - B_1^T C_1^T (\bar{\Lambda} - \Delta) - B_2^T (\bar{Q}_2 - Q_2) - (\bar{Q}_2 - Q_2)^T B_2.
\]

Pre-multiplying and post-multiplying (21) by
\[
\begin{bmatrix}
I & 0 & 0 \\
0 & -B^T_2 & I \\
0 & I & 0
\end{bmatrix}
\]
and it is transposition, respectively, gives
\[
\begin{bmatrix}
\tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} \\
\tilde{\Pi}_{12}^T & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} \\
\tilde{\Pi}_{13}^T & \tilde{\Pi}_{23}^T & \tilde{\Pi}_{33}
\end{bmatrix} < 0 \quad (22)
\]
where
\[
\tilde{\Pi}_{11} = \Pi_{11},
\]
\[
\tilde{\Pi}_{12} = A_1^T C_1^T \Delta K C_2 B_2 - (X_{11} + C_1^T \Delta K C_1)^T B_1 + A_1^T C_1^T (\bar{\Lambda} - \Delta) + C_1^T,
\]
\[
\tilde{\Pi}_{13} = \Pi_{12},
\]
\[
\tilde{\Pi}_{22} = -2K^{-1} - (\bar{\Lambda} - \Delta) C_1 B_1 - B_1^T C_1^T (\bar{\Lambda} - \Delta) - C_2 B_2 - B_2^T C_2^T,
\]
\[
\tilde{\Pi}_{23} = -B_2^T X_{22} - B_2^T C_2^T K Q_2^T + B_1^T C_1^T \Delta K C_2 - (\bar{Q}_2 - Q_2)^T C_2,
\]
\[
\tilde{\Pi}_{33} = \Pi_{22}.
\]

Consider system (20) and define a Lur’e Lyapunov function as
\[
V(x_1) = x_1^T X_{11} x_1 + 2 \sum_{i=1}^m \bar{\xi}_i \int_0^{\sigma_i} \phi_i(\sigma) d\sigma + 2 \sum_{i=1}^m \bar{\lambda}_i \int_0^{\sigma_i} (k_i \sigma_i - \phi_i(\sigma)) d\sigma.
\]

Calculating the derivative of \( V(x_1) \) along the trajectory of system (20), we have
\[
\dot{V}(x_1)|_{(20)} = \dot{x}_1^T X_{11} x_1 + x_1^T \dot{X}_{11} \dot{x}_1 + 2 \dot{x}_1^T X_{11} \dot{x}_1 + 2 \dot{x}_1^T \dot{x}_1 \Omega (\tilde{\Lambda} - \Delta)\phi + 2 \dot{x}_1^T \dot{x}_1 \tilde{\Delta} K(K\sigma - \phi)
\]
\[
= (A_1 x_1 + B_1 u)^T X_{11} x_1 + x_1^T X_{11} (A_1 x_1 + B_1 u) + 2(A_1 x_1 + B_1 u)^T C_1^T \tilde{\Lambda} \phi
\]
\[
+ 2(A_1 x_1 + B_1 u)^T C_1^T \tilde{\Delta} K(C_1 x_1 - C_2 B_2 u) - 2\phi^T (K^{-1} \phi - \sigma)
\]
\[
= x_1^T (A_1^T X_{11} + X_{11} A_1) x_1 + 2x_1^T X_{11} B_1 u + 2(A_1 x_1 + B_1 u)^T C_1^T (\bar{\Lambda} - \Delta)\phi
\]
\[
+ 2(A_1 x_1 + B_1 u)^T C_1^T \tilde{\Delta} K(C_1 x_1 - C_2 B_2 u) - 2\phi^T (K^{-1} \phi - \sigma)
\]
\[
= x_1^T (A_1^T X_{11} + X_{11} A_1) x_1 - 2x_1^T X_{11} B_1 \phi + 2(A_1 x_1 + B_1 u)^T C_1^T (\bar{\Lambda} - \Delta)\phi
\]
\[
+ 2(A_1 x_1 + B_1 u)^T C_1^T \tilde{\Delta} K(C_1 x_1 - C_2 B_2 u) - 2\phi^T (K^{-1} \phi - \sigma)
\]
\[
+ 2\phi^T (C_1 x_1 + C_2 B_2 \phi)
\]
\[
= \begin{bmatrix}
  x_1^T & \phi^T \\
\end{bmatrix}
\begin{bmatrix}
  \hat{\Pi}_{11} & \hat{\Pi}_{12} \\
  \hat{\Pi}_{21} & \hat{\Pi}_{22}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  \phi
\end{bmatrix}.
\]

(23)

From part (i), \( x_1 = 0 \) indicates \( \phi = 0 \), then LMI (22) guarantees \( \dot{V}(x_1)|_{(20)} < 0 \) for any \( x_1 \neq 0 \). Hence, system (20) is absolutely stable with respect to \( F[0, K] \).

In addition, part (i) has shown that \( x_1 = 0 \) indicates \( \phi = 0 \) and \( x_2 = 0 \), then absolute stability of system (20) implies

\[
\lim_{t \to +\infty} \phi(t) = 0, \quad \lim_{t \to +\infty} x_2(t) = 0.
\]

At the same time, note that \( \phi(\cdot) \) is continuous with respect to \( x_1 \). Then LDS (16) is globally asymptotically stable for any \( \phi \in F[0, K] \).

So far, we have proved that LDS (5) is strongly absolutely stable with respect to \( F[0, K] \). \( \Box \)

From Theorem 11, we can get the following corollaries.

**Corollary 12.** LDS (5) is strongly absolutely stable with respect to \( F[0, K] \) if there exist

\[
\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_m\} \succeq 0
\]

and matrices \( X \in \mathbb{R}^{n \times n}, \overline{Q} \in \mathbb{R}^{n \times m} \), such that

\[
E^T X = X^T E \succeq 0,
\]

(24)

\[
C^T \Lambda = E^T \overline{Q},
\]

(25)

\[
\begin{bmatrix}
  A^T X + X^T A & C^T + A^T \overline{Q} - X^T B \\
  C + \overline{Q}^T A - B^T X & -2K^{-1} - \overline{Q}^T B - B^T \overline{Q}
\end{bmatrix} < 0.
\]

(26)

**Corollary 13.** LDS (5) is strongly absolutely stable with respect to \( F[0, K] \) if there exists \( X \in \mathbb{R}^{n \times n} \), such that

\[
E^T X = X^T E \succeq 0,
\]

(27)

\[
\begin{bmatrix}
  A^T X + X^T A & C^T - X^T B \\
  C - B^T X & -2K^{-1}
\end{bmatrix} < 0.
\]

(28)

**Remark 14.** Corollary 12 is obtained from Theorem 11 by letting \( \Lambda = 0 \). Corollary 13 is obtained from Theorem 11 by letting \( \Lambda = 0 \) and \( \overline{\Lambda} = 0 \). When \( E = I \), that is, LDS (5) reduces to an LSS, Theorem 11 coincides with Theorem 1 in [20] and Corollaries 12 and 13 reduce to the LMIs based absolute stability criterion for LSS given in [21].

4. Frequency-domain interpretation

In this section, we will present frequency-domain interpretations for the LMI based criterions given in Section 3.

**Theorem 15** (Frequency-domain interpretation of Theorem 11). Conditions of Theorem 11 hold if and only if one of the following statements hold.
(i) \((E, A)\) is admissible and there exist \(\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_m\}\) and \(Q \in \mathbb{R}^{n \times m}\) with \(C^T \Lambda = E^T Q\) such that
\[
K^{-1} + (I + s\Lambda)G(s)
\]

is ESPR, where \(G(s) = C(sE - A)^{-1}B\).

(ii) There exist \(\hat{X} \in \mathbb{R}^{n \times n}\), \(\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_m\}\) and \(Q \in \mathbb{R}^{n \times m}\) such that the following LMIs are feasible
\[
E^T \hat{X} = \hat{X}^T E \succeq 0, \quad (29)
\]
\[
C^T \Lambda = E^T Q, \quad (30)
\]
\[
\begin{bmatrix}
A^T \hat{X} + \hat{X}^T A & C^T + A^T Q - \hat{X}^T B \\
C + Q^T A - B^T \hat{X} & -2K^{-1} - Q^T B - B^T Q
\end{bmatrix} < 0. \quad (31)
\]

**Proof.** Computing
\[
K^{-1} + Q^T B + (C + Q^T A)(sE - A)^{-1}B
\]
\[
= K^{-1} + Q^T B + C(sE - A)^{-1}B + Q^T A(sE - A)^{-1}B
\]
\[
= K^{-1} + Q^T B + C(sE - A)^{-1}B + Q^T (-I + sE(sE - A)^{-1})B
\]
\[
= K^{-1} + C(sE - A)^{-1}B + s\Lambda C(sE - A)^{-1}B
\]
\[
= K^{-1} + (I + s\Lambda)G(s) \quad (32)
\]
indicates (i) is equivalent to (ii) by Lemma 10.

**Necessity.** Assume that conditions of Theorem 11 hold. Define \(\Lambda = \bar{\Lambda} - \Lambda, Q = \bar{Q} - Q\) and \(\hat{X} = X + QKC\), then it easy to show that (ii) holds.

**Sufficiency.** Assume (ii) holds. Let \(K_{\Delta}\) is an arbitrary diagonal matrix with \(0 \leq K_{\Delta} \leq K\). Pre-multiplying and post-multiplying (31) by
\[
\begin{bmatrix}
I & C^T K_{\Delta} \\
0 & I
\end{bmatrix}
\]
and it is transposition, respectively, gives
\[
\begin{bmatrix}
\tilde{\Pi}_{11} & \tilde{\Pi}_{12} \\
\tilde{\Pi}_{12}^T & \tilde{\Pi}_{22}
\end{bmatrix} < 0 \quad (33)
\]
where
\[
\tilde{\Pi}_{11} = (A - BK_{\Delta}C)^T (\hat{X} + QK_{\Delta}C) + (\hat{X} + QK_{\Delta}C)^T (A - BK_{\Delta}C) + 2C^T (K_{\Delta} - K_{\Delta}K^{-1}K_{\Delta})C,
\]
\[
\tilde{\Pi}_{12} = C^T (I - 2K_{\Delta}K^{-1}) + (A - BK_{\Delta}C)^T Q - (\hat{X} + QK_{\Delta}C)^T B,
\]
\[
\tilde{\Pi}_{22} = -2K^{-1} - Q^T B - B^T Q.
\]
Then
\[
(A - BK_{\Delta}C)^T (\hat{X} + QK_{\Delta}C) + (\hat{X} + QK_{\Delta}C)^T (A - BK_{\Delta}C) < 0. \quad (34)
\]
In addition, by (29) and (30), it is easy to check
\[ E^T (\hat{X} + Q K_{\Delta} C) = (\hat{X} + Q K_{\Delta} C)^T E. \]

Then, by Lemma 7, we have \((E, A - B K_{\Delta} C)\) is regular and impulsive-free.

Furthermore, we claim that \((E, A - B K_{\Delta} C)\) with \(0 \leq K_{\Delta} \leq K\) is admissible, otherwise,
\[ \det(I + K_{\Delta} G(s_0)) = 0 \]
should hold for some \(s_0\) with \(\text{Re}(s_0) \geq 0\). Since \((E, A)\) is admissible, then there must exist \(\alpha \in (0, 1]\) and \(\omega_0 \in \mathbb{R}\) for a fixed \(K_{\Delta}\) such that
\[ \det(I + \alpha K_{\Delta} G(j\omega_0)) = 0. \]

Then there exists a nonzero vector \(\xi\) such that
\[ (I + \alpha K_{\Delta} G(j\omega_0))\xi = 0 \]
which indicates
\[ \xi = -\alpha K_{\Delta} G(j\omega_0)\xi. \]

In this case
\[ \xi^H (2K^{-1} + G^H(j\omega_0) + G(j\omega_0))\xi = 2\alpha \xi^H G^H(j\omega_0)(\alpha K_{\Delta} K^{-1}K_{\Delta} - K_{\Delta})G(j\omega_0)\xi \leq 0. \]

On the other hand, it holds that
\[
0 < \xi^H \left[ 2K^{-1} + G^H(j\omega_0) + G(j\omega_0) + (j\omega_0 G(j\omega_0))^H \Lambda + \Lambda(j\omega_0 G(j\omega_0)) \right] \xi \\
= \xi^H (2K^{-1} + G^H(j\omega_0) + G(j\omega_0))\xi - j\omega_0 \alpha \xi^H G^T(j\omega_0)(\Lambda K_{\Delta} - K_{\Delta}\Lambda)G(j\omega_0)\xi \\
= \xi^H (2K^{-1} + G^H(j\omega_0) + G(j\omega_0))\xi
\]
which contradicts (38). Therefore, \((E, A - B K_{\Delta} C)\) with \(0 \leq K_{\Delta} \leq K\) is admissible.

By Lemma 6, in view of (34) and (35) and the admissibility of \((E, A - B K_{\Delta} C)\), we have
\[ E^T (\hat{X} + Q K_{\Delta} C) = (\hat{X} + Q K_{\Delta} C)^T E \geq 0 \]
for any \(K_{\Delta}\) with \(0 \leq K_{\Delta} \leq K\).

Next, we shall construct \(X, \overline{\Lambda}, \Lambda, \overline{Q}, Q\) shown in the Theorem 11. Partition \(Q\) as
\[ Q = [q_1, \ldots, q_m]. \]

Define
\[ I_+ = \{i \mid \lambda_i \geq 0, \ 1 \leq i \leq m\}, \quad I_- = \{i \mid \lambda_i < 0, \ 1 \leq i \leq m\} \]
and choose
\[ \overline{\Lambda} = \text{diag}\{\overline{\lambda}_1, \ldots, \overline{\lambda}_m\}, \quad \overline{\lambda}_i = \begin{cases} \overline{\lambda}_i, & i \in I_+, \\ 0, & \text{otherwise,} \end{cases} \]
\[ \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_m\}, \quad \lambda_i = \begin{cases} \lambda_i, & i \in I_, \\ 0, & \text{otherwise,} \end{cases} \]
\[ \overline{Q} = [\overline{q}_1, \ldots, \overline{q}_m], \quad \overline{q}_i = \begin{cases} q_i, & i \in I_+, \\ 0, & \text{otherwise,} \end{cases} \]
\( Q = [q_1, \ldots, q_m], \quad q_i = \begin{cases} -q_i, & i \in I_-, \\ 0, & \text{otherwise}, \end{cases} \)

\( K_\Delta = \text{diag}\{k_{\Delta 1}, \ldots, k_{\Delta m}\}, \quad k_{\Delta i} = \begin{cases} k_i, & i \in I_-, \\ 0, & \text{otherwise}, \end{cases} \)

Then \( \Lambda = \overline{\Lambda} - \Delta, \quad Q = \overline{Q} - Q, \quad C^T \overline{\Lambda} = E^T \overline{Q}, \quad C^T \Lambda = E^T Q \) and \( \Lambda K_\Delta = -\Lambda K \).

By (39),

\[
0 \leq E^T (\hat{X} + QK_{\Delta}C) = E^T \hat{X} + E^T QK_{\Delta}C \\
= E^T \hat{X} + C^T \Lambda K_{\Delta}C \\
= E^T \hat{X} - C^T \Delta KC \\
= E^T \hat{X} - E^T QKC \\
= E^T (\hat{X} - QKC) \\
= (\hat{X} - QKC)E
\]

thus we can choose \( X = \hat{X} - QKC \).

Then conditions of Theorem 11 hold. \( \square \)

**Remark 16.** Condition (ii) of Theorem 15 presents a more simpler and convenient criterion for LDS (5) to be strongly absolutely stable.

By Theorem 15, the frequency-domain interpretations of Corollaries 12 and 13 are obvious.

**Corollary 17** (Frequency-domain interpretation of Corollary 12). Conditions of Corollary 12 hold if and only if \((E, A)\) is admissible and there exist \( \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_m\} \) and \( Q \) with \( \Lambda \geq 0 \) and \( C^T \Lambda = E^T \overline{Q} \) such that

\[
K^{-1} + (I + s\overline{\Lambda})G(s)
\]

is ESPR, where \( G(s) = C(sE - A)^{-1}B \).

**Corollary 18** (Frequency-domain interpretation of Corollary 13). Conditions of Corollary 13 hold if and only if \((E, A)\) is admissible and

\[
K^{-1} + G(s)
\]

is to is ESPR, where \( G(s) = C(sE - A)^{-1}B \).

So far, we have presented three criterions:

(i) **Generalized Popov criterion:** Theorems 11 and 15 represent this class, and they are the generalization of the well-known Popov criterion [20].

(ii) **Generalized quasi-Popov criterion:** Corollaries 12 and 17 belong to this class. In Corollary 17, we require \( \Lambda \geq 0 \) while Theorem 15 does not restrict the nonnegativity of \( \Lambda \). Since the criterion are more conservative than Popov criterion, we call them quasi-Popov criterion.

(iii) **Generalized circle criterion:** Corollaries 13 and 18 compose this class. In fact, they are obtained from Theorem 11 or 15 by letting \( \Lambda = 0 \), so they can be interpreted as conditions for the existence of a fixed quadratic Lyapunov function, which shows that they are the generalization of the well-known circle criterion for LSS [21].
5. Computational issues and numerical example

This section will address the computational issues and show how to deal with the nonstrict LMI conditions involved in our main results. A strict LMI based algorithm is obtained without any additional conservatism and a numerical example is given to illustrate the effectiveness of our method. Matlab 6.5 is used to check the LMI feasibility problems.

Let orthogonal matrix $U = [U_1 \ U_2]$ and $V = [V_1 \ V_2]$ be such that

$$E = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^T$$

from which it can be seen that $E V_2 = 0$ and $U_2^T E = 0$, where $\Sigma_r = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ with $\sigma_i > 0$ for $i = 1, 2, \ldots, r$.

Define

$$Y_1 = \{ \hat{X} \in \mathbb{R}^{n \times n} \mid E^T \hat{X} = \hat{X}^T E \geq 0, \ \text{rank}(E^T \hat{X}) = r \}$$

and

$$Y_2 = \{ \hat{X} = P E + U_2 S \mid P \in \mathbb{R}^{n \times n}, \ P > 0, \ S \in \mathbb{R}^{(n-r) \times n} \}.$$ 

It is shown in [6] that

$$Y_1 = Y_2. \quad (40)$$

It should be pointed out that constraint $\text{rank}(E^T \hat{X}) = r$ is implied by LMI (31).

Consider the sets

$$Z_1 = \{(\Lambda, Q) \mid C^T \Lambda = E^T Q \}$$

and

$$Z_2 = \{(\Lambda, Q) \mid V_2^T C^T \Lambda = 0, \ Q = U \begin{bmatrix} \Sigma_r^{-1} V_1^T C^T \Lambda \\ Q_2 \end{bmatrix}, \ Q_2 \in \mathbb{R}^{(n-r) \times m} \}.$$ 

To see $Z_1 \subseteq Z_2$, note that $C^T \Lambda = E^T Q$ yields

$$C^T \Lambda = V \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} U^T Q$$

which gives

$$\begin{bmatrix} V_1^T C^T \Lambda \\ V_2^T C^T \Lambda \end{bmatrix} = \begin{bmatrix} \Sigma_r U_1^T Q \\ 0 \end{bmatrix}$$

by pre-multiplying $V^T$. Thus we have

$$V_2^T C^T \Lambda = 0, \quad U_1^T Q = \Sigma_r^{-1} V_1^T C^T \Lambda$$

then there exists $Q_2 \in \mathbb{R}^{(n-r) \times m}$ such that

$$Q = U \begin{bmatrix} \Sigma_r^{-1} V_1^T C^T \Lambda \\ Q_2 \end{bmatrix}.$$ 

Hence, $Z_1 \subseteq Z_2$. 

On the other hand, assume \((\Lambda, Q) \in Z_2\),
\[
E^T Q = V \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} U^T \begin{bmatrix} \Sigma^{-1}_r V_1^T C^T A \\ Q_2 \end{bmatrix} = V \begin{bmatrix} V_1^T C^T A \\ V_2^T C^T A \end{bmatrix} = C^T A \tag{41}
\]
which implies that \(Z_2 \subseteq Z_1\).

Hence,
\[
Z_1 = Z_2. \tag{42}
\]

By (40), (42), the feasibility problem of the nonstrict LMI conditions in (ii) of Theorem 15 reduces to the following strict LMI based algorithm without any additional conservatism.

**Algorithm 1.**

1. **Step 1.** Determine the set \(\Omega = \{ \Lambda = \text{diag} \{ \lambda_1, \ldots, \lambda_m \} \mid V_2^T C^T \Lambda = 0 \} \).
2. **Step 2.** Find matrices \(\Lambda \in \Omega, P \in R^{n \times n}\) with \(P > 0, S \in R^{(n-r) \times n}, Q_2 \in R^{(n-r) \times m}\) such that
\[
\begin{bmatrix}
AT (PE + U_2 S) + (PE + U_2 S)^T A & CT + AT Q - (PE + U_2 S)^T B \\
C + QT A - BT (PE + U_2 S) & -2K^{-1} - Q^T B - BT Q
\end{bmatrix} < 0 \tag{43}
\]
where
\[
Q = U \begin{bmatrix} \Sigma^{-1}_r V_1^T C^T A \\ Q_2 \end{bmatrix} = U_1 \Sigma^{-1}_r V_1^T C^T \Lambda + U_2 Q_2.
\]

**Remark 19.** Since the matrix variable \(\Lambda\) is diagonal, the set \(\Omega\) can be described by setting some elements of \(\Lambda\) to be zero, which is easy to perform.

**Remark 20.** For Corollaries 1 and 2, we can also obtain strict LMI based algorithms that are special cases of Algorithm 1.

**Numerical example.** Consider LDS (5) with system matrices
\[
E = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A = \begin{bmatrix}
-2 & 0 & 1 & 0 \\
0 & -4 & 2 & 0 \\
0 & 0 & -1 & 1 \\
0 & 2 & 0 & 1
\end{bmatrix},
\]
\[
C = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 1 & 4 & 0
\end{bmatrix}, \quad B^T = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1
\end{bmatrix},
\]
\[
K = \begin{bmatrix}
k_1 & 0 \\
0 & k_2
\end{bmatrix}.
\]

In this example, we consider three cases: \(k_1 = k_2 = \delta, k_1 = 0.5k_2 = \delta\) and \(k_1 = 2k_2 = \delta\), and calculate the maximal sector bounds \(\max(\delta)\) by choosing a big, by checking out the feasibility, and by reselecting a smaller (if feasible) or a bigger (otherwise).
Table 1
Maximal sector bounds

<table>
<thead>
<tr>
<th>max(δ)</th>
<th>$k_1 = k_2 = δ$</th>
<th>$k_1 = 0.5k_2 = δ$</th>
<th>$k_1 = 2k_2 = δ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized Popov criterion</td>
<td>2.7030</td>
<td>2.6242</td>
<td>2.8193</td>
</tr>
<tr>
<td>Generalized quasi-Popov criterion</td>
<td>2.0134</td>
<td>1.9119</td>
<td>2.1615</td>
</tr>
<tr>
<td>Generalized Circle criterion</td>
<td>1.4595</td>
<td>1.0663</td>
<td>1.8375</td>
</tr>
</tbody>
</table>

By computation, $U_1, U_2, V_1, V_2$ and $Σ_r$ can be obtained by the singular value decomposition of $E$ as

$$U_1 = \begin{bmatrix} -0.4318 & 0.7917 & -0.4321 \\ -0.7331 & -0.5872 & -0.3432 \\ -0.5255 & 0.1686 & 0.8339 \\ -0.0000 & -0.0000 & 0.0000 \end{bmatrix}, \quad U_2 = \begin{bmatrix} -0.0000 \\ -0.0000 \\ 0.0000 \\ 1.0000 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} -0.5477 & 0.8328 & -0.0810 \\ -0.2890 & -0.2791 & -0.9157 \\ -0.7852 & -0.4781 & 0.3935 \\ -0.0000 & -0.0000 & 0.0000 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -0.0000 \\ -0.0000 \\ 0.0000 \\ -1.0000 \end{bmatrix}$$

and

$$Σ_r = \begin{bmatrix} 2.5365 & 0 & 0 \\ 0 & 2.1038 & 0 \\ 0 & 0 & 0.3748 \end{bmatrix}.$$

Since

$$V_2^T C^T = [0 \ 0]$$

then

$$Ω = \left\{ Λ \left| Λ = \begin{bmatrix} λ_1 & 0 \\ 0 & λ_2 \end{bmatrix} \right. \right\}.$$

The obtained criterions are performed by the corresponding algorithm for each case and the resulting maximal bounds are summarized in Table 1.

6. Conclusions

This paper has considered Lur’e type differential-algebraic systems (LDS) and introduced strongly absolute stability. Such a notion is a generalization of absolute stability for LSS and admissibility of linear time-invariant differential-algebraic systems. The classical Popov criterion for LSS has been generalized to LDS. The computational issues have been reduced strict LMI based algorithms and the presented numerical example has illustrated our results.

Appendix A. Proof of Lemma 7

From (7), $A$ is nonsingular, then $(E, A)$ is regular. Thus, without loss of generality, let $(E, A)$ is in the Weierstrass canonical form (2) and partition

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$
compatible with (2). Then, (7) indicates
\[ X_{22}^T J = J^T X_{22} \]  \hspace{1cm} (A.1)
and
\[
\begin{bmatrix}
A_1^T X_{11} + X_{11}^T A_1 & A_1^T X_{12} + X_{21}^T \\
X_{12}^T A_1 + X_{21} & X_{22} + X_{22}^T
\end{bmatrix} < 0
\]  \hspace{1cm} (A.2)
which implies that
\[ X_{22} + X_{22}^T < 0. \]  \hspace{1cm} (A.3)
Since \( J \) is nilpotent, there exists \( v > 0 \) such that \( J^v = 0 \) while \( J^{v-1} \neq 0 \). Assume \( v > 1 \). Premultiplying (A.1) by \( (J^{v-1})^T \) gives
\[
J^{v-1} X_{22}^T J = (J^v)^T X_{22} = 0.
\]  \hspace{1cm} (A.4)
Post-multiplying (A.4) by \( (J^{v-2}) \) indicates
\[
J^{v-1} X_{22}^T J^{v-1} = 0
\]  \hspace{1cm} (A.5)
then we have
\[
J^{v-1} (X_{22} + X_{22}^T) J^{v-1} = 0
\]
by which, taking into account (A.3), we have \( J^{v-1} = 0 \) which results in a contradiction. Thus \( v = 1 \), that is, \( J = 0 \). Hence, \((E,A)\) is regular and impulsive-free. \( \square \)

References


