Uniform Distribution, Discrepancy, and Reproducing Kernel Hilbert Spaces

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In this paper we define a notion of uniform distribution and discrepancy of sequences in an abstract set $E$ through reproducing kernel Hilbert spaces of functions on $E$. In the case of the finite-dimensional unit cube these discrepancies are very closely related to the worst case error obtained for numerical integration of functions in a reproducing kernel Hilbert space. In the compact case we show that the discrepancy tends to zero if and only if the sequence is uniformly distributed in our sense. Next we prove an existence theorem for such uniformly distributed sequences and investigate the relation to the classical notion of uniform distribution. Some examples conclude this paper.

Key Words: abstract uniform distribution; discrepancy; numerical integration; reproducing kernel Hilbert spaces.

1. INTRODUCTION

Numerical integration of functions in a reproducing kernel Hilbert space has been investigated in recent times by several authors (see for example [5, 7, 8, 13] and the references therein).

If $E = [0, 1)^s$ and $H$ is a reproducing kernel Hilbert space of functions defined on $E$ with reproducing kernel $K$ and norm $\| \cdot \|_H$, it can be shown that

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_E f(x) \, dx \right| \leq D_N(P, K) \| f \|_H$$
for all \( f \in H \), where the quantity
\[
D_N(P, K) = \left\| \frac{1}{N} \sum_{n=1}^{N} K(., x_n) - \int_E K(., x) \, dx \right\|_H
\]
\[
= \sup_{f \in H, \|f\|_H \leq 1} \left\| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_E f(x) \, dx \right\|
\]
is called the discrepancy of the point set \( P = \{x_1, \ldots, x_N\} \), which is the worst case error of this quadrature method (see [5, 7, 8, 13]).

In many cases (see the examples at the end of this paper), this notion of discrepancy \( D_N(P, K) \) coincides with a classical \( L^2 \)-discrepancy or diaphony. From the theory of uniform distribution it is known that a sequence is uniformly distributed modulo 1 if and only if the \( L^2 \)-discrepancy, or the diaphony, tends to zero as \( N \) tends to infinity.

Since in general the discrepancy \( D_N(P, K) \) strongly depends on the reproducing kernel \( K \) and therefore on the reproducing kernel Hilbert space \( H \), we can ask under which assumptions on \( H \) do we have the result that
\[
\lim_{N \to \infty} D_N(P, K) = 0
\]
if and only if the sequence \( (x_n) \) is uniformly distributed with respect to a probability measure.

In order to study this problem, we use a more abstract approach and define a notion of uniform distribution, discrepancy, and diaphony via reproducing kernel Hilbert spaces.

Let \( H \) be an arbitrary reproducing kernel Hilbert space of complex valued functions defined on an abstract set \( E \) with inner product \( \langle ., . \rangle \).

For a fixed \( g \in H \) and a sequence \( (x_n) \) in \( E \) we consider for every \( N \in \mathbb{N} \) the elements
\[
r_n = \frac{1}{N} \sum_{n=1}^{N} K(., x_n) - g
\]
of \( H \). We call the sequence \( (x_n) \) \( g \)-uniformly distributed if and only if \( r_N \) converges weakly to zero as \( N \) tends to infinity (see Definition 1 and Theorem 3 in Section 3).

If \( E \) is a compact metric space and if all elements of \( H \) are continuous, we show in Theorem 7 that \( g \)-uniform distribution coincides with uniform distribution with respect to a probability measure if and only if the linear space generated by \( H \) and the constant function 1 is dense in the Banach
space \( C(E) \). In this case, the element \( g \in H \) can be interpreted as a probability measure on \( E \) and the elements \( K(x, x_n) \in H \) as the point measures \( \delta_{x_n} \).

Nevertheless, we develop our theory without the use of measure theory.

In the theory of uniform distribution in compact spaces there are several possibilities for defining a notion of discrepancy for sequences, which are uniformly distributed with respect to a probability measure (see for example [4] and the references therein).

In our approach, we call \( \|r_N\|_H \), the Hilbert space norm of the element \( r_N \), the \( g \)-diaphony and \( \|r_N\|_\infty = \sup_{x \in E} |r_N(x)| \) the \( g \)-discrepancy of the sequence \((x_n)\) in \( E \) (see Definition 2 in Section 3).

If \( E \) is a compact metric space, we prove in Theorem 3 that a sequence \((x_n)\) in \( E \) is \( g \)-uniformly distributed if and only if \( \lim_{N \to \infty} \|r_N\|_H = 0 \) or equivalently \( \lim_{N \to \infty} \|r_N\|_\infty = 0 \).

We can also generalize the inequality for numerical integration given at the beginning of this section to arbitrary reproducing kernel Hilbert spaces. If we consider instead of the linear functional \( f \mapsto \int_E f(x) \, dx \) an arbitrary bounded linear functional on \( H \), which can always be written in the form \( \langle \cdot , g \rangle \) with \( g \in H \), we obtain in Section 3 the analogous inequality

\[
\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \langle f, g \rangle \right| \leq \|r_N\|_H \|f\|_H
\]

for all \( f \in H \).

Concerning the existence of \( g \)-uniformly distributed sequences, we prove in Theorem 5 that there exists a \( g \)-uniformly distributed sequence if and only if \( g \) is in the closure of the convex hull of the set \( \{ K(x, \cdot) : x \in E \} \). In this theorem we do not assume that the underlying space \( E \) is compact. Combining Theorem 5 and Theorem 7, we obtain as a corollary the existence of uniformly distributed sequences with respect to a probability measure in a compact metric space.

The paper concludes with some examples, where we also show that classical definitions of uniform distribution and discrepancy (such as the classical diaphony, the weighted spectral test or a \( L^2 \)-discrepancy in compact groups) can be obtained as special cases of \( g \)-uniform distribution and \( g \)-diaphony.

2. NOTATION

In this section we summarize some basic facts about reproducing kernel Hilbert spaces. For a more thorough discussion see [3].
Let $E$ be a nonempty set and $H$ a Hilbert space of functions $f: E \to \mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_H$.

A function

$$K: E \times E \to \mathbb{C}$$

is called a reproducing kernel of $H$, if

- for every $y \in E$ the function $K_y: E \to \mathbb{C}$,
  $$K_y(x) = K(x, y),$$
  is an element of $H$,

- for every $f \in H$ and for every $y \in E$,
  $$f(y) = \langle f, K_y \rangle.$$

If there exists a reproducing kernel of $H$, then $H$ is called a reproducing kernel Hilbert space (hereafter, r.k. Hilbert space).

The reproducing kernel $K$ of a r.k. Hilbert space is a positive definite mapping,

$$\sum_{i=1}^{N} \alpha_i \alpha_j K(x_j, x_i) \geq 0,$$

for all $N \geq 1$; $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$; $x_1, \ldots, x_N \in E$.

Conversely, any positive definite function $K: E \times E \to \mathbb{C}$ uniquely defines a reproducing kernel Hilbert space with reproducing kernel $K$.

Another property of r.k. Hilbert spaces, which is important for numerical integration, is the following.

A Hilbert space of functions on a set $E$ has a reproducing kernel if and only if for all $y \in E$ the linear functionals $l_y: H \to \mathbb{C}$,

$$l_y(f) = f(y),$$

are continuous.

Throughout this paper we use the following notation: $\text{conv } A$ denotes the convex hull of the set $A$ and $\text{lin } A$ denotes the linear space generated by the set $A$. Furthermore, if $A$ is a subset of a Banach space, then $\overline{A}$ denotes the norm closure of $A$ and $\overline{A}^*$ the weak closure of $A$. 
3. THE RESULTS

Definition 1. Let \( H \) be a r.k. Hilbert space of complex valued functions on a set \( E \) with reproducing kernel \( K \). Furthermore, let \( g \) be a fixed element of \( H \).

A sequence \( (x_n)_{n \geq 1} \) of points in \( E \) is said to be \( g \)-uniformly distribution if for every \( f \in H \),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \langle f, g \rangle.
\]

Definition 2. For every \( N \geq 1 \) let

\[
r_N := \frac{1}{N} \sum_{n=1}^{N} K_{x_n} - g.
\]

The number \( \|r_N\|_H \) is called the \( g \)-diaphony of \( (x_n)_{n \geq 1} \) and \( \|r_N\|_\infty = \sup_{x \in E} |r_N(x)| \) the \( g \)-discrepancy of \( (x_n)_{n \geq 1} \).

Remark. If the function \( x \mapsto K(x, x) \) is bounded, then

\[
\|r_N\|_\infty \leq c \|r_N\|_H
\]

with some \( c > 0 \).

Proof. From the reproducing property of the kernel \( K \) it follows immediately that

\[
|r_N(x)| = |\langle r_N, K_x \rangle| \leq \|r_N\|_H \sqrt{K(x, x)} \leq c \|r_N\|_H,
\]

where \( c = \sup_{x \in E} \sqrt{K(x, x)} \).

Remark. For all \( f \in H \) we have

\[
\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \langle f, g \rangle \right| \leq \|f\|_H \|r_N\|_H.
\]

Proof.

\[
\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \langle f, g \rangle \right| = \left| \frac{1}{N} \sum_{n=1}^{N} \langle f, K_{x_n} \rangle - \langle f, g \rangle \right|
\]

\[
= |\langle f, r_N \rangle| \leq \|f\|_H \|r_N\|_H
\]

for all \( f \in H \).
Theorem 3. Let \( E \) be a compact metric space, \( H \) a r.k. Hilbert space of functions on \( E \), \( g \in H \), and \((x_n)_{n \geq 1}\) a sequence in \( E \). Furthermore, we assume that for all \( x \in E \) the functions \( x \mapsto K_x \) from \( E \to H \) are continuous. Then the following three conditions are equivalent:

(i) \((x_n)_{n \geq 1}\) is \( g \)-uniformly distributed

(ii) \( \lim_{N \to \infty} \| r_N \|_H = 0 \)

(iii) \( \lim_{N \to \infty} \| r_N \|_\infty = 0 \).

Proof. Let \((x_n)_{n \geq 1}\) be \( g \)-uniformly distributed. By definition this is equivalent to

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \langle f, g \rangle
\]

for all \( f \in H \), and since \( K \) is the reproducing kernel of \( H \)

\[
\lim_{N \to \infty} \left\langle f, \frac{1}{N} \sum_{n=1}^{N} K_{x_n} - g \right\rangle = 0
\]

for all \( f \in H \). Therefore \( r_N \to 0 \) weakly as \( N \to \infty \).

From the assumption that \( E \) is compact and \( x \mapsto K_x \) is continuous for all \( x \in E \) it is easy to see that \( \{K_x : x \in E\} \subseteq C(E) \) and compact. Since every weakly convergent sequence in \( H \) converges uniformly on the compact set \( E \) (see [3, p. 344]), it follows that

\[
\| r_N \|_\infty = \sup_{x \in E} |r_N(x)| \to 0
\]

as \( N \to \infty \), and we conclude that (i) implies (iii).

On the other side, assume that \( \lim_{N \to \infty} \| r_N \|_\infty = 0 \). For every \( x \in E \) we have

\[
r_N(x) = \langle r_N, K_x \rangle \to 0.
\]

It follows that

\[
\langle r_N, f \rangle \to 0
\]

for every \( f = \sum_{i=1}^{k} x_i K_{x_i} \in H \).
From the reproducing property of the kernel $K$ it is easy to see that $\text{lin}\{K_x : x \in E\}$ is dense in $H$ and therefore
\[ \langle r_N, f \rangle \to 0 \quad (N \to \infty) \]
for all $f \in H$. Thus (iii) implies (i).

From (i) and (iii) we obtain that
\[ \|r_N\|^2_H = \left\langle \frac{1}{N} \sum_{n=1}^{N} K_{x_n} - g, \frac{1}{N} \sum_{m=1}^{N} K_{x_m} - g \right\rangle = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{N} \sum_{m=1}^{N} \langle K_{x_n}, K_{x_m} \rangle - \frac{1}{N} \sum_{n=1}^{N} \langle K_{x_n}, g \rangle - \frac{1}{N} \sum_{m=1}^{N} \langle g, K_{x_m} \rangle + \|g\|^2 \]
\[ \leq \frac{1}{N} \sum_{n=1}^{N} \left( \frac{1}{N} \sum_{m=1}^{N} |K_{x_m}(x_n) - g(x_m)| + \frac{1}{N} \sum_{m=1}^{N} \overline{\langle g, K_{x_m} \rangle} - \|g\|^2 \right) \]
\[ = \frac{1}{N} \sum_{m=1}^{N} |r_N(x_m)| + \frac{1}{N} \sum_{n=1}^{N} |r_N(x_n)| - \langle g, g \rangle \]
\[ \leq \sup_{x \in E} |r_N(x)| + \frac{1}{N} \sum_{n=1}^{N} |g(x_n)| - \langle g, g \rangle \]
\[ \to 0 \]
as $N \to \infty$, and therefore (ii) holds.

Since every norm convergent sequence is weakly convergent, (i) follows from (ii).

Now we characterize those $g \in H$ for which there exists a $g$-uniformly distributed sequence. This characterization is a corollary of the following proposition.

**Proposition 4.** Let $E$ be a Banach space with norm $\|\cdot\|$ and $A \subseteq E$ be a bounded subset of $E$.

For all $x \in \text{conv} A$ there exists a sequence $(x_n)_{n \geq 1}$ in $A$, such that
\[ \lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n - x \right\| = 0. \]

Since this proposition is more a result in functional analysis rather than a key result of the paper and the proof is very similar to that of Theorem 1 in [10], we shall give the proof of Proposition 4 as an appendix.
Theorem 5. Let $H$ be a r.k. Hilbert space with reproducing kernel $K$. Suppose that $\{K_y: y \in E\}$ is bounded. There exists a g-uniformly distributed sequence $(x_n)_{n \geq 1}$ in $E$ if and only if $g \in \text{conv}\{K_y: y \in E\}$.

Proof. If $(x_n)_{n \geq 1}$ is g-uniformly distributed, $\frac{1}{n} \sum_{n=1}^{N} K_{x_n} \to g$ weakly. Therefore $g$ is in the weak closure of $\text{conv}\{K_y: y \in E\}$. Since for a subset $A$ of a Banach space $\text{conv} A^* = \text{conv} \overline{A}$, we obtain $g \in \text{conv}\{K_y: y \in E\}$.

The other implication follows from Proposition 4 with $A = \{K_y: y \in E\}$ and the fact that every norm convergent sequence is weakly convergent.

The next theorem shows the relationship between g-uniformly distributed sequences in a r.k. Hilbert space $H$ and uniformly distributed sequences with respect to probability measures on a compact metric space. We start off with a proposition.

Proposition 6. Let $E$ be a compact metric space, $\mathcal{P}(E)$ the space of all probability measures on $E$, $K: E \times E \to \mathbb{C}$ a continuous positive definite function, and $H$ the r.k. Hilbert space with reproducing kernel $K$. Then we have:

(i) For every $g \in \text{conv}\{K_y: y \in E\}$ there exists a $\mu \in \mathcal{P}(E)$ with

$$\langle f, g \rangle = \int_{E} f \, d\mu$$

for all $f \in H$.

(ii) For every $\mu \in \mathcal{P}(E)$ there exists a unique $g \in \text{conv}\{K_y: y \in E\}$ with

$$\langle f, g \rangle = \int_{E} f \, d\mu$$

for all $f \in H$.

Proof. Since $K: E \times E \to \mathbb{C}$ is continuous, it follows that the mapping $x \mapsto K_x$ from $E \to H$ is continuous and $H \subseteq \mathcal{C}(E)$.

(i) Let $g \in \text{conv}\{K_y: y \in E\}$. Then there exists a sequence $(g_N) \subseteq \text{conv}\{K_y: y \in E\}$ with

$$\|g - g_N\|_H \to 0$$

as $N \to \infty$.

From this it follows that

$$\langle f, g_N \rangle \to \langle f, g \rangle$$
for all \( f \in H \). Since each \( g_N \) is of the form 
\[
g_N = \sum_{n=1}^{N} \alpha_n \delta_{x_n}\]
with \( \sum_{n=1}^{N} \alpha_n = 1 \), it follows from the reproducing property of \( K \) that
\[
\sum_{n=1}^{N} \alpha_n f(x_n) \to \langle f, g \rangle \quad (N \to \infty)
\]
for all \( f \in H \).

Define
\[
\mu_N := \sum_{n=1}^{N} \alpha_n \delta_{x_n}
\]
with \( \delta_x(f) = f(x) \) for all \( f \in C(E) \). Then \( \mu_N \in P(E) \), and since \( P(E) \) is a w*-compact metric space (see [11, p. 45]) there exists a subsequence \( (\mu_{N_k}) \) of \( (\mu_N) \) and a \( \mu \in P(E) \) with
\[
w^* - \lim_{k \to \infty} \mu_{N_k} = \mu.
\]
This means that for all \( f \in C(E) \) and in particular for all \( f \in H \),
\[
\sum_{n=1}^{N_k} \alpha_n f(x_N) \to \mu(f) = \int_E f \, d\mu \quad (k \to \infty).
\]
Therefore
\[
\langle f, g \rangle = \int_E f \, d\mu
\]
for all \( f \in H \).

(ii) Let \( \mu \in P(E) \). By the Krein–Milman theorem (see [12, p. 75]), there exists a sequence \( \mu_N := \sum_{n=1}^{N} \alpha_n \delta_{x_n} \) (\( \sum_{n=1}^{N} \alpha_n = 1 \)) with
\[
w^* - \lim_{N \to \infty} \mu_N = \mu
\]
or
\[
\mu_N(f) \to \mu(f)
\]
for all $f \in C(E)$. Applying again the reproducing property of $K$, we obtain
\[ \left\langle f, \sum_{n=1}^{N} \alpha_n^N K_{x_n^N} \right\rangle \to \mu(f) \]
for all $f \in H$.

Let $g_N := \sum_{n=1}^{N} \alpha_n^N K_{x_n^N} \in \text{conv}\{ K_y : y \in E \}$.

Since $\{ K_x : x \in E \}$ is compact, it follows that $\text{conv}\{ K_y : y \in E \}$ is compact. Hence there exist a subsequence $(g_{N_k})$ of $(g_N)$ and a $g \in \text{conv}\{ K_y : y \in E \}$ with
\[ \| g_{N_k} - g \|_H \to 0. \]

Then for all $f \in H$,
\[ \langle f, g_{N_k} \rangle \to \langle f, g \rangle \]
and we obtain
\[ \mu(f) = \langle f, g \rangle \]
for all $f \in H$ with $g \in \text{conv}\{ K_y : y \in E \}$.

If $\mu(f) = \langle f, g_1 \rangle = \langle f, g_2 \rangle$ for all $f \in H$, then $\langle f, g_1 - g_2 \rangle = 0$ and therefore $g_1 = g_2$. 

Remember that a sequence $(x_n)$ in a compact metric space is called uniformly distributed with respect to a probability measure $\mu$ if and only if
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \mu(f) \]
for all $f \in C(E)$ (see [9] and [4]).

**Theorem 7.** Let $E$ be a compact metric space, $(x_n)$ a sequence in $E$, $K : E \times E \to \mathbb{C}$ a continuous positive definite function, and $H$ the r.k. Hilbert space with reproducing kernel $K$.

1. If $(x_n)$ is uniformly distributed with respect to a probability measure $\mu$, then $(x_n)$ is $g$-uniformly distributed for the uniquely determined $g \in \text{conv}\{ K_y : y \in E \}$ according to Proposition 6.
The following two conditions are equivalent:

(i) Let $\mu$ be a probability measure on $E$ and $g$ the uniquely determined element of $\text{conv}\{K_y : y \in E\}$ according to Proposition 6. If $(x_n)$ is $g$-uniformly distributed, then $(x_n)$ is uniformly distributed with respect to the probability measure $\mu$.

(ii) $\lim_{\|\cdot\|_1} \mathcal{H} = C(E)$.

Proof. Part (1) follows immediately from Proposition 6.

(2) Assume that $\lim_{\|\cdot\|_1} \mathcal{H} = C(E)$. Let $\mu$ be a probability measure on $E$ and $g$ the uniquely determined element of $\text{conv}\{K_y : y \in E\}$ according to Proposition 6. If $(x_n)$ is $g$-uniformly distributed, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \langle f, g \rangle = \mu(f)$$

for all $f \in \mathcal{H}$. Since

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} 1 = 1 = \mu(1)$$

and $\lim_{\|\cdot\|_1} \mathcal{H} = C(E)$, it follows from Theorem 1.1 in [9] that the sequence $(x_n)$ is uniformly distributed with respect to the probability measure $\mu$.

Now we show that (i) implies (ii).

Assume that $\lim_{\|\cdot\|_1} \mathcal{H} \neq C(E)$. Then, by the Hahn–Banach theorem, there exists $\lambda \in C(E)^*$ with $\lambda \neq 0$ and $\lambda(f) = 0$ for all $f \in \lim_{\|\cdot\|_1} \mathcal{H}$. By the Riesz representation theorem and the Hahn decomposition, $\lambda$ can be expressed as

$$\lambda = \alpha \mu_1 - \beta \mu_2$$

with $\alpha, \beta \geq 0$ and $\mu_1, \mu_2$ two probability measures on $E$.

Since

$$0 = \lambda(1) = \alpha \mu_1(1) - \beta \mu_2(1) = \alpha - \beta,$$

it follows that $\alpha = \beta$ and we can assume

$$\lambda = \mu_1 - \mu_2.$$
Furthermore, $\lambda(f) = 0$ for all $f \in \overline{\text{lin}\{H, 1\}}$, and therefore

$$\mu_1(f) = \mu_2(f)$$

for all $f \in H$.

As we have shown in Proposition 6, there exist uniquely determined elements $g_1, g_2 \in \text{conv}\{K, y \in E\}$ with $\langle f, g_1 \rangle = \mu_1(f)$ and $\langle f, g_2 \rangle = \mu_2(f)$ for all $f \in H$. Let $(x_n)$ be a $g_1$-uniformly distributed sequence (see Theorem 5). Then for all $f \in H$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \langle f, g_1 \rangle = \mu_1(f) = \mu_2(f) = \langle f, g_2 \rangle.$$ 

Therefore $(x_n)$ is $g_2$-uniformly distributed. From (i) it follows that $(x_n)$ is uniformly distributed with respect to the probability measures $\mu_1$ and $\mu_2$.

From the definition of uniform distribution with respect to a probability measure we obtain

$$\mu_1(f) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \mu_2(f)$$

for all $f \in C(E)$. It follows that $\lambda = \mu_1 - \mu_2 = 0$, but this is a contradiction to $\lambda \neq 0$. 

4. EXAMPLES

Let $E = [0, 1)^s$, with $s \geq 1$, the $s$-dimensional unit cube. Furthermore, let $\omega: \mathbb{Z}^s \to \mathbb{R}$ be a function with

- $\omega(k) > 0$ for all $k \in \mathbb{Z}^s$, and
- $\sum_{k \in \mathbb{Z}^s} \frac{1}{\omega(k)} < \infty$.

We define the r.k. Hilbert space

$$H = \left\{ f: [0, 1)^s \to \mathbb{C} \mid \sum_{k \in \mathbb{Z}^s} \omega(k) |\hat{f}(k)|^2 < \infty \right\}$$

with inner product

$$\langle f, g \rangle = \sum_{k \in \mathbb{Z}^s} \omega(k) \hat{f}(k) \overline{\hat{g}(k)}$$
and reproducing kernel

\[ K(x, y) = \sum_{k \in \mathbb{Z}^s} \frac{e^{2\pi ik \cdot (x - y)}}{\omega(k)} \]

(where \( x \cdot y \) denotes the inner product in \( \mathbb{R}^s \) and \( \hat{f}(k) \) the Fourier coefficient of \( f \) at \( k \in \mathbb{Z} \)).

Some examples of \( \omega \) are given by:

(a) For \( \alpha > 1 \) let

\[ \omega(k) = \omega((k_1, ..., k_s)) = (k_1 \cdots k_s)^{\alpha} \]

where

\[ k_i = \begin{cases} |k_i|, & k_i \neq 0 \\ 1, & k_i = 0. \end{cases} \]

(b) For \( \alpha > 1 \) let

\[ \omega(k) = \left( \frac{k_1}{\beta_1} \cdots \frac{k_s}{\beta_s} \right)^{\alpha} \]

with \( \beta_i > 0 \).

(c) For \( \alpha > \frac{1}{2} \) let

\[ \omega(k) = \begin{cases} (k_1^2 + \cdots k_s^2)^{\alpha}, & k \neq 0 \\ 1, & k = 0. \end{cases} \]

These spaces (with further examples of \( \omega \)) are studied by several authors (see for example \([7]\)).
Since every $f \in H$ has an absolutely and uniformly convergent Fourier expansion, it follows that a sequence $(x_n)_{n \geq 1}$ in $[0, 1)^s$ is $g$-uniformly distributed if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left( \sum_{k \in \mathbb{Z}^s} \hat{f}(k) e^{2\pi ik \cdot x_n} \right) = \sum_{k \in \mathbb{Z}^s} \omega(k) \hat{f}(k) \overline{g(k)}$$

or

$$\sum_{k \in \mathbb{Z}^s} \hat{f}(k) \left\{ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi ik \cdot x_n} - \omega(k) \overline{g(k)} \right\} = 0.$$

In particular, if we choose $f_k(x) = e^{2\pi i k \cdot x}$ it follows that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi ik \cdot x_n} = \omega(k) \overline{g(k)}$$

for all $k \in \mathbb{Z}^s$.

Therefore, we obtain the following Weyl criterion for $g$-uniformly distributed sequences.

**Proposition 8.** Let $H$ be the r.k. Hilbert space defined above. A sequence $(x_n)_{n \geq 1}$ in $[0, 1)^s$ is $g$-uniformly distributed if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi ik \cdot x_n} = \omega(k) \overline{g(k)}$$

for all $k \in \mathbb{Z}^s$.

**Corollary 9.** Let $H$ be the r.k. Hilbert space defined above with $\omega(k) = \omega(k_1, \ldots, k_s) = (\langle k_1 \rangle \cdots \langle k_s \rangle)^\alpha$ $(\alpha > 1)$ and $g \equiv 1$. A sequence $(x_n)_{n \geq 1}$ in $[0, 1)^s$ is 1-uniformly distributed if and only if $(x_n)_{n \geq 1}$ is uniformly distributed in the usual sense.

**Proof.** $(x_n)_{n \geq 1}$ is 1-uniformly distributed if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi ik \cdot x_n} = \omega(k) \overline{1(k)}.$$
Since
\[ \hat{1}(k) = \begin{cases} 1, & k = (0, \ldots, 0) \\ 0, & k \neq (0, \ldots, 0), \end{cases} \]
the corollary follows from the proposition above and the Weyl criterion for uniformly distributed sequences (see [9, p. 7]).

It is easy to see that the 1-diaphony is given by
\[ \|r_N\|_H = \left( \sum_{k \in \mathbb{Z}^s \setminus \{0\}} \frac{1}{o(k)} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \cdot x_n} \right|^2 \right)^{1/2}. \]
This quantity is also called the weighted spectral test (see [6]). In particular, for \( o(k) = o((k_1, \ldots, k_s)) = (k_1 \cdots k_s)^2 \) we obtain for \( \|r_N\|_H \) the classical diaphony \( F_N(x_n) \) (see [14] and [15]).

As we have seen in the proof of Theorem 3, \( \|r_N\|_H \leq \frac{1}{N} \sum_{n=1}^{N} g(x_n) - \langle g, g \rangle \), and since
\[ \|r_N\|_\infty \leq \sup_{x \in I} \sqrt{K(x, x)} \|r_N\|_H \]
we obtain the inequalities for the 1-diaphony and 1-discrepancy,
\[ \|r_N\|_2^2 \leq \|r_N\|_\infty \leq C \|r_N\|_H, \]
with \( C = (\sum_{k \in \mathbb{Z}^s \setminus \{0\}} o(k))^{1/2} \).

Remark. The previous example can be generalized easily to an arbitrary compact abelian group \( G \).
Let \( \varphi \in L^2(\hat{G}) \) with \( \hat{\varphi}(\chi) \neq 0 \) for all characters \( \chi \in \hat{G}, \chi \neq \chi_0, \) and \( \hat{\varphi}(\chi_0) = \int_{\hat{G}} \varphi(x) \, dx = 0. \)
Then
\[ H = \left\{ f : [0, 1)^s \to C \left| \sum_{\chi \in \hat{G}} \frac{1}{|\hat{\varphi}(\chi)|^2} |\hat{f}(\chi)|^2 < \infty \right. \right\}. \]
is a r.k. Hilbert space with reproducing kernel
\[ K(x, y) = \varphi^* \ast \varphi(y - x) = \sum_{x \in G} |\hat{\varphi}(x)|^2 \chi(y - x) \]
(where \( \varphi^*(x) = \overline{\varphi(-x)} \)).

Furthermore, the 1-diaphony is given by
\[ \|r_N\|_H^2 = \frac{1}{N^2} \sum_{n, m=1}^{N} \varphi^* \ast \varphi(x_n - x_{n+m}) \]
\[ = \sum_{x \in G} |\hat{\varphi}(x)|^2 \left| \frac{1}{N} \sum_{n=1}^{N} \chi(x_n) \right|^2. \]

This diaphony was investigated in [1] and [2], where also the non-abelian case was considered.

APPENDIX

Proof of Proposition 4. Suppose first that \( y \in E \) is of the form \( y = \frac{1}{k} \sum_{i=1}^{k} y_i \) with \( y_i \in A \). We claim that there exists a sequence \((z_n)_{n \geq 1}\) in \( A \) and a constant \( c(y) > 0 \) such that
\[ \left| \frac{1}{N} \sum_{n=1}^{N} z_n - y \right| \leq \frac{c(y)}{N} \quad (1) \]
for all \( N \geq 1 \).

To prove this, we consider the sequence
\[ (z_n) = (y_1, y_2, ..., y_k, y_1, y_2, ..., y_k, ...). \]

Let \( N \geq 1 \), then there exists \( s \geq 0 \) with \( s_k \leq N < (s+1)k \). Thus \( N = sk + r \) with \( 0 \leq r < k \) and
\[ \sum_{n=1}^{N} z_n - Ny = s \sum_{n=1}^{k} y_n - sy + \sum_{n=sk+1}^{N} y_n - ry. \]
Therefore
\[
\left\| \frac{1}{N} \sum_{n=1}^{N} z_n - y \right\| \leq \frac{sk}{N} \left\| \frac{1}{k} \sum_{n=1}^{k} y_n - y \right\| + \frac{r}{N} \sup_{z \in A} \|z\| + \frac{r}{N} \|y\|
\]
\[
\leq \frac{2(k-1)}{N} \sup_{z \in A} \|z\|.
\]

Now let \( x \in \text{conv} A \), then there exists a sequence \((y_j)_{j \geq 1}\) in \(\text{conv} A\) with
\[
\| x - y_j \| \to 0 \quad (j \to \infty).
\]
Without loss of generality we can suppose that \(y_j\) is of the form
\[
y_j = \frac{1}{K_j} \sum_{m=1}^{K_j} y_{j,m}^t
\]
with \(y_{j,m}^t \in A\).

From (1) we see that there exist sequences \((z_n^t)_{n \geq 1}\) in \(A\) and constants \(c_j = c(y_j) > 0\) with
\[
\left\| \frac{1}{M} \sum_{n=1}^{M} z_n^t - y_j \right\| \leq \frac{c_j}{M}
\]
for all \(M \geq 1\).

For each \(j \geq 1\) choose a positive integer \(t_j \geq 1\) with \(t_j \geq j(c_1 + c_2 + \cdots + c_{j+1})\). Let \(N > t_j\). Then \(N\) can be written as \(N = t_1 + t_2 + \cdots + t_k + l\) with \(0 < l < t_{k+1}\), where \(k\) is the greatest positive integer with \(t_1 + t_2 + \cdots + t_k < N\).

Consider the sequence \((x_n)_{n \geq 1}\),
\[
x_1 = z_1^1, \quad x_2 = z_2^1, \quad \ldots, \quad x_{t_j} = z_{t_j}^1,
\]
\[
x_{t_j+1} = z_1^2, \quad x_{t_j+2} = z_2^2, \quad \ldots, \quad x_{t_1+t_2} = z_{t_2}^2,
\]
\[
x_{t_1+t_2+1} = z_1^3, \quad \ldots
\]
\[
\vdots
\]
Since
\[
\sum_{n=1}^{N} x_n = \sum_{j=1}^{k} \sum_{m=1}^{t_j} z_m^j + \sum_{m=1}^{l} z_m^{k+1},
\]
we obtain

\[
\left| \frac{1}{N} \sum_{n=1}^{N} s_n - x \right| = \left| \sum_{j=1}^{k} \frac{t_j}{N} \left( \frac{1}{l} \sum_{m=1}^{l} z_m - y_j \right) + \frac{l}{N} \left( \frac{1}{l} \sum_{m=1}^{l} z_{m+1} - y_{k+1} \right) \\
+ \sum_{j=1}^{k} \frac{t_j}{N} y_j + \frac{l}{N} y_{k+1} - x \right| \\
\leq \sum_{j=1}^{k} \frac{t_j}{N} c_j + \frac{l}{N} c_{k+1} + \frac{1}{N} \left( \sum_{j=1}^{k} t_j y_j + l y_{k+1} \right) - x \\
= \frac{1}{N} \sum_{j=1}^{k} c_j + \frac{1}{N} \left( \sum_{j=1}^{k} t_j y_j + l y_{k+1} \right) - x \\
\leq \frac{1}{N} + \frac{1}{N} \left( \sum_{j=1}^{k} t_j y_j + l y_{k+1} \right) - x ,
\]

where the last step follows from \( c_1 + c_2 + \cdots + c_{k+1} \leq \frac{k}{2} < \frac{N}{2} \).

If \( N \to \infty \), then \( k \to \infty \), and the first term in the above sum tends to zero. Furthermore, from the assumption that \( y_j \to x \), it follows that the second term tends to zero.

REFERENCES


