Decentralized Harmonic Active Vibration Control of a Flexible Plate using Piezoelectric Actuator - Sensor Pairs.

Matthieu Baudry, Philippe Micheau and Alain Berry
G.A.U.S, Mechanical Engineering Department
Université de Sherbrooke, Sherbrooke, Québec, Canada J1K 2R1

Number of pages : 36
Number of figures : 10

Running title : Decentralized Control of Flexible Plate

Received :

August 19, 2005
Abstract

We have investigated decentralized active control of periodic panel vibration using multiple pairs combining PZT actuators and PVDF sensors distributed on the panel. By contrast with centralized MIMO controllers used to actively control the vibrations or the sound radiation of extended structures, decentralized control using independent local control loops does only require identification of the diagonal terms in the plant matrix. However, it is difficult to \textit{a priori} predict the global stability of such decentralized control. In this study, the general situation of non-collocated actuator-sensor pairs was considered. Frequency domain gradient and Newton-Raphson adaptation of decentralized control were analyzed, both in terms of performance and stability conditions. The stability conditions are especially derived in terms of the adaptation coefficient and a control effort weighting coefficient. Simulations and experimental results are presented in the case of a simply-supported panel with 4 PZT-PVDF pairs distributed on it. Decentralized vibration control is shown to be highly dependent of the frequency, but can be as effective as a fully centralized control even when the plant matrix is not diagonal-dominant or is not strictly positive real (not dissipative).

Keywords: active control, vibration, decentralized control, piezoelectric materials, complex envelope, closed-loop stability, dissipative systems
1 Introduction

Active control is an efficient tool for attenuating low frequency vibrations in structures whose vibration response or sound radiation needs to be globally reduced. This technique has been applied to control lumped and distributed parameter structures [9]. Usually, the vibration response is sensed at a number of locations on the structure and modified by a number of local actuators using a centralized controller. The sensors and the actuators must be located in order to have sufficient authority to respectively, sense and control structural vibrations. In simple situations, such as the active damping of a limited number of weakly damped structural modes, this approach is very effective. However, its application to more complicated situations involving forced response of a large number of structural modes, is more challenging: in such a case, a centralized control strategy involves modelling a large number of secondary transfer functions, requires cumbersome wiring and is prone to instability due to plant uncertainty or individual actuator or sensor failure.

The problem under study in this article is the active control of bending vibrations of a panel and the control strategy investigated is the use of independent control loops between an individual PVDF sensor and an individual PZT actuator instead of a centralized controller [19]. Piezoelectric materials are good candidates for decentralized active vibration control because under the pure bending assumption they can form collocated, dual actuator-sensor pairs [4, 23]. Decentralized control approaches were also recently applied to the active control of free-field sound radiation using loudspeaker - microphone pairs [14, 2]. The main advantage of such a decentralized control strategy is its reduced complexity, reduced processing requirement, ease of implementation and robustness to individual control unit failure. However, performance and stability of decentralized control are difficult to predict a priori.

Decentralized control strategies in the context of active vibration or vibroacoustic control has recently been studied by Elliott and his colleagues [5, 4, 11, 7, 1]. Under the assumption of
collocated and dual actuator-sensor pairs, decentralized control has the very attractive property that each local feedback loop (with the other feedback loops being active) is stable regardless of the local feedback gains applied [4], leading to a globally stable and robust implementation. When applied to globally reducing the vibration response of - or sound transmission through panels, decentralized control leads to control performance very similar to a fully centralized control structure [11, 10, 7, 1]. However, piezoelectric strain actuators (PZT) and strain sensors (PVDF) cannot, strictly speaking, be collocated and dual because of coupling through extensional excitation [6]. Hence, the main objective of this article is to analyze performance and stability of decentralized vibration control using the general situation of non-collocated actuator-sensor PZT-PVDF pairs.

The specific situation investigated here is the active control of a reverberant system (a weakly damped bending plate) under a periodic excitation. For such disturbance, the feedback controller and the x-LMS MIMO feedforward controller can both be seen as an equivalent resonant controller [21]. The similarity between feedback and feedforward comes from the fact that the equivalent compensator has 1 undamped mode at the disturbance frequency to provide the high loop gain necessary for the rejection of the disturbance frequency. Hence, without loss of generality, the control problem can be addressed with an harmonic controller which adjusts complex gains (amplitude and the phase of the sinusoidal control inputs with respect to the amplitude and phase of the sensor signals). The feedback law, can be tuned with two parameters: the adaptation coefficient which specifies the convergence rate, and the control effort weighting which limits the amplitude of the control input. Decentralized control is the special case where the gain matrix is diagonal, in contrast with a fully centralized control where the gain matrix is fully populated.

The problem of decentralized controller is to ensure the stability of the whole system [18]. Decentralized control was studied in the context of large structures, and it was established that a solution to the decentralized problem exists if and only if a solution exists to the centralized
problem [24]. For x-lms feedforward, the Gershgorin theorem is useful to derive a *sufficient* condition of stabilization [5]: the diagonal dominance of the matrix gain. The same condition can be derived from the Small Gain theorem for feedback controllers. However, this diagonal dominance condition proves to be too conservative to be applied in practice. More adequate necessary and sufficient stability conditions were derived in the active free-field sound control problem [14]. The objective of this article is to extend this previous analysis by providing a set of simple analytical tools to *a priori* predict closed loop stability of active vibration control using piezoelectric actuators and sensors.

Section 2 introduces the problem, and section 3 details the plant modelling. The controller synthesis in term of minimization of a quadratic criterion is presented in section 4, together with the conditions of stability related to the requirement of a positive definite plant matrix at the disturbance frequency. Section 5 presents the implementation of the harmonic controller using a complex envelope controller [16]. Finally, experimental results presented in section 6 illustrate the effectiveness of the proposed analytical tools.

## 2 The Problem

The physical system under study is a flexible panel under forced harmonic oscillation, whose vibration response or sound radiation needs to be globally reduced. To this end, the vibration response is sensed at a number of locations on the structure, and modified by a number of local actuators on it (figure 1). The control strategy investigated here is the use of $N$ independent control loops $\Gamma_i$ between an individual sensor and an individual actuator instead of a $N \times N$ centralized controller. The controller is implemented in the frequency domain. A general block-diagram form of the controller is shown on figure 2 ($\omega_0$ is the angular frequency of the disturbance, $d$ is the disturbance vector measured by the $N$ sensors, $y$ is the error signal at the sensors, $u$ is the control inputs at the $N$ actuators, and $H$ is the $N \times N$ transfer function
matrix between actuators and sensors). In order to reject the frequency $\omega_0$, the controller $\Gamma$ restricts to a matrix of complex gains at $\omega_0$. In the general case of a central controller, the control matrix has off-diagonal components $\Gamma_{ij}$. In the case of decentralized control, the control matrix is diagonal, $\Gamma = \text{diag}(\Gamma_{ii})$. In the following, the controller $\Gamma$ is iteratively adapted in order to minimize a given error criterion. Since it is assumed that the disturbance has a fixed frequency and that the error is slowly varying in time (slow convergence), the time variations of the error phasor are slow, and it is therefore possible to implement the controller adaptation at a much slower rate than the disturbance frequency. The demodulation and modulation blocks shown on figure 2 are used to respectively extract the phasor of the error $y$, and synthesize the oscillatory control input $u$ at the disturbance frequency. Since the error and control phasors have slow time variations (compared to the disturbance period), the adaptation of the controller $\Gamma$ can be done at a much slower rate than the disturbance frequency. The demodulation and modulation blocks are described in more details in section 5.

3 Plant Modelling

3.1 A model of the transfer functions between PZT actuators and PVDF sensors

We consider a rectangular, simply-supported panel equipped with $N$ surface-mounted, identical actuator-sensor pairs. Each pair consists of a rectangular piezoceramic (PZT) actuator and a rectangular polyvinylidene fluoride (PVDF) sensor, which are not necessarily collocated on the panel. In the following analysis, pure bending response is assumed, therefore, the effect of extensional deformation of the panel on the actuator-sensor transfer functions is not considered. Applying an oscillatory voltage (of angular frequency $\omega_0$) on an individual PZT actuator generates forced bending vibrations of the panel which are sensed by all PVDF films. The transfer function between actuator $j$ and sensor $i$ is defined by
\[ H_{ij}(\omega_0) = \frac{V_i^{(s)}(\omega_0)}{V_j^{(a)}(\omega_0)} \]  

where \( V_i^{(s)} \) is the output voltage of the PVDF sensor and \( V_j^{(a)} \) is the input voltage of the PZT actuator. The transverse displacement of the panel \( w(x, y, \omega_0) \) is decomposed over its eigenfunctions \( \Phi_{mn} \),

\[ w(x, y, \omega_0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(\omega_0) \phi_{mn}(x, y) \]  

\[ \phi_{mn}(x, y) = \sin(\gamma_m x) \sin(\gamma_n y) \]  

\[ \omega_{mn} = \sqrt{\frac{\tilde{E}_p h^3}{12\mu(1-\nu_p^2)} (\gamma_m^2 + \gamma_n^2)} \]  

\[ \gamma_m = \frac{m\pi}{L_x} \quad \text{and} \quad \gamma_n = \frac{n\pi}{L_y} \]  

where \( \omega_{mn} \) is the natural angular frequency of bending mode \( (m, n) \). Equation 2 assumes a homogeneous, isotropic panel. Also, \( \tilde{E}_p = E_p(1 + \eta) \) (\( E_p \) is the complex Young’s modulus of the panel, \( \eta \) is the structural loss factor), \( \mu, \nu_p, L_x, L_y \) are the mass per unit area, Poisson’s ratio, and dimensions of the panel, respectively. The complex modal amplitudes \( W_{mn} \) under the action of an individual actuator \( j \) are given by [9, 3]

\[ W_{mn}(\omega_0) = -\frac{\gamma_m^2 + \gamma_n^2}{\gamma_m \gamma_n} \frac{d_{31} C_0 V_j^{(a)}(\omega_0)}{4 M_p t_a (\omega_{mn}^2 - \omega_0^2)} \left[ \cos(\gamma_m x_{1j}^{(a)}) - \cos(\gamma_m x_{2j}^{(a)}) \right] \left[ \cos(\gamma_n y_{1j}^{(a)}) - \cos(\gamma_n y_{2j}^{(a)}) \right] \]  

where \( M_p = L_x L_y \mu \) is the mass of the panel, \( d_{31} \) is the strain coefficient of the piezoceramic (it is assumed that the actuator provides equal free strains in the \( x \)- and \( y \)-directions, that is \( d_{31} = d_{32} \)), \( t_a \) is its thickness, \( x_{1j}^{(a)}, x_{2j}^{(a)}, y_{1j}^{(a)}, y_{2j}^{(a)} \) are the positions of the limits of the piezoceramic on the panel and \( C_0 \) is a coefficient related to the bending moment applied by the PZT actuator, \( C_0 = E_p K f^2 h^2 / 6 \) where [9, 3]
and $r = \frac{2ta}{h}$, $K = \frac{E_a(1-\nu_a)}{E_p(1-\nu_a)}$ where $E_a$ and $\nu_a$ are the Young’s modulus and Poisson’s ratio of the piezoelectric actuator, respectively. It should be noted that Equations 3 and 4 are based on several assumptions related to the piezoelectric actuation of the panel: (i) the applied moment distribution is constant over the area covered by the PZT actuator (in reality, the moment vanishes at the edges of the actuator, modifying stress distribution in the panel within about four actuator thicknesses from the boundary, [3, 13]. This edge effect may also slightly affect the strain measured by a PVDF sensor located in the immediate vicinity of the PZT actuator). (ii) The preceding formulation assumes a PZT actuator on one side of the panel only; the asymmetric actuation of the panel generates not only bending but also extensional response of the panel (the extensional component is not taken into account in the following analysis, especially in terms of the PVDF sensor response). (iii) Finally, the PZT actuator is assumed to be perfectly bonded to the panel and the bonding layer is assumed to have a negligible thickness.

The closed circuit charge equation of an extended piezoelectric PVDF sensor $i$ bonded to the flexible panel is

$$q(\omega_0) = -\int_{x_{1i}}^{x_{2i}} \int_{y_{1i}}^{y_{2i}} e_{3l} \epsilon_{l}^{(s)} dxdy$$

(5)

where $x_{1i}, x_{2i}, y_{1i}, y_{2i}$ are the positions of the limits of the piezoelectric sensor on the panel, $e_{3l}$ are the piezoelectric coefficients and $\epsilon_{l}^{(s)}$ are the strain components in the sensor. If pure bending strain is assumed in the panel (that is, the extensional strain response of the panel is neglected), Equation 5 takes the form [15]

$$q(\omega_0) = \frac{h + t}{2} \int_{x_{1i}}^{x_{2i}} \int_{y_{1i}}^{y_{2i}} e_{31} \frac{\partial^2 w(x, y, \omega_0)}{\partial x^2} + e_{32} \frac{\partial^2 w(x, y, \omega_0)}{\partial y^2} + 2e_{36} \frac{\partial^2 w(x, y, \omega_0)}{\partial x\partial y} dxdy$$

(6)
where $t_s$ is the thickness of the PVDF film. Assuming a zero skew angle ($e_{36} = 0$) and identical sensitivity in the $x$- and $y$-directions ($e_{31} = e_{32}$), and using Equations 2 and 3, Equation 6 becomes

$$ q(\omega_0) = -e_{31} \frac{h + t_s}{2} \sum_{m,n} W_{mn} \frac{\gamma_m^2 + \gamma_n^2}{\gamma_m \gamma_n} \left[ \cos(\gamma_m x_{1i}^{(s)}) - \cos(\gamma_m x_{2i}^{(s)}) \right] \left[ \cos(\gamma_n y_{1i}^{(s)}) - \cos(\gamma_n y_{2i}^{(s)}) \right] $$

(7)

Finally, the voltage output of the PVDF sensor measured through a high impedance circuit is given by

$$ V_i^{(s)}(\omega_0) = -\frac{q(\omega_0)}{C_s} $$

(8)

where $C_s = \xi_s \frac{S}{t_s}$ is the sensor capacitance and $\xi_s$, $S$ are its dielectric permittivity and area, respectively. Finally, combining Equations 3, 7 and 8, the PZT-PVDF transfer function $H_{ij}$ of Equation 1 can be calculated. The application of decentralized vibration control using PZT-PVDF pairs empirically depends on the relative magnitude of off-diagonal coefficients $H_{ij}$ ($i \neq j$) and diagonal coefficients $H_{ii}$ of the matrix $H$. In section 6, both diagonal and off-diagonal coefficients of $H$ are examined and compared to measured values.

Note that in the case of an isolated, collocated PZT-PVDF pair (that is $(x_{1j}^{(a)}, x_{2j}^{(a)}, y_{1j}^{(a)}, y_{2j}^{(a)}) = (x_{1i}^{(s)}, x_{2i}^{(s)}, y_{1i}^{(s)}, y_{2i}^{(s)})$), and under the preceding assumption of pure bending, it can be easily shown that the product of the actuator input voltage $V_j^{(a)}$ and the sensor output voltage rate $\dot{V}_i^{(s)}$ is proportional to the power supplied to the panel: such a collocated PZT-PVDF pair is therefore dual. In such a case a simple proportional feedback loop between each collocated pair would provide an unconditionally stable feedback control [4]. However, the coupling of both the PZT and PVDF with extensional deformation of the panel makes the analysis more complicated. When considering extensional deformation, it turns out that a collocated PZT-PVDF pair is
no longer dual in the general case. In any case, the following analysis involves non-collocated actuator-sensor pairs, therefore the duality of actuators and sensors is not \textit{a priori} required.

4 Controller Synthesis

4.1 Control objective

When the system is in steady state (all transients have vanished) the error vector $y = [y_1 y_2 \ldots y_N]^T$ measured by the $N$ PVDF sensors is given by

$$y = d + H u$$  \hspace{1cm} (9)$$

where $u = [u_1 u_2 \ldots u_N]^T$ is the complex control input vector at the $N$ PZT actuators and $d = [d_1 d_2 \ldots d_N]^T$ is the disturbance at the $N$ PVDF sensors. The problem is to adjust the complex inputs to the $N$ PZT actuators to minimize the measured signal from each of the $N$ PVDF sensors. The trivial solution of this problem, $u_{opt}^p = -H^{-1}d$, cannot be implemented in practice because of uncertainty on the matrix $H$, the disturbance $d$, and measurement noise. The usual approach is to rather derive an optimal command $u_{opt}$ with respect to an error criterion, $J(u, y)$, in order to apply iterative methods of minimization. A quadratic criterion can be defined as the sum of power outputs plus the power of the control weighted effort:

$$J(u) = y^H y + u^H R u$$  \hspace{1cm} (10)$$

where $^H$ denotes the Hermitian transpose and $R$ is a positive definite matrix. The introduction of the weighting control matrix, $R$ in the criterion leads to a trade off of the optimal command between the active attenuation and the ”magnitude” of the control input. Two types of weighting matrices $R$ will be considered in this paper: The first type penalizes the power of the control inputs, $R = \beta I$, while the second type penalizes the power of the control signals that are measured by the error sensors: $R = \beta H^H H \ (\beta > 0)$. 
For the quadratic criterion (10) and the linear system (9), the gradient vector is \( \nabla_u J = 2H^H y + 2Ru \) and the Hessian matrix is \( [\Delta_u J] = 2H^H H + 2R \). The optimal command \( u_{opt} \) is defined as the command that minimizes \( J \). The minimum of \( J \) is obtained when \( \nabla_u J(u_{opt}) = 0 \) and \( [\Delta_u J] > 0 \). It can be easily established that

\[
u_{opt} = -(H^H H + R)^{-1} H^H d
\]

\( J_{min} = d^H [I - H(H^H H + R)^{-1} H^H] d \)

If the weighting matrix is in the form \( R = \beta H^H H \), Equation 12 takes the form \( J_{min} = \frac{\beta}{1+\beta} d^H d \). Hence, the attenuation level at the error sensors provided by the active control is

\[
Att_{dB} = 10 \log_{10}(\frac{\beta}{1+\beta})
\]

Equation 13 clearly reveals the effect of the weighting \( \beta \) on the attenuation performance: A large weighting coefficient \( \beta \) has the effect of decreasing the optimal attenuation \( Att_{dB} \) while decreasing the control effort \( u_{opt} \). Hence, the weighting coefficient should be chosen small (\( \beta << 1 \)) in order to obtain a significant active attenuation. The limiting case of \( \beta = 1 \) leads to an attenuation of 3 dB of the criterion. In addition, the following sections will demonstrate that the introduction of a small control effort weighting in the criterion can have a stabilizing effect on decentralized vibration control.

### 4.2 Design of the controller

One of the most useful iterative methods of minimization is the steepest-descent (or gradient) algorithm. It is used to iteratively adjust the command \( u \) in the opposite direction of the gradient of the criterion in order to reach a minimum: \( u(k+1) = u(k) - \mu_g \nabla_u J \) where \( \mu_g \) is the adaptation coefficient of the algorithm. For the criterion defined by Equation 10, the gradient algorithm takes the form

\[
u(k+1) = u(k) - 2\mu_g (H^H y(k) + Ru(k))
\]
The main problem of the steepest-descent algorithm is that the adaptation rate directly depends on the local gradient: when the gradient is small, the adaptation becomes very slow. A faster convergence to the optimal command can be obtained with the Newton algorithm by multiplying the gradient by the inverse of the Hessian matrix: 

\[ u(k+1) = u(k) - \mu_N [\Delta_u J]^{-1} \nabla_u J \]

where \( \mu_N \) is the adaptation coefficient of the Newton algorithm. For the criterion defined by Equation 10, the Newton algorithm takes the form

\[ u(k+1) = u(k) - \mu_N (H^H H + R)^{-1} (H^H y(k) + Ru(k)) \quad (15) \]

Both the gradient (14) and Newton algorithms (15) are therefore special forms of the generic update equation

\[ u(k+1) = (I - \mu W)u(k) - \mu C y(k) \quad (16) \]

where \( \mu = 2\mu_g \), \( C = H^H \) and \( W = 2R \) for the gradient algorithm; \( \mu = \mu_N \), \( C = (H^H H + R)^{-1} H^H \), \( W = (H^H H + R)^{-1} R \) for the Newton algorithm. Therefore, it is appropriate to define distinct weighting matrices \( R \) for the gradient and Newton algorithms: \( R = \beta I \) with \( \beta \) a real positive number for the gradient algorithm, and \( R = \frac{\beta}{1-\beta} H^H H \) for the Newton algorithm. In such a way, the matrix \( W \) in Equation 16 is the same diagonal matrix for both the gradient and the Newton algorithms: \( W = \beta I \). However, the penalization is in terms of the control inputs for the gradient algorithm, while it is in terms of the control signals that are measured by the error sensors for the Newton algorithm. Moreover, for the Newton algorithm, the matrix \( C \) is simplified to \( C = H^{-1} \). In the following, centralized and decentralized versions of the update equation 16 are analyzed.

### 4.3 Proposed controllers

A perfect measurement of the plant matrix \( H \) is necessary to rigourously implement the iterative algorithms (16), however only an estimation \( \hat{H} \) of \( H \) is available in practice. When the matrix \( C \) is built from the complete estimate of the plant matrix \( \hat{H} \) (both diagonal and off-diagonal
coefficients), then Equation 16 provides the update of a centralized controller. In what follows, it is assumed that for centralized control, both the diagonal and off-diagonal coefficients of $\mathbf{H}$ are perfectly estimated, therefore $\hat{\mathbf{H}} = \mathbf{H}$ for centralized control.

The case of a decentralized controller corresponds to $N$ independent update equations that take into account only the diagonal coefficients of the plant matrix. This is formally equivalent to a biased estimation of the complete matrix $\mathbf{H}$, $\hat{\mathbf{H}} = \text{diag}(H_{ii})$ for all $i = 1, \ldots, N$; in other words, the off-diagonal coefficients are forced to 0 and the diagonal coefficients are assumed to be perfectly estimated. Therefore, decentralized control derived from the gradient algorithm corresponds to $\mathbf{C} = \text{diag}(H_{ii}^*)$, where $H_{ii}^*$ denotes the complex conjugate of $H_{ii}$. On the other hand, decentralized control derived from the Newton algorithm corresponds to $\mathbf{C} = \text{diag}(1/H_{ii})$. Note that decentralized control derived from the Newton algorithm does not need the inversion of the complete plant matrix, but only the inverse of its diagonal elements.

In summary, Equation 16 is the general update equation that applies to the four cases investigated in this article: the control matrix is $\mathbf{C} = \hat{\mathbf{H}}^H$ for the gradient algorithm, $\mathbf{C} = \hat{\mathbf{H}}^{-1}$ for the Newton algorithm, $\hat{\mathbf{H}} = \mathbf{H}$ for centralized control and $\hat{\mathbf{H}} = \text{diag}(H_{ii})$ for decentralized control. For the four cases, the weighting matrix is $\mathbf{W} = \beta \mathbf{I}$.

The main limitation of decentralized control is that both the Hessian and the gradient of the criterion are approximated from only the diagonal elements of the plant matrix. Hence, the estimated gradient computes a biased direction for searching the optimal command. In the worst case, the controller may search the minimum of the criterion in its opposite direction and never reach it: the system is unstable. The following sections address the conditions of stability of the controlled system.

### 4.4 Conditions of stability

It is possible to establish the condition of asymptotic stability of the closed loop system described by Equations 16 and 9. If we consider that the closed loop system is stable, then the command
\( u(k) \) converges to \( u_\infty = -(CH + W)^{-1}Cd \) (obtained by imposing \( u(k+1) = u(k) \) in equation 16). It is then possible to re-express Equations 16 and 9 as an autonomous linear discrete-time system:

\[
x(k+1) = Ax(k)
\]

where \( x(k) = u(k) - u_\infty \) (or \( x(k) = y(k) - y_\infty \)) and \( A = I - \mu(CH + W) \). Such a discrete-time dynamic system is called stable when \( x(k) \) tends exponentially to zero when \( k \to \infty \) for any initial condition such that \( x(0) \neq 0 \). The necessary and sufficient condition of stability of the closed loop system is that the matrix \( A \) is Schur stable: all its eigenvalues lie within the interior of the unit circle in the complex plane: \( |\lambda_i(A)| < 1 \) for all \( i \).

In the special case of \( A = I - \mu B \) with \( B = CH + W \) and \( \mu > 0 \), the condition of Schur stability becomes \( |1 - \mu \lambda_i(B)| < 1 \) for all \( i = 1, ..., N \). Hence, the Schur stability condition leads to the necessary and sufficient condition of stability

\[
\mu < \frac{2\Re(\lambda_i(B))}{ |\lambda_i(B)|^2 }, \quad \forall i = 1, ..., N
\]

where \( \Re(\lambda_i) \) designates the real part of the complex eigenvalue \( \lambda_i \). An important interpretation of the above condition is that it introduces an upper bound for the adaptation coefficient in the iterative algorithm, \( \mu_{\text{max}} = \frac{2\Re(\lambda_i(B))}{ |\lambda_i(B)|^2 } \). The main implication of Equation 18 is that the real part of all eigenvalues of \( B \) must necessarily be positive, \( \Re(\Lambda_i) > 0 \) for all \( i \). In the following, and in the context of this paper, the term "\( \mu \)-stabilization" refers to the necessary condition of stability: if \( \Re(\lambda_i(B)) > 0 \) for all \( i \) then it is possible to obtain stable control with an appropriate tuning of the adaptation coefficient \( \mu \). When all eigenvalues of \( B \) are located in the open right complex half plane, the matrix theory literature often refers to the concept of positive stability. In control theory literature, the matrix \(-B\) is said to be Hurwitz stable: the associated autonomous continuous time system \( \dot{x}(t) = -Bx(t) \) is stable.

An alternative and simpler method of analyzing the "\( \mu \)-stabilization" of the closed loop system is derived from the following theorem [12] (p. 88): If \( B \in C^{N\times N} \) with \( n_0(B) = 0 \)
and if the Hermitian nonsingular matrix $P \in \mathbb{C}^{N \times N}$ satisfies the condition $PB + B^H P = -Q$ with $Q \geq 0$ then $\text{In} B = \text{In} P$. The inertia of a matrix $B$, denoted $\text{In} B$ is defined by the triple $(n_+(B), n_0(B), n_-(B))$ where $n_+(B)$ is the number of eigenvalues of $B$ in the open right complex half-plane, $n_0(B)$ is the number of purely imaginary eigenvalues of $B$, $n_-(B)$ is the number of eigenvalues of $B$ in the open left complex half-plane. According to the theorem, choosing $P = I$ and assuming that $B^H + B$ is positive definite, then $\text{In} B = (N, 0, 0)$ (meaning that $\Re(\lambda_i(B)) > 0$). Consequently, if $B$ is a “strictly positive-real” (SPR) matrix (that is, $B^H + B$ is positive definite) it is possible to find a $\mu$ which achieves stable control of the overall closed-loop system. Moreover, when $B$ is a SPR matrix, this implies that $V(x) = x^H x$ is a quadratic Liapunov function for the system $\dot{x}(t) = -Bx(t)$. Therefore $dV/dt \leq 0$, the matrix $-B$ is said to be Hurwitz diagonally stable (this is also referred to as Volterra stability, Volterra-Liapunov stability or VL-stability, or dissipativity) [12]. Hence, the necessary condition of $B$ SPR means that in the limiting case of $\mu \to 0$, $-B$ must be Hurwitz diagonally stable. The diagonal stability is a stronger requirement than just stability; in our case diagonal stability means that the sum of power outputs measured by the PVDF sensors monotonously decreases with time when $\mu$ is very small.

From a purely mathematical point of view, the concept of strictly positive-real matrix $B$ has the advantage of simplifying the stability analysis and of providing some physical insight. For example, we consider the case of centralized control, a perfectly estimated plant matrix and no effort weighting ($R = 0$). In the case of the centralized Newton algorithm, $C = H^{-1}$, then $CH = I$ is SPR (its eigenvalues are 1). In the case of the centralized gradient algorithm, $C = H^H$, then $CH = H^H H$ is a SPR matrix (because $H^H H$ is a hermitian positive definite matrix). Consequently, centralized control with perfect system estimation will always be stable.

The case of decentralized control is more interesting. The consideration of only the diagonal coefficients of the plant matrix $H$ is formally equivalent to a biased estimation of the complete matrix $H$. In the case of imperfect plant estimation, the eigenvalues of $CH$ are not necessarily
real and positive, therefore the condition $\Re(\lambda_i(B)) > 0$ suggests that control effort weighting $W$ may be necessary to stabilize the control. The following sections investigate the effect of effort weighting on the stability of decentralized control.

4.5 Decentralized control without effort weighting, $W = 0$

When $W = 0$, the disturbance rejection is theoretically perfect but "$\mu$-stabilization" of the closed loop system is possible if and only if $\Re(\lambda_i(CH)) > 0$, this condition is verified if $CH$ is a strictly positive-real (SPR) matrix. In order to illustrate the role of the decentralized compensator $C$, 2 cases are considered in this section for the plant matrix: $H$ is a SPR matrix and $H$ is not SPR.

- When $H$ is a SPR matrix, the "$\mu$-stabilization" in decentralized or centralized is guaranteed. Disturbance rejection is possible without any estimation of the plant matrix $H$: with $C = I$, the matrix $CH = H$ is SPR, therefore a sufficiently small adaptation coefficient $\mu$ will necessarily ensure closed loop stability. A SPR plant matrix $H$ for all frequencies physically corresponds to a dissipative plant [22]. The SPR condition is obtained in a linear undamped flexible structure with dual-collocated actuator-sensor pairs (passive operator), without poles on the imaginary axis (strictly stable linear system) and without null eigenvalues (dissipativity condition) [20]. In other words, the collocation of actuators and sensors and the intrinsic damping in the plant imply the SPR condition at any frequency. In contrast, the configuration of distributed and non-collocated actuator-sensor pairs results in a non PR plant matrix $H$ (and consequently to a non SPR $H$), leading to more difficult control situations.

- When $H$ is not SPR, the role of the diagonal compensator matrix $C$ can be to ensure a SPR matrix $CH$. A necessary (but not sufficient) condition for $CH$ to be SPR is that the real part of diagonal elements of $CH$ are strictly positive: $\Re(C_iH_{ii}) > 0$. With the
assumption of perfect estimation of the diagonal coefficients of the plant matrix, \( \hat{H}_{ii} = H_{ii} \), this necessary condition is always verified in decentralized control because \( C_i H_{ii} = |h_{ii}|^2 \) for the gradient algorithm and \( C_i H_{ii} = 1 \) for the Newton algorithm. However, in the case of a biased estimation of the diagonal coefficients of the plant matrix, this condition may be not verified. For example, if we consider the Newton algorithm with a phase-shift error \( \phi \) of the diagonal coefficient, \( H_{ii} = \hat{H}_{ii} \exp^{j\phi} \), then the diagonal coefficients of \( CH \) have a real part \( \Re(H_{ii}/\hat{H}_{ii}) = \cos(\phi) \) which is negative when the phase error is such that \( \pi/2 > \phi > 3\pi/2 \) (mod \( 2\pi \)): therefore, an individual control unit can be unstable (with the other units being inactive). In this case, since the corresponding diagonal coefficient \( C_i H_{ii} \) is not strictly positive, the whole system is not SPR (it may still be stable, though).

To summarize, the main role of the diagonal compensator \( C \) in decentralized control is to compensate the phase of the diagonal coefficients of the plant matrix \( H \) in order to ensure a SPR matrix \( \text{diag}(C_i H_{ii}) \). Along these lines, we can conclude that any compensator of the form \( C = K\hat{H}^H \) with \( \hat{H} = \text{diag}(H_{ii}) \), and \( K \) a diagonal positive matrix is a candidate for decentralized control. For example, \( K_i = 1 \) corresponds to the gradient algorithm and \( K_i = 1/|H_{ii}|^2 \) corresponds to the Newton algorithm.

### 4.6 Decentralized control with effort weighting, \( W \neq 0 \)

In situations where it exists at least one negative eigenvalue \( \Re(\lambda_i(CH)) < 0 \), then, according to Equation 18, the weighting matrix \( W = \beta I \) must be included in the criterion to ensure a stable closed loop. The weighting coefficient \( \beta \) that ensures theoretical \( \mu \)-stabilization of the convergence process should satisfy:

\[
\beta > -\Re(\lambda_i(CH)) \quad \forall i
\]

Therefore, the minimum value of \( \beta \) is \( \beta = \max(-\Re(\lambda_i(CH))) \). The main drawback of weighting the control effort is to decrease the active attenuation. For example, Equation 13 shows that
\(\beta = 1\) leads to a 3 dB attenuation of the criterion after control. In practical situations, values of \(\beta\) should be kept much lower than unity in order to maintain satisfactory control performance.

### 4.7 Positive definite and diagonal dominance

In case of a perfect estimation of the diagonal coefficients of the plant matrix, the Gershgorin theorem can be applied to derive a sufficient condition of \(\mu\)-stabilization. The Gershgorin theorem states that "the eigenvalues of a complex matrix \(B\) are within disks in the complex plane whose centers are the diagonal coefficients \(B_{ii}\) of \(B\) and whose radii are the sums of the modulus of off-diagonal coefficients \(B_{ij}\) with \(i \neq j\)." When \(B_{ii}\) is a real positive number and \(\sum_{j=1,j\neq i}^{m}|B_{ij}| < B_{ii}\), then all these disks lie within the right-hand side of the complex plane, therefore all eigenvalues of \(B\) have a positive real part: \(\Re(\lambda_i(B)) > 0\); therefore, the system is \(\mu\)-stabilizable. A matrix such that \(\sum_{j=1,j\neq i}^{m}|B_{ij}| < B_{ii}\) for all \(i\) is called diagonal dominant. Consequently, a sufficient condition of \(\mu\)-stabilization is that the matrix \(B = CH + W\) is diagonal dominant and \(\text{diag}(B_{ii}) > 0\).

It can be easily established that the condition \(CH\) diagonal dominant is equivalent to the condition \(H\) diagonal dominant for the gradient and Newton algorithms, \(|H_{ii}| > \sum_{j=1,j\neq i}^{m}|H_{ij}|\) for all \(i\). Therefore, if \(H\) is diagonal dominant then \(CH\) is SPR and the decentralized control is \(\mu\)-stabilizable without control effort weighting. The condition that \(H\) is diagonal dominant is easy to test in practice. However this condition is fairly restrictive because the phase of transfer functions between actuators and sensors is not considered. For example, in the case where the number of actuator-sensor pairs \(N \to \infty\), the diagonal dominant condition will be never satisfied even if the matrix \(CH\) is SPR or the system is dissipative. The plant \(H\) does not necessarily need to be diagonal dominant in order to make decentralized control effective.

To summarize, the various conditions of \(\mu\)-stabilization (to obtain \(A = I - \mu(CH + W)\) Schur stable by tuning \(\mu\)) are presented in Table I as a function of the plant matrix \(H\), the control matrix \(C\), the effort weighting coefficient \(\beta\) and the adaptation coefficient \(\mu\).
5 Control Implementation

5.1 Implemented harmonic controller

In the preceding sections, the inherent delay of the control system in processing the error signal \( y \) and generating the next control input \( u \) has not been taken into account. In practice, the implementation of a frequency-domain control used demodulation and modulation blocks that extract the phasor of the error \( y \) and generate the oscillatory control input, respectively (figure 2). This section presents the practical signal processing to implement the decentralized harmonic controllers.

In order to extract the phasor \( Y_i \) from the real oscillatory signal \( y_i(t) \) provided by sensor \( i \), a complex demodulation is applied. The method involves translating the signal \( y_i \) in the frequency domain, from the harmonic frequency \( \omega_0 \), to 0 (rad/s), by multiplication with the complex sine wave of frequency \( \omega_0 \). A low-pass filter \( f \), is applied to this complex signal in order to extract the phasor of interest:

\[
Y_i(t) = f(t) \ast (y_i(t) \exp(-i\omega_0 t))
\]  

where \( \ast \) denotes the convolution operator. The cut-off frequency of the low-pass filter is set equal to \( \omega_c << \omega_0 \) and its static gain is set to 1, \( |f(0)| = 1 \). Because of the low-pass filtering, the complex signal \( Y_i(t) \), called the complex envelope, is localized in the low frequency domain. Hence, according to the Shannon criterion, \( Y_i(t) \) can be sampled without aliasing at a rate \( \omega_e < 2\omega_0 \). The discrete down-sampled output of the complex envelope is \( Y_i[k] = Y_i(kT_e) \) with \( T_e = 2\pi/\omega_e \) the sampling period and \( k \) the discrete time.

In order to generate each command signal \( u_i(t) \) from each phasor \( U_i \) (figure 2), a complex modulation is used. First, the causal low-pass filtering \( f \) is applied to generate \( U_i(t) \) from the down-sampled discrete-time signal \( U_i[k] \). Then, the signal is translated in the frequency domain from 0 (rad/s) to the harmonic frequency, \( \omega_0 \), and the real part is extracted to generate the
command signal:

\[ u_i(t) = 2\Re(U_i(t) \exp(i\omega_0 t)) \]  

(21)

In order to provide insight in the controller implementation, its equivalent form is developed. In the continuous time domain, Equation 16 can be written by considering the finite difference approximation with \( T_e \to 0 \) and \( \mu \to 0 \):

\[ \frac{U(k + 1) - U(k)}{T_e} \approx \frac{dU(t)}{dt} = -aU(k) - bCY(k) \]  

(22)

where \( b = \frac{\mu}{T_e} \) and \( a = b\beta \). The Laplace Transform of Equation 22, gives the equivalent compensator in terms of the complex envelopes \( U \) and \( Y \):

\[ U(s) = -\frac{b}{s + a}CY(s) \]  

(23)

By introducing Laplace Transforms of Equations 20 and 21 in Equation 23, the equivalent decentralized compensator is obtained in terms of the real oscillatory signals \( u_i \) and \( y_i \):

\[ \frac{u_i(s)}{y_i(s)} = -\frac{bN(s)}{(s + a)^2 + \omega_0^2} \]  

(24)

with \( N(s) = Cf^2(s - j\omega_0)(s + a + j\omega_0) + C^*f^2(s + j\omega_0)(s + a - j\omega_0) \).

The numerator \( N(s) \) includes bandpass filtering \( f \), centered at \( \omega_0 \); the denominator includes two weakly damped poles close to the harmonic frequency \( \omega_0 \): \( s_0 = -a + j\omega_0 \) and \( s_0^* = -a - j\omega_0 \). The feedback loop gain at the disturbance frequency, \( \omega_0 \), is approximately \( bC_{ii}/a \). It is tuned with the \( \mu \) parameter: with \( \mu = 0 \) there is no feedback loop (\( b = 0 \)) and no control. The maximal adaptation coefficient, \( \mu_{\text{max}} \) defined by Equation 18, is equivalent to the critical feedback gain: the stability is marginal for this value. When \( \beta = 0 \), Equation 24 is the expression of a resonant controller because the added pole is undamped: the compensator is characterized by an infinite gain at \( \omega_0 \). According to the Wohan’s Principle [8], this allows perfect rejection of the harmonic disturbance because the compensator includes a model of the harmonic generator: it is able to generate a sinusoidal waveform at the disturbance frequency even when the error is null. When
it exists at least one negative eigenvalue $\Re(\lambda_i(CH)) < 0$, the compensator poles should be damped for the purpose of adding stability robustness: the increase of $\beta$ in Equation 24 moves the compensator poles from the imaginary axis to the left-side of the complex plane. According to Siever [21], the equivalent controller is independent of the methodology chosen (analog, adaptive, classical, modern method); for example a MIMO adaptive feedforward algorithm (usually called x-lms) will also present complex poles centered at the disturbance frequency [14].

5.2 Stability analysis of the implemented controller

When the controller slowly adapts the control inputs ($\mu \to 0$) in comparison to the dynamics of the low-pass filters, the influence of modulation and demodulation blocks is negligible and closed loop stability can be analyzed using Equation 9. However, in the general case, the low-pass filtering operations induce transients in the modulation and demodulation processes and they must be taken into account by modifying Equation 9 as follows:

$$y(k) = d + H \phi(q^{-1})u(k)$$

(25)

where $\phi(q^{-1}) = \phi_0 + \phi_1 q^{-1} + ... + \phi_d q^{-d}$ is the impulse response of the filters and $q^{-1}$ is the delay operator. The impulse response coefficients are computed from the down-sampled discrete-time impulse response of the two lowpass filters in series: $\phi_n = (f \ast f)(nT_e)$ for $n = 0, 1, 2, ..., d$. In other words, the impulse response of the combined modulation and demodulation blocks are represented by a Finite Impulse Response (FIR) filter with $d+1$ coefficients. This FIR filter has an unitary static gain, $\phi(1) = \phi_0 + \phi_1 + ... + \phi_d = 1$, because $|f(\omega)| \approx 1$ for $\omega \approx 0$. Consequently, the steady state version of Equation 25 is Equation 9. In other words, Equation 25 describes the transients due to the signal processing operations (modulation and demodulation), but not the transients due to the physical system.

Similar to Equation 17, Equation 25 can be re-expressed as a discrete time autonomous
system of state variable $x(k)$:

$$x(k + 1) = (1 - \mu \beta)x(k) - \mu CH\phi(q^{-1})x(k)$$  \hspace{1cm} (26)

where $x(k) = u(k) - u_\infty$. By introducing the expression of $\phi(q^{-1})$, Equation 26 takes the form of a difference equation

$$x(k + 1) = [(1 - \mu \beta)I - \mu \phi_0 CH]x(k) - \mu \sum_{i=1}^{d} \phi_i CHx(k - i)$$  \hspace{1cm} (27)

The condition for which $x$ converges to $x_\infty$ is more difficult to establish in the case of Equation 27 than in the case of Equation 17. It is necessary to re-express Equation 27 in an augmented state space form

$$z(k + 1) = A_d z(k)$$  \hspace{1cm} (28)

where $z(k) = [x(k) \ x(k - 1) \ ... \ x(k - d)]^t$ is the new state vector, and the evolution matrix is:

$$A_d = \begin{bmatrix}
(I - \mu W) - \mu F_0 & -\mu F_1 & ... & -\mu F_{d-1} & -\mu F_d \\
I & O & ... & O & O \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
O & O & ... & I & O
\end{bmatrix} \text{ with } F_i = \phi_i CH.$$

The necessary and sufficient condition of stability is that the matrix $A_d$ is Schur stable: all its eigenvalues are inside the unit circle in the complex plane.

When the weighting coefficient $\beta$ is fixed, the stability condition consists of finding the maximum adaptation coefficient $\mu_{max}$ such that $\forall \mu < \mu_{max} : \sup |\lambda_i(A_d(\mu))| < 1$ and $|\lambda_i(A_d(\mu_{max}))| = 1$. Considering the complexity of the matrix $A_d$, this search was done numerically.
6 Simulation and experimental results

6.1 Comparison of plant model with measured data

The prediction of $\mathbf{H}$, presented in section 3 was first compared to experimental data in the case of a simply-supported 48cm $\times$ 42cm aluminium panel instrumented with 4 BM-500 2.54cm $\times$ 2.54cm, 1mm thick PZT actuators and 4 1.5cm $\times$ 1cm, 28 $\mu$m thick PVDF sensors. Tables II and III list the panel, actuator and sensor data. Table IV shows the calculated and measured natural frequencies of the panel in the $[0 - 800Hz]$ range. The positions of actuator and sensor pairs on the panel (subsequently referred to as positions 2, 4, 6 and 8) are shown on figure 3. The actuator positions were selected randomly in order to excite as many panel modes as possible in the $[0 - 800Hz]$ frequency range. Note however that in this configuration, PZT 2 and 4 do not effectively excite the panel modes $n = 3$, and PZT 6 and 8 are not effective for panel modes $n = 4$. A PVDF sensor was bonded close to each PZT to form a pair (their separation was of the order of 1mm). The PZT-PVDF transfer functions were measured using a $[0 - 800Hz]$ white noise input signal sequentially driving the PZT actuators; the charge response of the PVDF sensors was amplified and converted to a voltage output through high input impedance Frequency Device filters. The following transfer functions do not include the input gain of the high voltage PZT amplifier and the output gain of the PVDF filters. On the other hand, a structural loss factor of the panel of $5 \times 10^{-3}$ was considered in the simulations; this value was adjusted to fit the measured response on resonance of selected panel modes.

Figure 4 shows examples of measured and predicted transfer functions $H_{28}, H_{48}, H_{66}, H_{88}$. The agreement is generally good for the off-diagonal terms $H_{28}, H_{48}$ (with the prediction slightly overestimating the measured values), but much less satisfactory for the diagonal terms $H_{66}, H_{88}$. In this case the prediction largely underestimates the magnitude of the transfer function off-resonance. This leads to a measured $\mathbf{H}$ which is more diagonally-dominant than the calculated
H, and therefore an a-priori more viable implementation of decentralized control. Possible explanations for the underestimation of direct transfer by the theory are bending near-field effects of the PZT actuators and the effect of extensional excitation of the panel by the PZT actuators, which were not accounted for in the model.

6.2 Predicted stability indicators for decentralized control

For the physical configuration with 4 control units shown on figure 3, the various stability conditions listed in Table I were tested for decentralized control. The stability conditions are derived from the theoretical plant matrix or the measured plant matrix H. In this case, the plant matrix was measured by driving the PZT actuators with a pure sine in the frequency range from 500Hz to 800 Hz, with a 10 Hz step. The amplitude of the actuator signal was automatically adjusted in order to obtain the largest possible output signal from the PVDF sensors without saturation. Figure 5 presents predicted stability of decentralized control derived from either the theoretical or measured plant matrix H in the frequency range 500 Hz - 800 Hz. The presented stability indicators are the diagonal dominant property of H, the condition of $\Re(\lambda_i(CH)) > 0$, the condition CH strictly positive real (SPR) and the minimum control effort weighting $\beta$ according to Equation 19.

In the case of stability derived from the theoretical plant matrix, the sufficient condition of H being diagonal dominant is never verified in the frequency range investigated: the interaction between PZT-PVDF pairs is large at all frequencies. In the case of stability derived from the measured plant matrix, the diagonal dominant condition is verified in some frequency intervals. This is consistent with the observations derived from the plant matrix components $H_{ij}$ in Figure 4. The measured plant matrix tends to be not diagonal dominant close to resonances of panel modes.

The condition that the matrix CH is SPR guarantees that $\mu$-stabilization of the closed loop system is possible without requiring control effort weighting ($\beta = 0$). This condition is presented
on Figure 5 for decentralized Newton, for which \( C = \text{diag}(1/H_{ii}) \). In the case of stability derived from the theoretical plant matrix and for the Newton algorithm, the \( CH \) matrix is SPR only in some frequency intervals included in those of the condition \( \Re(\lambda_i(CH)) > 0 \); accordingly, the minimum value of \( \beta \) is predicted to be larger than 0 only when \( \Re(\lambda_i(CH)) < 0 \). Note that for the gradient algorithm, the condition \( \Re(\lambda_i(CH)) > 0 \) is verified at all frequencies, therefore the minimal value of \( \beta \) is predicted to be null (not shown on the figure). Stability derived from the measured \( H \) is less stringent: for decentralized Newton, \( CH \) always verifies \( \Re(\lambda_i(CH)) > 0 \) except at 500Hz and in the frequency range 670 Hz - 690 Hz. At these frequencies, control effort weighting is necessary in order to stabilize the closed-loop system, \( \beta_{min} > 0 \) for both the gradient and Newton algorithms. Note that at frequencies for which \( \beta_{min} > 0 \), \( H \) is not diagonal dominant, but the inverse is not true: for example, at 800Hz, the experimental and theoretical matrices \( H \) are not diagonal dominant but the associated \( CH \) matrix is SPR for the gradient algorithm, and both gradient and Newton algorithms are \( \mu \)-stable because \( \Re(\lambda_i(CH)) > 0 \). The hierarchy between the conditions is verified: the diagonal dominance of \( H \) or the SPR of \( CH \) are sufficient, but not necessary conditions of closed-loop stability.

Note also that the minimum value of the effort weighting coefficient \( \beta \) required to stabilize the Newton algorithm is large, and usually larger than for the gradient algorithm (a value of \( \beta = 1 \) reduces the attenuation at the error sensors to 3 dB). Hence, the control performance obtained with the weighted Newton is expected to be smaller than the performance of the weighted gradient.

### 6.3 Predicted and measured stability limits

The decentralized and centralized controllers were implemented in a rapid prototyping dSPACE system. The sampling frequency was set to 5kHz. For the analysis and synthesis of the complex envelopes, the low-pass FIR filters were implemented with 4 Butterworth second order filters in order to ensure an attenuation of 40dB outside \( \pm 7.5 \) Hz. Consequently, the complex
envelopes were under-sampled at 25Hz without aliasing. The identification of the plant matrix was performed for each frequency with an harmonic input of the PZT actuators (with the disturbance off) and under steady-state response of the system. The identified plant matrix was used to perform the active centralized/decentralized control of the plate. Steady-state harmonic response of the system can be assumed for the control due to the slow dynamics of the FIR filters (and the slow dynamics of the controller - low value of $\mu$) against the dynamics of a low damped mode in the considered narrow frequency band [17]. This point is experimentally verified with the results presented in Figure 6 where the dynamics of the filters limit the maximum value of the adaptation coefficient.

Figure 6 shows the predicted and measured stability limits of the centralized and decentralized controllers in the case of the 4 PZT-PVDF control pairs shown on figure 3, in the frequency range $[500\text{Hz} - 800\text{Hz}]$. Note that the predicted stability limit was obtained from the measured plant matrix $H$. The maximum adaptation coefficient, $\mu_{\text{max}}$, observed experimentally for the various controllers is also reported. Two cases of low-pass filter dynamic response were examined in the prediction: (i) a unitary frequency response, $\phi_0 = 1$ and $\phi_i = 0$ for $i > 0$ (in this case, no filter delay is assumed, and the stability limit was obtained from Equation 18); (ii) the measured impulse response of the low-pass filters $\phi(q^{-1})$ (in this case, the stability limit was derived from the eigenvalues of the matrix $A_d$ in Equation 28).

In the case of the centralized gradient and Newton algorithms, the $CH$ matrix is always SPR, therefore the control effort weighting was set to zero: $\beta = 0$. The maximum adaptation coefficient $\mu_{\text{max}}$ turns to be frequency-independent for the centralized Newton, its predicted value being 2 in the case of a zero loop delay and close to 0.02 in the case of the loop delay induced by the low-pass filters. The negative impact of the time-delay, or filtering, is clearly apparent in this computation. The value of $\mu_{\text{max}} = 2$ in the case of a zero loop delay is consistent with convergence analysis derived from Equation 16. As expected, when loop delay is increased, the critical feedback gain is reduced. The experimental values of $\mu_{\text{max}}$ are close to the values
predicted when the measured impulse response of low-pass filters is included in the loop. It is clear that $\mu_{max}$ is largely overestimated if no loop delay is included. In the case of centralized gradient, the values of $\mu_{max}$ are frequency-dependent: when $C = H^H$, Equation 18 becomes $\mu_{max} < 2/\sigma_{max}^2$, where $\sigma_{max}^2$ is the largest eigenvalue of $H^H H$. Therefore, at resonance of panel modes, the system presents a high gain and the adaptation coefficient must be reduced. The time-delay induced by low-pass filters still decreases the critical value of $\mu$. The agreement between experimental and predicted values of $\mu_{max}$ when the measured impulse response of low-pass filters is included in the loop, is again very good.

In the case of decentralized control, the matrix $CH$ is not SPR for all frequencies in the [500 Hz − 800 Hz] range, therefore a strictly positive control effort weighting coefficient $\beta$ has to be introduced at certain frequencies. The same value of $\beta$ was used in the experiments and the prediction (1.5 time the minimum value computed with Equation 18). It appears that in this case $\mu_{max}$ is frequency-dependent for both the gradient and Newton algorithms.

The following sections detail the control results obtained at frequencies where: (i) $H$ is diagonal dominant, (ii) $\Re(\lambda_i(CH)) > 0$, the closed loop is $\mu$-stable without effort weighting; (iii) $\Re(\lambda_i(CH)) < 0$, control effort weighting is necessary. In all cases, the primary disturbance is a transverse point force provided by an electrodynamic shaker at location $x = 35$ cm, $y = 17$ cm in the coordinate system of Figure 3.

6.4 Experimental results for a case H diagonal dominant

At the disturbance frequency of 550 Hz, the measured plant matrix $H$ is diagonal dominant (see figure 5), and SPR. The diagonal dominant condition of $H$ is a sufficient condition to guarantee that the matrix $CH$ is positive definite, and that the system is $\mu$-stable without control effort weighting, $\beta = 0$. Consequently, the eigenvalues associated to the decentralized gradient algorithm, $\lambda_i(\text{diag}(H^H H)) = \{0.70, 0.41, 0.27, 0.10\}$, are close to the eigenvalues of the centralized gradient $\lambda_i(H^H H) = \{0.85, 0.41, 0.29, 0.06\}$. Similarly, the eigenvalues associated
to the decentralized Newton algorithm, \( \Re(\lambda_i(CH)) = \{1.36, 1.26, 0.78, 0.59\} \), are close to the eigenvalues of the centralized Newton (equal to unity).

Figure 7 shows the measured convergence of the error measured by each of the PVDF sensors, for centralized Newton, decentralized Newton and decentralized gradient. In this case, \( \beta = 0 \) since the closed-loop system is \( \mu \)-stable without control effort weighting. For equal adaptation coefficients, the decentralized Newton algorithm converges at a rate similar to the centralized Newton, because the associated eigenvalues are similar. Moreover, the convergence rate is similar for all PVDF sensors in the case of the centralized and decentralized Newton. For the decentralized gradient, the convergence rate varies from one PVDF sensor to another: the largest eigenvalue of \( CH \) is associated in this case to the fastest convergence mode and the smallest eigenvalue of \( CH \) corresponds to the slowest convergence mode. The ratio of the largest to the smallest eigenvalue defines the condition number of \( CH \), and is representative of the convergence rate of the algorithm. It was experimentally verified that in this case, decentralized Newton converges faster (\( \text{cond}(CH) = 2.43 \)) than decentralized gradient (\( \text{cond}(CH) = 7.23 \)) for equal adaptation coefficients \( \mu \).

6.5 Experimental results for a case \( CH \) is SPR

At the frequency of 520 Hz, the disturbance is close to the natural frequency of the (2,3) plate mode. The plant matrix \( H \) measured at this frequency is not diagonal dominant: the interaction between control units is significant. The transverse velocity field of the panel due to the primary point force driving the panel at 520 Hz is presented on the right-hand side of figure 9. It appears that at this frequency, PZT actuator 2 is close to a vibration node and leads to a low response of PVDF sensor 2 \( |H_{22}| = 0.36 \), but PZT actuator 8, close to an antinode, leads to a high response of PVDF sensor 2: \( |H_{82}| = 0.63 \). Consequently, the diagonal dominant condition can not be respected. Moreover, the plant matrix is not SPR, \( \lambda_i(H^H + H) = \{5.41, 0.97, 0.51, -0.67\} \), the system is not dissipative: it is necessary to introduce the diagonal compensator matrix \( C \).
Consequently, it is not possible to conclude on the closed-loop stability without investigation of the positiveness of $\mathbf{CH}$. The matrices $\mathbf{CH}$ associated to the decentralized gradient and newton algorithms are both SPR; consequently, the stability can be diagonal for $\mu \to 0$: the decentralized algorithms are predicted $\mu$-stable.

Figure 8 shows the measured convergence of the error measured by each of the PVDF sensors, for centralized Newton, decentralized Newton and decentralized gradient. $\beta$ is set to 0 since the closed-loop system is $\mu$-stable without control effort weighting. The centralized gradient algorithm converges less rapidly than the decentralized gradient because the condition number of $\mathbf{H}^H \mathbf{H}$ is large in comparison to the condition number of $\text{diag}((\mathbf{H})^H \mathbf{H})$. In other words, the panel resonance leads to a ill-conditioned matrix, which implies that the centralized gradient (cond($\mathbf{CH}$) = 97) is slower than the decentralized gradient (cond($\mathbf{CH}$) = 45). On the other hand, the Newton algorithm is seen to have the largest convergence rates, cond($\mathbf{CH}$) = 6 for the decentralized algorithm and cond($\mathbf{CH}$) = 1 for the centralized algorithm.

Figure 9 shows the transverse vibration field of the panel at 520 Hz with the control off and after convergence of the decentralized gradient algorithm. A significant decrease of the panel response is observed not only at the sensor locations, but also over the whole panel area.

6.6 Experimental results for a case with effort weighting $\beta$

At the frequency of 680 Hz, the measured plant matrix $\mathbf{H}$, is not diagonal dominant (for example, $|H_{11}| = 0.26$ but $|H_{14}| = 0.59$), $\mathbf{H}$ is not SPR, and the matrices $\mathbf{CH}$ corresponding to decentralized gradient and decentralized Newton are not SPR: there exists eigenvalues such that $\Re(\lambda_i(\mathbf{CH})) = -0.05$ for the gradient algorithm, and $\Re(\lambda_i(\mathbf{CH})) = -0.5$ for the Newton algorithm. In order to guarantee that the matrix $\mathbf{B} = \mathbf{CH} + \beta \mathbf{I}$ is SPR, a control effort weighting coefficient must be introduced: a value of $\beta = 0.3$ was chosen for the gradient algorithm and $\beta = 0.7$ for the Newton algorithm. Figure 10 shows the measured convergence of the error measured by each of the PVDF sensors, for various control algorithms. Decentralized gradient
is seen to diverge when $\beta = 0$ and when a relatively small value $\mu = 0.02$ of the adaptation coefficient is chosen. On the other hand, applying an effort weighting $\beta = 0.3$ has the effect of stabilizing the decentralized gradient when $\mu = 0.05$. This is done however at the detriment of a smaller attenuation of the error signals (of the order of 10 dB). On the other hand, the performance of decentralized Newton is marginal in this case, because of the relatively large effort weighting applied.

7 Conclusions

We have analyzed the performance and stability of decentralized active control of panel vibration using multiple pairs combining PZT actuators and PVDF sensors distributed (and not necessarily collocated) on the panel. The stabilization condition of the closed loop by adjusting the convergence coefficient were especially investigated through an adjustable control effort term in the quadratic cost function. Various necessary or sufficient conditions derived from the plant matrix have been obtained to analyze the stability of decentralized gradient and decentralized Newton-Raphson algorithms; these conditions, were summarized in Table I from the most restrictive to the less restrictive condition. Figure 11 illustrates the hierarchy between the three cases investigated in the experiments: at 550 Hz the plant matrix is diagonal dominant, hence the stability is ensured by the low interaction between units; at 520 Hz the plant matrix is not SPR, hence the stability is ensured by the diagonal compensator; at 680 Hz the stability requires a control effort weighting term, consequently perfect rejection can not be reached.

While the diagonal dominance of the plant matrix $H$ provides a sufficient condition for closed loop $\mu$-stability, it is not a necessary condition. Therefore, decentralized vibration control is achievable even in the presence of strong interaction between control units. Also, the collocation of dual actuator-sensor pairs has the very attractive property to theoretically provide a SPR plant matrix $H$, but this is also a sufficient condition of $\mu$-stability. Finally, the necessary condition
of $\mu$-stability (the real part of the eigenvalues of $\mathbf{CH}$ being positive), can be interpreted as a tolerance on the collocation of dual actuator-sensor pairs.
References


machinery noise from resonant substructures. *Journal of Acoustical Society of America*,

[21] L.A. Sievers and A.H. Flotow. Comparison and extensions of control methods for narrow-
band disturbance rejection. *IEEE Transactions on signal processing*, 40(10):2377–2391,


[23] J.Q. Sun. Some observations on physical duality and collocation of structural control sensors

[24] G. West-Vukovich and E.J Davison. The decentralized control of large flexible space
List of Tables

I  The various conditions of stability of the closed loop system 37
II  Panel data 37
III BM-500 PZT actuator and PVDF sensor data 38
IV  Natural frequencies of the panel 38

List of Figures

1  Physical configuration under study: a flexible panel controlled by $N$ independent loops 39
2  Block diagram of the frequency domain controller 39
3  Positions of the 4 PZT-PVDF pairs on the panel 40
4  Magnitude of transfer function matrix coefficients between PZT actuators and PVDF sensors. Off-diagonal coefficients (a): $H_{28}$, (b): $H_{48}$; Diagonal coefficients (c): $H_{66}$, (d): $H_{88}$. Solid line: Experimental; Dashed line: Predicted. 41
5  Stability indicators of decentralized control derived from theoretical and measured plant matrix $H$: condition of $H$ diagonal dominant, condition of $CH$ SPR, minimum control effort weighting $\beta$ 42
6  Maximum value of the adaptation coefficient $\mu_{max}$ for centralized or decentralized gradient and Newton-Raphson adaptation in the case of 4 PZT-PVDF pairs on the panel. 43
7  Experimental results at 550Hz for the case of $H$ SPR and diagonal dominant. No control effort weighting ($\beta = 0$). The levels are normalized by $\|y(0)\|_2$. 44
8  Experimental results at 520Hz for the case of $H$ not SPR and not diagonal dominant, but $CH$ SPR. No control effort weighting ($\beta = 0$). The levels are normalized by $\|y(0)\|_2$. 45
transverse velocity field of the panel measured by Doppler Laser Vibrometry at 520 Hz. Note that the amplitude scale is increased by a factor about 3 on the right-hand side.

Experimental results at 680Hz for the case of CH not SPR and not diagonal dominant. The levels are normalized by $\|y(0)\|_2$.

The hierarchy of stability conditions shows that at 550 Hz the system is diagonal dominant and SPR (in C6 and C7). At 520 Hz H is not SPR and not dd but CH is SPR (in C4 but not in C6 nor C7). At 680 Hz only C3 is satisfied, control effort weighting is necessary to stabilize the closed loop.
Table I: The various conditions of stability of the closed loop system

<table>
<thead>
<tr>
<th>Ref</th>
<th>Condition</th>
<th>Type</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>C0</td>
<td>$</td>
<td>\lambda_i(I - \mu(CH + \beta I)</td>
<td>&lt; 1$</td>
</tr>
<tr>
<td>C1</td>
<td>$\Re(\lambda_i(CH + \beta I)) &gt; 0$</td>
<td>Necessary for C0</td>
<td>the system is stable if $\mu \to 0$</td>
</tr>
<tr>
<td>C2</td>
<td>$\Re(\lambda_i(CH)) &gt; 0$</td>
<td>Sufficient for C1</td>
<td>Effort weighting $\beta$ not necessary</td>
</tr>
<tr>
<td>C3</td>
<td>$CH + \beta I$ SPR</td>
<td>Sufficient for C1</td>
<td>diagonal stability when $\mu \to 0$</td>
</tr>
<tr>
<td>C4</td>
<td>$CH$ SPR</td>
<td>Sufficient for C2</td>
<td>diagonal stability and $\beta$ not necessary</td>
</tr>
<tr>
<td>C5</td>
<td>$CH_{ii}$ SPR</td>
<td>Necessary for C4</td>
<td>Each individual unit is stable</td>
</tr>
<tr>
<td>C6</td>
<td>$H$ SPR</td>
<td>Sufficient for C4</td>
<td>The system is dissipative</td>
</tr>
<tr>
<td>C7</td>
<td>$H$ dd</td>
<td>Sufficient for C4</td>
<td>Low interaction between units</td>
</tr>
</tbody>
</table>

Table II: Panel data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness</td>
<td>$h = 3.18 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>Length</td>
<td>$L_x = 48 \times 10^{-2}$ m</td>
</tr>
<tr>
<td>Width</td>
<td>$L_y = 42 \times 10^{-2}$ m</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E_p = 68.5 \times 10^9$ Pa</td>
</tr>
<tr>
<td>Poisson’s coefficient</td>
<td>$\nu_p = 0.33$</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho = 2700$ kg.m$^{-3}$</td>
</tr>
<tr>
<td>Loss factor</td>
<td>$\eta = 5.0 \times 10^{-3}$</td>
</tr>
</tbody>
</table>
Table III: BM-500 PZT actuator and PVDF sensor data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>PZT actuator</th>
<th>PVDF sensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness</td>
<td>$t_a = 1.02 \times 10^{-3}$ m</td>
<td>$t_s = 28 \times 10^{-6}$ m</td>
</tr>
<tr>
<td>Length</td>
<td>$L_x^{(a)} = 2.54 \times 10^{-2}$ m</td>
<td>$L_x^{(s)} = 1 \times 10^{-2}$ m</td>
</tr>
<tr>
<td>Width</td>
<td>$L_y^{(a)} = 2.54 \times 10^{-2}$ m</td>
<td>$L_y^{(s)} = 1.5 \times 10^{-2}$ m</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E_a = 61.1 \times 10^9$ Pa</td>
<td></td>
</tr>
<tr>
<td>Poisson’s coefficient</td>
<td>$\nu_a = 0.29$</td>
<td></td>
</tr>
<tr>
<td>Piezoelectric coefficient</td>
<td>$d_{31} = -1.9 \times 10^{-10}$ C/N</td>
<td></td>
</tr>
<tr>
<td>Piezoelectric coefficient</td>
<td>$e_{31} = 46 \times 10^{-3}$ C.m$^{-2}$</td>
<td></td>
</tr>
<tr>
<td>Capacitance</td>
<td>$C_s = 5 \times 10^{-10}$ F</td>
<td></td>
</tr>
</tbody>
</table>

Table IV: Natural frequencies of the panel

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>1</th>
<th>3</th>
<th>2</th>
<th>4</th>
<th>3</th>
<th>4</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$f_{mn}$, calculated (Hz)</td>
<td>77</td>
<td>177</td>
<td>208</td>
<td>308</td>
<td>344</td>
<td>425</td>
<td>474</td>
<td>525</td>
<td>577</td>
<td>692</td>
<td>708</td>
<td>730</td>
</tr>
<tr>
<td>$f_{mn}$, measured (Hz)</td>
<td>80</td>
<td>176</td>
<td>206</td>
<td>292</td>
<td>337</td>
<td>425</td>
<td>455</td>
<td>517</td>
<td>571</td>
<td>676</td>
<td>705</td>
<td>719</td>
</tr>
</tbody>
</table>
Figure 1: Physical configuration under study: a flexible panel controlled by $N$ independent loops

Figure 2: Block diagram of the frequency domain controller
Figure 3: Positions of the 4 PZT-PVDF pairs on the panel
Figure 4: Magnitude of transfer function matrix coefficients between PZT actuators and PVDF sensors. Off-diagonal coefficients (a): $H_{28}$, (b): $H_{48}$; Diagonal coefficients (c): $H_{66}$, (d): $H_{88}$. Solid line: Experimental; Dashed line: Predicted.
Figure 5: Stability indicators of decentralized control derived from theoretical and measured plant matrix $H$: condition of $H$ diagonal dominant, condition of $CH$ SPR, minimum control effort weighting $\beta$
Figure 6: Maximum value of the adaptation coefficient $\mu_{max}$ for centralized or decentralized gradient and Newton-Raphson adaptation in the case of 4 PZT-PVDF pairs on the panel.
Figure 7: Experimental results at 550Hz for the case of H SPR and diagonal dominant. No control effort weighting ($\beta = 0$). The levels are normalized by $||y(0)||_2$. 
Figure 8: Experimental results at 520Hz for the case of $H$ not SPR and not diagonal dominant, but $CH$ SPR. No control effort weighting ($\beta = 0$). The levels are normalized by $\|y(0)\|_2$. 
Figure 9: transverse velocity field of the panel measured by Doppler Laser Vibrometry at 520 Hz. Note that the amplitude scale is increased by a factor about 3 on the right-hand side.
Figure 10: Experimental results at 680Hz for the case of CH not SPR and not diagonal dominant. The levels are normalized by $\|y(0)\|_2$. 
Figure 11: The hierarchy of stability conditions shows that at 550 Hz the system is diagonal dominant and SPR (in C6 and C7). At 520 Hz $\textbf{H}$ is not SPR and not dd but $\textbf{CH}$ is SPR (in C4 but not in C6 nor C7). At 680 Hz only C3 is satisfied, control effort weighting is necessary to stabilize the closed loop.