Brief note on the variation of constants formula for fuzzy differential equations

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Abstract

This note gives a theory of state transition matrices for linear systems of fuzzy differential equations. This is used to give a fuzzy version of the classical variation of constants formula. A simple example of a time-independent control system is used to illustrate the methods. While similar problems to the crisp case arise for time-dependent systems, in time-independent cases the calculations are elementary solutions of eigenvalue–eigenvector problems. In particular, for nonnegative or nonpositive matrices, the problems at each level set, can easily be solved in MATLAB to give the level sets of the fuzzy solution. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recall the classical variation of constants formula for a nonhomogeneous n-dimensional system of first order differential equations (DEs)

\[ x'(t) = A(t)x(t) + f(t), \quad x(0) = x_0, \]

for example, the control system \( x' = A(t)x + B(t)u(t) \). The solution may be written as

\[ x(t) = \Phi(t)x_0 + \int_0^t \Phi(t)\Phi(s)^{-1}f(s)\,ds, \tag{1} \]

where \( \Phi(t,s) = \Phi(t)\Phi(s)^{-1} \) is the state transition matrix and \( \Phi(t) \) satisfies the matrix DE

\[ \Phi'(t) = A(t)\Phi(t), \quad \Phi(0) = I. \tag{2} \]

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In the special case when \( A \) is time-independent, \( \Phi(t) = e^{At} \) involves the matrix exponential and \( \Phi(t, s) = \Phi(t - s) = e^{(t - s)A} \).

In the traditional theory of fuzzy DEs (FDEs) using the Hukuhara derivative [11,15], significant problems arise with any attempted development of formulas such as (1), largely because the solutions of FDEs have quite different properties from those of crisp DEs, lacking observed properties of physical systems such as stability, periodicity and bifurcation.

A simple one-dimensional example serves to illustrate the problem. Let \((c; d)_{S}\) denote the symmetric triangular fuzzy number with the interval \([c, d]\) as its support and let \(x(t)\) be a fuzzy number valued function of time. Consider the FDE initial value problem

\[
x'(t) = -2x, \quad x(0) = X^0 = (0; 1)_{S}.
\]

Write the \( \beta \)-level set of \( x(t) \) as the compact interval \( x_\beta(t) = [x^R_\beta(t), x^L_\beta(t)] \) and note that \( -2x_\beta = [-2x^R_\beta, -2x^L_\beta] \), while \( x_\beta(0) = [\beta/2, 1 - \beta/2] \). Writing \( \xi_\beta(t) \) as the vector with components \( x^L_\beta(t), x^R_\beta(t) \), obtain the ordinary initial value problem

\[
\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \xi_\beta(t) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad \xi_\beta(0) = \begin{pmatrix} \beta/2 \\ 1 - \beta/2 \end{pmatrix}
\]

for \( 0 \leq \beta \leq 1 \). It is easy to see that

\[
\xi_\beta(t) = \frac{1}{2} e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\beta - 1}{2} e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

that is,

\[
x_\beta(t) = [e^{-2t}/2 + (\beta - 1)e^{2t}/2, e^{-2t}/2 + (1 - \beta)e^{2t}/2].
\]

Hence, the solution to (3) is \( x(t) = e^{-2t}/2 + (-e^{2t}/2; e^{2t}/2)_S \). Thus, (3) has an unstable solution, in stark contrast to the behaviour of the associated crisp problem \( z'(t) = -2z(t), \quad z(0) = \frac{1}{2} \), which has solution \( z(t) = e^{-2t}/2 \). So giving the initial condition some uncertainty by fuzzification has totally changed the qualitative behaviour of the solution. Indeed, an arbitrarily small fuzzification \( x(0) = (\frac{1}{2} - \varepsilon; \frac{1}{2} + \varepsilon \)_S \) has the same effect, the solution being \( x(t) = e^{-2t}/2 + (-e^{2t}; e^{2t})_S \), although, as \( \varepsilon \to 0+ \), the crisp solution is the limit.

Hüllermeier [10] largely overcame this undesirable property by interpreting an FDE \( x'(t) = G(t, x(t)) \) as a family of differential inclusions

\[
x'_\beta(t) \in G_\beta(t, x_\beta(t)), \quad 0 \leq \beta \leq 1
\]

and the approach has been further exploited in [6,7,9]. Note that [16] studies a not dissimilar methodology. A solution to the \( \beta \)th inclusion is a function \( x_\beta(\cdot) \) which is absolutely continuous and satisfies the \( \beta \)th inclusion almost everywhere (a.e.) [1,5]. Under fairly natural conditions, the solution sets and attainability sets of the family are fuzzy sets and, moreover, have properties that are similar to solutions of ordinary differential equations. Consequently, this interpretation is much better adapted to modelling under uncertainty than the other formalism. In particular, if the FDE (3) is replaced by the family

\[
x'_\beta \in [-2x^R_\beta, -2x^L_\beta], \quad x_\beta(0) \in X^0 = [\beta/2, 1 - \beta/2], \quad 0 \leq \beta \leq 1,
\]

then it has a fuzzy solution set \( \Sigma(X^0, \tau) \) and a fuzzy attainability set \( \mathcal{A}(X^0, t) \) respectively, defined by the \( \beta \)-level sets

\[
\begin{align*}
\Sigma_\beta(X^0, \tau) &= \{ x(\cdot): x(t) \in [\beta e^{-2t}, (1 - \beta/2)e^{-2t}], \quad 0 \leq t \leq \tau \}, \\
\mathcal{A}_\beta(X^0, t) &= [\beta e^{-2t}, (1 - \beta/2)e^{-2t}].
\end{align*}
\]
This matches the sort of desirable behaviour a fuzzification of the crisp DE should have, namely it is asymptotically stable and approaches the crisp limit as the uncertainty becomes negligible.

This note obtains fuzzy state transition matrices for systems of FDEs by interpreting the matrix FDE (2) as a family of inclusions. The variation of constants formula (1) follows as an easy consequence. Although the theorems are enunciated in terms of a general continuous fuzzy matrix valued function \( A(\cdot) \), for conciseness and clarity the proofs are done for the time-independent case. The details for time-dependence are very much the same, both cases being essentially already known in general [6,7,9]. A two-dimensional example is given. Very many calculations can be performed in MATLAB, using only the functions \( \text{EIG} \) and \( \text{INV} \).

2. State transition matrices

Let \( \mathcal{E}^n \) be the space of all normal, uppersemicontinuous and fuzzy convex sets, with compact support, on \( \mathbb{R}^n \), endowed with the sup metric on level sets \( d_\infty \) (see [8] for more detail). Denote by \( \mathcal{M}^n \) the space of \( n \times n \) matrices with entries in \( \mathcal{E}^1 \), that is whose entries are fuzzy numbers. If \( A \in \mathcal{M}^n \), it also lies in \( \mathcal{E}^{n \times n} \).

Consequently, \( A:[0,T] \to \mathcal{M}^n \) is said to be continuous if it is continuous in \( e, \delta \) sense in the \( |\cdot| \) and \( d_\infty \) metrics.

Following [10,6,7,9], interpret (2) as the family of differential inclusions

\[
\phi^\beta(t) \in A(t)\phi^\beta(t), \quad \phi^\beta(0) = I, \quad 0 \leq \beta \leq 1,
\]

where \( A(t) \) denotes the level set of \( A(t) \). That is, if \( U(\cdot) \) is a measurable matrix valued selection [3] in the set of matrices \( A(\cdot) \), \( \phi^\beta(t) = U(t)\phi^\beta(t) \) a.e. and \( \phi^\beta(0) = I \). Such solutions are guaranteed by Carathéodory’s Theorem, over an interval \([0,T], 0 < T \leq \infty\), provided the endpoint matrix valued functions \( A(\cdot), A' \) are majorized by an integrable function on \([0,T] \) [4, pp. 42–48]. Denote

\[
\Phi^\beta(t) = \{ Y(t): Y' = U(t)Y, \; Y(0) = I, \; U(\cdot) \in A(\cdot) \}. 
\]

Clearly, \( \Phi^\beta(t) \) is nonempty. By the results of [9], the \( \Phi^\beta(\cdot) \) are the level sets of a \( \mathcal{E}^{n \times n} \)-valued fuzzy function \( \Phi(\cdot) \).

In particular, if \( A \) is a constant fuzzy matrix and the FDE is time-independent, define

\[
\Phi(t) = \{ Y(t): Y' = U(t)Y, \; Y(0) = I \}
\]

for measurable matrix valued selections \( U(\cdot) \in A(\cdot) \). Then, the \( Y(t) \) are found in the usual way: find a basis of vector solutions \( u_1(t), u_2(t), \ldots, u_n(t) \) for the differential equation \( y' = U(t)y \) and form the matrix \( Z(t) = [u_1(t) \; u_2(t) \cdots \; u_n(t)] \). Put \( Y(t) = Z(t)Z(0)^{-1} \).

The \( \Phi^\beta(t) \) now form the level sets of a \( \mathcal{E}^{n \times n} \)-valued function, since an \( n \times n \) interval matrix is a convex set in \( \mathbb{R}^{n \times n} \). This has outlined a proof of the following result.

**Theorem 1.** Let \( A:[0,T] \to \mathcal{E}^{n \times n} \) be a continuous fuzzy matrix valued function. Then (5) has a solution set \( \Phi(t) \) which is a \( \mathcal{E}^{n \times n} \) valued fuzzy matrix function. If \( A \) is constant, \( \Phi(t) \) is an \( \mathcal{E}^{n \times n} \) valued functions whose level sets are interval matrices.

**Remarks.** (1) Note that \( \Phi(t) \) is invertible in the sense that each trajectory \( \phi^\beta_\beta \) of the \( \beta \)-th inclusion (5) is invertible. Indeed, it is well known that \( \psi^\beta_\beta = \phi^\beta_\beta^{-1} \) satisfies the matrix DE \( \psi^\beta_\beta = -\psi^\beta_\beta U(t) \) a.e., \( \psi^\beta_\beta(0) = I \).

(2) If the initial time is \( t_0 \), the notation \( \Phi(t,t_0) \) should be used, \( \Phi(t_0,t_0) = I \). Here we have abbreviated \( \Phi(t):= \Phi(t,0) \). Note that \( \Phi^{-1}(t,t_0) = \Phi(t_0,t) \). For constant \( A \), \( \Phi^{-1}(t) = \Phi(-t) \), sharing this property with the classical matrix exponential solution.

(3) Carathéodory’s Theorem is existence only and solutions may not be unique. However, there exist maximum and minimum solutions [4, Theorem 1.2].
Since $A$ is an interval matrix, that is, has compact real intervals as entries, $U$ belongs to the interval $[A, \tilde{A}]$. Here, $A$, $\tilde{A}$ denote ordinary matrices whose elements are, respectively, the lower and upper endpoints of the real intervals (see [14] for notation and theory). This gives a method for evaluating $\Phi_t(\cdot)$ as an interval matrix.

In the case where $A$ is a nonnegative matrix, that is all elements of the matrix are nonnegative, or $\tilde{A}$ is a nonpositive matrix, the computation is especially simple. If $B - A$ is a nonnegative matrix, write $B \succeq A$.

**Lemma.** Let $A$ be a nonnegative matrix and suppose that $B \succeq A$. If
\[
X'(t) = AX, \quad X(0) = I, \\
Y'(t) = BY, \quad Y(0) = I,
\]
then $Y(t) \succeq X(t)$, $t \geq 0$. In particular, if $A = [A, \tilde{A}]$ is an interval matrix with $0 \leq A \leq U \leq \tilde{A}$ and
\[
X'(t) = AX(t), \quad X'(t) = UX(t), \quad \tilde{X}'(t) = \tilde{A}X(t), \\
X(0) = X(0) = \tilde{X}(0) = I, \quad \text{then } X(t) \leq X(t) \leq \tilde{X}(t), \quad t \geq 0.
\]

**Proof.** This is a simple consequence of Theorem 5.1.1, pp. 315, 316 of [12], since the matrix differential equations are equivalent to
\[
X(t) = I + \int_0^t AX(s) \, ds,
\]
\[
Y(t) = I + \int_0^t BY(s) \, ds \geq I + \int_0^t AY(s) \, ds
\]
and the function $X \mapsto AX$ is monotonic nondecreasing in the partial order induced by the positive orthant. \qed

Clearly, a similar result holds for nonpositive matrices. Indeed, it could be generalized to $K$-monotone matrices [2], where $K$ is a cone, but for simplicity this is omitted. In the case where the interval matrix is not of these types, the interval matrix function $AX(t)$ will have extreme points corresponding to matrices internal to the interval matrix $A$. This can be estimated numerically by solving the matrix DEs on a grid, but obviously significantly increases the computation involved.

**Example 1.** As a numerical illustration, consider the system where $n = 2$ and $A$ is given by
\[
A = \begin{pmatrix}
1 & 2 \\
0.5 & 1.5
\end{pmatrix}
\]
Expressing the $\beta$-levels of $A$ as an interval matrix,
\[
A_\beta = \begin{pmatrix}
[1 + 0.5\beta, 2 - 0.5\beta] & [0.5 + 0.5\beta, 1.5 - 0.5\beta] \\
[0.5 + 0.5\beta, 1.5 - 0.5\beta] & [2 + 0.5\beta, 3 - 0.5\beta]
\end{pmatrix},
\]
this interval system matrix is a positive family, because every matrix contained in the interval is a positive matrix and, applying the lemma, only the endpoint equations need be solved. So, for example, if $\beta = 0.0$, $\Phi_{0.0}(t) = [\Phi_{0.0}(t), \tilde{\Phi}_{0.0}(t)]$. Using the MATLAB function EIG for the eigenvalue–eigenvector calculations (INV is not needed because $A$, $A$ are here symmetric),
\[
\Phi_{0.0}(t) = \begin{pmatrix}
0.8536e^{2t} + 0.1465e^{2t} & -0.3536e^{2t} + 0.3536e^{2t} \\
-0.3536e^{2t} + 0.3536e^{2t} & 0.1465e^{2t} + 0.8536e^{2t}
\end{pmatrix},
\]
(6)
where \( \kappa = 0.7929 \), \( \varphi = 2.2071 \), and

\[
\Phi_{[0,0]}(t) = \begin{pmatrix}
0.6580 e^{\varphi t} + 0.3419 e^{\kappa t} & -0.4743 e^{\varphi t} + 0.4743 e^{\kappa t} \\
-0.4743 e^{\varphi t} + 0.4743 e^{\kappa t} & 0.3419 e^{\varphi t} + 0.6580 e^{\kappa t}
\end{pmatrix},
\]

where \( \kappa = 0.9189 \), \( \varphi = 4.0811 \).

### 3. Variation of constants formula

In practice, the crisp fundamental matrix \( \Phi(t_0) \) is difficult to find explicitly if the system matrix is time-dependent. Apart from theoretical discussions, the variation of constants formula is usually only useful when \( A \) is constant. For brevity, the discussion of this section will be for constant \( A \), but the general case is not difficult and the more general proof obvious.

Let \( A \in \mathbb{E}^{n \times n} \) be a matrix with fuzzy number entries, \( X^0 \in \mathbb{E}^n \) and let \( F: [0, T] \rightarrow \mathbb{E}^n \) be a continuous function (and hence strongly integrably bounded, with integrable selections). Consider the fuzzy initial value problem

\[
x'(t) = Ax + F(t), \quad x(0) = X^0,
\]

that is, for \( 0 \leq \beta \leq 1 \), the family of inclusions

\[
x^\beta_\beta(t) \in A_\beta x_\beta(t) + F_\beta(t), \quad x_\beta(0) \in X^0_\beta.
\]

Here, suffixes \( \beta \) indicate level sets. Let \( \Phi(t) \) indicate the solution set of the time-independent matrix FDE \( \Phi' = A \Phi, \quad \Phi(0) = I \). From the inclusion (8), there is a measurable matrix valued function \( U_\beta(\cdot) \in A_\beta, \) an integrable selection \( f_\beta(\cdot) \in F_\beta(\cdot) \) and \( y_0 \in X^0_\beta \) such that

\[
x^\beta_\beta(t) = U(t)x_\beta(t) + f_\beta(t) \text{ a.e.,} \quad x_\beta(0) = y_0.
\]

Denote by \( X_U(\cdot) \) a solution to the equation \( X_U' = U(t)X_U, \quad X_U(0) = I \).

From the classical formula (1),

\[
x_\beta(t) = X_U(t)y_0 + \int_0^t X_U(t-s)f_\beta(s) ds.
\]

Consequently, the solution set of (8) is

\[
\Sigma_\beta(t) = \bigcup \left\{ X_U(t)y_0 + \int_0^t X_U(t-s)f_\beta(s) ds : U \in A, f_\beta \in F_\beta, y_0 \in X^0_\beta \right\},
\]

\[
\Sigma_\beta(t) = \Phi_\beta(t)X^0_\beta + \int_0^t \Phi_\beta(t-s)F_\beta(s) ds,
\]

by Theorem 1 and the definition of the Aumann integral, where the union has been taken over all maximal and minimal Carathéodory solutions, so as to give (10). Since for \( 0 \leq t \leq T \), \( \Phi_\beta(t), \) \( X^0_\beta \) and \( F_\beta(t) \) are all compact, so also is \( \Sigma_\beta(t) \). Moreover, clearly \( \Sigma_\beta(t) \subseteq \Sigma_\beta' \) for \( 0 \leq \beta' \leq \beta \leq 1 \).

**Theorem 2.** Let \( A \in \mathbb{E}^{n \times n} \) be a matrix with fuzzy number entries, \( X^0 \in \mathbb{E}^n \) and let \( F: [0, T] \rightarrow \mathbb{E}^n \) be a continuous function. The fuzzy initial value problem

\[
x'(t) = Ax + F(t), \quad x(0) = X^0,
\]
has an absolutely continuous $\delta^n$ valued solution $\Sigma(t)$, a.e. $t \in [0, T]$, given by the variation of constants formula

$$\Sigma(t) = \Phi(t)X^0 + \int_0^t \Phi(t - s)F(s) \, ds.$$ 

Here, $\Phi(t)$ is the fuzzy matrix valued solution of

$$\Phi'(t) = A\Phi(t), \quad \Phi(0) = I.$$

Proof. From the comments above, to prove that the family $\Sigma_\beta(t)$ are the level sets of a fuzzy set $\Sigma(t)$ over $C([0, T], \mathbb{R}^n)$, it only remains to show uppersemicontinuity (usc). That is, if $\beta_n \to \beta$ is a nondecreasing sequence, then $\Sigma_{\beta_n}(t) \to \Sigma_\beta(t)$. To see this, note that $\Sigma_{\beta_n}(t) = \cap_n \Sigma_{\beta_n}(t)$ is nonempty, since the intersection is of compact, nonempty nested sets. Clearly, $\Sigma_{\beta_n}(t) \subseteq \Sigma_{\beta_n}(t)$. Thus, $X^0_n \to X^0_\beta$ and $\Phi_{\beta_n}(t) \to \Phi_\beta(t)$ by usc. Let $x_n \in \Sigma_{\beta_n}$. Then there are $f_{\beta_n} \in F_{\beta_n}$, $U_n \in A_{\beta_n}$ and $y_n \in X^0_{\beta_n}$ satisfying (9). The sequence $\{y_n\}$ is bounded in $\mathbb{R}^n$ since $X^0$ has compact support and, similarly, $U_n$ is bounded in $\mathbb{R}^{n \times n}$. Each $f_{\beta_n} \in L^\infty([0, T], \mathbb{R}^n)$ and, since $F$ is bounded, there is a constant $C$, such that $\|f_{\beta_n}(t)/C\| \leq 1$. Thus, by Alaoglu’s theorem, $\{f_{\beta_n}(t)/C\}$ is weakly* compact and there is a subsequence of $\{f_{\beta_n}\}$ weakly* converging to $h \in L^\infty([0, T], \mathbb{R}^n)$. Clearly, the mapping

$$T_x : L^\infty([0, T], \mathbb{R}^n) \to C([0, T], \mathbb{R}^n) : g(\cdot) \mapsto \int_0^t \Phi_x(t - s)g(s) \, ds$$

is a compact operator, so the sequence $\{T_{f_{\beta_n}}f_{\beta_n}\}$ is compact. Hence, there is a subsequence $\beta_{n(k)}$ such that $T_{f_{\beta_n(k)}}f_{\beta_n(k)} \to T_\beta h$, where $h(\cdot) \in F_\beta(\cdot)$. Along with the other convergences, clearly $x_{\beta_{n(k)}} \to \xi$ and $\xi \in \Sigma_\beta$ from (9). That is, $\Sigma_{\beta_n}(t) \subseteq \Sigma_{\beta_n}(t)$. By the Negoita–Ralescu characterization [13], the $\Sigma_\beta(t)$ are thus the level sets of a fuzzy set $\Sigma(t)$. $\square$

Example 1 (Ctd). For $0 \leq t \leq T < \infty$, let the nonhomogeneous term be given by

$$F(t) = \begin{pmatrix} (1 + t; 2 + 3t)s \\ (e^{-t}; e^t)s \end{pmatrix}$$

and consider the FDE

$$x'(t) = Ax(t) + F(t), \quad x(0) = X^0 = \begin{pmatrix} (1; 2)s \\ (0; 1)s \end{pmatrix},$$

where $A$ is as in the first part of the example. Note that the level set transition matrices $\Phi_\beta, \bar{\Phi}_\beta$ are positive matrices for $t > 0$ and when multiplying interval vectors the order of endpoints is preserved. Writing $\Sigma_{\beta}(t) = [\Sigma_{\beta_l}(t), \Sigma_{\beta_u}(t)]$, from Theorem 2

$$\Sigma_{\beta_l}(t) = \Phi_{\beta_l}(t)X^0_{\beta_l} + \int_0^t \Phi_{\beta_l}(t - s)F_{\beta_l}(s) \, ds,$$

$$\Sigma_{\beta_u}(t) = \bar{\Phi}_{\beta_u}(t)X^0_{\beta_u} + \int_0^t \bar{\Phi}_{\beta_u}(t - s)\bar{F}_{\beta_u}(s) \, ds.$$
For $\beta = 0.0$, using the results in the first part of the example, a tedious calculation produces

$$
\Sigma_{0.0}(t) = \begin{pmatrix}
-2.5308 - 1.1430t + 0.0826e^{-t} + 3.1007e^{2t} + 0.3476e^{3t} \\
0.2040 + 0.3734t + 0.3479e^{-t} - 0.5517e^{2t} - 0.0002e^{3t}
\end{pmatrix},
$$

$$
\tilde{\Sigma}_{0.0}(t) = \begin{pmatrix}
-3.9942 - 2.3797t - 6.0022e^{t} + 10.4599e^{2t} + 1.5363e^{3t} \\
2.3996 + 1.1998t + 4.0022e^{t} - 7.5399e^{2t} + 2.1380e^{3t}
\end{pmatrix},
$$

where $\kappa = 0.7929$, $\nu = 2.2071$, $\tilde{\kappa} = 0.9189$, $\tilde{\nu} = 4.0811$. The same can be done at any $\beta$-level and, using interpolation, an estimation of $\Sigma(t)$ obtained. Closed form solutions for the $\beta$-levels can be obtained, but are quite complicated to write out.

4. Summary

Meaning was given to a fuzzy analogue of the classical crisp variation of constants formula. First, fuzzy matrix transfer functions were defined in terms of solutions sets of families of matrix differential inclusions. Then, nonhomogeneous DEs were similarly interpreted and the classical result used to obtain fuzzy solution sets satisfying a fuzzy variation of constants formula. An example was given.

References