Credit financing in economic ordering policies for defective items with allowable shortages

Chandra K. Jaggia,⇑ Satish K. Goel, Mandeep Mittal

⇑ Corresponding author.
E-mail address: ckjaggi@yahoo.com (C.K. Jaggi).

In the classical inventory models, the common unrealistic assumption is that all the items produced are of good quality in nature. However, in realistic environment, it can be observed that there may be some defective items in an ordered lot. These items are usually picked up during the screening process and are sold as a single lot at the end of screening process. Further, it is tacitly assumed that the supplier must be paid for the items as soon as the items are received. Whereas, in today business transaction, it is common to see that the retailer is allowed some grace period before they settle the account with the supplier. Under this scenario, a new inventory model for imperfect quality items has been developed under permissible delay in payments. Shortages are allowed and fully backlogged, which are eliminated during screening process as it has been assumed that screening rate is greater than the demand rate. This model jointly optimizes the order quantity and shortages by maximizing the expected total profit. Results have been validated with the help of numerical example using Matlab 7.0.1. Comprehensive sensitivity analysis has also been presented.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

In today’s technology driven world, despite of efficient planning of manufacturing system and emergence of sophisticated production methods and control systems; the items produced may have some fraction of defectives. By considering this fact, researchers devoted a great amount of effort to develop EPQ/EOQ models for defective items [1–5]. In 2000, Salameh and Jaber [6] extended the traditional EPQ/EOQ model for the imperfect quality items. They also considered that the imperfect – quality items are sold at a discounted price as a single batch by the end of the screening process. Cárdenas-Barrón [7] corrected the optimum order size formula obtained by Salameh and Jaber [6] by adding constant parameter which was missing in their optimum order size formula. Further, Goyal and Cárdenas-Barrón [8] presented a simple approach for determining economic production quantity for imperfect quality items and compare the results based on the simple approach with optimal method suggested by Salameh and Jaber [6], which results in almost zero penalty. Papachristos and Konstantaras [9] examined the Salameh and Jaber [6] paper closely and rectify the proposed conditions to ensure that shortages will not occur. They extended their model to the case in which withdrawing takes place at the end of the planning horizon. Further, Wee et al. [10] extended the model of Salameh and Jaber [6] for the case where shortages are back ordered in each cycle. They also studied the effect of varying backordering cost value and showed that as the backordering cost increases then the rate of change in their annual profit decreases as compared to that in Salameh and Jaber [6] model. Maddah and Jaber...
corrected Salameh and Jaber [6] work related to the method of evaluating the expected total profit per unit time. They applied renewal-reward theorem [12] which leads to simple expressions of the optimal order quantity and expected profit per unit time. In addition, they have extended their model by allowing for several batches of imperfect quality items to be consolidated and shipped in one lot. Further, Ergulu and Ozdemir [13] also extended the model of Salameh and Jaber [6] by allowing shortages to be backordered. They suggest that a fraction of good quality items in each cycle not only fulfills current demand but also fulfills backorders during screening process. They also examined effects of defective items on lot size and optimal total profit and showed that the optimal total profit per unit time decreases when percentage of defective items increases. Thereafter, several interesting papers for controlling imperfect quality items have appeared in different journals (e.g. [14–19]).

In all the above mentioned papers, it is tacitly assumed that payment will be made to the supplier for the goods immediately after receiving the consignment. However, in day-to-day dealing, it is found that the supplier allows a certain fixed period to settle the account. During this period, no interest is charged by the supplier, but beyond this period interest is charged under certain terms and conditions agreed upon, since inventories are usually financed through debt or equity. Owing to this fact, during the past few years, a lot of research work has been done on inventory models with permissible delay in payments, which has been summarized by Soni et al. [20].

Further, Chung and Huang [21] incorporated the concept of inspection of imperfect items with trade credit. Jaggi et al. [22] formulated an inventory model with imperfect quality deteriorating items with the assumption that the screening rate is more than the demand rate. This assumption helps one to meet his demand parallel to the screening process, out of the items which are of perfect quality. Recently, Sarkar [23,24] discussed inventory models with delay in payments having time-varying deterioration rate and stock dependent demand.

In this paper, an inventory model is developed for imperfect quality items under permissible delay in payments where shortages are allowed and fully backlogged. The screening rate is assumed to be more than the demand rate. This assumption helps one to meet his demand and backorders parallel to the screening process, out of the items which are of perfect quality. The proposed model jointly optimizes retailer’s order quantity and shortages by maximizing his expected total profit. A comprehensive sensitivity analysis has also been performed to study the impact of $M, I_o, I_p$ and $E[\alpha]$ on $Q, B$ and expected total profit.

2. Assumption and notations

The following assumptions are used to develop the model:

1. Demand rate is known with certainty and is uniform.
2. The replenishment is instantaneous.
3. Shortages are allowed and are full backlogged.
4. The lead-time is negligible.
5. The supplier provides a fixed credit period to settle the accounts to the retailer.
6. Both screening as well as demand proceeds simultaneously, but the screening rate is assumed to be greater than demand rate.
7. It is assumed that each lot contains percentage defectives of ‘$\alpha$’ with known uniform probability function, $f(\alpha)$.

The following notations are used in developing the model:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>demand rate in units per unit time</td>
</tr>
<tr>
<td>$Q$</td>
<td>order size for each cycle</td>
</tr>
<tr>
<td>$B$</td>
<td>maximum backorder level allowed</td>
</tr>
<tr>
<td>$K$</td>
<td>fixed cost of placing an order</td>
</tr>
<tr>
<td>$c$</td>
<td>unit cost</td>
</tr>
<tr>
<td>$p$</td>
<td>unit selling price of good quality items</td>
</tr>
<tr>
<td>$c_s$</td>
<td>unit selling price of imperfect quality items, $c_s &lt; p$</td>
</tr>
<tr>
<td>$h$</td>
<td>holding cost per unit time</td>
</tr>
<tr>
<td>$b$</td>
<td>unit screening cost</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>screening rate in units per unit time, $\lambda &gt; D$</td>
</tr>
<tr>
<td>$t_1$</td>
<td>time to build up backorder level of ‘$B$’ units</td>
</tr>
<tr>
<td>$t_2$</td>
<td>time to eliminate the backorder level of ‘$B$’ units</td>
</tr>
<tr>
<td>$t_3$</td>
<td>time to screen Q units ordered per cycle</td>
</tr>
<tr>
<td>$T_1$</td>
<td>time where inventory level will become zero</td>
</tr>
<tr>
<td>$T$</td>
<td>cycle length</td>
</tr>
<tr>
<td>$C_2$</td>
<td>shortage cost per unit per unit time</td>
</tr>
<tr>
<td>$I_e$</td>
<td>interest earned per unit per unit time</td>
</tr>
</tbody>
</table>

(continued on next page)
3. Mathematical model

This paper develops a mathematical model for defective items under permissible delay in payments. Shortages are allowed and fully backlogged which are eliminated during the screening process as it has been assumed that screening rate is greater than the demand rate. The behavior of the inventory model is illustrated in Fig. 1. It is assumed that a lot of $Q$ units enter in the inventory system at time $t = 0$ and contains $\alpha$ percent defective items with a probability density function, $f(x)$, which can be estimated from the past data. Screening is done for the entire received lot at a rate of $k$ units per unit time to separate good and defective quality items. Further, it is assumed that the good quality items are screened at the rate of $(1 - \alpha)k$ during time $t_2$ and a fraction of good quality items fulfill the demand at the rate $D$ and the rest is used to eliminate backorders with a rate of $(1 - \alpha)k - D$. After the end of screening process at time $t_3$, defective items i.e. $\alpha Q$ are sold immediately as a single batch at discounted price $c_s$. Thereafter, inventory level gradually decreases due to demand and reaches zero at time $T_1$.

The cycle length $T$ for the proposed inventory model is given by

$$T = \frac{(1 - \alpha)Q}{D}. \quad (1)$$

As the percent defective items ($\alpha$), is a random variable, the expected value of cycle length is given by

$$E[T] = \frac{(1 - E[\alpha])Q}{D}. \quad (2)$$

The time to build up a backorder of '$B$' units is,

$$t_1 = \frac{B}{D}, \quad (3)$$

and the time to eliminate the '$B$' units is

$$t_2 = \frac{B}{(1 - \alpha)k - D} = \frac{B}{\alpha A}, \quad \text{where} \quad A = 1 - \alpha - D/k. \quad (4)$$

Fig. 1. Inventory system with inspection for the cases: $M \leq t_2 \leq T, t_2 \leq M \leq t_1 \leq T, M \leq T_1 \leq T$ and $T_1 \leq M \leq T$. 

---

$l_p$ interest paid per unit per unit time,
$z$ inventory level at time $t_2$
$z_1$ inventory level at time $t_3$
$\alpha$ percentage of defective items in $Q$
$f(x)$ probability density function of $\alpha$
$E(\cdot)$ expected value operator
$E(\alpha)$ expected value of $\alpha$, which is equal to $\int_a^b \alpha f(\alpha)d\alpha, \quad 0 < a < b < 1$
$(1 - \alpha)\lambda$ rate of good quality items during $t_2$
$(1 - \alpha)\lambda - D$ rate of good quality items to eliminate backorder, $(1 - \alpha)\lambda - D > 0$
and also
\[ t_2 = \frac{Q - z}{(1 - \alpha)A}. \]  
(5)

The value of \( z \) is obtained by using Eqs. (4) and (5) as follows:
\[ z = Q - \frac{(1 - \alpha)B}{A}. \]  
(6)

The screening time, \( t_3 \), is obtained as
\[ t_3 = \frac{Q}{z}. \]  
(7)

Using Fig. 1, \((t_3 - t_2)\) can be written as
\[ t_3 - t_2 = \frac{(z - z_1 - \alpha Q)}{D}. \]  
(8)

and \( z_1 \) and \( T_1 \) are obtained as
\[ z_1 = AQ - B \]
and
\[ T_1 = \frac{AQ - B + Q}{D} \]  
(9)

The present model has been developed under the condition of permissible delay in payments, therefore, depending upon the credit period, there would be five distinct possible cases for the retailer's total profit, \( \pi_j(Q, B) \), \( j = 1, 2, 3, 4, 5 \), viz.

(i) \( M \leq t_2 \leq t_3 \leq T_1 \),
(ii) \( t_2 \leq M \leq t_3 \leq T_1 \),
(iii) \( t_2 \leq t_3 \leq M \leq T_1 \),
(iv) \( T_1 \leq M \leq T \) and,
(v) \( T \leq M \).

Since, the retailer's total profit consists of the following components:
\[ \pi_j(Q, B) = \text{Sales revenue} - \text{Ordering cost} - \text{Purchasing cost} - \text{Screening cost} - \text{Holding cost} \]
+ Interest earned – Interest paid.
(11)

Therefore, various individual components are evaluated as follows:

(a) Sales revenue is the sum of revenue generated by the demand met during the time period \((0, T)\) and by the sale of imperfect quality items is
\[ = p(1 - \alpha)Q + c_I x Q. \]  
(12)

(b) Ordering Cost = \( K \).
(13)

(c) Purchase Cost = \( cQ \).
(14)

(d) Screening Cost = \( \beta Q \).
(15)

(e) Holding Cost during the time period \( 0 \) to \( T_1 \),
\[ T_1 = h \left[ \frac{t_2(Q + z)}{2} + \frac{(t_3 - t_2)(z + z_1 + 2Q)}{2} + \frac{(T_1 - t_3)z_1}{2} \right]. \]  
(16)

where \( t_1 = \frac{b}{a} \), \( t_2 = \frac{b}{a} \), \( t_3 = \frac{b}{a} \), \( z_1 = AQ - B \), \( z = Q - \frac{(1 - \alpha)B}{A} \) and \( T_1 = \frac{AQ - B + Q}{D} \) and

(f) Shortage Cost = \( \frac{C_2(t_1 + t_2)B}{2} \).  
(17)

The interest earned and paid is calculated for the five different cases as follows:

**Case (i):** \( M \leq t_2 \leq t_3 \leq T_1 \)

The retailer earns interest on average sales revenue generated for the time period \( 0 \) to \( M \). At \( M \) account is settled and the finances are to be arranged to make the payment to the supplier for the remaining stock at some specified rate of interest for the period \( M \) to \( T_1 \), (Fig. 2).

Therefore, the interest earned for the cycle = \[ \frac{(1 - \alpha) \hat{d} - D}{2} M^2 I d + \frac{DM^2 I d}{2}. \]  
(18)
And the interest payable per cycle for the inventory not sold after M is given as

\[ \frac{1}{2} \left( \frac{1}{C_0} a \right) Q \left( \frac{1}{C_0} a \right) D \left( \frac{1}{C_0} a \right) \left( T_1 - M \right) + c_i l_p \alpha Q (t_3 - M). \]  

(19)

Substitute the values from Eqs. (12)-(19) in Eq. (11), the total profit for case (i), \( \pi_1(Q,B) \) becomes

\[
\pi_1(Q,B) = p(1-\alpha)Q + c_2 xQ - K - cQ - \beta Q - h \left[ \frac{t_2(Q + z)}{2} + \frac{(t_3 - t_2)(z + z_1 + \alpha Q)}{2} + \frac{(T_1 - t_2)z_1}{2} \right] - \frac{C_2(t_1 + t_2)B}{2} \\
+ \left[ \frac{(1-\alpha)\lambda - D}{2} \right] M^2 l_p \left[ \frac{t_2}{2} + \frac{DM^2 l_p}{2} \right] - \left[ \frac{(1-\alpha)Q - DM - \{(1-\alpha)\lambda - D\} M |c_l p(T_1 - M)|}{2} + c_i l_p \alpha Q (t_3 - M) \right].
\]

(20)

**Case (ii): \( t_2 \leq M \leq t_3 \leq T_1 \)**

In this case, the retailer can earn interest on revenue generated from the sales up to M. Although, he has to settle the account at M, for that, he has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period M to T_1. Retailer will also earn interest due to shortages meet for the time period \( (M - t_2) \) (Fig. 3).

**Interest earned**

\[
\text{Interest earned} = \frac{[(1-\alpha)\lambda - D]t_2^2 l_p}{2} + B p l_r (M - t_2) + \frac{DM^2 l_p}{2}.
\]

(21)

**Interest payable**

\[
\text{Interest payable} = \frac{[(1-\alpha)Q - DM - \{(1-\alpha)\lambda - D\} t_2 |c_l p(T_1 - M)|}{2} + c_i l_p \alpha Q (t_3 - M).
\]

(22)
Substitute the values from Eqs. (12)–(17), (21) and (22) in Eq. (11), the total profit for case (ii), $\pi_2(Q, B)$ becomes

$$
\pi_2(Q, B) = p(1 - \alpha)Q + c_0 Q - K - cQ - \beta Q - h\left[\frac{t_2(Q + z)}{2} + \frac{(t_3 - t_2)(z + z_1 + \alpha Q)}{2} + (T_1 - t_2)z_1\right] - \frac{C_2(t_1 + t_2)B}{2} 
+ \left[\frac{(1 - \alpha)\lambda - D}{2}t_2^2I_pB + Bpl_c(M - t_2) + \frac{DM^2I_p}{2}\right] 
- \left[\frac{(1 - \alpha)Q - DM - \{(1 - \alpha)\lambda - D\}t_2cI_p(T_1 - M)}{2} + cI_p\alpha Q(t_3 - M)\right] \quad (23)
$$

**Case (iii):** $t_2 \leq t_3 \leq M \leq T_1$

Here, in addition to interest earned and paid as in case (i), retailer not only earn interest from the sales of defective items, $\alpha Q$, after screening process for the time period ($M - t_3$) but also earn interest from shortages which are backlogged during ($M - t_2$) (Fig. 4).

**Interest earned**

$$
\text{Interest earned} = \frac{[(1 - \alpha)\lambda - D]t_2^2I_pB}{2} + Bpl_c(M - t_2) + \frac{DM^2I_p}{2} + l_c\alpha Q(M - t_3), \quad (24)
$$

**Interest payable**

$$
\text{Interest payable} = \frac{[(1 - \alpha)Q - DM - \{(1 - \alpha)\lambda - D\}t_2cI_p(T_1 - M)}{2} \quad (25)
$$

Substitute the values from Eqs. (12)–(17), (24) and (25) in Eq. (11), the total profit for case (iii), $\pi_3(Q, B)$ becomes

$$
\pi_3(Q, B) = p(1 - \alpha)Q + c_0 Q - K - cQ - \beta Q - h\left[\frac{t_2(Q + z)}{2} + \frac{(t_3 - t_2)(z + z_1 + \alpha Q)}{2} + (T_1 - t_2)z_1\right] - \frac{C_2(t_1 + t_2)B}{2} 
+ \left[\frac{(1 - \alpha)\lambda - D}{2}t_2^2I_pB + Bpl_c(M - t_2) + \frac{DM^2I_p}{2}\right] 
- \left[\frac{(1 - \alpha)Q - DM - \{(1 - \alpha)\lambda - D\}t_2cI_p(T_1 - M)}{2} + cI_p\alpha Q(t_3 - M)\right] \quad (26)
$$

**Case (iv):** $T_1 \leq M \leq T$

This case discusses the situation when inventory cycle is less than or equal to permissible delay. Thus, in this scenario no interest is payable by the retailer, where as he earns interest on the revenue generated from the sales from time 0 to M. Further, not only retailer earns interest due to sales of defective items, $\alpha Q$, for the time period ($M - t_3$) but also earn interest from the shortages which are backlogged for the time period ($M - t_2$) (Fig. 5).

**Interest earned**

$$
\text{Interest earned} = \frac{[(1 - \alpha)\lambda - D]t_2^2I_pB}{2} + Bpl_c(M - t_2) + \frac{DM^2I_p}{2} + DT_1I_p(M - T_1) + l_c\alpha Q(M - t_3), \quad (27)
$$

**Interest payable**

$$
\text{Interest payable} = 0
$$

![Inventory system with inspection for the case 3: $t_2 \leq t_3 \leq M \leq T_1$.](image-url)
Substitute the values from Eqs. (12)–(17) and (27) in Eq. (11), the total profit for case (iv), \( \pi_4(Q, B) \) becomes

\[
\pi_4(Q, B) = p(1-x)Q + c_a x Q - K - cQ - \beta Q - h \left[ \frac{t_2(Q+z)}{2} + \frac{(t_3-t_2)(z+z_1+xQ)}{2} + \frac{(T_1-t_3)z_1}{2} \right] \\
- \frac{C_2(t_1+t_2)B}{2} + \left[ \frac{(1-x)\lambda - D}{2} M^2 I_p + \frac{DM^2 I_p}{2} \right] \\
- \left[ \frac{(1-x)Q - DM - \{(1-x)\lambda - D\}c Q(T_1-M)}{2} + c_a E(x)Q(t_3-M) \right]
\]  \( (28) \)

**Case (v):** \( T \leq M \)

In this case, all the expressions for interest earned and paid coincide with that of case (iv).

Hence, effectively we have four different cases for the retailer's total profit per cycle, \( \pi(Q, B) \), which can be expressed as

\[
\pi(Q, B) = \begin{cases} 
\pi_1(Q, B), & M \leq t_2 \leq t_3 \leq T_1, \text{ case 1,} \\
\pi_2(Q, B), & t_2 \leq M \leq t_3 \leq T_1, \text{ case 2,} \\
\pi_3(Q, B), & t_2 \leq M \leq t_3 \leq M \leq T_1, \text{ case 3,} \\
\pi_4(Q, B), & M \geq T_1, \text{ case 4.}
\end{cases}
\]  \( (29) \)

To determine the expected total profit per unit time, we apply renewal-reward theorem Ross [12] as \( x \) is a random variable with known probability density function, \( f(x) \) and get the expected total profit per unit time for different cases which follows:

\[
E[\pi^T(Q, B)] = E \left[ \frac{\pi(Q, B)}{T} \right] = E[\pi(Q, B)]/E[T].
\]  \( (30) \)

\[
\pi(Q, B) = \begin{cases} 
E[\pi_1(Q, B)]/E[T], & M \leq t_2 \leq t_3 \leq T_1, \text{ case 1} \\
E[\pi_2(Q, B)]/E[T], & t_2 \leq M \leq t_3 \leq T_1, \text{ case 2} \\
E[\pi_3(Q, B)]/E[T], & t_2 \leq M \leq t_3 \leq T_1, \text{ case 3} \\
E[\pi_4(Q, B)]/E[T], & M \geq T_1, \text{ case 4}
\end{cases}
\]  \( (31) \)

where

\[
E[\pi_1(Q, B)] = p(1-E(x))Q + c_a E(x)Q - K - cQ - \beta Q - h \left[ \frac{t_2(Q+z)}{2} + \frac{(t_3-t_2)(z+z_1+xQ)}{2} + \frac{(T_1-t_3)z_1}{2} \right] \\
- \frac{C_2(t_1+t_2)B}{2} + \left[ \frac{(1-E(x)\lambda - D)M^2 I_p}{2} + \frac{DM^2 I_p}{2} \right] \\
- \left[ \frac{(1-E(x)Q - DM - \{(1-E(x)\lambda - D\}c Q(T_1-M)}{2} + c_a E(x)Q(T_3-M) \right].
\]  \( (32) \)
Our objective is to find the optimal values of Q and B which maximize the expected total profit function, \( E[\pi(Q,B)] \), therefore, the necessary conditions for \( E[\pi(Q,B)] \) to be optimal are \( \frac{\partial E[\pi(Q,B)\cdot Q]}{\partial Q} = 0 \) and \( \frac{\partial E[\pi(Q,B)\cdot B]}{\partial B} = 0 \) which follows as case wise:

**Case (i):** \( M \leq t_2 \leq t_1 \leq T_1 \)
The optimal values of \( Q = Q_1 \) (say) and \( B = B_1 \) (say), which maximizes, \( E[\pi^1(Q,B)] \) can be obtained by solving the Eq. (31a),
\[
Q^2(-hX_1 + X_5) - B^2Q^2hX_2 + B^2X_6 + QX_9 + (X_4 + h - X_7 - X_{10}) = 0,
\]
(37)
\[
B = QX_{14}.
\]
(38)
All \( X_i \) (i = 1, 2, ..., 14) are elaborated in Appendix A.

**Case (ii):** \( t_2 \leq M \leq t_3 \leq T_1 \)
The optimal values of \( Q = Q_2 \) (say) and \( B = B_2 \) (say), which maximizes, \( E[\pi^2(Q,B)] \), are

\[
B^2Y_{19} + Y_{20} - BY_{21} + B^2Q^2Y_{10} + Q^2Y_{22} = 0,
\]
(39)
\[
BQY_{12} + B^2Y_7 + QY_{13} + Q^2Y_{14} = 0.
\]
(40)
All \( Y_i \)'s are elaborated in Appendix B.

**Case (iii):** \( t_2 \leq t_3 \leq M \leq T_1 \)
The optimal values of \( Q = Q_3 \) (say) and \( B = B_3 \) (say), which maximizes, \( E[\pi^3(Q,B)] \), are

\[
B^2Y_{19} + Y_{20} - BY_{21} + B^2Q^2Y_{10} + Q^2Y_{22} + Q^2\left(Mw_1 - \frac{W_1}{z}\right) = 0,
\]
(41)
\[
BQY_{12} + B^2Y_7 + QY_{13} + Q^2Y_{14} = 0.
\]
(42)
Case (iv): $M \geq T_1$

The optimal values of $Q = Q_4$ (say) and $B = B_4$ (say), which maximizes, $E[p_{T1}(Q, B)]$, are

$$3Q^4L_1 = 2Q^3L_3B + Q^2B^2(L_2 + L_{15}) + Q^2(-L_{14} + L_{12}) + B^2(-L_{17} + L_{11}) + B(-L_{10} + L_6) - L_{13} - L_4 - 1 = 0, \quad (43)$$

$$-Q^2L_3 + 2Q^2BL_2 - QL_{16} + B(-2L_{15} - 2L_{17} + 2L_{11} + L_9) + L_{16} - L_6 = 0. \quad (44)$$

All $L_i$’s are elaborated in Appendix C.

Further, for the expected profit function to be concave, the following sufficient conditions must be satisfied:

$$\left(\frac{\partial^2 E[p_{T1}(Q, B)]}{\partial Q \partial B}\right)^2 - \left(\frac{\partial^2 E[p_{T1}(Q, B)]}{\partial Q^2}\right) \times \left(\frac{\partial^2 E[p_{T1}(Q, B)]}{\partial B^2}\right) \leq 0, \quad (45)$$

and

$$\left(\frac{\partial^2 E[p_{T1}(Q, B)]}{\partial Q^2}\right) \leq 0, \quad \left(\frac{\partial^2 E[p_{T1}(Q, B)]}{\partial B^2}\right) \leq 0. \quad (46)$$

Second order derivatives have been calculated and these are very complicated (Appendix D). Therefore, it is very difficult to prove the concavity mathematically; hence the concavity of all the expected total profit functions have been established graphically and for one of the case is shown below (Fig. 6).

Now, in order to find the optimal values of $Q_*$ and $B_*$, which maximizes the retailer’s expected profit, we propose the following algorithm:

**Algorithm:**

**Step 1:** Determine $Q_0 = Q_1$ (say) and $B_0 = B_1$ (say) from Eqs. (37) and (38). Now, using the values of $Q_1$ and $B_1$, calculate the values of $t_2$, $t_3$ and $T_1$ from Eqs. (5), (7) and (10). If $M \leq t_2 \leq t_3 \leq T_1$ then the optimal value of expected total profit is obtained from Eq. (31a).

**Step 2:** Determine $Q_0 = Q_2$ (say) and $B_0 = B_2$ (say) from Eqs. (39) and (40). Now, using the values of $Q_2$ and $B_2$, calculate the values of $t_2$, $t_3$ and $T_1$ from Eqs. (5), (7) and (10). If $t_2 \leq M \leq t_3 \leq T_1$ then the optimal values of expected profit is obtained from Eq. (31b).

**Step 3:** Determine $Q_0 = Q_3$ (say) and $B_0 = B_3$ (say) from Eqs. (41) and (42). Now, using the values of $Q_3$ and $B_3$, calculate the values of $t_2$, $t_3$ and $T_1$ from Eqs. (5), (7) and (10). If $t_2 \leq t_3 \leq M \leq T_1$ then the optimal values of expected profit is obtained from Eq. (31c).

**Step 4:** Determine $Q_0 = Q_4$ (say) and $B_0 = B_4$ (say) from Eqs. (43) and (44). Now, using the values of $Q_4$ and $B_4$, calculate the values of $t_2$, $t_3$ and $T_1$ from Eqs. (5), (7) and (10). If $M \geq T_1$ then the optimal values of expected profit is obtained from Eq. (31d).
Step 5: Determine the profit at $T = M$ from Eq. (36).
Step 6: Compare the calculated expected profit for cases 1, 2, 3, 4 and at $M$ and select the optimal values of $Q$ and $B$ associated with the maximum expected profit.

5. An illustrative example

Example 1. An example is devised to illustrate the effect of permissible delay in payments on the retailer’s ordering policy for the developed model using the following data.

$D = 15000$ units/year, $A = $400/cycle, $h = $4/unit/year, $\lambda = 60000$ unit/year, $c = $35/unit, $p = $60/unit, $c_s = $25/unit, $d = $1.0/unit and $C_2 = $6/year, percentage defective random variable $x$ with its pdf, $f(x) = \begin{cases} 10, & 0 \leq x \leq 0.1 \\ 0, & \text{otherwise} \end{cases}$, $E[x] = 0.05$, $M = 18$ days.

Two cases are considered.

(a) Let $I_r = 0.08$/year and $I_p = 0.10$/year, $(pl_r = 4.8 > 3.5 = cl_p)$

Results are obtained using the proposed algorithm as: $Q_s = 1642$ units, $B_s = 674$. Substituting the optimal values of $Q_s$ in Eqs. (2) and (7), we get, $T_s = 0.104$ year, $t_{3s} = 0.0274$ year and $E[p^T(Q^*, B^*)] = $347086.

(b) Let $I_r = 0.04$ and $I_p = 0.07$, $(pl_r = 2.4 < 2.45 = cl_p)$

Using the proposed algorithm, results are obtained as: $Q_s = 1804$ units, $B_s = 653$, $T_s = 0.114$ year and $t_{3s} = 0.0301$ year and $E[p^T(Q^*, B^*)] = $346095.

Now, if supplier offers no trade credit, i.e. $M = 0$, we get:

$Q_s = 2130$ units, $B_s = 596$, $T_s = 0.135$ year, $t_{3s} = 0.0355$ year and $E[p^T(Q^*, B^*)] = $345384.

Example 2. An example is devised to illustrate the effect of permissible delay in payments on the retailer’s ordering policy for the developed model using the following data.

$D = 25000$ units/year, $A = $1000/cycle, $h = $5/unit/year, $\lambda = 60000$ unit/year, $c = $45/unit, $p = $70/unit, $c_s = $30/unit, $d = $1.0/unit and $C_2 = $7/year, percentage defective random variable $x$ with its pdf, $f(x) = \begin{cases} 10, & 0 \leq x \leq 0.1 \\ 0, & \text{otherwise} \end{cases}$, $E[x] = 0.05$, $M = 20$ days.

Two cases are considered:

(a) Let $I_r = 0.10$/year and $I_p = 0.12$/year, $(pl_r = 7 > 5.4 = cl_p)$

Results are obtained using the proposed algorithm as: $Q_s = 2671$ units, $B_s = 886$. Substituting the optimal values of $Q_s$ in Eqs. (2) and (7), we get, $T_s = 0.101$/year, $t_{3s} = 0.0445$ year and $E[p^T(Q^*, B^*)] = $568588.

(b) Let $I_r = 0.05$ and $I_p = 0.08$, $(pl_r = 3.5 < 3.6 = cl_p)$

Using the proposed algorithm, results are obtained as: $Q_s = 3004$ units, $B_s = 881$, $T_s = 0.114$ year and $t_{3s} = 0.0501$ year and $E[p^T(Q^*, B^*)] = $566251.

Now, if supplier offers no trade credit, i.e. $M = 0$, we get:

$Q_s = 3693$ units, $B_s = 821$, $T_s = 0.140$ year, $t_{3s} = 0.0616$ year and $E[p^T(Q^*, B^*)] = $564697.

Example 3. $D = 40000$ units/year, $A = $1000/cycle, $h = $5/unit/year, $\lambda = 60000$ unit/year, $c = $45/unit, $p = $70/unit, $c_s = $30/ unit, $d = $1.0/unit and $C_2 = $7/year, percentage defective random variable $x$ with its pdf, $f(x) = \begin{cases} 10, & 0 \leq x \leq 0.1 \\ 0, & \text{otherwise} \end{cases}$, $E[x] = 0.05$, $M = 20$ days.

Two cases are considered:

(a) Let $I_r = 0.10$/year and $I_p = 0.12$/year, $(pl_r = 7 > 5.4 = cl_p)$

Results are obtained using the proposed algorithm as: $Q_s = 2941$ units, $B_s = 530$. Substituting the optimal values of $Q_s$ in Eqs. (2) and (7), we get, $T_s = 0.070$/year, $t_{3s} = 0.049$ year and $E[p^T(Q^*, B^*)] = $914760.

(b) Let $I_r = 0.05$ and $I_p = 0.08$, $(pl_r = 3.5 < 3.6 = cl_p)$

Using the proposed algorithm, results are obtained as: $Q_s = 3439$ units, $B_s = 535$, $T_s = 0.082$ year and $t_{3s} = 0.0573$ year and $E[p^T(Q^*, B^*)] = $910292.

Now, if supplier offers no trade credit, i.e. $M = 0$, we get:

$Q_s = 4321$ units, $B_s = 510$, $T_s = 0.103$ year, $t_{3s} = 0.072$ year and $E[p^T(Q^*, B^*)] = $906826.

6. Sensitivity analysis

Sensitivity analysis has been performed to study the impact of permissible delay ($M$), interest earned ($I_r$), interest paid ($I_p$) and expected number of imperfect quality items ($E[x]$) on the lot size ($Q_s$), backorders ($B_s$) and the retailer’s expected total profit ($E[p^T(Q^*, B^*)]$). Results are summarized in Tables 1–3. Based on the computational results as shown in the Tables 1–3, we obtain the following managerial phenomena:
It is observed from Table 1, when \( pI_e < cIp \) the cycle length increases with permissible delay period which implies that the retailer should procure more quantity to avoid higher interest charges after the credit period that helps him to increase his expected profit e.g. in case of credit cards, usually banks charges very high rate of interest after the grace period. Further, when \( pI_e > cIp \), then both cycle length as well as order quantity decreases with increase in permissible delay period which means that the trade credit offered to the retailer has a positive impact on the economic ordering policy for imperfect quality items. In this scenario, results reveal that the retailer should order less quantity and take the advantage of permissible delay in payments more frequently.

Table 2
Impact of \( l_e \) and \( I_p \) on the optimal replenishment policy (\( M = 20 \) days).

<table>
<thead>
<tr>
<th>( l_e \rightarrow I_p )</th>
<th>0.03</th>
<th>0.05</th>
<th>0.07</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E[pI_e(Q, B)] )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q )</td>
<td>1686</td>
<td>1643</td>
<td>1596</td>
</tr>
<tr>
<td>( B )</td>
<td>671</td>
<td>683</td>
<td>692</td>
</tr>
<tr>
<td>( E[pI_e(Q, B)] )</td>
<td>353075</td>
<td>353670</td>
<td>354285</td>
</tr>
<tr>
<td>( Q )</td>
<td>1716</td>
<td>1665</td>
<td>1608</td>
</tr>
<tr>
<td>( B )</td>
<td>669</td>
<td>682</td>
<td>691</td>
</tr>
<tr>
<td>( E[pI_e(Q, B)] )</td>
<td>353094</td>
<td>353679</td>
<td>354288</td>
</tr>
<tr>
<td>( Q )</td>
<td>1758</td>
<td>1695</td>
<td>1625</td>
</tr>
<tr>
<td>( B )</td>
<td>667</td>
<td>681</td>
<td>690</td>
</tr>
<tr>
<td>( E[pI_e(Q, B)] )</td>
<td>353120</td>
<td>353692</td>
<td>354292</td>
</tr>
</tbody>
</table>

Table 3
Impact of \( E[x] \) on \( Q, B \) and \( E[pI_e(Q, B)] \) (\( M = 18 \) days, \( l_e = 0.08 \) and \( I_p = 0.1 \)).

<table>
<thead>
<tr>
<th>( E[x] )</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E[pI_e(Q, B)] )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q )</td>
<td>1600</td>
<td>1610</td>
<td>1621</td>
<td>1631</td>
<td>1642</td>
</tr>
<tr>
<td>( B )</td>
<td>694</td>
<td>689</td>
<td>684</td>
<td>679</td>
<td>674</td>
</tr>
<tr>
<td>( E[pI_e(Q, B)] )</td>
<td>354217</td>
<td>352489</td>
<td>350725</td>
<td>348924</td>
<td>347086</td>
</tr>
</tbody>
</table>

(i) It is observed from Table 1, when \( pI_e < cIp \) the cycle length increases with permissible delay period which implies that the retailer should procure more quantity to avoid higher interest charges after the credit period that helps him to increase his expected profit e.g. in case of credit cards, usually banks charges very high rate of interest after the grace period. Further, when \( pI_e > cIp \), then both cycle length as well as order quantity decreases with increase in permissible delay period which means that the trade credit offered to the retailer has a positive impact on the economic ordering policy for imperfect quality items. In this scenario, results reveal that the retailer should order less quantity and take the advantage of permissible delay in payments more frequently.

(ii) Table 2 reveals as \( l_e \) increases then order quantity decreases but expected profit increases which implies that when the interest earned per dollar is high, the expected total cost is low, which results in higher expected profit. Further, increase in \( I_p \) results a decrease in order quantity as well as the expected profit because the expected total cost increases when the capital opportunity cost in stock per dollar is high. From retailer point of view, it implies that when the capital opportunity cost per dollar is high, the retailer should order less but more frequently.

(iii) It is clearly evident from Table 3 that as the percentage of defective items increases the optimal order quantity increases but the backorders decreases significantly, which results in less expected profit. In this scenario, the retailer should allow less backorders and be more vigilant while ordering.

7. Conclusions

In this paper, we have developed an EOQ-based inventory model for imperfect quality items to determine the optimal ordering policies of a retailer under permissible delay in payments with allowable shortages. Screening rate is assumed to be more than the demand rate. During the screening process, retailer not only fulfills his current demand but also fulfills the back orders. In the above scenario, the model jointly optimizes the order quantity and shortages with the help of proposed algorithm. A comprehensive sensitivity analysis is also conducted to explore the effects of the key parameters (viz. \( M, l_e, I_p \) and \( E[x] \)) on the optimal results. Findings clearly suggest that the presence of trade credit has got affirmative effect on retailer ordering policy. The retailer should order more to avoid higher interest charged after the grace period that eventually increases his expected profit under the situation, \( pI_e < cIp \), whereas in other situation, i.e. \( pI_e > cIp \), the retailer should order less to avail the benefit of permissible delay more frequently. Further, when the interest payable rate increases, the retailer should order less but more frequently. However, increase in percentage defective items alert the retailer to look into source of supply and take the corrective measures in order to improve the quality of supply.

Further, the proposed model can be extended for more realistic situations such as deteriorating items, stock dependent and stochastic demand with partial- trade credit, inflation etc.
Acknowledgments

The authors are thankful to anonymous referees and the editor for their constructive comments and valuable suggestions that improved the presentation of the paper. The first author would like to acknowledge the support of Research Grant No. Dean (R)/R&D/2012/917, provided by University of Delhi (India) for conducting this research.

Appendix A

\[ X_1 = \frac{D(1 + \alpha + A + A^2)}{(1 - E[x])}, \quad X_2 = \frac{D(3(1 - E[x]) + 2A + A^2)}{(1 - E[x])}, \quad X_3 = \frac{C_2[1/D + 1/I]A}{2}, \]

\[ X_4 = \frac{KD}{(1 - E[x])}, \quad X_5 = \frac{1}{(1 - E[x])}, \quad X_6 = \frac{X_3D}{(1 - E[x])}, \quad X_7 = \frac{X_5D}{(1 - E[x])}. \]

\[ X_8 = \frac{1}{2\lambda} \frac{D}{(1 - E[x])}, \quad X_9 = \frac{1}{(1 - E[x])}, \quad X_{10} = \frac{DC_2}{2(1 - E[x])} \frac{1}{(1 + 1/I)A}, \quad X_{11} = \frac{X_{12} - X_{11}}{X_{12}}. \]

\[ X_{13} = \frac{1}{(1 - E[x])} \left[ -D \frac{clp}{2\lambda} - \frac{clp}{2\lambda} DM + (1 - E[x]) \frac{1}{I} \right]. \]

Appendix B

\[ Y_1 = \frac{clp}{2\lambda} \frac{D}{(1 - E[x])}, \quad Y_2 = \left( \frac{1 + A}{2\lambda A} \right) [(1 - E[x]) \lambda - D] \frac{1}{(1 - E[x])}, \quad Y_3 = \frac{clpDM}{2\lambda}. \]

\[ Y_4 = \left( \frac{clp}{2\lambda^2 A} \right) [(1 - E[x]) \lambda - D], \quad Y_5 = \frac{pleMD}{(1 - E[x])}, \quad Y_7 = \frac{1}{(1 - E[x])} \frac{(1 - E[x]) \lambda - D}{(1 + 1/I)A}. \]

\[ Y_8 = \frac{2D}{(1 - E[x])}, \quad Y_9 = \frac{C_2D}{2(1 - E[x])} \left( 1 + 1/I \right) \frac{1}{(1 + 1/I)A}, \quad Y_{10} = \frac{DC_2[1/D + 1/I]A}{2(1 - E[x])}. \]

\[ Y_{11} = \frac{D}{(1 - E[x])} \left[ 3(1 - E(x) + 2A + A^2), \quad Y_{12} = \frac{D}{(1 - E(x))} \left[ 1 - 2E(x) + A^2 \right], \right. \]

\[ Y_{13} = \frac{D}{(1 - E(x))} \left[ 1 + E(x) + A + A^2 \right], \quad Y_{14} = \frac{KD}{(1 - E(x))}, \quad w_1 = lec, E[x], \right. \]

\[ Y_{15} = 2Y_{10} + 2Y_9 - Y_8 - 2Y_4, \quad Y_{16} = Y_5 - Y_3, \quad Y_{17} = Y_1Y_2 + Y_1, \]

\[ Y_{18} = 2Y_{10} + 2Y_9 - Y_8 - 2Y_4, \quad Y_{19} = -Y_{24} - Y_{16} - 1, \quad Y_{20} = Y_3 + Y_5, \]

\[ Y_{21} = Y_2 + Y_1Y_7 - (2Y_{10} + 2Y_9 - Y_{16} - 2Y_4). \]

Appendix C

\[ L_1 = \frac{D(1 + A)^2}{(1 - E[x])^2}, \quad L_2 = \frac{D}{(1 - E[x])^2}, \quad L_3 = \frac{2D(1 + A)}{(1 - E[x])^2}, \quad L_4 = \frac{pleMD^2}{(1 - E[x])}, \]

\[ L_5 = \frac{D^2 pleM(1 + A)}{\lambda(1 - E[x])}, \quad L_6 = \frac{D^2 lep}{(1 - E[x])}, \quad L_7 = \frac{(1 - E[x]) \lambda - D}{2\lambda^2 A(1 - E[x])}. \]
\[ L_9 = \frac{[(1 - E[x])\lambda - D]*D^\text{lep}}{\lambda^2 A^2(1 - E[x])}, \quad L_{10} = \frac{p\text{leMD}}{1 - E[x]}, \quad L_{11} = \frac{D}{2\lambda A(1 - E[x])}, \]
\[ L_{12} = \frac{iE D^\text{lep}_E[x]}{(1 - E[x])}, \quad L_{14} = \frac{(1 + \alpha + A + A^2)D}{(1 - E[x])}, \quad L_{15} = \frac{(3(1 - x) + 2A + A^2)D}{(1 - E[x])}, \]
\[ L_{16} = \frac{(1 - 2\alpha - A^2)D}{(1 - E[x])}, \quad L_{17} = \frac{C_2(1/D + 1/iA)D}{(1 - E[x])}, \]

Appendix D

Find below second derivatives of expected total profit functions for each case:

\textbf{Case (i)}: \( M \leq t_2 \leq t_3 \leq T_1 \)

\[ \frac{\partial^2 E\pi^*_1(Q, B)}{\partial Q^2} = 2Q(-hX_1 + X_8) - 2B^2 QhX_2 + X_9, \quad (D.1) \]
\[ \frac{\partial^2 E\pi^*_1(Q, B)}{\partial B^2} = 1, \quad (D.2) \]

where

\[ X_1 = \frac{D(1 + \alpha + A + A^2)}{(1 - E[x])}, \quad X_2 = \frac{D(3(1 - E[x]) + 2A + A^2)}{(1 - E[x])}, \quad X_3 = \frac{C_2(1/D + 1/iA)}{2}, \]
\[ X_4 = \frac{KD}{(1 - E[x])}, \quad X_5 = \frac{[(1 - E[x])\lambda - D]*M^2^\text{lep}}{2} + \frac{DM^2^\text{lep}}{2}, \]
\[ X_6 = \frac{X_9 D}{(1 - E[x])}, \quad X_7 = \frac{X_9 D}{(1 - E[x])}, \quad X_8 = \frac{clp}{2\lambda} \frac{D}{(1 - E[x])}, \]
\[ X_9 = \left(\frac{clp}{2\lambda} [DM + ((1 - E[x])\lambda - D)M]\right) \frac{D}{(1 - E[x])}, \quad X_{10} = \left(\frac{clp E[x]}{\lambda}\right) \frac{D}{(1 - E[x])}, \]
\[ X_{11} = \left[\frac{2D(3(1 - E[x]) + 2A + A^2)}{(1 - E[x])}\right]^{1/2}, \quad X_{12} = \frac{DC_2}{2(1 - E[x])} \left[1 + \frac{1}{2A}\right], \]
\[ X_{13} = \frac{1}{(1 - E[x])} \left[-D \frac{clp}{2\lambda} - D \frac{clp}{2\lambda} [DM + ((1 - E[x])\lambda - D)M]\right], \quad X_{14} = \frac{X_{13}}{X_{12} - X_{11}}. \]

\textbf{Case (ii)}: \( t_2 \leq M \leq t_3 \leq T_1 \)

\[ \frac{\partial^2 E\pi^*_1(Q, B)}{\partial Q^2} = 2B^2 QY_{10} + 2QY_{22}, \quad (D.3) \]
\[ \frac{\partial^2 E\pi^*_1(Q, B)}{\partial B^2} = QY_{12} + 2BY_7, \quad (D.4) \]

where

\[ Y_1 = \left(\frac{clp}{2\lambda}\right) \frac{D}{(1 - E[x])}, \quad Y_2 = \left(\frac{1 + A}{2A}\right) \left[\frac{((1 - E[x])\lambda - D) + (1 - \alpha)}{1 - E[x]}\right], \quad Y_3 = \frac{clp DM}{2\lambda}, \]
\[ Y_4 = \left(\frac{clp}{2\lambda A}\right) \left[(1 - E[x])\lambda - D]\right), \quad Y_5 = \frac{p\text{leMD}}{(1 - E[x])}, \quad Y_7 = \frac{[(1 - E[x])\lambda - D]*D^\text{lep}}{2\lambda^2 A^2(1 - \alpha)}, \]
\[ Y_8 = \frac{2D}{(1 - E[x])}, \quad Y_9 = \frac{C_2 D}{2(1 - E[x])} \left[1 + \frac{1}{2A}\right], \quad Y_9 = \frac{DC_2(1/D + 1/iA)}{2(1 - \alpha)}}, \]
\[ Y_{10} = \frac{D}{(1 - E(x))} [3(1 - E(x) + 2A + A^2)], \quad Y_{11} = \frac{D}{(1 - E(x))} [1 - 2E(x) + A^2], \]

\[ Y_{23} = \frac{D}{(1 - E(x))} [(1 + E(x) + A + A^2)], \quad Y_{24} = \frac{KD}{(1 - E(x))}, \]

\[ Y_{12} = 2Y_{10} + 2Y_9 - Y_8 - 2Y_4, \quad Y_{13} = Y_5 - Y_3, \quad Y_{14} = Y_1Y_2 + Y_{11}, \]

\[ Y_{19} = 2Y_{10} + 2Y_9 - Y_8 - 2Y_4, \quad Y_{20} = -Y_{24} - Y_{16} - 1, \quad Y_{21} = Y_3 + Y_5, \]

\[ Y_{22} = Y_{23} - Y_1Y_{17} - (2Y_{10} + 2Y_9 - Y_{16} - 2Y_4). \]

**Case (iii):** \( t_2 \leq t_3 \leq M \leq T_1 \)

\[ \frac{\partial^2 E[\pi_1^r(Q, B)]}{\partial Q^2} = 2B^2QY_{10} + 2QY_{22} + 2Q \left( Mw_1 - \frac{w_1}{\lambda} \right), \quad (D.5) \]

\[ \frac{\partial^2 E[\pi_1^r(Q, B)]}{\partial B^2} = QY_{12} + 2BY_7, \quad (D.6) \]

where

\[ Y_1 = \frac{clp}{2\lambda} \frac{D}{(1 - E(x))}, \quad Y_2 = \left( \frac{1 + A}{2A} \right) \left[ ((1 - E(x)) \lambda - D) + (1 - E(x)) \right], \quad Y_3 = \frac{clpDM}{2\lambda}, \]

\[ Y_4 = \left( \frac{clp}{2\lambda^2A} \right) [(1 - E(x)) \lambda - D], \quad Y_5 = \frac{plesMD}{(1 - E(x))}, \quad Y_7 = \frac{[(1 - E(x)) \lambda - D]Dlep}{2\lambda^2A^2(1 - E(x))}, \]

\[ Y_8 = \frac{2D}{(1 - E(x))}, \quad Y_9 = \frac{C_2D}{2(1 - E(x))} \left[ \frac{1}{D} + \frac{1}{\lambda A} \right], \quad Y_9 = \frac{DC_2[1/D + 1/\lambda A]}{2(1 - E(x))}, \]

\[ Y_{10} = \frac{D}{(1 - E(x))} [3(1 - E(x) + 2A + A^2)], \quad Y_{11} = \frac{D}{(1 - E(x))} [1 - 2E(x) + A^2], \]

\[ Y_{23} = \frac{D}{(1 - E(x))} [(1 + E(x) + A + A^2)], \quad Y_{24} = \frac{KD}{(1 - E(x))}, \quad w_1 = lcsE[x], \]

\[ Y_{12} = 2Y_{10} + 2Y_9 - Y_8 - 2Y_4, \quad Y_{13} = Y_5 - Y_3, \quad Y_{14} = Y_1Y_2 + Y_{11}, \]

\[ Y_{19} = 2Y_{10} + 2Y_9 - Y_8 - 2Y_4, \quad Y_{20} = -Y_{24} - Y_{16} - 1, \quad Y_{21} = Y_3 + Y_5, \]

\[ Y_{22} = Y_{23} - Y_1Y_{17} - (2Y_{10} + 2Y_9 - Y_{16} - 2Y_4). \]

**Case (iv):** \( M \geq T_1 \)

\[ \frac{\partial^2 E[\pi_1^r(Q, B)]}{\partial Q^2} = 12Q^2L_1 - 6Q^2L_3B + 2QB^2(L_2 + L_{15}) + 2Q(-L_{14} + L_{12}), \quad (D.7) \]

\[ \frac{\partial^2 E[\pi_1^r(Q, B)]}{\partial B^2} = 2Q^2L_2 + (-2L_{15} - 2L_{17} + 2L_{11} + L_9), \quad (D.8) \]

where

\[ L_1 = \frac{D(1 + A)^2}{(1 - E(x)) \lambda^2}, \quad L_2 = \frac{D}{(1 - E(x)) \lambda^2}, \quad L_3 = \frac{2D(1 + A)}{(1 - E(x)) \lambda^2}, \quad L_4 = \frac{plesMD^2}{(1 - E(x))}, \]

\[ L_5 = \frac{D^2 plesM(1 + A)}{4(1 - E(x))}, \quad L_6 = \frac{D^2 lep}{4(1 - E(x))}, \quad L_7 = \frac{[(1 - E(x)) \lambda - D]Dlep}{2\lambda^2A^2(1 - E(x))}, \]

\[ L_9 = \frac{[(1 - E(x)) \lambda - D]Dlep}{\lambda^2A^2(1 - E(x))}, \quad L_{10} = \frac{plesMD}{(1 - E(x))}, \quad L_{11} = \frac{D}{\lambda A(1 - E(x))}. \]
\[ L_{12} = \frac{IeDcE[x]}{(1 - E[x])}, \quad L_{14} = \frac{(1 + x + A^2)D}{(1 - E[x])}, \quad L_{15} = \frac{(3(1 - x) + 2A + A^2)D}{(1 - E[x])}, \]

\[ L_{16} = \frac{(1 - 2x - A^2)D}{(1 - E[x])}, \quad L_{17} = \frac{C^2(1/D + 1/\lambda A)D}{(1 - E[x])}. \]

References