A mean field approximation for the capacity of server-limited, gate-limited multi-server polling systems

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ABSTRACT
We propose a mean field approximation for the capacity of a limited-gated multiserver polling system with a limit on the number of servers a given station can use simultaneously. The approximation provides an expression for the stability limit under very general assumptions about the traffic process and system configuration. The result is expressed in terms of a certain parameter that is shown to be the unique solution of a simple fixed point equation. We sketch a proof that the approximation is asymptotically exact by showing that the succession of server visits converges to a Poisson process.

Categories and Subject Descriptors
C.2.5 [Local and Wide-Area Networks]: Access schemes; G.3 [Probability and Statistics]: Queuing theory

Keywords
Polling system, multiserver, traffic capacity

1. INTRODUCTION
In this short paper, we revisit the determination of stability limits for multiserver polling systems. The study is motivated by the requirement to evaluate the traffic capacity of wavelength division multiplexing (WDM), passive optical networks (PONs) where user stations (optical network units) are equipped with tunable transmitters [1]. This implies users can use any of the multiple wavelengths to transmit their data but only within the limit determined by the number of transmitters they possess.

The literature on multiserver polling systems recognizes that the multiserver polling system with limited grants (each station can only transmit a limited amount of data on each server visit) and limited server capacity (each station can only use a limited number of servers at any one time) is intractable in general [3]. In [1] we proposed a mean field approximation for the capacity of large WDM PONs that was shown to be very accurate. However, we were not then able to provide a proof of convergence.

Our objective in the present paper is to outline one possible proof. In the interests of generality, we present the demonstration in terms of a general multiserver polling system where, unlike the PON, servers have a non-zero walk time between successive visits to stations. The proof is based on the convergence of a superposition of many sparse point processes to a Poisson process.

2. SYSTEM AND ASSUMPTIONS
We consider a multiserver polling system with S servers and N service stations. Station i contains n_i queues with independent stationary work processes \( \rho_{ij}(t) \). We write \( \rho_i = \sum_j \rho_{ij} \) and \( \rho = \sum_i \rho_i \).

Servers have identical service rates of one unit of work per second. Server s visits stations in the order determined by a Markov chain. The probability server s visits station j on leaving station i is \( r_{ij} \). The dependence on s is such that the stationary visit probabilities \( \pi_i \) are the same for all s: \( \pi_i = \sum_j r_{ji} \pi_j \). This allows the order of visits to be different for every server as long as the relative visit frequency for any given station is the same. Mean walk time from station i to station j is \( w_{ij} \) and successive walk times are independent.

When server s visits station i, it serves each station queue applying the following limited-gated policy. When the work load of queue (i, j) is w at the instant s arrives, discounting work already assigned to other servers, the amount of work assigned \( u_{ij}(w) \) satisfies the following conditions [3]:

S1. \( \lim_{w \to \infty} \mathbb{E}[u_{ij}(w)] = \tilde{u}_{ij} < \infty \),

S2. \( \mathbb{E}[u_{ij}(w)] \leq \tilde{u}_{ij} \) for \( w \geq 0 \).

In addition to serving the queues, the server remains at station i for an additional “overhead” of mean \( \Delta_i \) seconds. In the PON application, the overhead corresponds to time used to report queue contents together with a physical guard time. We denote by \( d_i \) the expected time a server spends at station i: \( d_i = \mathbb{E} \left[ \sum_j u_{ij} \right] + \Delta_i \).

The number of servers that can simultaneously attend station i is limited to \( s_i \). We assume the station has \( s_i \) “places” that can be occupied by a server. When a server visits station i it tests a place at random and stays to serve only if the place is free. Otherwise, the server proceeds to the next station determined by the routing matrix.

For technical reasons (proof of convergence), we require the time between visits to successive stations to be greater than some \( T_{min} > 0 \).

3. STATION PROCESS
Fig. 1 depicts the activity of station i assumed to have two service places (\( s_i = 2 \)). Each server visit performs some work for each non-empty queue and remains for an overhead \( \Delta_i \). The \( n^{th} \) visit to place j lasts \( d_{ij}(n) \) seconds and \( \mathbb{E}[d_{ij}(n)] = d_i \), for \( 1 \leq j \leq s_i \). After a server visit there is a vacation period \( \Theta_{ij}(n) \). The \( \Theta_{ij}(n) \) constitute a stationary sequence and, by symmetry of the routing to places, we have \( \mathbb{E}[\Theta_{ij}(n)] = \theta_i \) for some undetermined \( \theta_i \), for \( 1 \leq j \leq s_i \). Note that S1 and S2 ensure \( \theta_i \) is always finite.
4. SERVER PROCESS

Each server is either performing work, occupied at a station by the overhead or in the process of walking between stations. Let \( b_i \) be the probability a visit to \( i \) is blocked because the station place is already occupied by another server. The mean inter-visit time of server \( s \), denoted \( \delta^s \), may then be written

\[
\delta^s = \sum_{1 \leq j, k \leq N} \pi_{jk} (d_j (1 - b_j) + w_{sk}) \tag{1}
\]

\[
= \sum_j \pi_j d_j (1 - b_j) + w^s, \tag{2}
\]

where the first term is the average sojourn time at any station and \( w^s \) is the average walk time of server \( s \).

Consider a realization of the polling conducted by server \( s \), as depicted in Fig. 2. The successive inter-visit intervals for station \( i \) are denoted \( T_i^s (n) \) with \( \mathbb{E}[T_i^s (n)] = \tau_i^s \). Without loss of generality, time 0 is chosen to coincide with a visit to station \( i \). The epoch of the \( k^{th} \) subsequent visit is denoted \( t_k \). The mean interval \( \tau_i^s \) is given by the following.

**Proposition 1.** The mean interval between visits by server \( s \) to station \( i \) is

\[
\tau_i^s = \delta^s / \pi_i, \tag{3}
\]

for \( 1 \leq i \leq N \) and \( 1 \leq s \leq S \).

**Proof.** The number \( m(k) \) of visits in \( (t_{k-1}, t_k) \) to any station is an iid sequence and \( \mathbb{E}[m(k)] = 1 / \pi_i \). Thus,

\[
\delta^s = \lim_{n \to \infty} \frac{t_n}{\sum_{k=1}^n m(k)} = \lim_{n \to \infty} \frac{t_n}{n / \pi_i} = \pi_i \tau_i^s.
\]

\( \square \)

Let \( \delta = (\sum 1/\delta^s)^{-1} \) be the mean time between visits to any station by any server. The mean time between visits to \( i \) by any server is then:

\[
\tau_i = \delta / \pi_i. \tag{4}
\]

5. MEAN FIELD APPROXIMATION

We assume the vacations \( \Theta_{iy} (n) \) defined above are independent and derive the stability condition in terms of the mean interval lengths \( \theta_i \). This is a mean field approximation in the sense that the service process of a given station can be viewed as an isolated multiserver polling system in which the impact of other stations is encapsulated in the value of parameter \( \theta_i \).

When the \( \Theta_{ij} (n) \) are iid, Fig. 1 can be seen to represent a classical \( s_i \)-server polling system without server limits where we assimilate vacation plus overhead to a walk time between the last and first queues. The traffic capacity is therefore defined by the following proposition deduced from results in [3] or [4].

**Proposition 2.** Under the assumption of iid vacation times, station \( i \) is stable if and only if

\[
\rho_i + \max_j \left\{ \frac{\rho_{ij}}{\pi_{ij}} \right\} (\Delta_i + \theta_i) < s_i. \tag{5}
\]

Assume now the system is stable. Equating the expected amount of work performed with the offered load,

\[
s_i \frac{d_i - \Delta_i}{\pi_i - \rho_i} = \rho_i,
\]

yielding

\[
d_i = \frac{s_i \Delta_i + \rho_i \theta_i}{\pi_i - \rho_i}. \tag{6}
\]

To proceed, we in fact need to make a stronger assumption. This consists in supposing the process of visits of any server to station \( i \) constitutes a Poisson process. Inter-visit intervals thus have an exponential distribution and, by the memoryless property, so have the vacation times \( \Theta_{ij} \) and we have \( \theta_i = s_i \tau_i \) where \( \tau_i \) is given by (4).

6. CONVERGENCE

We now demonstrate that the approximation is valid as \( N, S \to \infty \) with \( \pi_i \to 0 \) for all stations \( i \).

6.1 Independent processes

Without loss of generality, suppose the \( n^{th} \) visit to station \( i \) at time \( t_{n-1} \) is by server 1, as depicted in Figure 3. Let \( B_i^n \) be the time backwards from \( t_{n-1} \) to the last visit to this station by server \( s \). Let \( F_i^n \) be the time forwards from \( t_{n-1} \) to the next visit by \( s \) \( i \). Inter-visit times are then \( T_i^n = T_i (n - 1) \) and \( T_i (n) = \min \{ F_i^n \} \). The figure, the respective minima correspond to servers \( s' \) and \( s'' \).

Suppose the times between tick marks or crosses in Fig. 3 are independent random variables. The \( B_i^n \) and \( F_i^n \) are then also independent and, for fixed \( s \), identically distributed
since routing is Markovian. As $N$, $S$ and the expectations of the $B_i^t$ and $F_i^t$ tend to infinity, we have a superposition of many independent sparse renewal processes. The latter is known to converge to a Poisson process (see [2, Chapter 6 and Ex 25], for example): $T_i(n - 1)$ and $T_i(n)$ are independent exponential random variables of mean $1/\tau_i$.

### 6.2 Dependent processes

Return now to the actual polling system where the considered times between tick marks or crosses in Fig. 3 are not independent. Consider a modification of the process depicted in the figure where, in defining the $B_i^t$ and $F_i^t$, we account only for the first $m$ visits of each server after $t_{n-1}$ and the last $m$ visits before $t_{n-1}$.

For most servers the variables are not defined since station $i$ will not be included in the considered visits. In the large system limit, the number that actually visit $i$ has a Poisson distribution of mean $2m(S-1)\pi_i$ and is therefore finite with probability 1. Denote this set of servers by $S_i$.

We now assume the routing probabilities are such that the stations other than $i$ visited by the servers in $S_i$ are distinct (with probability 1). More generally, we assume the stations visited by any finite subset of servers, at one of a given finite subset of visits, are distinct with probability 1. This would be the case if $r_{ij} \to 0$ for all $i$, $j$ and $s$. It would also be so if each server has a deterministic routing cycle as long as this cycle was chosen randomly and independently for each server.

Since dependence derives from servers visiting the same stations within a relatively short time frame and this is excluded in the considered limit, we (informally) deduce the following.

**Lemma 1.** In the considered large system limit, the variables $B_i^t$ and $F_i^t$ defined for $s \in S_i$ are independent.

The following proposition states the required convergence result.

**Proposition 3.** In the considered large system limit, the inter-visit times to a given station constitute a sequence of independent exponentially distributed variables of mean $\tau_i$.

**Proof.** Consider the respective superpositions of the following sets of processes defined with reference to Fig. 3:

- **P1:** all intervals between visits of a given server (tick marks and crosses in Fig. 3) are drawn independently from their respective stationary distributions for given server and visited stations.

- **P2:** all intervals for $s \in S_i$ are determined as above for P1; the value of the corresponding intervals for $s \notin S_i$ is unspecified except that it is greater than $T_{\min} > 0$.

The superposition of processes P1 for $1 \leq s \leq S$ corresponds to that considered in Section 6.1. It follows that the time from $t_{n-1}$ backwards to the end of the last visit to $i$ and the time forwards from $t_{n-1}$ to the first visit to $i$ are independent and exponentially distributed with mean $\tau_i$.

Now consider the superposition of processes P2. By Lemma 1, under the large system limit, the description of P2 applies (with high probability) to the considered polling system. By construction, for any $t, u < mT_{\min}$, the probability the time to the first visit to $i$ by any server after $t_{n-1}$ is less than $t$ and the time from the end of the last visit by any server to $i$ to $t_{n_1}$ is less than $u$, is exactly the same in both superpositions P1 and P2. For values less than $mT_{\min}$, these intervals can thus be considered independent and exponentially distributed with mean $\tau_i$. The relevant time range can be made as large as necessary by the choice of $m$.

### 7. FIXED POINT EQUATION

To exploit Proposition 2, it remains to evaluate the $\theta_i$ which are defined in terms of an unknown $\delta$. We have so far not specified the blocking probability $b_i$ appearing in expression (2). Observe that, in the considered limit, the occupancy of a station place is independent of the activity of any one server since that server visits it extremely rarely. Each server thus sees the place occupancy in its stationary state and the blocking probability is the time average,

$$b_i = d_i/(d_i + \theta_i).$$

Using (6) and (7) in (2), we deduce,

$$\delta^s = \sum_i \frac{\pi_i \theta_i (s_i \Delta_i + \rho \theta_i)}{s_i \theta_i + s_i \Delta_i} + w^s.$$

Recalling that $1/\delta = \sum_i 1/\delta^s$ and using $\theta_i = s_i / \pi_i$, we have the following equation for $\delta$.

$$\sum_i \left( \frac{s_i (s_i \Delta_i + \rho \Delta_i)}{s_i \theta_i + s_i \Delta_i} + w^s / \delta \right)^{-1} = 1. \tag{8}$$

It may be verified that the left hand side of (8) increases monotonically with $\delta$ from 0 to $s_i / \rho$. Since $\rho < s_i$ is necessary for stability, (8) has a unique solution from which the $\theta_i$ can be derived.

### 8. CONCLUDING REMARKS

Conditions (5) with $\theta_i$ defined in terms of $\delta$, the unique solution of (8), constitute an approximation for the traffic capacity of the considered server-limited, gate-limited multiserver polling system that is asymptotically exact in the large system limit. The present demonstration of this result is not entirely satisfactory in that we have not provided a formal proof of Lemma 1. It would also be preferable to remove the requirement that inter-visit times are lower bounded away from zero. These refinements are the subject of ongoing work.

The considered server limit manifested by the notion of $s_i$ “places” at station $i$ is not natural in all applications. It might be considered more realistic that a visiting server be blocked only if the current number of servers present is equal to $s_i$. Unfortunately, this variant of the multiserver polling system appears to remain intractable, even in the large system limit.

### 9. REFERENCES


