UNIDIGRAPHIC AND UNIGRAPHIC DEGREE SEQUENCES THROUGH UNIQUELY REALIZABLE INTEGER-PAIR SEQUENCES

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In this paper we use the concept of integer-pair sequences, an invariant of graphs and digraphs introduced in Hakimi and Patrinos [9], and results on its unique realizability, in Das [4, 5], to obtain results on the unique realizability of degree sequences, another invariant of graphs and digraphs. We thus present a unified approach to solving the problem of unique realizability of these two invariant sequences of graphs and digraphs.

1. Introduction and definitions

The concept of integer-pair sequences, abbreviated as i.p.s., was introduced by Hakimi and Patrinos in [9], where it was considered to extend the concept and results of degree sequences. The intimate connection between these two invariant sequences of graphs and digraphs is further demonstrated in this paper where the concept of i.p.s. and results on unigraphic i.p.s., given in Das [4, 5], are used to characterize unidigraphic and unigraphic degree sequences. We thus present a unified approach to solving the problem of unique realizability of these two invariant sequences of graphs and digraphs.

Further results on i.p.s. appear in [1] and [18]. For other results on degree sequences one is referred to the recent survey [17].

In Section 2 of this paper we characterize unidigraphic degree sequences, thus solving a problem posed in Rao [17]. In Section 3 we characterize unigraphic degree sequences. Alternate characterizations of unigraphic degree sequences have been obtained earlier in [11, 13–15]. However our approach, as already explained, and results are different. In both sections results are extended to the bipartite case.

All graphs (digraphs) considered in this paper are finite, without isolated vertices and without loops of multiple edges (arcs). For definitions and notation not explained here the reader is referred to Harary [10].

Let $G$ be a graph (digraph) and $A, B \subseteq V(G)$. Then $G[A, B]$ is defined by the following: $V(G[A, B]) = A \cup B$ and $E(G[A, B]) = \{uv \in E(G): u \in A, v \in B\}$. $G[A, A]$ is sometimes denoted by $G[A]$.  

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For a digraph $G$ the outdegree (indegree) of a vertex $u$ is denoted by $d^+_G(u)$ ($d^-_G(u)$) and then the degree of $u$, denoted $d_G(u)$, is the ordered pair $(d^+_G(u), d^-_G(u))$. For a graph $G$ the degree of a vertex $u$, denoted $d_G(u)$, is the number of edges incident at $u$. Hence, whether $G$ is a graph or digraph, the degree sequence of $G$, denoted $\Pi(G)$, is the sequence of the degrees of the vertices. Two degree sequences are considered equal if one is a reordering of the other.

Similarly given a pair of sequences $[\phi_1, \phi_2]$ we say it has a realization by bipartite graph (digraph) if there is a bipartite graph (digraph) $G$ with bipartition $V_1 \cup V_2$, such that the degrees in $G$ of the vertices of $V_m$ are given by $\phi_m$, for $m = 1, 2$. Then we also write $\Pi(G) = [\phi_1, \phi_2]$ when there is no ambiguity about the bipartition being considered. $[\phi_1, \phi_2]$ is said to have unique realization by bipartite graph (digraph) if for any two realizations $G, H$ on $V_1 \cup V_2$ there is an isomorphism $\sigma$ with $\sigma(V_1) = V_1$.

Let $G$ be a graph with $E(G) = \{u_1v_1, \ldots, u_qv_q\}$ where $q = |E(G)|$. Then by the integer-pair sequence, abbreviated as i.p.s., $S_G$ of $G$ we mean the sequence $((a_1, b_1), \ldots, (a_q, b_q))$ where $a_i, b_i$ are the degrees of $u_i, v_i$ respectively. Also, given such a finite sequence $S$ of ordered pairs of positive integers we say that $S$ is graphic if there is a graph $G$ such that $S_G = G$ and $G$ is then called a realization of $S$. Further if any two realizations of $S$ are isomorphic, then $S$ is said to be a unigraphic i.p.s.

$x \mid y$ ($x \nmid y$) means $x$ divides (does not divide) $y$.

The $\delta_i$'s used are the Kronecker deltas.

$[x]^*$ denotes the least integer not less than $x$ and $[x]$ denotes the greatest integer not greater than $x$.

A $(G, \overline{G})$ trail $I = I(x_1, x_2, \ldots, x_{2m})$ is a sequence of vertices of $G$ such that for all odd $i, j, i \neq j$, we have $x_ix_{i+1} \in E(G)$, $x_ix_{i-1} \notin E(G)$, $(x_i, x_{i+1}) \neq (x_j, x_{j+1})$ and $(x_i, x_{i-1}) \neq (x_j, x_{j-1})$ where the subscripts are taken modulo $2m$. By writing $G \rightarrow I \rightarrow H$ we mean that $I$ is a $(G, \overline{G})$ trail and that we obtain $H$ from $G$ by deleting the edges (arcs) $x_ix_{i+1}$ and adding $x_ix_{i-1}$, $i$ odd and $1 \leq i \leq 2m$. Then clearly $H$ is also a graph (digraph) and for $v \in V(G) = V(H)$ we have $d_G(v) = d_{\overline{G}}(v)$. Hence $\Pi(H) = \Pi(G)$.

2 Unigraphic degree sequences

(2.1) Let $\Pi = (r_1, s_1)^{t_1}, \ldots, (r_n, s_n)^{t_n}$ be a sequence of ordered pairs of non-negative integers where $(r_i, s_i)^{t_i}$ denotes that $(r_i, s_i)$ occurs exactly $t_i$ times in $\Pi$ and for $i \neq j$, $1 \leq i, j \leq n$, we have $t_i > 0$ and $(r_i, s_i) \neq (r_j, s_j)$.

Let $\Pi$ be as in (2.1). If $\Pi$ is digraphic, then we define an i.p.s. $S^*(\Pi)$ according to the following algorithm based on the one of [12].

Step 1. Put $S^*(\Pi) = \emptyset$ and $V = \emptyset$ where $\emptyset$ denotes the empty sequence. For all $1 \leq i \leq n$, add $t_i$ members $(i, r_i, s_i)$ to $V$. Go to Step 2.
Step 2. Order $V$ such that if $(i, r, s)$ occurs before $(j, r, s)$, then either $r_i > r_j$ or $r_i = r_j$ and $s_i > s_j$. Let $(k, r_k, s_k)$ be the first member of $V$ with $s_k$ non-zero. Then add $(i, k)$ to $S^*(II)$ and put $n_i = n_i - 1$ for the first $s_k$ members $(i, r_i, s_i)$, other then the chosen $(k, r_k, s_k)$, of $V$. Put $s_k = 0$. Proceed to Step 3.

Step 3. If for any member $(j, r_j, s_j)$ of $V$, $r_j = s_j = 0$ then remove it from $V$. If $V = \emptyset$ stop. Otherwise go to Step 2.

Now we make the following definitions for $1 \leq i \neq j \leq n$:

$k'(i, j)$ is the number of times $(i, j)$ occurs in $S^*(II)$,

$$X_i^+ = \sum_r k'(i, r) \pmod{t_i}, \quad Y_i^+ = \sum_r (-k'(i, r)) \pmod{t_i},$$

$$X_i^- = \sum_r k'(r, i) \pmod{t_i}, \quad Y_i^- = \sum_r (-k'(r, i)) \pmod{t_i},$$

$\Pi_i$ is $\{(r_1, \ldots, r_n), (r_1, \ldots, r_n)\}$ where the first $k'(i, j)$ of the $r_{mk}$'s are $[k'(i, j)/t_m]^+$ and the remaining are $[k'(i, j)/t_m]^-$ for $m = i, j$.

$\Pi_i$ is of Type 1 if one of the following holds:

(a) $k'(i, j) \leq t_i$ or $t_i$ or $k'(i, j) \geq t_i t_j - t_i$ or $t_i t_j - t_i$,

(b) $k'(i, j) = t_i + 1$ or $t_i - 1 - t_j$ and $t_j \mid k'(i, j)$,

(c) $k'(i, j) = t_i + 1$ or $t_i - 1 - t_j$ and $t_j \mid k'(i, j)$,

(d) $\Pi_i$ is one of $[(2^4, 1), (3^3), [3^3, (2^4, 1)], [(4, 3^2), (2^5)], [(2^5), (4, 3^2)]$.

Now we give the characterizing theorem.

**Theorem 2.1.** Let $\Pi$ be as in (2.1). Then $\Pi$ is unidigraphic if and only if $\Pi$ is digraphic and $S^*(II)$ satisfies the following three sets of conditions:

(A) **A1.** If $i, j, r, s$ are such that $i \neq r, j \neq s$ and $0 < \min \{k'(i, j), k'(r, s)\}$, then either $k'(i, s) = t_i (t_i - \delta_u)$ or $k'(r, j) = t_r (t_r - \delta_u)$.

(A2. Either $k'(i, i) \in \{0, 1, t_i (t_i - 1) - 1, t_i (t_i - 1)\}$ or $k'(i, i) = t_i = 3$.

(A3. If $k'(i, i) = t_i = 3$, then for all $j \neq i$ we have $\{k'(i, j), k'(j, i)\} \subset \{0, 3 t_i\}$.

(B) Let $I = \{i: t_i \geq 2 \text{ and } k'(i, i) = 0 \text{ or } t_i (t_i - 1)\}$. Then for all $i \in I$ we have the following:

**B1.** $(X_i^+, Y_i^+, X_i^-, Y_i^-) \in \{(0, 0, 0, 0), (t_i, m_i, 0, 0), (m_i, t_i, 0, 0), (0, 0, t_i, m_i), (0, 0, 0, m_i), (t_i, t_i - \delta, \delta)\}$.

**B2.** If $i \neq j$ and $t_j < k'(i, j) < t_j (t_j - t_i)$, then $\Pi_{ij}$ is of Type 1; $X_i^+, Y_i^- < t_i$; and for all $m \in I, i \neq j$, we have $k'(i, m) \in \{0, t_j (t_j - t_i) - \delta, \delta = 0, 1\}$.

Further if $t_j \mid k'(i, j)$, then for all $m \neq j, i$ we have $k'(i, m) \in \{0, t_j (t_j - t_i)\}$.

(B3. If $t_i t_j - t_i < k'(i, j) < t_i t_j$ (resp. $0 < k'(i, j) < t_i$), then $Y_i^- = t_i$ (resp. $X_i^+ = t_i$).

**B4.** If $t_i t_j - t_i < k'(i, j) < t_i t_j - 1$ (resp. $1 < k'(i, j) < t_i$), then for all $m \neq j, i$ either $k'(i, m) = 0 \pmod{t_j (t_j - t_i)}$ or $k'(i, m) = t_j (t_j - t_i - 1) \pmod{t_j (t_j - t_i)}$.

**B5.** If $i, j, r$ are all-distinct, $0 < k'(i, j) < t_i$ and $0 < k'(i, r) < t_i$. If $Y_i^- = t_i$ (resp. $X_i^- = t_i$), then for all $m \neq i, j, r$ either $k'(i, m) > t_j t_i - t_i$ (resp. $k'(i, m) < t_i$) or $k'(i, m) = 0 \pmod{t_j (t_j - t_i)}$.

**B6.** Conditions B2 to B5 hold with $X_r^+, Y_r^+, k'(r, s)$ and $\Pi_{rs}$ replaced by $X_r^-, Y_r^-, k'(s, r)$ and $\Pi_{sr}$, respectively, for $1 \leq r, s \leq n$. 

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(C) Let \( G(II) \) be the digraph defined by the following: \( V(G(II)) = I, E(G(II)) = \{(i, j) : 2 \leq k'(i, j) \leq t_t - 2\} \). Then the following conditions are satisfied:

**Cl.** If there is a directed path from \( i \) to \( j \) in \( G(II) \) then at least one of the following holds for \( i, j \):

1. \( X_i^+ = Y_i^- = 0 \) and for all \( m \) either \( k'(m, i) \leq t_i \) or \( k'(i, m) \geq t_{t_m} - t_i \).
2. \( X_i^+ = Y_i^- = 0 \) and for all \( m \) either \( k'(i, m) \leq t_i \) or \( k'(m, i) \geq t_{t_m} - t_i \).
3. \( X_i^+ = Y_i^- = X_j^+ = Y_j^- = 0 \), \( X_i^+ = Y_i^- = t_i \), for all \( m \) either \( k'(m, j) \leq t_j \) or \( k'(j, m) \leq t_{t_m} - t_j \), and there exist \( m_1, m_2 \) such that \( t_i < k'(i, m_1) < t_{t_m} - t_i \) and \( (k'(j, m_2) \mod t_j) = t_j - 1 \).
4. \( X_i^+ = Y_i^- = X_j^+ = Y_j^- = 0 \), for all \( m \) either \( k'(i, m) \leq t_i \) or \( k'(i, m) \geq t_{t_m} - t_i \), and there exist \( m_1, m_2 \) such that \( t_i < k'(i, m_1) < t_{t_m} - t_i \) and \( (k'(m_2, i) \mod t_j) = t_j - 1 \).
5. \( \max(t_i, t_j) < k(i, j) < \max(4, (k'(i, j) \mod 4)) = t_j - 1 \).

**C2.** There is no directed path in \( G(II) \) from \( i \) to \( m \) via \( j \) (\( i, j, m \) all distinct) if either each of \( i, j \) or \( i, m \) satisfy Cl(c) or each of \( i, m \) or \( i, j \) satisfy Cl(d).

**C3.** There is no directed cycle in \( G(II) \).

Now to prove the characterizing theorem we require the following set-up. Whenever \( II \) is digraphic we write it as in (2.1). Then we consider all realizations \( G \) of \( II \) to be on the vertex set \( V = \bigcup_{i=1}^n V_i \) such that \( \forall u \in V_i \), \( d_G(u) = (r_u, s_u) \) for \( 1 \leq i \leq n \). Hence \( |V_i| = t_i \). \( G[V_i, V_j] \) is denoted by \( G_{ij} \) and \( G[V_i] \) by \( G_i \) or \( G_{ii} \). For \( i \neq j \), note that \( G_{ij} \) is an asymmetric, bipartite digraph (which may hence sometimes considered as a bipartite graph) and the bipartition is always taken as \( V_i \cup V_i \).

For \( 1 \leq i, j \leq n, \Delta_{ij}^+(G) \) and \( \varepsilon_{ij}^+(G) \) (resp. \( \Delta_{ij}^-(G) \) and \( \varepsilon_{ij}^-(G) \)) denote the maximum and minimum outdegree (resp. indegree) in \( G_{ij} \) of a vertex in \( V_i \) (resp. \( V_j \)). \( G_{ij} \) is said to be semi-regular if \( \Delta_{ij}^+(G) - \varepsilon_{ij}^+(G) \leq 1 \) and \( \Delta_{ij}^+(G) - \varepsilon_{ij}^-(G) \leq 1 \).

We will use the following canonical realization of a digraphic degree sequence.

**Theorem 2.2 (Rao [16]).** Let \( II \) be digraphic. Then there is a realization \( G \) of \( II \) such that for \( 1 \leq i, j \leq n, G_{ii} \) is semi-regular.

In what follows till the proof of the characterizing theorem, we take \( II \) to be a unidigraphic degree sequence, \( S = S^*(II) \) and \( G \) to be the canonical realization of Theorem 2.2. Note that for \( i \neq j, 1 \leq i, j \leq n \) we have \( II(G_{ij}) = II_{ij} \) where \( G_{ij} \) is considered as an undirected bipartite graph. We also make the following notational simplifications: We omit the qualifying symbol \( G \) and write \( \Delta_{ij}^+, \varepsilon_{ij}^+, d_{ij}^+(x), d(x) \) and \( d^+(x) \) for \( \Delta_{ij}^+(G) \) and so on.

We now state and prove a series of assertions, which will be required to establish the necessity in Theorem 2.1.

**Assertion 1.** If \( i, j, r, s \) are such that \( i \neq r, j \neq s \) and \( 0 < \min\{k'(i, j), k'(r, s)\} \), then either \( k'(i, s) = t_i(t_i - \delta_{i\alpha}) \) or \( k'(r, j) = t_j(t_j - \delta_{r\alpha}) \).
Proof. Suppose not. Then there are the following four cases to consider: (a) $i = s, j = r$; (b) $i = s, j \neq r$; (c) $i \neq s, j = r$; and (d) $i \neq s, j \neq r$.

Case (a). As $k'(i, i) < t_i(t_i - 1)$ and $k'(i, j) < t_i(t_i - 1)$ so we can get $x, y \in V_i, x \neq y$ and $u, v \in V_i, u \neq v$ such that $xy, uv \notin E(G)$. Now if there is no $w \in V_i$ such that $xw \in E(G)$, then let $a \in V_i, w \in V_i$ be such that $aw \in E(G)$. So if there exists $b \neq a$ such that $xb \in E(G), ab \notin E(G)$ (note $b \neq y$), then let $G \rightarrow I(awxb) \rightarrow G'$ and now $xw \in E(G')$. If there is no such $b$, then $xa \in E(G)$ and $ax \notin E(G)$. So there is $d \neq a, x$ such that $dx \in E(G), da \notin E(G)$. Let $G \rightarrow I(dxawxa) \rightarrow G'$. Note then $a \neq y$ and $xw \in E(G')$. So anyway we have $xw \in E(G')$ and $xy \notin E(G')$. As $G' \cong G$ so, without loss of generality, we can assume that there is a $w \in V_i$ such that $xw \in E(G)$.

Now if $xy \notin E(G)$ and if there is a $c \neq w$ such that $cw \in E(G), cw \notin E(G)$ (note $c \neq u$), then let $G \rightarrow I(xwcv) \rightarrow G'$ and $xc \in E(G')$. If, however, $xy \notin E(G)$ but there is no such $c$, then $xc \in E(G)$ and $xc \notin E(G)$. Hence there is $e \neq w, v$ such that $ve \in E(G), ve \notin E(G)$. Let $G \rightarrow I(xwcv) \rightarrow G'$. Then again $xy \in E(G')$. As $G' \cong G$ so, again, without loss of generality, we can assume that $xy \in E(G)$ and $xy \notin E(G')$. It can be shown similarly that we can further assume that $uy \in E(G)$. Now let $G \rightarrow I(xwuy) \rightarrow G'$ and then $|E(G[V_i])| \neq |E(G[V_i])|$. Contradiction. Hence this case is proved. The other cases are similar.

Assertion 2. For $1 \leq i \leq n$ either $k'(i, i) \in \{0, 1, t_i(t_i - 1) - 1, t_i(t_i - 1)\}$ or $k'(i, i) = t_i - 3$.

Proof. Suppose not. Then we consider the following two exhaustive cases.

Case (a). $\Delta_i^+ = \Delta_i^-$. Hence $\Delta_i^+ = \Delta_i^- = \Delta_i^+ = \Delta_i^- = r$ (say). Clearly we need only consider $t_i \geq 3$.

Let $t_i \geq 4$. As $\Pi_i$ must be unidigraphic so $r \geq 2$ and $r \leq t_i - 3$. Now $K_n$ consists of $\lfloor \frac{1}{2}(n - 1) \rfloor$ disjoint 2-factors. If $r$ is even there are at least two possibilities for $G_i$. In one case we can have the symmetric arcs of $G_i$ forming $\frac{1}{2}r$ disjoint 2-factors; and in another the symmetric arcs of $G_i$ forming $\frac{1}{2}(r - 2)$ disjoint 2-factors and the asymmetric arcs forming two disjoint 2-factors. Thus $\Pi_i$, and hence $\Pi$, is not unidigraphic. Contradiction. Thus $r$ is odd. If $3 \leq r \leq t_i - 4$, then there are at least two possible non-isomorphic realizations for $\Pi_i$: in one case the symmetric arcs of $G_i$ form $\frac{1}{2}(r - 1)$ disjoint 2-factors and the asymmetric arc one 2-factor; and in another case the symmetric arcs form $\frac{1}{2}(r - 3)$ disjoint 2-factors and the asymmetric arcs three disjoint 2-factors. Again a contradiction. So $3 \leq r = t_i - 3$. In this case we consider the complements of the realizations obtained for $r = 2$.

So if $\Pi$ is unidigraphic Case (a) cannot hold unless $t_i = 3$ and then $r = \Delta_i^+ = \Delta_i^- = 1$ and thus $k'(i, i) = t_i = 3$.

Case (b): $\Delta_i^+ \neq \Delta_i^-$. We prove this case through a series of claims.

Claim 1. There do not exist $x, y \in V_i$ such that $d_i(x) = (\Delta_i^+, \Delta_i^-)$ and $d_i(y) = (\Delta_i^-, \Delta_i^+)$.

Suppose not. Then there is $w \in V_i, w \neq x, y$, such that either $yw \notin E(G), xw \in$
E(G) or wy$ E(G), wx E(G). Say yw$ E(G), xw E(G). Then there is z E V \setminus V_i such that yz E (G), xz E (G). Let G \rightarrow I(yzxw) \rightarrow H. Then H[V_i] has one vertex less than G ii with degree ($\Delta^+_w, \Delta^-_w$). But $\Pi(H) = \Pi$. Contradiction. This proves the claim.

So we can get x, y E V_i such that $d_i(x) = (\Delta^+_w, \epsilon^-_w)$ and $d_i(y) = (\epsilon^+_w, \Delta^-_w)$.

Claim 2. For all z \neq x, y, z E V_i, either xz, yz E (G) or xz, yz$ E (G).

Suppose not. Suppose there is z \neq x, y, z E V_i such that xz E (G), zy$ E (G).

Then we can get w \neq x, y, w E V_i such that wx$ E (G) and wy E (G) as $d_i(y) > d_i(x)$. Hence we suppose, without loss of generality, that there is z' E V_i, z' \neq x, y and z'y E (G), z'x$ E (G). So there is w' E V \setminus V_i such that w'x E E(G), w'y$ E(G). Let G \rightarrow I(w'xy) \rightarrow H. Then H[V_i] contradicts Claim 1. This proves Claim 2.

Similarly we have the following:

Claim 3. For all z \neq x, y, z E V_i either xz, yz E (G) or xz, yz$ E (G).

From Claim 2 and Claim 3 we see that xy E (G) and yx$ E (G).

Claim 4. For all z E V_i, z \neq x, y $d_i(z) = d_i(z)$.

Suppose not. Then, without loss of generality, we can get z E V_i, z \neq x, y such that $d(z) = (\Delta^-_w, \epsilon^+_w)$. Then as before we see that zy E (G), zy$ E (G). Now there is w E V \setminus V_i such that xw$ E (G), yw E (G). So if xz E (G), let G \rightarrow I(ywzx) \rightarrow H and in H[V_i] we have $d(y) = (\Delta^+_w, \Delta^-_w)$ and $d(x) = (\epsilon^+_w, \epsilon^-_w)$ contradicting Claim 1. So xz$ E (G). Now there is w' E V \setminus V_i such that w'x E E(G), w'y$ E(G). Let G \rightarrow I(w'xy) \rightarrow H and we get a contradiction to Claim 1 as above. Hence Claim 4 is proved.

So we suppose, without loss of generality, that for all z E V_i, z \neq x, y we have $d_i(z) = (\epsilon^+_w, \epsilon^-_w)$. (Otherwise we consider the complement.)

Claim 5. Either $\epsilon^-_w = 0$ or $t_i - 2$.

Suppose not. Then by Claim 2 there exist z, w E V_i, {z, w} \cap \{x, y\} = \emptyset such that xz, zy E (G) and wx, wy$ E (G). By Claim 4 we know $d_i(z) = d_i(w)$. So there is u E V_i, u \neq z such that wu E (G), zu$ E (G). Let G \rightarrow I(wuzx) \rightarrow H. Then w and u contradict Claim 2 in H[V_i]. This proves Claim 5.

Now Claims 4 and 5, together with the only possibility in Case (a), yield the assertion.

Assertion 3. If $k'(i, i) = t_i = 3$, then for all j \neq i, 1 \leq j \leq n, \{k'(i, j), k'(j, i)\} \subseteq \{0, 3t_j\}.

Proof. Let x, y E V_i. Suppose there exists w E V \setminus V_i such that xw E (G), yw$ E (G). Then, without loss of generality, we can have xz \notin E(G ii) and yz E (G ii) where z is the third vertex of G ii. Let G \rightarrow I(xwyz) \rightarrow H. Then $\Pi(H[V_i]) \neq \Pi(G ii)$, a contradiction. Hence there can be no such w for any pair of vertices of V_i. So if u E V_i(j \neq i) and u is joined from (not joined from) all vertices of V_i then all vertices of V_i are joined from (not joined from) all vertices of V_i as G ii is semiregular. Hence $k'(i, j) \in \{0, 3t_j\}$. Similarly $k'(j, i) \in \{0, 3t_i\}$. This proves the assertion.
Let $I = \{ i : t_i \geq 2 \text{ and } k'(i, i) = 0 \text{ or } t_i(t_i - 1) \}$. We now make two observations for all $i \in I$. These will be used repeatedly, and sometimes implicitly, in our arguments.

**Observation 1.** As $k'(i, i) = 0$ or $t_i(t_i - 1)$, so for $x, y \in V_i$, if we have $xy \not\in E(G)$, $uy \not\in E(G)$ (resp. $ux \not\in E(G)$, $vy \not\in E(G)$), then we can get a $u, v \not= x, y$, such that $yu \in E(G), xv \not\in E(G)$ (resp. $vy \in E(G), ux \not\in E(G)$).

**Observation 2.** For any $j_1, j_2 \not= i$ and $x \in V_i, d_{j_1}^+(x)$ is independent of $d_{j_2}^-(x)$. That is if there is $y \in V_i$ such that $d_{j_2}^-(y) = \alpha$, then without loss of generality we may assume that $d_{j_1}^+(x) = \alpha$ and $d_{j_2}^-(x)$ is unaffected.

**Assertion 4.** For all $i \in I$ conditions B1 to B6 are satisfied.

**Proof.** Either $X_i^+ = Y_i^+ = 0$ or $X_i^+ = t_i$ or $Y_i^+ = t_i$ and either $X_i^- = Y_i^- = 0$ or $X_i^- = t_i$ or $Y_i^- = t_i$. This can be proved exactly in the same way, using Observation 1, as Assertion 1 of [5] was proved, though we considered graphs there. So now if B1 does not hold then, without loss of generality, we have $i \in I$ such that $X_i^+ = t_i$ and $X_i^- = t_i$. Let $j_1, j_2$ be such that $\Delta_{j_1}^+ \not\in \varepsilon_{j_1}^+ \text{ and } \Delta_{j_2}^- \not\in \varepsilon_{j_2}^-$. Then using Observation 2 we see that there are at least two non-isomorphic realizations of $\Pi$; in one we have the maximum possible vertices of $V_i$ with $d_{j_1}^+ = \Delta_{j_1}^+$ and $d_{j_2}^- = \Delta_{j_2}^-$ and in another the minimum possible number of such vertices. This proves that B1 is satisfied.

Using Observation 1 we can prove that B2 to B5 (and hence also B6 which is the converse) hold just as Assertions 2 to 8 of [5] were proved. The difference in formulation of the conditions arise as now, for $m = i$, we know that $k'(i, i) = 0$ or $t_i(t_i - 1)$ and $i \in I$ implies $t_i \geq 2$.

For the next three assertions we define the following. $G(\Pi)$ is the following digraph:

$$V(G(\Pi)) = I, \quad E(G(\Pi)) = \{ ij : 2 \leq k'(i, j) \leq t_it_j - 2 \}.$$

Let $i \in I$. We say that $i$ is

- $A^+$ if there exists $m \in I$ such that $t_i < k'(i, m) < t_mm - t_i$ and $t_i \mid k'(i, m)$.
- $B^+$ if there exists $m \in I$ such that $t_i < k'(i, m) < t_mm - t_i$ and $t_i \mid k'(i, m)$.
- $C^+$ if for all $h \in I$ either $k'(i, h) \leq t_i$ or $k'(i, h) \geq t_h - t_i$; and either $(X_i^+, Y_i^-) \not\in \{(0, 0), (t_i, t_i)\}$ or there exists $m$ such that $\min\{ k'(m, i) \pmod{t_i}, -k'(m, i) \pmod{t_i} \} \geq 2$.
- $D^+$ if $(X_i^+, Y_i^-) = (t_i, t_i)$, there exists $m$ such that $k'(m, i) \pmod{t_i} = t_i - 1$, and for all $h \in I$ either $k'(i, h) \leq t_i$ or $k'(i, h) \geq t_h - t_i$.
- $E^+$ if $(X_i^+, Y_i^-) = (0, 0)$ and for all $h \in I$ either $k'(i, h) \leq t_i$ or $k'(i, h) \geq t_h - t_i$.

It may be seen that the above classification of $I$ is mutually exclusive and exhaustive. Similarly we define types $A^-, B^-, C^-, D^-$ and $E^-$ by considering the converse of the above.
In the above context we have the following:

**Assertion 5.** If there is a directed path from $i$ to $j$ ($i$ and $j$ are not necessarily distinct) in $G(II)$, then at least one of the following holds:

(a) $i$ is $E^+$,
(b) $j$ is $E^-$.
(c) $i$ is $A^+$, $j$ is $D^-$ and $X_i = Y_i^- = 0$,
(d) $i$ is $D^+$, $j$ is $A^-$ and $X_j = Y_j^- = 0$,
(e) $i$ is $A^+$ or $B^+$, $j$ is $A^-$ or $B^-$, $X_i = Y_i^- = X_j = Y_j^- = 0$ and $ij \in E(G(II))$.

**Proof.** Suppose not. By Observation 2 we see that we need not distinguish the cases $i, j$ are distinct and are not distinct. Then of all the pairs $i, j$ that violate the assertion we choose one $i_0, j_0$ with the shortest directed path from $i_0$ to $j_0$. Then $i_0$ is one of $A^+, B^+, C^+, D^+$ and $j_0$ is one of $A^-, B^-, C^-, D^-$. We will consider all the sixteen possibilities. However, here we will give the proof in only one case, as the proofs of the others are similar. Without any loss of generality we take the directed path from $i_0$ to $j_0$ to be $\mu = [i_0 = 1, 2, \ldots, j_0]$.

**Case (i).** $i_0$ is $A^+$ and $j_0$ is $A^-$. If $j_0 = 2$, then $II_{i_0j_0}$ is not of Type 1 as it is regular on both sides, and hence $B2$ is not satisfied. Contradiction. So $j_0 \geq 3$. Now by choice of $i_0$ and $j_0$ we know that for $2 \leq k \leq j_0 - 2$, $k_{j_0} \notin E(G(II))$ and hence $k$ is either $E^+$ or $D^+$. Similarly for $3 \leq k \leq j_0 - 1$, $k$ is either $E^-$ or $D^-$. In case $2 \leq k \leq j_0 - 2$ then either $k'(k, k + 1) \leq \min\{k, k + 1\}$ or $k'(k, k + 1) \geq t_{k}t_{k + 1} - \min\{k, k + 1\}$. So some vertices of, say $\alpha_k$, of $V_k$ are matched in a 1-1 fashion onto $\alpha_k$ vertices of $V_{k + 1}$ either through arcs, if $k'(k, k + 1) \leq \min\{k, k + 1\}$, or through non-arcs, if $k'(k, k + 1) \geq t_{k}t_{k + 1} - \min\{k, k + 1\}$. Also $\alpha_k \geq 2$ as $2 \leq k'(k, k + 1) \leq t_{k}t_{k + 1} - 2$. By Observation 2 we see that we can get $\alpha$ ($\alpha \geq 2$) vertices of $V_2$ matches in a 1-1 fashion, through a series of arcs and non-arcs, to $\alpha$ vertices of $V_{j_0 - 1}$. (If $j_0 = 3$, then we take the identity map.)

Now we have three subcases to consider.

**Subcase (1):** $t_2 < k'(1, 2) < t_1t_2 - t_2$ and $t_{j_0 - 1} < k'(j_0 - 1, j_0) < t_{j_0 - 1}t_{j_0} - t_{j_0 - 1}$. As $B2$ is satisfied so $II_{12}$ and $II_{j_0 - 1, j_0}$ are of Type 1. Hence $\Delta_{12} \neq \varepsilon_{12}$ and $\Delta_{j_0 - 1, j_0} \neq \varepsilon_{j_0 - 1, j_0}$. Hence if $\alpha = t_2$, by Observation 2, we know we can get at least two non-isomorphic realizations of $\Pi$; in one we leave out of the matching the maximum possible and in another the minimum possible number of vertices of $V_2$ with $d_{12} = \varepsilon_{12}$. Contradiction. Hence $\alpha = t_2$. Similarly $\alpha = t_{j_0 - 1}$. Again as before we can get at least two non-isomorphic realizations by matching in one case the maximum possible and in another the minimum possible of vertices in $V_2$ with $d_{12} = \varepsilon_{12}$ to vertices in $V_{j_0 - 1}$ with $d_{j_0 - 1, j_0} = \varepsilon_{j_0 - 1, j_0}$. Contradiction. Hence this subcase cannot hold.

**Subcase (2):** $t_2 < k'(1, 2) < t_1t_2 - t_2$ and either $k'(j_0 - 1, j_0) \leq t_{j_0 - 1}t_{j_0} - t_{j_0 - 1}$ or $k'(j_0 - 1, j_0) \geq t_{j_0 - 1}t_{j_0} - t_{j_0 - 1}$ (resp. $t_{j_0 - 1} < k'(j_0 - 1, j_0) < t_{j_0 - 1}t_{j_0} - t_{j_0 - 1}$ and either $k'(1, 2) \leq t_2$ or $k'(1, 2) \geq t_{1}t_{2} - t_2$). As in previous subcase we get that $\alpha = t_2$. Also $k'(j_0 - 1, j_0)/t_{j_0} (\geq 2)$ vertices of $V_{j_0 - 1}$ are joined through arcs of $G_{j_0 - 1, j_0}$ to a single
vertex of $V_{j_0}$ as $j_0$ is $A^-$. The vertices of $V_{i_0-1}$, which are not in the matching from $V_2$, may be made adjacent, in one case, from as many distinct vertices of $V_{j_0}$ as possible and, in another case, from as few as possible. So if $\alpha = k'(j_0 - 1, j_0) - 2$ then these two numbers are distinct and hence $\Pi$ is not unidigraphic. Contradiction.

If $\alpha = k'(j_0 - 1, j_0) - 1$, then let $u$ be the only vertex of $V_{i_0-1}$ which is not in the matching from $V_2$ and let $v$ be the only vertex of $V_{i_0}$ to which $u$ is matched. (That is $u$ is only adjacent from $v$ or non-adjacent from $v$ according as $k'(i_0 - 1, i_0) = t_{i_0}$ or $k'(j_0 - 1, j_0) = t_{i_0-1}t_{j_0-1} - 1$.) (Note we can get such $u$ and $v$ from Observation 2.) Now as $1 < d_{i_0-1, v}(v) < t_{i_0-1} - 1$, so we can have $v$ matched to as many as possible vertices of $V_2$ with $d_{i_2} = e_{i_2}$, in one case, and to as few as possible, in another case, to give two non-isomorphic realizations of $\Pi$. Contradiction. Hence $a = k'(j_0 - 1, j_0) = \alpha$. So $t_2$ is not a prime number as $j_0$ is $A^-$, $t_{i_0} > 2$ and $t_2 = k'(i_0 - 1, j_0)$.

Now by choice of $i_0, j_0$ we know 1, 2 satisfy condition (e) and $\Pi_{i_2}$ is of Type 1 as B2 is satisfied. From definition of Type 1 we can see that, as $\Delta_{i_2} = \delta_{i_2}$ and $t_2$ is not a prime number, we can always get $U \subset V_2$ with $|U| \geq 2$, $|V_2 - U| \geq 2$ such that $x \in U, y \in V_2 - U, \alpha$ is an automorphism of $G_{i_2}$ implies that $\alpha(x) \neq y$. So the vertices of $U$ can be matched to as many distinct vertices of $V_i$, as possible in one case, and to as few as possible in another case to give non-isomorphic realizations of $\Pi$. Contradiction. Hence this subcase cannot hold.

Subcase (3): Not in any of the previous subcases. Again, as before, we see that $k'(1, 2)/t_1$ vertices of $V_2$ are matched through arcs to a single vertex of $V_1$ and $k'(j_0 - 1, j_0)/t_{i_0}$ of $V_{i_0-1}$ to a single vertex of $V_{j_0}$. Let $V_1 = \{v_1, \ldots, v_{t_1}\}$ and let $g_s$ denote the number of distinct vertices of $V_{i_0}$ to which $v_s$ is matched in the above matching for $1 \leq s \leq t_1$. Then, as $\alpha \geq 2$, we can get distinct unordered $t_1$-tuples $(g_1, \ldots, g_{t_1})$. This implies, by Observation 2, that we can get non-isomorphic realizations of $\Pi$. Contradiction. This completes the proof for Case (i).

Assertion 6. There is no directed path from $i$ to $m$ via $j$ in $G(\Pi)$ if either $i$ is $A^+$ and $j, m$ are $D^-$ or $i, j$ are $D^+$ and $m$ is $A^-$. 

Proof. Suppose not. Then of all triples violating the assertion we choose $i_0, j_0, m_0$ such that the directed path from $i_0$ to $m_0$ via $j_0$ is shortest. Without loss of generality we may take $i_0$ to be $A^+$ and $j_0, m_0$ to be $D^-$. So as seen in Case (i) of proof of Assertion 5 we can get $\alpha (\geq 2)$ vertices of $V_{i_0}$ matches through a series of arcs and non-arcs to at least $\alpha$ vertices of $V_{i_0}$ and then to at least $\alpha$ vertices of $V_{m_0}$. Let $z$ (resp $z'$) be the vertex of $V_{i_0}$ (resp. $V_{m_0}$) which has its outdegree distinct from all other vertices of $V_{i_0}$ (resp. $V_{m_0}$) in some $G_{i_0r}$ (resp. $G_{m_0r}$), then it may be seen, by Observation 2, that the same or different vertices of $V_{i_0}$ may be matched to $z$ and $z'$ to yield non-isomorphic realizations of $\Pi$. Contradiction. This proves the assertion.

We will require the following result in the proof of the next assertion.
Lemma 2.3. (Das [5]). Let \( G \) be a bipartite graph with bipartition \( V_1 \cup V_2 \), which is semi-regular and has \( q \) edges. If \( 2 \leq q \leq mn - 2 \), where \( |V_1| = n \geq 2 \) and \( |V_2| = m \geq 2 \), then there exist \( x, y \in V_1 \) and \( u, v \in V_2 \) such that \( xu, yv \in E(G) \) and \( xu, yu \notin E(G) \).

Assertion 7. There is no directed cycle in \( G(II) \).

Proof. Suppose not. Without loss of generality we take \((1, \ldots, r)\) to be the shortest directed cycle in \( G(II) \). For all \( k, 1 \leq k \leq r \), we can get \( x_k, y_k \in V_k \) such that \( x_k \) (resp. \( y_k \)) is matched (through a non-arc if \( k'(k, k + 1) \geq \max\{k_1, k_1 + 1\} \) and through an arc otherwise) to \( x_{k+1}\) (resp. \( y_{k+1}\)) and not matched to \( y_{k-1}\) (resp. \( x_{k-1}\)), where the subscripts are taken modulo \( r \). This is so because of Observation 2 and Lemma 2.3, which is satisfied since \((k, k+1) \in E(G(II))\).

We call \((x_1, \ldots, x_r)\) and \((y_1, \ldots, y_r)\) \( r \)-cycles of the matching. Let \( G \rightarrow I(x_1, x_2, y_2) \rightarrow G' \). Now in \( G' \) the total number of \( r \)-cycles of the matching must be same as in \( G' \). Hence there exist \( k_1, k_2, 1 \leq k_1, k_2 \leq r \) such that \( x_{k_1} \) or \( y_{k_1} \), is matched to more than one vertex of \( V_{k_1+1} \) and \( x_{k_2} \) or \( y_{k_2} \) is matched from more than one vertex of \( V_{k_2-1} \). Hence, \( k_1 \) is \( A^+ \) or \( B^+ \), \( k_2 \) is \( A^- \) or \( B^- \) and there is a directed path from \( k_1 \) to \( k_2 \) in \( G(II) \). Thus, by Assertion 5, we get that \( k_2 = k_1 + 1 \).

Now let \( G \rightarrow I(x_k, x_{k+1}, y_k, y_{k+1}) \rightarrow G' \) and as above we get \( m_1, m_2, 1 \leq m_1, m_2 \leq r \) such that \( m_1 \) is \( A^+ \) or \( B^+ \), \( m_2 \) is \( A^- \) or \( B^- \) and \( m_1 \neq k_1, m_2 \neq k_2 \). So now we have a directed path of length greater than one from \( k_1 \) to \( m_2 \), contradicting Assertion 5. Hence this assertion is proved.

Proof of Theorem 2.1. The necessity of (A) follows from Assertions 1 to 3 and that of (B) from Assertion 4. Now note that conditions C1(a) to C1(e) of Theorem 2.1 are a restatement of conditions (a) to (e) of Assertion 5, and C2 is a restatement of Assertion 6. Hence the necessity of (C) follows from Assertions 5 to 7.

To prove the sufficiency we require the following definitions. Let \( H \) be a digraph with \( \{W_1, \ldots, W_n\} \) a fixed partition of \( V(H) \) and \( E(H) = \{u_1v_1, \ldots, u_nv_n\} \) where \( q = |E(H)| \). Then we associate with \( H \) the i.p.s. \( S_H = \{(a_1, b_1), \ldots, (a_n, b_n)\} \) where \( a_i = r, b_i = s \) if and only if \( u_i \in W_s, v_i \in W_t \). Then for \( 1 \leq i \leq n \) we define \( t_i = 1/|W_i| \).

Let \( G \) and \( H \) be two digraphs with the same partition \( \{W_1, \ldots, W_n\} \) of \( V(G) = V(H) \). For \( x \in W_i \) we define \( d_i^+(x) = d_{GI[W_i]}^+(x) \) and \( d_i^-(x) = d_{GI[W_i]}^-(x) \); \( e_i^+(x) \) and \( e_i^-(x) \) are similarly defined with \( H \) replacing \( G \).

For \( x \in W_i \) let

\[
\begin{align*}
f(x) &= ((d_{i_1}(x), \ldots, d_{i_n}(x)), (d_{i_l}(x), \ldots, d_{i_m}(x))), \\
g(x) &= ((e_{i_1}(x), \ldots, e_{i_n}(x)), (e_{i_l}(x), \ldots, e_{i_m}(x))).
\end{align*}
\]

Then a map \( \phi : V(H) \rightarrow V(G) \) is said to be permissible of type 3 (PT3) if \( \phi \) is 1-1 and for \( 1 \leq i \leq n, x \in W_i \) implies \( \phi(x) \in W_i \) and \( g(x) = f(\phi(x)) \).
Now we will consider all realizations of $\Pi$ to be on the vertex set $V$ with fixed partition \{\(V_1, \ldots, V_n\)\} such that all members of $V_i$ have degree \((r_i, s_i)\) in any realization of $\Pi$. We take $G$ to be the canonical realization of Theorem 2.2 and $H$ to be any realization of $\Pi$. Then we will show that we can get a PT3 map from $V(H)$ onto $V(G)$, which will moreover be an isomorphism.

We first note from the construction of $S^*(\Pi)$ that $S^*(\Pi) = S_D$ where $D$ is a realization of $\Pi$. Now we know (see [3], [8]) that we can obtain $H$ from $D$ as follows:

$$D = D_0 \rightarrow I_1 \rightarrow D_1 \cdots \rightarrow I_k \rightarrow D_k = H$$

where $I_m$ is a $(D_{m-1}, D_{m-1})$ trail of length four for $1 \leq m \leq k$. As A1 is satisfied by $S_D$, so $S_{I_m} = S_D = S^*(\Pi)$. Hence $S_{I_m} = S_{G} = S^*(\Pi)$.

We obtain a PT3 map $\phi$ from $V(H)$ onto $V(G)$ as follows: for each $i, 1 \leq i \leq n$, we define $\phi$ on $V_i$ according to one of the following four steps, depending on the conditions satisfied by $i$. These are exhaustive as A2 is satisfied.

**Step 1:** $i$ is such that $t_i = 1$. Then we define $\phi$ to be the identity map. Clearly for $x \in V_i, \phi(x) \in V_i$ and, as $S_{I_m} = S_{G}$ it follows that $g(x) = f(\phi(x))$.

**Step 2.** $i$ is such that $t_i \geq 2$ and $k'(i, i) = 1$ (resp. $t_i(t_i - 1) - 1$). We define $\phi$ on $V_i$ to be an isomorphism from $H_i$ onto $G_i$, which clearly exists. To show that this definition of $\phi$ will serve we have the following:

**Claim 1.** There is an $h$ such that $t_h = 1, k'(i, h) = k'(h, i) = t_i t_h - 1$ (resp. 1) and for all $m \neq i, h$ we have \(k'(i, m), k'(m, i) \subset \{0, t_i t_m\}\).

To prove the claim we observe that there do not exist $e, f$ such that $i \neq e, f$, $t_e \geq 2$ if $e = f$, $0 < k'(i, f) < t_i t_f$ and $0 < k'(e, i) < t_e t_i$. (If such $e, f$ exist then we get a contradiction as we can choose $j, r, s$ from amongst $i, e, f$ such that $i, j, r, s$ do not satisfy A1.)

However, from the construction of $S^*(\Pi)$ we know that $\sum_{i=1}^{n} k'(i, r) = t_i t_r$ and so $t_i \mid \sum_{r} k'(i, r)$. Similarly, $t_h \mid \sum_{r} k'(h, r)$. As $t_h \mid \sum_{r} k'(i, r)$ so there exist $e, f$ such that $0 < k'(i, f) < t_i t_f$ and $0 < k'(e, i) < t_e t_i$. Hence, by the above observation, we get that $e = f = h$(say) and $t_h = 1$.

Also if for any $m \neq i, h$ we have $0 < k'(i, m) < t_i t_m$ or $0 < k'(m, i) < t_m t_i$, then we can choose $j, r, s$ from amongst $i, h, m$ such that $i, j, r, s$ do not satisfy A1. Contradiction. So for all $m \neq i, h$ we have \{k'(i, m), k'(m, i)\} $\subset \{0, t_i t_m\}$. This moreover implies, as $t_i \mid \sum_{r} k'(i, r)$ and $t_h \mid \sum_{r} k'(h, r)$, that $k'(i, h) = k'(h, i) = t_i t_h - 1$ (resp. 1). This proves the claim.

Now it follows immediately from Claim 1 that for all $x \in V_i, \phi(x) \in V_i$ and $g(x) = f(\phi(x))$.

**Step 3:** $k'(i, i) = t_i = 3$. As A3 is satisfied so $\Pi(G_H) = \Pi(H_H) = (1, 1)^3$. Hence there is an isomorphism $\sigma$ from $G_H$ onto $H_H$. We take $\phi = \sigma$ on $V_i$. Thus for all $x \in V_i, \phi(x) \in V_i$ and, as A3 is satisfied, $g(x) = f(\phi(x))$.

**Step 4:** $i$ does not satisfy the conditions of any of the previous steps. We first make the following claim.

**Claim 2.** For all $j, 1 \leq j \leq n, \Pi(H_{ij}) = \Pi(G_{ij}) = \Pi_{ij}$ and $\Pi(H_{ii}) = \Pi(G_{ii}) = \Pi_{ii}$. 
First note that \( t_i \geq 2 \) and \( k'(i, i) = 0 \) or \( t_i(t_i - 1) \) as we are in this step and A2 is satisfied. Hence B2 to B6 is satisfied by \( i \). The rest of the proof is omitted as it is similar to the proof of the corresponding claim, given in the sufficiency of Theorem 2.1 in [5].

Now from Claim 2 and B1 we see that we can define \( \phi \) on \( V_i \) such that for \( x \in V_i, \phi(x) \in V_i \) and \( g(x) = f(\phi(x)) \).

Hence we have shown that \( \phi \), obtained above, is indeed a PT3 map. Now, if \( x \in V_i \), where \( k'(i, i) \neq 0 \) or \( t_i(t_i - 1) \), then by A2 we know that either \( k'(i, i) = 1 \) or \( t_i(t_i - 1) - 1 \) or \( k'(i, i) = t_i = 3 \). Hence if \( \psi \) is any PT3 map from \( V(H) \) onto \( V(G) \) obtained as above then we see from A3 and Claim 1 that \( x \in E(H) \) if and only if \( \psi(x) \psi(y) \in E(G) \). Similarly \( yx \in E(H) \) if and only if \( \psi(y) \psi(x) \in E(G) \). Let \( U = \{x \in V(H) : x \in V_i \text{ where } k'(i, i) = 0 \text{ or } t_i(t_i - 1)\} \) and let \( \psi \) be an isomorphism from \( H[U] \) onto \( G[U] \). Then \( \psi \) may be extended to an isomorphism \( \phi \) from \( H \) onto \( G \) by defining \( \phi \) on \( V(H) - U \) according to Steps 2 and 3 and defining \( \phi - \psi \) on \( U \). Hence it is sufficient to show that \( H[U] \) is isomorphic to \( G[U] \) to complete the proof of this theorem. This is shown in the following claim as \( H[U] \) and \( G[U] \) satisfy the conditions of the claim.

Claim 3. If \( D_1 \) and \( D_2 \) are two digraphs with a fixed partition \( \{W_1, \ldots, W_n\} \) of \( V(D_1) = V(D_2) \) such that

(i) for \( 1 \leq i \leq n \), \( y \in W_i \) imply \( d_{D_1}(x) = d_{D_1}(y) = d_{D_2}(x) = d_{D_2}(y) \) (ii) there is a PT3 map \( \phi \) from \( V(D_1) \) onto \( V(D_2) \); (iii) \( S_{D_1} \) satisfies B and C and, for \( 1 \leq i \leq n \), \( k'(i, i) = 0 \) or \( t_i(t_i - 1) \),

then there is an isomorphism \( \psi \) from \( V(D_1) \) onto \( V(D_2) \) such that \( \psi(W_i) = W_i \) for \( 1 \leq i \leq n \).

Before proving the claim we note that \( H[U] \) and \( G[U] \) satisfy (i) because of A3 and Claim 1.

We prove the claim by induction on \( |E(G(II))| \). Clearly the claim is true for \( |E(G(II))| = 0 \) (and for all \( n \)) as then \( \phi \) is the required isomorphism from \( D_1 \) onto \( D_2 \). Our induction hypothesis is that the claim is true if \( |E(G(II))| \leq m \).

Let \( |E(G(II))| = m + 1 \). We first note that \( S_{D_2} = S_{D_1} \) as there is a PT3 map. As C3 is satisfied so we can get in \( G(II) \) a maximal directed path from \( i \) to \( j \), say. Then we have the following three exhaustive cases:

Case 1. \( i \) satisfies C1(a) (respectively \( j \) satisfies C1(b) by considering the converse).

Case 2. \( i, j \) satisfy C1(c) (respectively C1(d) by considering the converse).

Case 3. \( i, j \) satisfy C1(e).

We give here the proof of Case 1 only as the proof of the other cases are similar.

Let \((i, m_1), \ldots, (i, m_r) \in E(G(II))\). Then we form \( D_i' \) from \( D_i \) as follows: we delete all arcs of \( G_{im} \) for \( 1 \leq k \leq r \). Let \( W_{r+1} = \{v\} \), where \( v \notin V(D_i) \), and join \( v \) to all \( x \in W_{m_k} \) such that \( d_{m_k}(x) = \Delta_{m_k}, 1 \leq k \leq r \). Similarly we form \( D_j' \). Then by the induction hypothesis we can get an isomorphism \( \phi' \) from \( V(D_i') \) onto \( V(D_j') \) such that \( \phi'(W_i) = W_j' \) for \( 1 \leq i \leq n + 1 \).
Then we define $\psi$ on $W_i$ as follows: for $1 \leq k \leq r$ if $x \in W_{m_k}$ is matched (through arcs or nonarcs as $i$ is $E^*$) to $x_1, x_2, \ldots, x_a \in W_i$ in $D_1$ and $\phi'(x)$ is matched similarly to $y_1, y_2, \ldots, y_a \in W_i$ in $D_2$ then we define $\psi(x_i) = y_s$ for $1 \leq s \leq a$. We see that $\psi$ is defined on all vertices of $W_i$ because $i$ is $E^+$ and $B1$ is satisfied. On $V(D_1) - W_i$ we define $\psi = \phi'$. Since the fact that a maximal path begins from $i$, which is $E^+$, implies that for all $m \neq i, k'(m, i) = 0$ or $t_{it}$, so it follows that $\psi$ serves as the required isomorphism.

This completes the proof of Theorem 2.1.

We immediately have the following corollary.

**Corollary 2.4.** \[ [(a_1, b_1), \ldots, (a_m, b_m)), ((c_1, d_1), \ldots, (c_m, d_n))] \] has unique realization by bipartite digraph if and only if it is bipartite digraphic and $S^*(II)$ satisfies conditions (A), (B) and (C), where $II$ is the sequence \((a_1 + m - 1, b_1 + m - 1), \ldots, (a_m + m - 1, b_m + m - 1), (c_1, d_1), \ldots, (c_m, d_n))\) written as in (2.1).

**Proof.** The proof follows from the observation that with each bipartite realization of \([(a_1, b_1), \ldots, (a_m, b_m)), ((c_1, d_1), \ldots, (c_m, d_n))] \) with bipartition $V_1 \cup V_2$ we can associate a realization of $II$ by adding arcs between all pairs of distinct vertices of $V_1$ and that the converse is possible since all realizations $G$ of $II$ on $V_1 \cup V_2$ will have $G[V_1]$ complete.

3. Unigraphic degree sequences

(3.1) Let $II = d'_1, \ldots, d'_n$ be a sequence of positive integers where $d'_i$ denotes that $d_i$ occurs exactly $t_i$ times in $II$ and for $1 \leq i \neq j \leq n$, we have $t_i > 0$ and $d_i \neq d_j$.

Let $II$ be as in (3.1). If $II$ is graphic, then we define an i.p.s. $S(II)$ according to the following algorithm based on the one of [7, 12].

**Step 1.** Put $S(II) = \emptyset$ and $V = \emptyset$ where $\emptyset$ denotes the empty sequence. For all $i, 1 \leq i \leq n$, add $t_i$ members $(d_i, d'_i)$, with $d'_i = d_i$, to $V$. Go to Step 2.

**Step 2.** Order $V$ such that the $d'_i$ sequence is non-increasing. Let $(d_k, d'_k)$ be the first member of $V$. Then add $(d_k, d'_k)$ to $S(II)$ and put $d'_i = d'_i - 1$ for the first $d'_i$ members $(d_j, d'_j)$ of $V$, other than the first member $(d_k, d'_k)$. Remove $(d_k, d'_k)$ from $V$. Proceed to Step 3.

**Step 3.** If for any member of $Vd'_i$ is zero, then remove it from $V$. If $V = \emptyset$ stop. Otherwise go to Step 2.

Then by definition $k(d_i, d_j)$ is the number of times $(d_i, d_j)$ or $(d_j, d_i)$ occurs in $S(II)$.

We now give a characterization of unigraphic degree sequences in terms of unigraphic i.p.s., characterized in [4, 5]. Alternate characterizations of unigraphic degree sequences have been obtained earlier in [11], [14] and [15].
Theorem 3.1. A graphic degree sequence II is unigraphic if and only if $S(II)$ is a unigraphic i.p.s. satisfying the following condition:

**P1.** If $i, j, r, s$ are such that $i \neq r, j \neq s$ and $\delta_i, \delta_r < \min\{k(d_i, d_j), k(d_r, d_s)\}$, then either

$$k(d_i, d_j) = \frac{t_i(t_i - \delta_i)}{1 + \delta_i} \quad \text{or} \quad k(d_r, d_s) = \frac{t_r(t_r - \delta_r)}{1 + \delta_r}.$$ 

**Proof.** *Necessity.* It is clear that $S(II)$ is a unigraphic i.p.s.. The proof that P1 is satisfied is omitted as it is similar to the proof of Assertion 1 of the previous section.

*Sufficiency.* Let $H$ be any realization of $II$. Now we see from the construction of $S(II)$ that $S(II) = S_G$ where $G$ is a canonical realization of $II$. We know (see [2, p. 153], cf. [6, 7]) that we can obtain $H$ from $G$ as follows:

$$G = G_0 \rightarrow I_1 \rightarrow G_1 \rightarrow \cdots \rightarrow I_k \rightarrow G_k = H$$

where $I_m$ is a $(G_{m-1}, \tilde{G}_{m-1})$ trail of length four for $1 \leq m \leq k$. As P1 is satisfied so we get that $S_H = S_G = S(II)$. Hence $H$ is isomorphic to $G$ as $S(II)$ is unigraphic. This proves the sufficiency.

We now obtain a characterization for pairs of sequences having unique realization by bipartite graphs in the following corollary, whose proof is similar to that of Corollary 2.4 and hence omitted. An alternate characterization has been obtained earlier in [13].

**Corollary 3.2.** $[(a_1, \ldots, a_m), (b_1, \ldots, b_n)]$ has unique realization by bipartite graph if and only if it is bigraphic and $S((a_1 + m - 1, \ldots, a_m + m - 1, b_1, \ldots, b_n))$ is a unigraphic i.p.s. satisfying P1.

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**References**


