Exponential time differencing Crank-Nicolson method with a quartic spline approximation for nonlinear Schrödinger equations

Xiao Liang\textsuperscript{a}, Abdul Q. M. Khaliq\textsuperscript{a}, Qin Sheng\textsuperscript{b}

\textsuperscript{a}Department of Mathematical Sciences and Center for Computational Science, Middle Tennessee State University, Murfreesboro, TN 37132-0001, USA
\textsuperscript{b}Department of Mathematics and Center for Astrophysics, Space Physics and Engineering Research, Baylor University, Waco, TX 76798-7328, USA

Abstract
This paper studies a central difference and quartic spline approximation based exponential time differencing Crank-Nicolson (ETD-CN) method for solving systems of one-dimensional nonlinear Schrödinger equations and two-dimensional nonlinear Schrödinger equations. A local extrapolation is employed to achieve a fourth order accuracy in time. The numerical method is proven to be highly efficient and stable for long-range soliton computations. Numerical examples associated with Dirichlet, Neumann and periodic boundary conditions are provided to illustrate the accuracy, efficiency and stability of the method proposed.

Keywords: quartic spline approximation, ETD-CN method, nonlinear Schrödinger equation, soliton, stability

1. Introduction
Nonlinear Schrödinger equations (NLSE) have been widely used to model a variety of important wave phenomena, such as solitary wave propagations [1], deep water turbulence [12], and laser beam transmissions [22]. In this paper, we are particularly interested in applications in fiber optics, where the NLSE models solitons that not only travels over a long distance without little optical losses, but also maintains the shape after wave collisions. Such an optical soliton is critical to optics and soliton-based communication systems [1]. The type of solitons owes to the balance of its nonlinear and dispersion...
effects, which keeps the wave shape of pulse either unchanged or changed periodically. This is also known as Kerr effect, or the quadratic electro-optic effect, in which the wave refractive index changes depending on its amplitude or strength. A fiber reshapes a pulse to form a soliton based on its total mass, which may be conserved over long distances [20].

Solving the NLSE numerically has been challenging due to their high non-linearity and solution complexity. Let $\psi(x,t)$ be the shape-phase movement and $|\psi|$ be the amplitude of an optical soliton. We consider an ETD-CN method [11] together with central difference and quartic spline approximations for the numerical solution of following equations:

\begin{equation}
 i \psi_t + \psi_{xx} + \lambda |\psi|^2 \psi = 0, \tag{1}
\end{equation}

and [26],

\begin{align*}
 i \psi_{1t} + \alpha \psi_{1xx} + \left( |\psi_1|^2 + \rho |\psi_2|^2 \right) \psi_1 &= 0, \\
 i \psi_{2t} + \alpha \psi_{2xx} + \left( \rho |\psi_1|^2 + |\psi_2|^2 \right) \psi_2 &= 0. \tag{2}
\end{align*}

and [2],

\begin{align*}
 i \psi_{1t} + \alpha_1 \psi_{1xx} + \left( \delta_{11} |\psi_1|^2 + \delta_{12} |\psi_2|^2 + \delta_{13} |\psi_3|^2 \right) \psi_1 &= 0, \\
 i \psi_{2t} + \alpha_2 \psi_{2xx} + \left( \delta_{21} |\psi_1|^2 + \delta_{22} |\psi_2|^2 + \delta_{23} |\psi_3|^2 \right) \psi_2 &= 0, \\
 i \psi_{3t} + \alpha_3 \psi_{3xx} + \left( \delta_{31} |\psi_1|^2 + \delta_{32} |\psi_2|^2 + \delta_{33} |\psi_3|^2 \right) \psi_3 &= 0. \tag{3}
\end{align*}

and finally [25]

\begin{equation}
 i \psi_t + \psi_{xx} + \psi_{yy} + |\psi|^2 \psi = 0. \tag{4}
\end{equation}

Equation (1) can be reformulated to

$$
\psi_t = (\mathcal{L} + \mathcal{N}) \psi,
$$

where

$$
\mathcal{L} \psi = i \psi_{xx}, \quad \mathcal{N} \psi = i \lambda |\psi|^2 \psi.
$$

The linear operator $\mathcal{L}$ contains a second-order spatial derivative and $i = \sqrt{-1}$. It governs the propagation phase of a dispersive wave and determines the phase traveling velocity and direction. The operator $\mathcal{N}$ balances the dispersion and nonlinearity of the optical wave. Such a balance determines a soliton which is mass conserved throughout propagations.

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Numerous numerical methods have been developed for solving the aforementioned equations, such as the split-step Fourier scheme [17], quadratic B-spline finite element method [20], and local discontinuous Galerkin method [25]. Exponential integrators have also been proposed by Berland et al. [3]. In the context of exponential integrators in general, the reader is referred to Hochbruck and Ostermann [7], and references therein. In this paper, an ETD-CN strategy is employed to deal with coupled nonlinearities in the NLSE. While our approach achieves expected accuracy and stability in computations, there is no need to solve nonlinear systems at every time step. The latter ensures a high efficiency.

To utilize an ETD-CN strategy, we decompose the complex function $\psi$ to real and imaginary parts in the vector form. This is different from any split-step Fourier scheme [17], where the Fourier transform and its inverse are applied directly to $\psi$. Our design is apparently more convenient when systems or multi-dimensional NLSE are considered.

Consider the following system [9]

$$
\begin{align*}
{i} \psi_{1t} + \alpha \psi_{1xx} + \left( |\psi_1|^2 + \varrho |\psi_2|^2 \right) \psi_1 &= 0, \\
{i} \psi_{2t} + \alpha \psi_{2xx} + \left( \varrho |\psi_1|^2 + |\psi_2|^2 \right) \psi_2 &= 0.
\end{align*}
$$

The equations (5) are often called the integrable Manakov equations. They model solitons in a birefringent optical fiber possessing a refractive index depending on the polarization and propagation direction of pulses [1, 26]. Two solitons can be formed by a decomposition of each light ray. We may show two situations of such a birefringence. One of them is for two solitons traveling together in the same direction, and another is for two solitons interacting and then traveling separately. Linearly implicit and Crank-Nicolson methods are used in [9] for solving (5). Their well-recognized results will be compared to show the superiority of our new method.

Set $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\delta_{11} = \delta_{13} = \delta_{22} = \delta_{31} = \delta_{33} = \sigma$, and let $\delta_{12} = \delta_{21} = \delta_{23} = \delta_{32} = \varrho$. Consider the following particular case of (3) with interesting applications [2],

$$
\begin{align*}
{i} \psi_{1t} + \psi_{1xx} + \left( \sigma |\psi_1|^2 + \varrho |\psi_2|^2 + \sigma |\psi_3|^2 \right) \psi_1 &= 0, \\
{i} \psi_{2t} + \psi_{2xx} + \left( \varrho |\psi_1|^2 + \sigma |\psi_2|^2 + \varrho |\psi_3|^2 \right) \psi_2 &= 0, \\
{i} \psi_{3t} + \psi_{3xx} + \left( \sigma |\psi_1|^2 + \varrho |\psi_2|^2 + \sigma |\psi_3|^2 \right) \psi_3 &= 0,
\end{align*}
$$

(6)

where $u_1$, $u_2$ and $u_3$ are solitary wave propagation phases in the optical system, $\sigma$ is the Landau constant describing the self modulation of soliton, and
$\varrho$ is the wave-wave interaction coefficient for the cross-modulations [2]. The system (6) represents an association of three perturbed high-order solitary waves, together with three pulses and multiple interactions enabling the solitons to reshape periodically. The total mass of each soliton is preserved. We shall be able to show this in later simulations.

A two-dimensional extension of (1),

\[ i \psi_t + \psi_{xx} + \psi_{yy} + |\psi|^2 \psi = 0, \]

is considered recently by Sulem and Patera [22], and Xu and Shu [25]. The investigators reveal details of a strong evidence of a singularity occurring in a finite time. With given coefficients and initial conditions, the solution of (7) also has a singularity in finite time [22]. We shall validate the singularity via our algorithms in new numerical experiments.

2. The ETD-CN Method

Consider the following nonlinear initial-boundary value problem:

\[
\begin{align*}
    u_t + Au &= F(u,t), \quad (x,t) \in \Omega \times (0, \infty); \\
    u &= u_b, \quad (x,t) \in \partial \Omega \times (0, \infty), \\
    u(x,0) &= u_0, \quad x \in \Omega,
\end{align*}
\]

where $\Omega \subset \mathbb{R}$ is bounded, $A$ is a linear operator in a Banach space and the function $F$ is bounded, see also [27]. Let $k = t_{n+1} - t_n$ be the temporal step size to be used on the mesh $\{t_n\}$ such that the approximate solution $u(t_n)$ is denoted by $u_n$. Then the formal solution [11, 19] of (8) is

\[
u(t_{n+1}) = e^{-kA}u(t_n) + k \int_0^1 e^{-kA(1-\tau)}F(u(t_n + \tau k), t_n + \tau k) d\tau.\]

While (9) serves as a foundation for the general ETD-CN method [11, 27], a different class of ETD schemes for problems with mildly nonlinear operator $A$ has also been studied [10, 24]. Herewith we employ a second order [1/1] Padé approximant $R_{1,1}$ to the matrix exponential in (9). This yields a standard ETD-CN scheme [11]:

\[
u_{n+1} = b_n + k(2I + kA)^{-1}[F(b_n, t_{n+1}) - F(u_n, t_n)], \]

\[
b_n = R_{1,1}(kA)u_n + 2k(2I + kA)^{-1}F(u_n, t_n),
\]

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in which
\[ R_{1,1}(kA) = 4(2I + kA)^{-1} - I, \]
where \( I \) is the identity operator.

Scheme (10) can be realized in two steps:

1. To acquire \( b_n \), we solve
   \[ (2I + kA)N_b = 4u_n + 2kF(u_n, t_n) \]
   for \( N_b \) first and then set
   \[ b_n = N_b - u_n. \]

2. We solve
   \[ (2I + kA)N_u = k[F(b_n, t_{n+1}) - F(u_n, t_n)] \]
   for \( N_u \) and subsequently,
   \[ u_{n+1} = b_n + N_u. \]

Kleefield et al. [11] have proved a quadratic convergence of the above-mentioned ETD-CN method. We are able to demonstrate computationally that its order of convergence in time is indeed two.

2.1. ETD-CN Method with Central Difference Approximation

Let \( re \) and \( im \) represent the real and imaginary parts of a complex number \( z \), respectively. Therefore, we may express \( z \) in the form of a real vector, that is,
\[ z \Rightarrow \begin{bmatrix} re \\ im \end{bmatrix} \]
and consequently,
\[ i \cdot z \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} z. \]

Now, to solve (1) numerically, we let \( 0 < h \ll 1 \) be a uniform spatial step size, \( n \) be the total number of mesh points in space, \( u = (u_1, u_2, \ldots, u_j, \ldots u_n)^\top \), where \( u_j \), are approximations of \( \psi \) at mesh points along \( x \), \( \tilde{u}_j \) be the vector form of \( \{u_j\}_1^n \), that is, \( \omega_j = \begin{bmatrix} v_j \\ w_j \end{bmatrix} \), in which \( v_j \) and \( w_j \) are real and imaginary parts of an approximation of that of \( u_j \), respectively. Then (1) under
homogeneous Dirichlet boundary conditions can be solved readily through the following semi-discretized system,

$$\omega_t + A_1 \omega = i f(|\omega|^2) \omega,$$  

(11)

where $A_1 \in \mathbb{R}^{2n \times 2n}$,

$$A_1 = -\frac{1}{h^2} \begin{bmatrix}
-2P & P & 0 & 0 & \cdots \\
P & -2P & P & 0 & \cdots \\
0 & P & -2P & P & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\cdots & 0 & 0 & P & -2P \\
\end{bmatrix}, \quad P = \begin{bmatrix} 0 & -1 \end{bmatrix},$$  

(12)

and

$$\omega = (\omega_1^T, \omega_2^T, \ldots, \omega_j^T, \ldots, \omega_n^T)^T,$$

based on a second order central finite difference approximation.

2.2. ETD-CN Method with a Quartic Spline Approximation

For any sufficiently smooth function $r(x_j) = r_j$, we define

$$\delta_x^2 r_j = r_{j-1} - 2r_j + r_{j+1}, \quad j = 1, 2, \ldots, n.$$  

(13)

Let $\tilde{u}(\xi_j, t)$ be a vector solution of (1) which is sufficiently smooth, and $s(x, t)$ be its quartic spline approximation. Then, according to Numerov condition [20], we have the following collocation relation,

$$m_{j-1} + 10m_j + m_{j+1} = \frac{12}{h^2} \delta_x^2 \omega_j + e_j, \quad j = 1, 2, \ldots, n,$$

where $\omega_j = \omega(x_j, t)$ are at least fourth order approximations of $\tilde{u}(x_j, t)$, $m_j = s_{xx}(x_j, t)$, $x_j \in \Omega$, and $e_j$ are local truncation errors given by

$$e_j = -\frac{h^4}{240} \tilde{u}_{x^6}(\xi_j, t),$$

where $\xi_j$ reside in neighborhoods of $x_j$, $j = 1, 2, \ldots, n$, respectively. Consequently, equation (1) together with homogeneous boundary conditions can be approximated by

$$\left(I_{2n} + \frac{1}{12} A_q\right) \omega_t + A_1 \omega = i \left(I_{2n} + \frac{1}{12} A_q\right) f(|\omega|^2) \omega,$$  

(14)
where $I_{2n} \in \mathbb{R}^{2n \times 2n}$ is the identity matrix and $A_q \in \mathbb{R}^{2n \times 2n}$, that is,

$$A_q = \begin{bmatrix}
-2I_2 & I_2 & 0 & 0 & \cdots \\
I_2 & -2I_2 & I_2 & 0 & \cdots \\
0 & I_2 & -2I_2 & I_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\cdots & 0 & 0 & I_2 & -2I_2 \\
\end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (15)$$

The equation (14) leads to

$$\omega_t + \left( I_{2n} + \frac{1}{12} A_q \right)^{-1} A_1 \omega = i f(|\omega|^2) \omega.$$

Denote $A_2 = (I_{2n} + (1/12) A_q)^{-1} A_1$. We acquire the desired ETD-CN method to use:

$$\omega_t + A_2 \omega = i f(|\omega|^2) \omega. \quad (16)$$

2.3. Mass Conservation

Since the mass of a numerical solution, $u$, can be evaluated via the spectral norm, for given $0 \leq \epsilon \ll 1$, its mass conservation can be defined by the following inequality [20],

$$\left| \|u\|_2^2 - c \right| \leq \epsilon |t - t_0|,$$

where $c > 0$ is a constant oriented from the mass of the analytic solution.

For any $u, v \in \mathbb{R}^{2n}$, we consider the inner product

$$\langle u, v \rangle = u^\top v = \sum_{j=1}^{2n} u_j v_j.$$

Therefore,

$$\|u\|_2 = \sqrt{h \langle u, u \rangle} = \sqrt{h \sum_{j=1}^{2n} u_j^2}.$$

Let $\omega$ be an approximation of $u$. Since $A_1$ is skew symmetric, we have [20]

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 \approx h \left\langle \frac{d\omega}{dt}, \omega \right\rangle = 0.$$
2.4. Stability Analysis

**Lemma 1.** [6, 21] Let
\[ E = \text{tridiag}(1, -2, 1) \in \mathbb{R}^{n \times n}. \]

Then its eigenvalues are
\[ \mu_j = -4 \sin^2 \frac{j\pi}{2(n+1)}, \quad j = 1, 2, \ldots, n. \]

Consequently, eigenvalues of the TST matrices \( T^\pm = I_n \pm rE \), where \( I_n \in \mathbb{R}^{n \times n} \) is the identity matrix, are
\[ \lambda_j^\pm = 1 \mp 4r \sin^2 \frac{j\pi}{2(n+1)}, \quad j = 1, 2, \ldots, n, \]
and
\[ \|T^\pm\|_2 = \max_{1 \leq j \leq n} |\lambda_j|. \]

Let \( \otimes \) be the Kronecker product of matrices [9, pp.137-141].

**Lemma 2.** [9] Let \( \lambda_i, \quad 1 \leq i \leq n, \) and \( \mu_j, \quad 1 \leq j \leq m, \) be eigenvalues of \( A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{m \times m}, \) respectively. Then the eigenvalues of \( A \otimes B \) are
\[ \lambda_1 \mu_1, \ldots, \lambda_1 \mu_m, \lambda_2 \mu_1, \ldots, \lambda_2 \mu_m, \ldots, \lambda_n \mu_1, \ldots, \lambda_n \mu_m. \]

**Theorem 1.** The ETD-CN schemes (11) and (14) are unconditionally linearly stable.

**Proof.** The linear stability of (11) and (14) depends on properties of
\[ M = 2I_{2n} + kA_1, \quad N = 2I_{2n} + kA_2, \]
respectively, where
\[ A_1 = -\frac{1}{h^2} E \otimes P, \quad A_2 = -\frac{1}{h^2} \left( I_{2n} + \frac{1}{12} E \otimes I_2 \right)^{-1} (E \otimes P), \]
and \( P, \ I_2 \) are given in (12) and (15), respectively.
We need to show that \( \|M^{-1}\|_2, \|N^{-1}\|_2 < 1 \). To this end, we observe that \( A_1^\top = -A_1, A_2^\top = -A_2 \) since \( P^\top = -P \). Recall the symmetry of \( E \) and \( I_2 \).

It follows that

\[
MM^\top = (2I_{2n} + kA_1)(2I_{2n} - kA_1) = 4I_{2n} - k^2A_1^2
\]

\[
= 4I_{2n} - \frac{k^2}{h^4}(E \otimes P)(E \otimes P) = 4I_{2n} - \frac{k^2}{h^4}E^2 \otimes P^2.
\]

Thus, utilizing Lemmas 2.1 and 2.2, eigenvalues of \( MM^\top \) are

\[
\text{eigen}(MM^\top)_j = 4 - \frac{k^2}{h^4} \text{eigen}(E^2 \otimes P^2)_j
\]

\[
= 4 + 16k^2 \sin^4 \frac{j\pi}{2(2n + 1)} > 1, \quad j = 1, 2, \ldots, 2n,
\]

due to the fact that \( \text{eigen}(P^2)_j = -1, j = 1, 2 \). The above inequalities imply that

\[
\|M\|_2 > 1.
\]

Hence,

\[
\|M^{-1}\|_2 < 1.
\]

This ensures the stability of (11).

On the other hand,

\[
NN^\top = 4I_{2n} - k^2A_2^2.
\]

Let \( \lambda \) be an eigenvalue and \( x \) be an eigenvector of \( NN^\top \). We have

\[
(4I_{2n} - k^2A_2^2)x = \lambda x.
\]

The above can be reformulated to

\[
A_2^2x = \mu^2x,
\]

where

\[
\mu^2 = \frac{4 - \lambda}{k^2}.
\]

Thus, \( A_2x = \mu x \) and this leads to

\[
\left( I_{2n} + \frac{1}{12}E \otimes I_2 \right)^{-1} (E \otimes P)x = \left( (E \otimes P)^{-1} \left( I_{2n} + \frac{1}{12}E \otimes I_2 \right) \right)^{-1} x = h^2 \mu x.
\]
Since \( I_{2n} + \frac{1}{12}E \otimes I_2 \) and \( E \otimes P \) are both nonsingular, \( A_2 \) is nonsingular and subsequently, \( \mu \neq 0 \). According to Lemmas 2.1 and 2.2, we have

\[
(E \otimes P)^{-1} \left( I_{2n} + \frac{1}{12}E \otimes I_2 \right) x = (E^{-1} \otimes P^{-1}) \left( I_{2n} + \frac{1}{12}E \otimes I_2 \right) x
\]

\[
= \left[ E^{-1} \otimes P^{-1} + \frac{1}{12} \left( E^{-1} \otimes P^{-1} \right) (E \otimes I_2) \right] x
\]

\[
= \left( E^{-1} \otimes P^{-1} + \frac{1}{12} I_n \otimes P^{-1} \right) x
\]

\[
= \left( E^{-1} + \frac{1}{12} I_n \right) \otimes P^{-1} x = \frac{1}{h^2 \mu} x.
\]

Recall that \( P^{-1} = P^\top \). We acquire immediately that

\[
\frac{1}{h^2 \mu} = \pm i \text{eigen} \left( E^{-1} + \frac{1}{12} I_n \right) = \pm i \left( -\frac{1}{4 \sin^2(j\pi/(2n+2))} + \frac{1}{12} \right),
\]

where \( i = \sqrt{-1} \) for \( j = 1, 2, \ldots, n \). It follows immediately that

\[
\mu^2 = \frac{1}{h^4} \left( \frac{1}{12} - \frac{1}{4 \sin^2(j\pi/(2n+2))} \right)^{-2} = -\frac{\sigma^2}{h^4}, \quad j = 1, 2, \ldots, n.
\]

Thus,

\[
\sigma_j^2, \sigma_{n+j}^2 = 2 \left( \frac{1}{3} - \frac{1}{\sin^2(j\pi/(2n+2))} \right)^{-2} > 0, \quad j = 1, 2, \ldots, n.
\]

Now, recall (17). We find that

\[
4 - \lambda_j = -\frac{k^2}{h^4} \sigma_j^2 < 0, \quad j = 1, 2, \ldots, 2n,
\]

and they imply that \( \max_{1 \leq j \leq 2n} \lambda_j > 4 > 1 \) which leads to

\[
\| N^{-1} \|_2 < \frac{1}{4} < 1.
\]

The proof is thus completed. \( \square \)
2.5. Stability Regions

In previous discussions, we have studied the linear stability of the ETD-CN method. Now, let us consider stability regions of the numerical method when applied to a nonlinear equation similar to that in [5]. Given an ordinary differential equation,

\[ u_t = cu + F(u), \]  

(18)

where \( F(u) \) is a nonlinear function. Assume that there exists a fixed point \( u_0 \) such that \( cu_0 + F(u_0) = 0 \). If \( u \) is a perturbation of \( u_0 \) and \( \lambda = F'(u_0) \), then after a linearization, we have

\[ u_t = cu + \lambda u. \]  

(19)

We say that the fixed point \( u_0 \) is stable if \( \text{Re}(c + \lambda) < 0 \). Denote \( x = \lambda k \) and \( y = ck \), where \( k \) is the time step size and apply the ETD-CN method (11) or (14) for solving (19). Then the corresponding amplification factor can be computed via

\[ \frac{u_{n+1}}{u_n} = r(x, y) = \frac{2 + y + 2x}{2 - y} + \frac{2x^2 + 2yx}{(2 - y)^2}. \]  

(20)

Figure 1: The stability region (light blue colored) of an ETD-CN method when \( x \) and \( y \) being real.

To obtain stability regions, we assume that \( r(x, y) < 1 \). When \( x \) and \( y \) are both real, the stability region of the ETD-CN method is shown in Figure
1. When $x$ is complex, we can fix $y$ with some non-positive values and plot
stability regions with the axes being real and imaginary parts of $x$ in Figure
2. According to Beylkin et al.[4], for a method to be useful, it is important
that stability regions grow as $|ck|$ becomes larger. We observed that the
stability region tends to the second order Runge-Kutta scheme as $y \to 0$;
and as $y$ decreases from $-5$ to $-20$, the stability region grows. This result
gives an indication of the stability of the ETD-CN method.

![ETD-CN Stability regions for different y values](image_url)

Figure 2: Stability regions of the ETD-CN method with $y$ fixed to some non-positive values.
2.6. Extrapolation of the ETD-CN Method

Following the local extrapolation procedure by Lawson and Morris [15], let us consider (10) over a temporal span of $2k$, that is,

$$
\begin{align*}
    u_{n+2} &= b_n + 2k(2I + 2kA)^{-1}[F(b_n, t_{n+2}) - F(u_n, t_n)], \\
    b_n &= R_{1,1}(2kA)u_n + 2k(2I + 2kA)^{-1}F(u_n, t_n),
\end{align*}
$$

where

$$
R_{1,1}(2kA) = 4(2I + 2kA)^{-1} - I.
$$

Alternately, when (10) is applied twice we have

$$
\begin{align*}
    u_{n+2} &= b_{n+1} + k(2I + kA)^{-1}[F(b_{n+1}, t_{n+2}) - F(u_{n+1}, t_{n+1})], \\
    b_{n+1} &= R_{1,1}(kA)u_{n+1} + 2k(2I + kA)^{-1}F(u_{n+1}, t_{n+1}), \\
    u_{n+1} &= b_n + k(2I + kA)^{-1}[F(b_n, t_{n+1}) - F(u_n, t_n)], \\
    b_n &= R_{1,1}(kA)u_n + 2k(2I + kA)^{-1}F(u_n, t_n),
\end{align*}
$$

in which

$$
R_{1,1}(kA) = 4(2I + kA)^{-1} - I.
$$

Let us denote solution procedures in (21) and (22) by $u_{n+2}^{(1)}$ and $u_{n+2}^{(2)}$, respectively. However,

$$
u_{n+2}^{(E)} = \frac{4}{3}u_{n+2}^{(2)} - \frac{1}{3}u_{n+2}^{(1)}
$$

is a fourth order approximation to the true solution at $t_{n+2}$ with a principal coefficient the truncation error $E_5 = 1/10$, see also [8, 13, 16] for the use of Richardson extrapolation to gain the same order of accuracy.

3. Numerical Experiments

In following numerical experiments, $h$ represents the spatial step size and $k$ represents the time step size. An error vector, $ER$, is measured by using the $\ell_\infty$ and $\ell_2$ norms defined by

$$
\|ER\|_\infty = \max_{1 \leq m < N} \left\{ \|\psi(x_m, t_n)\| - \|\tilde{u}_{1,m}^n + i\tilde{u}_{2,m}^n\| \right\},
$$

and

$$
\|ER\|_2 = \left[ \sum_{m=1}^N \|\psi(x_m, t_n)\| - \|\tilde{u}_{1,m}^n + i\tilde{u}_{2,m}^n\|^2 \right]^{1/2},
$$

where $N$ is the dimension of $\tilde{u}$. 
3.1. Single Soliton of the NLSE

We consider the following NLSE:
\[
\psi_t = i \psi_{xx} + 2i |\psi|^2 \psi, \quad -10 < x < 10, \quad t > 0,
\]
(24)
together with the same initial and boundary conditions in [17],
\[
\psi(x, 0) = \text{sech}(x), \quad -10 < x < 10,
\]
\[
\psi(x, t)_x = 0, \quad \text{at } x = \pm 10, \quad t > 0.
\]
The analytic solution of the problem is
\[
\psi_A(x, t) = \exp(i t) \text{sech}(x).
\]
(25)

In Table 1, errors and the CPU time of the ETD-CN method with central difference approximation and with a quartic spline approximation are listed. A comparison of the results reveals that the latter out performs the former without a considerable additional cost in terms of the CPU time. The extrapolated ETD-CN schemes with central difference approximation and with a quartic spline approximation seem to follow a similar pattern in Table 2. The CPU data are based on computations on a Matlab 7.9.0 platform operated on an Intel Core i5-2410M 2.30GHz workstation.

<table>
<thead>
<tr>
<th>( T )</th>
<th>CPU time (s)</th>
<th>( \ell_\infty ) error</th>
<th>( \ell_2 ) error</th>
<th>CPU time (s)</th>
<th>( \ell_\infty ) error</th>
<th>( \ell_2 ) error</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.7956</td>
<td>0.0014</td>
<td>0.0017</td>
<td>1.1372</td>
<td>2.553e-004</td>
<td>3.219e-004</td>
</tr>
<tr>
<td>10</td>
<td>1.8408</td>
<td>0.0019</td>
<td>0.0028</td>
<td>2.1560</td>
<td>4.074e-004</td>
<td>4.320e-004</td>
</tr>
<tr>
<td>15</td>
<td>2.2464</td>
<td>0.0027</td>
<td>0.0039</td>
<td>3.0093</td>
<td>5.914e-004</td>
<td>6.092e-004</td>
</tr>
<tr>
<td>20</td>
<td>3.2916</td>
<td>0.0011</td>
<td>0.0020</td>
<td>3.7801</td>
<td>7.675e-004</td>
<td>8.035e-004</td>
</tr>
</tbody>
</table>

Rates of convergence of the ETD-CN and extrapolated ETD-CN algorithms with quartic spline approximations are shown in Figures 3 and 4, respectively. It can be seen in Figure 4 that the extrapolated ETD-CN method with a quartic spline demonstrates a fourth order accuracy both in space and time.
Table 2: Efficiency comparisons for the solution of (24) with \( k = 0.01, \ h = 0.1 \).

<table>
<thead>
<tr>
<th>( T )</th>
<th>CPU time</th>
<th>( \ell_\infty ) error</th>
<th>( \ell_2 ) error</th>
<th>Extrapolated ETD-CN with Quartic Spline</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.5462</td>
<td>2.237e-004</td>
<td>3.144e-004</td>
<td>1.6536 8.382e-005 8.819e-005</td>
</tr>
<tr>
<td>10</td>
<td>3.4246</td>
<td>4.237e-004</td>
<td>5.534e-004</td>
<td>3.7128 7.309e-005 8.876e-005</td>
</tr>
<tr>
<td>15</td>
<td>5.3820</td>
<td>7.260e-004</td>
<td>8.335e-004</td>
<td>5.5380 1.055e-004 1.379e-004</td>
</tr>
<tr>
<td>20</td>
<td>7.9405</td>
<td>9.860e-004</td>
<td>0.0018</td>
<td>8.2837 1.122e-004 1.540e-004</td>
</tr>
</tbody>
</table>

Figure 3: The profile of \( \ell_2 \) errors versus the spatial step \( h \) (left) and temporal step \( k \) (right) for equation (24) using the ETD-CN method with quartic spline approximation.

Figure 4: The profile of \( \ell_2 \) errors versus the spatial step \( h \) (left) and temporal step \( k \) (right) for equation (24) using the extrapolated ETD-CN method with quartic spline approximation.

3.2. Collison of Two Solitons
Figure 5: A collision of two solitons (26) \( (h = 0.1, k = 0.01 \text{ and } T = [0, 5]). \)

Table 3: Mass conservations of the collision of two solitons (26).

<table>
<thead>
<tr>
<th>( T )</th>
<th>( |\psi|_2^2 )</th>
<th>Absolute Error in Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.000000</td>
<td>0</td>
</tr>
<tr>
<td>2.5</td>
<td>4.000556</td>
<td>0.000556</td>
</tr>
<tr>
<td>5</td>
<td>3.999805</td>
<td>0.000195</td>
</tr>
</tbody>
</table>

We replace the conditions in last section by

\[
\psi(x, 0) = \exp(2i(x + 10)) \text{sech}(x + 10) + \exp(2i(x - 10))\text{sech}(x - 10),
\]

\[
\psi(x, t)_x = 0 \quad \text{at} \quad x = \pm 25. \tag{26}
\]

Figure 5 demonstrates the numerical solution under (26), obtained via the ETD-CN method with quartic spline approximation. We observe in Figure 5 that the solitons merge to reach the peak, and then separate from each other with their original shapes and amplitudes. The absolute error of the mass in Table 3 indicates that these solitons are not affected by each other. This is
a good indication that the solitary waves may travel for a long distance in optical fibers without a significant loss of mass.

3.3. System of Two NLS Equations

We consider the following system \[9, 26\]

\[
\begin{align*}
i \psi_1 t + & \alpha \psi_{1xx} + (|\psi_1|^2 + \varrho |\psi_2|^2) \psi_1 = 0, \\
i \psi_2 t + & \alpha \psi_{2xx} + (\varrho |\psi_1|^2 + |\psi_2|^2) \psi_2 = 0.
\end{align*}
\]

(27)

We consider two key cases. In Case One, the following initial and boundary conditions are utilized.

\[
\begin{align*}
\psi_1(x,0) &= \sqrt{\frac{2\alpha}{1 + \varrho}} \sech(\sqrt{\alpha} x) \exp(i \upsilon x), \\
\psi_2(x,0) &= \sqrt{\frac{2\alpha}{1 + \varrho}} \sech(\sqrt{\alpha} x) \exp(i \upsilon x), \\
\psi_1(x,t)_x &= \psi_2(x,t)_x = 0 \text{ at } x = -20, 60,
\end{align*}
\]

(28)

where \(\varrho = 1\), \(\upsilon = 1\) and \(\alpha = 1\). Corresponding analytic solutions of (27) are given in [9]:

\[
\psi_{A_j}(x,t) = \sqrt{\frac{2\alpha}{1 + \varrho}} \sech(\sqrt{\alpha} (x - \upsilon t)) \exp(i (\upsilon x - (\upsilon^2 - \alpha) t)) \text{, } j = 1, 2.
\]

(29)

The soliton solution of (27) together with (28) obtained via the ETD-CN method with quartic spline approximation \((h = 0.1, k = 0.01 \text{ and } T = [0, 30])\) is shown in Figure 6. The modulus \(|\psi_2|\) is identical to \(|\psi_1|\) for the same initial values. We observe that the phase of the soliton moves to the right at a constant speed of 1. The solitons indicate that, when the amplitudes of the underlying pulses are equal, the waves should propagate with the same shape. Their phase traveling directions and speeds are identical, respectively.

In Table 4, we compare the convergence of the ETD-CN method with central difference approximation and the ETD-CN method with quartic spline approximation to the accuracy of linearly implicit and Crank-Nicolson methods [9]. The linearly implicit method loses its theoretical order of convergence when the nonlinear term is treated explicitly, while the ETD-CN method with central difference approximation maintains its theoretical order of convergence with much less errors as shown in Table 4. The same \(h, k\).
Figure 6: Numerical simulation of system (27) with initial conditions (28).
Table 4: A comparison of $L_\infty$ errors of solutions to system (27)-(28) ($h = 0.1, k = 0.01$ and $T = [0,30]$).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.047690</td>
<td>0.01972</td>
<td>0.00595</td>
<td>5.60342e-5</td>
</tr>
<tr>
<td>10</td>
<td>0.089070</td>
<td>0.03777</td>
<td>0.01125</td>
<td>9.08083e-5</td>
</tr>
<tr>
<td>15</td>
<td>0.126621</td>
<td>0.05582</td>
<td>0.01663</td>
<td>2.17458e-4</td>
</tr>
<tr>
<td>20</td>
<td>0.188602</td>
<td>0.07379</td>
<td>0.02204</td>
<td>4.49535e-4</td>
</tr>
<tr>
<td>25</td>
<td>0.182201</td>
<td>0.09149</td>
<td>0.02768</td>
<td>7.73105e-4</td>
</tr>
<tr>
<td>30</td>
<td>0.276529</td>
<td>0.109349</td>
<td>0.03320</td>
<td>0.0011933</td>
</tr>
</tbody>
</table>

and other parameters are used. The ETD-CN method is more accurate and efficient as compared with Crank-Nicolson method because the nonlinear equations do not need to be solved at each time step. When a quartic spline approximation is used, the method reaches a high accuracy in space. This is due to the fact that the system of two NLSE generates errors in both amplitude and horizontal position due to the horizontal movement of the wave. The precision of the ETD-CN method with a quartic spline approximation makes it better suited in this situation. During the computation, we only employ the $LU$ decomposition once, and matrix multiplications for several times within the loop of updating values of $\psi(\cdot, t + 1)$. These demonstrate the satisfactory efficiency of our newly constructed ETD-CN method.

Table 5 is devoted to comparisons of the numerical results by the extrapolated ETD-CN method with quartic spline approximation, and results given in [8] via Richardson extrapolations. The same parameters given by [8] are used: $\alpha = 0.5$, $\rho = 2/3$, $\nu = 1$, $h = 0.2$, $k = 0.05$.

Table 5: The $\ell_\infty$ error comparison of the extrapolated ETD-CN method with quartic spline approximation, with the Richardson extrapolation method [8].

<table>
<thead>
<tr>
<th>$T$</th>
<th>Ismail and Alamri [8]</th>
<th>Extrapolated ETD-CN</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.001321</td>
<td>0.001126</td>
</tr>
<tr>
<td>8</td>
<td>0.002408</td>
<td>0.002168</td>
</tr>
<tr>
<td>12</td>
<td>0.003750</td>
<td>0.003545</td>
</tr>
<tr>
<td>16</td>
<td>0.005429</td>
<td>0.005112</td>
</tr>
<tr>
<td>20</td>
<td>0.007226</td>
<td>0.006955</td>
</tr>
</tbody>
</table>

In Case Two, we consider the birefringent situation as given in [26]. We
replace the initial and boundary conditions for (27) by the following:

\[
\begin{align*}
\psi_1(x, 0) &= \sqrt{2} \alpha_1 \sech(\alpha_1 x + x_0) \exp(i \nu_1 x), \\
\psi_2(x, 0) &= \sqrt{2} \alpha_2 \sech(\alpha_2 x - x_0) \exp(i \nu_2 x), \\
\psi_1(x, t) &= \psi_2(x, t), \quad \text{at } x = \pm 40,
\end{align*}
\]  

(30)

where \( \rho = \frac{2}{3}, \ \nu_1 = 0.2, \ \nu_2 = -0.2, \ \alpha_1 = 0.6, \ \alpha_2 = 0.5 \) and \( x_0 = 20 \).

Table 6: Mass conservations of (27), (30) by using the ETD-CN method with a quartic spline approximation.

<table>
<thead>
<tr>
<th>T</th>
<th>[|\psi_1|^2]</th>
<th>Error in mass</th>
<th>[|\psi_2|^2]</th>
<th>Error in mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.549193</td>
<td>0</td>
<td>1.414214</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>1.549263</td>
<td>0.000070</td>
<td>1.414228</td>
<td>0.000014</td>
</tr>
<tr>
<td>40</td>
<td>1.549331</td>
<td>0.000138</td>
<td>1.414245</td>
<td>0.000031</td>
</tr>
<tr>
<td>60</td>
<td>1.549392</td>
<td>0.000199</td>
<td>1.414247</td>
<td>0.000033</td>
</tr>
<tr>
<td>80</td>
<td>1.549475</td>
<td>0.000282</td>
<td>1.414316</td>
<td>0.000102</td>
</tr>
<tr>
<td>100</td>
<td>1.549538</td>
<td>0.000345</td>
<td>1.414317</td>
<td>0.000103</td>
</tr>
</tbody>
</table>

This initial-boundary value problem generates two optical waves propagating and interacting in a birefringent fiber. We may observe in Figure 7 obtained via the ETD-CN method based on a quartic spline approximation \((h = 0.1, \ k = 0.05, \ T = [0, 100])\) that in the situation with (30), the two solitons \(\psi_1\) and \(\psi_2\) start their propagation at \(x = -20\) and \(x = 20\) respectively. After the solitons interact at \(t \approx 40\), their phase travel directions are slightly altered and there appears to be a daughter wave for each of them.
The amplitude of such a daughter wave of $|\psi_2|$ is larger since the amplitude of $|\psi_1|$ is larger, which means the effect of $|\psi_1|$ to $|\psi_2|$ in the interaction is larger. Table 5 gives mass profiles of the two solitons as $t_n$ increases. It is found that after the collision, mass errors of the numerical solutions grow only mildly, therefore the daughter waves are not consequences of numerical errors.

3.4. System of Three NLS Equations

In this section we consider the following system of three NLS equations as suggested in [2]

\begin{align}
    i\psi_1 + \psi_{1xx} + (\sigma|\psi_1|^2 + \varrho|\psi_2|^2 + \sigma|\psi_3|^2) \psi_1 &= 0, \\
    i\psi_2 + \psi_{2xx} + (\varrho|\psi_1|^2 + \sigma|\psi_2|^2 + \varrho|\psi_3|^2) \psi_2 &= 0, \\
    i\psi_3 + \psi_{3xx} + (\sigma|\psi_1|^2 + \varrho|\psi_2|^2 + \sigma|\psi_3|^2) \psi_3 &= 0,
\end{align}

(31)
together with initial and boundary conditions

\begin{align}
    \psi_1(x, 0) &= a_0[1 - \epsilon \cos(lx)], \\
    \psi_2(x, 0) &= b_0[1 - \epsilon \cos(l(x + \theta))], \\
    \psi_3(x, 0) &= c_0[1 - \epsilon \cos(lx)], \\
    u_j(x + L, t) &= u_j(x, t), \ j = 1, 2, 3 \text{ and } L = 8\pi,
\end{align}

where $\varrho = 1$, $\sigma = 1$, $l = 0.5$, $a_0 = 0.2$, $b_0 = 0.3$, $c_0 = 0.2$, $\epsilon = 0.1$, $\theta = 0$.

Surface plots of the solutions obtained via the extrapolated ETD-CN method with quartic spline approximation ($N = 128$, $k = 0.005$ and $T = [0, 80]$) are shown in Figure 8. Since the solution $\psi_3$ is identical to $\psi_1$, we only provide surfaces of $\psi_1$ and $\psi_2$. We may observe from the simulations that this problem models solitons that change their shapes periodically. The two solitons have identical period of 40. However, the amplitudes of the two solitons differ. While the soliton of $\psi_1$ has its amplitude around 0.5, the amplitude of the other one remains around 0.7. As pulses enter the optical fiber, they are perturbed and interact with each other [1, 20]. These interactions help the solitons maintain their mass.

We may observe in Table 7 that as time increases from 0 to 80, the mass, $\|\psi_1\|^2_2$ or $\|\psi_3\|^2_2$, stays around 1.005159, which is the same as the initial mass. The errors are calculated as $\|\|\psi(\cdot, T)\|^2_2 - \|\psi(\cdot, 0)\|^2_2\|$. The maximum error is 0.000027. The mass $\|\psi_2\|^2_2$ preserves at the level of initial mass 1.507738.
Figure 8: Surfaces of destabilized wave solutions of system (31).

Table 7: Mass conservations of the system of three NLSEs (31) via the extrapolated ETD-CN method with quartic spline approximation ($N = 128$ and $k = 0.005$).

<table>
<thead>
<tr>
<th>$T$</th>
<th>$|\psi_1|_2^2$</th>
<th>Error</th>
<th>$|\psi_2|_2^2$</th>
<th>Error</th>
<th>$|\psi_3|_2^2$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.005159</td>
<td>0</td>
<td>1.507738</td>
<td>0</td>
<td>1.005159</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>1.005143</td>
<td>0.000016</td>
<td>1.507678</td>
<td>0.000060</td>
<td>1.005143</td>
<td>0.000016</td>
</tr>
<tr>
<td>50</td>
<td>1.005159</td>
<td>0.000000</td>
<td>1.507738</td>
<td>0.000000</td>
<td>1.005159</td>
<td>0.000000</td>
</tr>
<tr>
<td>80</td>
<td>1.005132</td>
<td>0.000027</td>
<td>1.507672</td>
<td>0.000066</td>
<td>1.005132</td>
<td>0.000027</td>
</tr>
</tbody>
</table>

Table 8: Mass conservations of the system of three NLSEs (31) via the multisymplectic six-point method [2] ($N = 128$ and $k = 0.005$).

<table>
<thead>
<tr>
<th>$T$</th>
<th>$|\psi_1|_2^2$</th>
<th>Error</th>
<th>$|\psi_2|_2^2$</th>
<th>Error</th>
<th>$|\psi_3|_2^2$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.005159</td>
<td>0</td>
<td>1.507738</td>
<td>0</td>
<td>1.005159</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>1.005734</td>
<td>0.000575</td>
<td>1.508609</td>
<td>0.000871</td>
<td>1.005734</td>
<td>0.000575</td>
</tr>
<tr>
<td>50</td>
<td>1.005159</td>
<td>0.000000</td>
<td>1.507738</td>
<td>0.000000</td>
<td>1.005159</td>
<td>0.000000</td>
</tr>
<tr>
<td>80</td>
<td>1.005187</td>
<td>0.000028</td>
<td>1.507813</td>
<td>0.000075</td>
<td>1.005187</td>
<td>0.000028</td>
</tr>
</tbody>
</table>

Table 8 provides results of the multisymplectic six-point method [2] for the same problem in current section. We may see that the maximum error of mass in the case is 0.000575 for $\psi_1$, $\psi_3$ and 0.000871 for $\psi_2$, respectively. From these tables we are confident that the extrapolated ETD-CN method with quartic spline approximation is more reliable in mass conservation than the multisymplectic six-point method.
3.5. Two-Dimensional NLS Equations

In this section, we first consider the following two-dimensional NLSE:

\[
i\psi_t + \psi_{xx} + \psi_{yy} = |\psi|^2\psi, \quad (x, y, t) \in \Omega \times \Omega \times (0, \infty),
\]

\[
\psi = \psi_b, \quad (x, y, t) \in \partial \Omega \times \partial \Omega \times (0, \infty), \tag{32}
\]

\[
\psi(x, 0) = \psi_0, \quad x, y \in \Omega,
\]

where \( \Omega \subset \mathbb{R} \) is bounded.

We consider the ETD-CN algorithm (10) and alternating direction implicit (ADI) method [18, 21] to solve problem (32). This indicates a split of the differential equation (32) to two sub-equations. One with the \( x \)-derivative and the other with the \( y \)-derivative.

First, we solve the following equation in the \( x \)-direction

\[
\nu_t + A_1 \nu = \frac{1}{2} i |\nu|^2 \nu, \tag{33}
\]

where matrix \( A_1 \) is same as in (12) and \( \nu \) is a \( 2N \times N \) matrix approximating \( \psi \), that is,

\[
\nu = \begin{bmatrix}
\nu_{11} & \nu_{12} & \nu_{13} & \cdots & \nu_{1N} \\
\nu_{21} & \nu_{22} & \nu_{23} & \cdots & \nu_{2N} \\
\nu_{31} & \nu_{32} & \nu_{33} & \cdots & \nu_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\nu_{N1} & \nu_{N2} & \nu_{N3} & \cdots & \nu_{NN}
\end{bmatrix},
\]

in which

\[
\nu_{ij} = \begin{bmatrix}
v_{ij} \\
w_{ij}
\end{bmatrix},
\]

where \( v_{ij} \) and \( w_{ij} \) are the real and imaginary parts of \( \nu_{ij} \) and \( |\nu_{ij}|^2 = v_{ij}^2 + w_{ij}^2 \).

Let us denote \( a_{ij} = |\nu_{ij}|^2 \), then

\[
|\nu|^2 \nu = \begin{bmatrix}
a_{11}\nu_{11} & a_{12}\nu_{12} & a_{13}\nu_{13} & \cdots & a_{1N}\nu_{1N} \\
a_{21}\nu_{21} & a_{22}\nu_{22} & a_{23}\nu_{23} & \cdots & a_{2N}\nu_{2N} \\
a_{31}\nu_{31} & a_{32}\nu_{32} & a_{33}\nu_{33} & \cdots & a_{3N}\nu_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{N1}\nu_{N1} & a_{N2}\nu_{N2} & a_{N3}\nu_{N3} & \cdots & a_{NN}\nu_{NN}
\end{bmatrix}.
\]

By solving equation (33) using the ETD-CN method with central difference approximation, we acquire \( \nu(t + k/2) \) as an intermediate value.
Then we repeat the procedure for (33) in the \( y \)-direction to obtain \( \nu(t + k) \). The particular design is often referred as the split-step finite difference (SSFD) method [23].

Next, we implement the above mentioned strategy to solve two-dimensional NLS equations.

3.5.1. Numerical results for a two-dimensional problem

We consider a two-dimensional NLS equation given in [23].

\[
i \psi_t + \frac{1}{2} \psi_{xx} + \frac{1}{2} \psi_{yy} = |\psi|^2 \psi + V(x, y) \psi, \quad 0 < x, y < 2\pi, \quad t > 0, \quad (34)
\]

with initial condition

\[
\psi(x, y, 0) = \sin x \sin y, \quad 0 < x, y < 2\pi,
\]

and homogeneous Dirichlet boundary conditions, where the potential function \( V(x, y) = 1 - \sin^2 x \sin^2 y \). The corresponding analytic solution is

\[
\psi(x, y, t) = \sin x \sin y \exp(-2i t), \quad 0 < x, y < 2\pi, \quad t > 0.
\]

We now compare the accuracy of the ETD-CN method with central difference approximations with the SSFD method [23]. The maximum errors of the two numerical solutions when \( T \) from 4 to 32 are shown in Table 9. We observe from the results that errors of the ETD-CN method with central difference approximations are almost at the same level of those of the SSFD. The performance of the ETD-CN method with central difference approximations is slightly better, which indicates a more favorable accuracy. The surface plot of the absolute error for (34) is given at \( T = 32 \) in Figure 9 with the parameters \( N = 128 \) and \( k = 0.01 \).

3.5.2. Numerical results for another two-dimensional problem

We consider two-dimensional NLS equation [25]:

\[
i \psi_t + \psi_{xx} + \psi_{yy} + |\psi|^2 \psi = 0, \quad 0 < x, y < 2\pi, \quad t > 0, \quad (35)
\]

together with the initial and homogeneous Neumann boundary conditions

\[
\psi(x, y, 0) = (1 + \sin x)(2 + \sin y), \quad 0 < x, y < 2\pi,
\]

\[
\psi(x, y, t)_x = \psi(x, y, t)_y = 0 \quad \text{at} \quad x = 0, \quad 2\pi; \quad y = 0, \quad 2\pi.
\]
Table 9: An accuracy comparison of the ETD-CN method with central difference approximations and the SSFD method for solving (34) with $N = 128$, $k = 0.01$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\ell_\infty$ error of SSFD [23]</th>
<th>$\ell_\infty$ error of ETD-CN</th>
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<td>32</td>
<td>6.492e-3</td>
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Figure 9: Surface plot of absolute error for (34).

By solving (35) using the ETD-CN with central difference approximation and blended with an ADI [18] method ($N = 128$, $k = 0.001$), we observe a strong evidence of a singularity occurring in a finite time in Figure 11. However, there is still no rigorous proof of a breakdown in this case according to [22, 25].
Figure 10: Modules of the initial function \((1 + \sin x)(2 + \sin y)\) in (35).

Figure 11: Singular solutions of (35) at \(T = 0.054\) and \(T = 0.108\), respectively.
4. Conclusions and Future Work

We have developed and analyzed a new central difference and quartic spline approximation based ETD-CN method for solving systems of one-dimensional nonlinear Schrödinger equations and two-dimensional nonlinear Schrödinger equations. The accuracy, efficiency and stability of the numerical method are investigated. It has been evident from the analysis and numerical experiments that the new numerical method is highly efficient and reliable. Our continuing explorations include the development of highly accurate ET-D schemes for nonlinear wave propagation simulations, semi and full mesh adaptations, and priori, posterior error analysis of the underlying numerical methods.

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References


