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Stabilisation of parameterically perturbed input-delay systems with saturating actuator

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In this article, the problem of delay-dependent robust stabilisation via dynamic output feedback controllers for time-delay saturating actuator systems with norm-bounded parameter uncertainty is addressed. Instead of applying the Lyapunov function, properties of the comparison theorem and the matrix measure with the model transformation technique are employed to investigate the problems. The sufficient condition for robust stabilisation of input-delay systems with saturating actuators subjected to parametric perturbation is also derived. Both results give an estimate of the maximum time delay, which preserves robust stabilisation. Moreover, an algorithm is proposed to synthesise a dynamic feedback controller for guaranteeing the asymptotic stabilisation of the uncertain time-delay saturating systems. Numerical examples are given to indicate significant improvements over some existing results.

Keywords: stabilisation; delay-dependent; time delay; saturating actuator

1. Introduction

Both time delay and saturating actuators are commonly encountered in various engineering systems; their existence are often the source of instability. Therefore, it is very desirable for the control system design to investigate the problem of stabilisation of systems with time delay and saturating actuators. Many methods for checking the stability of time delay systems (Halanay 1966; Lakshmikantham and Leela 1969; Lu, Tsai, and Su 2002) or uncertain systems with saturating controls have been proposed (Glattfelder and Schaufelberger 1983; Chou, Horng, and Chen 1989). A different idea exploited in the literature (Goubet, Dambrine and Richard 1997) is to find some decomposition of the delayed term in order to improve the delay bounds. Furthermore, the problem of stabilisation of uncertain time-delay systems with saturating actuators has attracted an important amount of interest in recent years (Niculescu, Dion, and Dugard 1996; Tsay and Liu 1996; Cao et al. 2002). The approach is based on a Lyapunov–Krasovskii technique for analysing the case of robust stabilisation of time-delay systems with saturating actuators (Niculescu et al. 1996; Tsay and Liu 1996; Cao et al. 2002). One of them is a lack of efficient algorithms for constructing the corresponding Lyapunov–Krasovskii functionals (Kharitonov and Zhabko 2003). Unfortunately, there is no direct method to compute the Lyapunov function. Therefore, the Lyapunov’s method for investigation of stability criteria of equations with delay encountered some difficulty. In this article, we will further exploit the idea in Liu (2001) to arrive at an estimate of the domain of attraction for parameterically perturbed input-delay systems subject to actuator saturation. The main objective of this article is to design linear dynamic output feedback controllers, which ensure global uniform asymptotic stabilisation for any time delay not larger than a given bound. The most striking feature of these results eliminates the need to involve any turning of parameters, as in the case with robust stabilisation (Su et al. 2002; Liu, 2005), and it can be computed effectively (Liu 1995). Two illustrate examples are worked out to show the usefulness of the theoretical results. Numerical examples show that the results obtained in this article are less conservative than the existing stabilisation criteria (Niculescu et al. 1996; Tsay and Liu 1996; Su et al. 2001; Liu 2005).

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2. Problem statement

Consider the linear constrained uncertain time-delay systems described by the differential difference equation of the form:

\[
\dot{x}(t) = A x(t) + \Delta A x(t) + A_1 x(t-\tau) + \Delta A_1 x(t-\tau) + B u(t) + \Delta B u(t),
\]

\[
y(t) = C x(t) + \Delta C x(t),
\]

\[
u_s(t) = \text{sat}[u(t)]
\]

with the initial condition

\[
x(t) = \phi(t), \quad -\tau \leq t \leq 0, \quad \tau \geq 0,
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(u(t) \in \mathbb{R}^m\) is the control input vector to the actuator, \(u_i(t) \in \mathbb{R}^m\) is the control input vector to the plant and \(y(t) \in \mathbb{R}^r\) is the measured output vector; \(\phi(t)\) is the continuous vector-valued initial function; \(A, A_1, B, \) and \(C\) are constant matrices of appropriate dimensions. Also, \(\Delta A, \Delta A_1, \Delta B, \) and \(\Delta C\) are nonlinear parametric uncertainties with bounds as follows:

\[
\|\Delta A\| \leq \alpha,
\]

\[
\|\Delta A_1\| \leq \alpha_1,
\]

\[
\|\Delta B\| \leq \beta,
\]

\[
\|\Delta C\| \leq \gamma.
\]

The saturation function is shown in Figure 1 and is mathematically defined as

\[
\text{Sat}(u_i(t)) = \begin{bmatrix} \text{sat}(u_1(t)) & \text{sat}(u_2(t)) & \cdots & \text{sat}(u_m(t)) \end{bmatrix}^T,
\]

in which the operation of \(\text{sat}(u_i(t))\) is linear for \(u_i \leq u_i \leq u_H\) as

\[
\text{sat}(u_i(t)) = \begin{cases} u_L & \text{if } u_i < u_L < 0, \\ u_i & \text{if } u_L \leq u_i \leq u_H, \\ u_H & \text{if } 0 < u_H < u_i, \quad i = 1, 2, \ldots, m. \end{cases}
\]

If the saturating actuator \(\text{sat}(u_i(t))\) saturates at \(u_H\) or \(u_L\) then the following inequality is satisfied (Liu 1995)

\[
\|\text{sat}[u(t)] - u(t)\| \leq \frac{\|u(t)\|}{2}.
\]

In this control system, \((A, B)\) is controllable, i.e. the process state \(x(t)\) can be determined on the basis of control input \(u(s)\) for \(s \leq t\).

For the constrained uncertain time-delay system (1), we propose the following dynamic output feedback controller with more design freedom to accommodate the more complicated feedback systems

\[
\dot{x}_r(t) = A_r x_r(t) + B_r y(t),
\]

\[
u(t) = K_1 x_r(t) + K_2 y(t), \quad x_r(0) = 0,
\]

where \(A_r, B_r, K_1\) and \(K_2\) are constant matrices of appropriate dimensions.

The control problem we shall address is as follows. Find a controller of the form (6) for system (1) such that the resulting closed-loop system is globally uniformly asymptotically stable for any constant time delay \(\tau\) not larger than a given positive scalar \(\bar{\tau}\).

3. Robust stabilisation

The problem being treated here is the determination of the stabilization condition of the saturating time-delay system (1) under the control law (6). Adding and subtracting \(Bu(t)/2\), (1) can be rewritten as

\[
\dot{x}_d(t) = A_d x_d(t) + A_{1d} x_d(t-\tau) + \Delta A_d x_d(t) + \Delta A_{1d} x_d(t-\tau),
\]

\[
y(t) = C x(t) + \Delta C x(t),
\]

\[
x_d(0) = x_d(0) = [x^T(0) \quad 0]^T,
\]

where

\[
x_d(t) = [x(t) \quad x_e(t)]^T, \quad A_d = \begin{bmatrix} A + \frac{BK_2 C}{2} & BK_1 \\ B r C & A_r \end{bmatrix},
\]

\[
A_{1d} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}.
\]
Consider the system

\[ \Delta A_d x_d(t) = \left[ \begin{array}{c} \Delta A x(t) + \Delta B u(t) \\ B \Delta C x(t) \end{array} \right], \]

where \( \Delta A = A_{1d} = A_{11} + A_{12} \) and \( A_{11} \) is chosen such that \( A_d + A_{11} \) is more stable than \( A_d \) (Goubet et al. 1997). Roughly, this decomposition corresponds to a decomposition of the delayed terms into two groups: the stabilising ones and destabilising ones. This technique enables one to take the stabilising effect of part of the delayed terms into account.

**Remark 1:** The key point of Theorem 1 is the decomposition of the matrix \( A_{1d} \) into \( A_{1d} = A_{11} + A_{12} \), where \( A_{11} \) is chosen such that \( A_d + A_{11} \) is more stable than \( A_d \) (Goubet et al. 1997). Roughly, this decomposition corresponds to a decomposition of the delayed terms into two groups: the stabilising ones and destabilising ones. This technique enables one to take the stabilising effect of part of the delayed terms into account.

**Remark 2:** If we choose the matrix \( A_{1d} = A_{11} \) and \( A_{12} = 0 \), condition (9) of Theorem 1 reduces to

\[ 0 < \tau \leq \tilde{\tau} = \frac{-m_d - \Omega - \|A_{12}\| - \alpha_1}{k_d(\|A_{11}\| + \|A_{11}\| + \|A_{11}\| + \Omega + \alpha_1)}. \]  

\[ (10) \]

**Algorithm:** From the above analysis, we introduce the procedures for finding the dynamic control parameters \( A_d, B, K_1 \) and \( K_2 \) to satisfy the stability criterion (9) as follows:

**Data:** The parameter \( A, A_1, B \) and \( C \) of the system (7) and the parametric uncertainties \( \Delta A(t), \Delta A_1(t), \Delta B(t) \) and \( \Delta C(t) \).

**Objective:** To choose a natural subdivide matrix of \( A_{1d} \) as \( A_{1d} = A_{11} + A_{12} \) and select an appropriate dynamic output feedback controller (6) such that the system (7) is asymptotically stable.

**Step 1:** Use the pole-placement method to design the eigenvalues of \( A_d + A_{11} \) to be \( \lambda_i \) for \( i = 1, 2, 3, \ldots, n \).

**Step 2:** Find the dynamic control parameters \( A_d, B, K_1 \) and \( K_2 \) such that the system \( A_d + A_{11} \) has the specified \( \lambda_i \).

**Step 3:** Find the values of \( m_d, \|A_{11}\|, \|A_{11}\|, \alpha_0, \alpha_1 \) and \( \Omega \).

**Step 4:** Check whether \( A_d, B, K_1 \) and \( K_2 \) satisfy the criterion (9). If so, the dynamic controller as (6) is obtained. If system (9) is not satisfied, then go to Step 5.

**Step 5:** Shift the eigenvalues of \( A_d + A_{11} \) to the left \( \lambda_i = \lambda_i - \Delta \lambda_i \), where \( \Delta \lambda_i > 0 \) for \( i = 1, 2, 3, \ldots, n \), then go to Step 2.

**Remark 3:** The inequality (9) is only a sufficient condition; therefore, even (9) is not satisfied, we cannot say that no robust dynamic controller exists.

### 4. The robust stability of input-delay constrained systems

Consider a multi-input delay constrained system described by the following differential-difference equation

\[ \dot{x}(t) = Ax(t) + B \text{sat}(u(t)) + \sum_{j=1}^{N} B_j \text{sat}[u(t - \tau)], \tag{11a} \]

\[ y(t) = Cx(t), \tag{11b} \]

\[ x(t) = \phi(t), \quad -\tau \leq t \leq 0, \quad \tau \geq 0, \tag{11c} \]
where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( u_i(t) \in \mathbb{R}^m \) and \( y(t) \in \mathbb{R}^r \) are the vectors. \( A \), \( B \) and \( B_i \) are constant matrices with appropriate dimensions and the saturation function is described in (3).

The problem being treated here is the determination of the stabilisation condition of the saturating time-delay system (11) under the control law (6). Adding and subtracting \( Bu(t)/2 + \Sigma_{i=1}^{N} B_i u_i(t - \tau)/2 \), (11) can be rewritten as

\[
\dot{x}_s(t) = A_s x_s(t) + A_{1s} x_s(t - \tau) + B_s(t) + B_{1s}(t),
\]

where

\[
A_s = \begin{bmatrix}
A + \frac{B K_2 C}{2} & \frac{B K_1}{2} \\
B C & A_s
\end{bmatrix},
\]

\[
A_{1s} = \begin{bmatrix}
\sum_{i=1}^{N} B_i K_2 C & \sum_{i=1}^{N} B_i K_2 C \\
0 & 0
\end{bmatrix},
\]

\[
B_s(t) = \begin{bmatrix}
\frac{B_s \text{sat}(u(t)) - u(t)}{2} \\
0
\end{bmatrix},
\]

\[
B_{1s}(t) = \begin{bmatrix}
\sum_{i=1}^{N} B_i \text{sat}(u_i(t - \tau)) - u_i(t - \tau) \\
0
\end{bmatrix}.
\]

Applying the decomposition ‘delayed term’ \( A_{1s} \) as \( A_{1s} = A_s + A_{12} \) and substituting (B1) into (12), our result is summarised in the following Theorem 2.

**Theorem 2:** Consider the system (12) and assume that \( A_s + A_{12} \) is a Hurwitz stable matrix satisfying

\[
\| \exp[(A_s + A_{12})t] \| \leq k_s \exp(m_s t), \quad t \geq 0
\]

for some real numbers \( k_s > 0 \) and \( m_s < 0 \). If the control parameters \( A_s \), \( B_s \), \( K_1 \) and \( K_2 \) are selected to satisfy the following inequality

\[
0 < \tau \leq \tilde{\tau} = \frac{-m_s - \| A_{12} \| - \beta_s - \beta_{12}}{k_s \| A_{12} A_{12} \| + \| A_{12} A_s \| + \| A_s \| (\beta_s + \beta_{12})},
\]

then, the time-delay constrained closed-loop system (12) is asymptotically stable.

**Remark 4:** If we select the matrix \( A_1 = A_s \) and \( A_2 = 0 \), condition (14) of Theorem 2 reduces to

\[
0 < \tau \leq \tilde{\tau} = \frac{-m_s - \beta_s - \beta_{12}}{k_s \| A_{12} A_{12} \| + \| A_{12} A_s \| + \| A_s \| (\beta_s + \beta_{12})}.
\]

**Proof:** From Theorem 1, the proof is similar to that of Theorem 1.

Now, consider the linear constrained time-delay systems (1), we propose the following state feedback controller to accommodate the feedback systems:

\[
u(t) = -Fx(t).
\]

The problem is to determine whether the parameters \( F \) are constant matrices of appropriate dimensions.

The problem being treated here is the determination of the stabilisation condition of the saturating time-delay system (1) under the control law (16). Adding and subtracting \( (B + \Delta B)u(t)/2 \), (1) can be rewritten as

\[
\dot{x}(t) = A_1 x(t) + A_{12} x(t - \tau) + \Delta A_{11} x(t) + \Delta A_{12} x(t - \tau)
\]

\[
+ (B + \Delta B) \left[ u(t) - \frac{u(t)}{2} \right],
\]

where \( A_1 = A - BF/2 \), \( \Delta A_1 = \Delta A - \Delta BF/2 \).

Then the problem becomes how to choose the control parameters \( F \) such that the closed-loop Equation (17) is asymptotically stable; in other words, the parametrical uncertainties can be tolerated. Applying the decomposition ‘delayed term’ \( A_1 \) as \( A_1 = A_{11} + A_{12} \) and substituting (B1) into (17), our result is summarised in the following corollary.

**Corollary:** Consider the system (17) and assume that \( A_{12} + A_{11} \) is a Hurwitz stable matrix satisfying

\[
\| \exp[(A_{12} + A_{11})t] \| \leq k_1 \exp(m_1 t), \quad t \geq 0
\]

for some real numbers \( k_1 > 0 \), \( m_1 < 0 \), and square matrix \( A_{11} \) of appropriate dimensions. If the control parameters \( F \) are selected to satisfy the following inequality:

\[
0 < \tau \leq \tilde{\tau} = \frac{-m_1 - \| A_{12} \| - \Xi - \alpha_1}{k_1 \| A_{11} A_{11} \| + \| A_{11} A_{12} \| + \| A_{11} \| (\Xi + \alpha_1)},
\]

where

\[
\Xi = \alpha + \frac{(\| B \| + \beta)\| F \|}{2},
\]
then, the parametric perturbed system (16) is asymptotically stable.

**Proof:** From Theorem 1, the proof is similar to that of Theorem 1. □

To demonstrate the applicability of the present schemes, we give the following examples.

**Remark 5:** In the special case (without any decomposition) when \( A_1 = A_{11} \) and \( A_{12} = 0 \), condition (19) of the corollary reduces to

\[
0 < \tau_1 = \frac{-m_1 - \Xi - \alpha_1}{k_1 \| A_{11} \| + \| A_{11} \| + \| A_1 \| (\Xi + \alpha_1)}.
\]

(20)

**5. Examples**

**Example 1:** Consider the following linear time-delay system of which control values are saturated at \( \pm 1 \) and

\[
\dot{x}(t) = Ax(t) + \Delta Ax(t) + A_1 x(t - \tau) + \Delta A_1 x(t - \tau) + B u_\tau(t) + \Delta Bu_\tau(t),
\]

\[
y(t) = C x(t) + \Delta C x(t),
\]

where

\[
A = \begin{bmatrix} -4 & 0 \\ 1 & -5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0.5 \sin(t) & 0 \\ 0 & 0.5 \cos(t) \end{bmatrix},
\]

\[
\Delta A_1 = \begin{bmatrix} 0.2 \sin(t) & 0 \\ 0 & 0.2 \cos(t) \end{bmatrix}, \quad \Delta B = \begin{bmatrix} 0.2 \sin(t) & 0 \\ 0 & 0.2 \cos(t) \end{bmatrix}
\]

and \( \Delta C(x(t)) = [0 \ 0]^T \).

The example shows that the ‘natural’ decomposition \( A_{11} = 0.5 A_{1d} \) if we place the poles of \( A_{d} + A_{11} \) in \(-4.6298, -4.8347, -6.1928 \) and \( 6.3427 \). From the above algorithm, it is verified that the robust stability (9) is satisfied. Then, the dynamic output feedback controller parameters are obtained as follows:

\[
\hat{x}_d(t) = \begin{bmatrix} -6 & 0.1 \\ 1 & -5 \end{bmatrix} x_d(t) + \begin{bmatrix} 0.5 & 0 \\ 0.1 & 0.1 \end{bmatrix} y(t).
\]

(22a)

\[
u(t) = \begin{bmatrix} -0.6 & 0 \\ 0.1 & -0.1 \end{bmatrix} x_d(t) + \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} y(t).
\]

(22b)

By applying Theorem 1 for \( k_{d} = 1 \), we obtain \( m_d = -4.6298, \alpha = 0.5, \alpha_1 = 0.2, \beta = 0.2, \end{equation}

\[\| A_{11} A_{1d} \| = 4.8012, \quad \| A_{11} A_{1d} \| = 1.2071, \quad \| A_{1d} \| = 0.8090, \Omega = 2.0586 \]

and

\[0 < \tau_1 = \frac{-m_d - \Omega - \| A_{1d} \| - \alpha_1}{k_1 \| A_{11} A_{1d} \| + \| A_{1d} \| + \| A_{1d} \| (\Omega + \alpha_1)} = 0.1994.
\]

The system (21) is asymptotically stable for \( \tau < \bar{\tau} = 0.1994 \).

From (10), we obtain \( m_d = -4.382, \alpha = 0.5, \alpha_1 = 2.0586 \), \( \| A_{1d} \| = 9.6024, \| A_{1d} A_{1d} \| = 2.4142, \| A_{1d} \| = 1.6180, \Omega = 2.0586 \)

and

\[0 < \tau_1 = \frac{-m_d - \Omega - \| A_{1d} \| - \alpha_1}{k_1 \| A_{11} A_{1d} \| + \| A_{1d} A_{1d} \| + \| A_{1d} \| (\Omega + \alpha_1)} = 0.1355.
\]

Therefore, for this example, it can be improved by employing the decomposition technique to get a larger bound for uncertain time-delay saturating actuator systems (22).

**Remark 6:** If the delay-dependent criterion (Tsay and Liu 1996) is considered, the stability is as follows:

\[0 < \tau_2 = \frac{-\mu (A_{d} + A_{1d}) - \Omega - \alpha_1}{\| A_{d} A_{1d} \| + \| A_{d} \| + \| A_{d} \| (\Omega + \alpha_1)} = 0.1053.
\]

The value is more conservative than the result in this example.

**Example 2:** Consider the following uncertain time-delay system of which control values are saturated at \( \pm 0.5 \) and

\[
\dot{x}(t) = Ax(t) + \Delta Ax(t) + A_1 x(t - \tau) + \Delta A_1 x(t - \tau) + B u_\tau(t),
\]

where

\[
A = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
\Delta A = \Delta A_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

The example shows that the ‘natural’ decomposition \( A_1 = A_{11} + A_{12} \) choosing

\[
A_{11} = \begin{bmatrix} -1 & 0 \\ -0.5 & -0.6 \end{bmatrix}.
\]
The eigenvalues of $A_f + A_{11}$ are assigned at $-3.1$ and $-3.7$ by selecting a state feedback

$$F = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$ 

From the condition of the corollary, let $k_1 = 1$, we obtain $m_1 = -3.1$, $\alpha = \alpha_1 = 0.2$, $\|A_{11}A_f\| = 2.2477$, $\|A_{11}A_f\| = 1.4674$, $\|A_{12}\| = 0.5$, $\Xi = 0.25$ and

$$0 < \tau \leq \tau_1 = \frac{-m_1 - \|A_{12}\| - \Xi - \alpha_1}{k_1[\|A_{11}A_f\| + \|A_{11}\| + \|A_{11}\|(\Xi + \alpha_1)]} = 0.5075.$$ 

From the corollary, we can declare that system (23) is robustly stable for $\tau < \tau = 0.5075$. Using the result of Su et al. (2001), it has been found that the system (23) is robustly stable for $\tau < 0.2841$. On the other hand, the stability criterion of Niculescu et al. (1996) guarantees robust stabilisation of system (23) for $\tau < 0.3819$. This upper bound $\tau$ reported by Liu (2005) for the same example is $\tau < 0.452$. Hence, for this example, the robust stability criterion of this article is less conservative than the existing results (Niculescu et al. 1996; Su et al. 2001; Liu 2005).

Let us apply the corollary of this article without any decomposition ($A_1 = A_{11}$ and $A_{12} = 0$). From the condition of (20), we obtain $m_1 = -3$, $\alpha = \alpha_1 = 0.2$, $\|A_{11}A_f\| = 3.2268$, $\|A_{11}A_f\| = 2.0806$, $\|A_{11}\| = 1.477$, $\Xi = 0.25$ and

$$0 < \tau \leq \tau_1 = \frac{-m_1 - \Xi - \alpha_1}{k_1[\|A_{11}A_f\| + \|A_{11}\| + \|A_{11}\|(\Xi + \alpha_1)]} = 0.4270.$$ 

Therefore, for this example, it can be improved by employing the decomposition technique to get a larger bound for uncertain time-delay saturating actuator systems (23).

### 6. Conclusions

This article focused on the design of a dynamic output feedback controller for time-delay systems containing saturating actuators with state delay and input delay. The approach adopted here is based on the comparison theorem and matrix measure via model transformation. The objective of this article is to guarantee the allowable bound of delay time such that if time delays less than the obtained constant delays bound, the saturating system with time delay can be tolerated. Some comparisons have been made with the same examples that appear in many recent papers. Our results are shown to be less conservative than those reports.

### Notes on contributors

**Pin-Lin Liu** was born in Taiwan, R.O.C., in 1962. He received his BS, MS and PhD degrees in Industrial Education and Engineering from the National Changhua University of Education, Taiwan, in 1976, 1990, and 2001, respectively. In 1990, he joined the Department of Electrical Engineering, Chienkuo Technology University, Taiwan, where he is currently an Associate Professor. His research interests include time-delay systems, robust control and green energy and its application. He has published over 200 refereed papers in technology journals and conference proceedings.

### References


Applying the result of (Liu 1995), the upper bound on the norm of (B2), we obtain
\[
\|x_d(t)\| \leq \|\exp((A_d + A_{11})(t - s))\| \|x_d(0)\|
\]
\[
+ \int_s^t \|\exp((A_d + A_{11})(t - s))\| \|A_{11}A_dx_d(\theta)\|d\theta
\]
\[
+ \|A_{11}A_dx_d(\theta - \tau)\| + \|A_{11}\|\|\Delta A_{1d}x_d(\theta)\|
\]
\[
+ \|\Delta A_{1d}x_d(\theta - \tau)\|d\theta\bigg]ds
\]
\[
+ \int_0^t \|\exp((A_d + A_{11})(t - s))\|\|\Delta A_{dx_d(s)}\|
\]
\[
+ (\|A_{12}\| + \|\Delta A_{1d}\|)\|x_d(s - \tau)\|ds. \quad (B3)
\]

Then, (B3) becomes
\[
\|x_d(t)\| \leq k_d\exp(m_d(t)) \|x_d(0)\| + \int_0^t k_d\exp(m_d(t - s))
\]
\[
\times \int_s^t \left[\|A_{11}A_dx_d\| + \|A_{11}\|\|\Omega\|\|x_d(\theta)\|
\]
\[
+ \|A_{11}A_dx_d\| + \|A_{11}\|\|x_d(\theta - \tau)\|\right]d\theta ds
\]
\[
+ \int_0^t k_d\exp(m_d(t - s))\|\Omega\|\|x_d(\theta)\|
\]
\[
+ (\|A_{12}\| + \|\Delta A_{1d}\|)\|x_d(s - \tau)\|ds. \quad (B4)
\]

where \(\theta - \tau \leq \xi_1 \leq \theta, \quad \theta - 2\tau \leq \xi_2 \leq \theta\) and \(\theta - 2\tau \leq \xi \leq \theta\).

Then, we consider the following scalar differential difference equation
\[
r(t) \leq (m + \|\Omega\| + \|A_{12}\| + \|\Delta A_{1d}\| + \|\|A_{11}\|\|\|\Omega\| + \|\|A_{11}\|\|\|\Delta A_{1d}\|\|\|x_d(\xi)\|
\]
\[
+ (\|A_{12}\| + \|\Delta A_{1d}\|)\|x_d(\xi)\|ds. \quad (B5)
\]

The solution of (B5) is as follows:
\[
r(t) \leq k_d\exp(m_d(t)) \|x_d(0)\| + \int_0^t k_d\exp(m_d(t - s))
\]
\[
\times \tau \left[\|A_{11}A_dx_d\| + \|A_{11}\|\|\Omega\| + \|A_{11}\|\|A_{12}\|\right]\|x_d(\xi_1)\|
\]
\[
+ \int_0^t k_d\exp(m_d(t - s))\|\Omega + \|A_{12}\| + \|\Delta A_{1d}\|\|x_d(\xi_2)\|ds
\]
\[
+ \int_0^t k_d\exp(m_d(t - s))\|\Omega + \|A_{12}\| + \|\Delta A_{1d}\|\|x_d(\xi)\|ds. \quad (B6)
\]
Using the comparison theorem (Halanay 1966; Lakshmikantham and Leela 1969) of lemma A, one obtains
\[ \| x_d(t) \| \leq r(t), \quad \text{for } t \geq 0. \]  
(B7)

Hence, the asymptotic stability of \( r(t) \) implies that of \( x_d(t) \) and
\[ \sup_{-2t \leq \omega \leq t} x_d(\omega) \leq \sup_{-2t \leq \omega \leq t} r(\omega) \]  
(B8)

and
\[ \| x_d(t - \tau) \| \leq r(t - \tau) \leq \sup_{-2t \leq \omega \leq t} r(\omega). \]  
(B9)

From Lemma A, we obtain
\[ -(m_d + \Omega + \| A_{12} \| + \alpha_1) > k_d r[\| A_{11} \| + \| A_{11} A_d \| + \| A_{11} (\Omega + \alpha_1) \|] \geq 0, \]  
(B10)

which is condition (8) of the theorem, then there exists a \( k_d > 0 \), such that
\[ r(t) \leq k_d \exp[\lambda(t - \tau)] \]  
(B11)

and the inequality holds for \( \| x_d(t) \| \) similarly.

We should also note \( \lambda \) is the unique solution of Lemma A
\[ -\lambda = m_d + \Omega + \| A_{12} \| + \alpha_1 - k_d r[\| A_{11} \| + \| A_{11} A_d \| + \| A_{11} (\Omega + \alpha_1) \|] \]  
(B12)

Consequently, the system (7) is asymptotically stable if the following equation is satisfied
\[ m_d + \Omega + \| A_{12} \| + \alpha_1 + k_d r[\| A_{11} \| + \| A_{11} A_d \| + \| A_{11} (\Omega + \alpha_1) \|] \]  
(B13)