Note on optimization of individual psychotherapeutic processes

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Abstract
Individual psychotherapy typically involves a sequence of recurrent sessions of client–therapist interaction. Accordingly, the psychotherapeutic process can be conceived of as evolving at least at two different time scales: a fast time scale pertaining to the on-going interaction within each session and a slow time scale associated with sequential regularities occurring between consecutive sessions. It is possible to exploit the sequential regularities between sessions in order to assess and optimize an ongoing individual psychotherapeutic process in real time. In this note a computational paradigm is outlined according to which this can be implemented and applied in flexible ways. Illustrations are given with simulated data. A heuristic summary is provided in the closing section.

1. Introduction

Individual psychotherapy typically involves a sequence of recurrent sessions of client–therapist interaction. Accordingly, the psychotherapeutic process can be conceived of as evolving at least at two different time scales: a fast time scale pertaining to the on-going interaction within each session and a slow time scale associated with sequential regularities occurring between consecutive sessions (cf. Orlinsky, Rønnestad, and Willutzki (2004, p. 315) for a more elaborate classification of such time scales). Following an early lead of Bellman and Smith (1973), it is possible to exploit the sequential regularities between sessions in order to assess and optimize an ongoing individual psychotherapeutic process in real time. In this note a computational paradigm is outlined according to which this assessment and optimization can be implemented.

Paraphrasing Frank and Frank (1991, p.1), psychotherapy is defined as the process in which a therapist tries to mobilize healing forces in a client by psychological means in order to relieve the client’s experienced state of suffering and disability. In what follows, the evolution of the state of the client is represented by a q-variate latent Gaussian process \( x(t) \), where \( t \) denotes the \( t \)-th session. The latent state process \( x(t) \) is indirectly measured by a \( p \)-variate Gaussian process \( y(t) \), which is assessed at the start of the \( t \)-th session. It is understood that \( y(t) \) contains self-ratings by the client of the level of clinically relevant variables since the previous session \( t-1 \). Accordingly, sessions can be assumed to take place at equidistant time intervals. The therapeutic manipulations are represented by an \( r \)-variate process \( u(t) \) involving ratings by the therapist made at the end of each session.

The psychotherapeutic process aims to induce structural change in the state of the client. Consequently \( x(t) \), and hence \( y(t) \), will have statistical characteristics which vary in time, i.e., across sessions (henceforth \( t \) will be referred to as time). For a vector-valued stochastic process \( x(t) \), defined as a q-variate Gaussian distributed random variable indexed by time \( t = 0, 1, \ldots \), the mean function is in general a q-variate time-varying trend function: \( E[x(t)] = \mu_a(t) \). The serial covariance function of \( x(t) \) is in general a time-varying matrix function: \( \text{cov}[x(t_1), x(t_2)] = \Sigma_{ax}(t_1, t_2) \), i.e., a two-dimensional function of times \( t_1 \) and \( t_2 \).

The following terminology is used. In case a Gaussian process \( a(t) \) has constant mean function, i.e., \( \mu_a(t) = \mu_a \), and its covariance function only depends upon the time lag \( u = t_1 - t_2 \), i.e., \( \text{cov}[a(t_1), a(t_2)] = \Sigma_{aa}(u) \), then \( a(t) \) is called a stationary process. In case either the mean function or the covariance function or both are time-dependent, \( a(t) \) is called non-stationary. In case \( \mu_a = 0 \) and \( \Sigma_{aa}(u) = \delta(u)\Sigma_{aa} \), where \( \delta(u) = 1 \) iff \( u = 0 \) and \( \delta(u) = 0 \) otherwise, then \( a(t) \) is called a white noise series with covariance \( \Sigma_{aa} \). As is customary in the time series literature, no notational distinction will be made between a stochastic process \( a(t), t = 0, 1, \ldots \) and its realization \( a(t), t = 1, 2, \ldots, T \).

2. Client model

Let \( y(t) \) denote the \( p \)-variate observed time series for a given client. Given that the state \( x(t) \) of the client will in general be non-stationary, a linear state-space model with arbitrarily time-varying parameters for \( y(t) \) is defined as follows (\( \text{diag} \cdot A \) denotes a square matrix):
matrix $A$ with zero off-diagonal elements; $A^*$ denotes the transpose of $A$:

$$y(t) = A\theta(t)x(t) + v(t);$$  \(1a\)

$$\text{cov}(v(t), v(t - u)) = \delta(u) \text{diag} - V$$  \(1b\)

$$x(t + 1) = B\theta(t)x(t) + \Gamma\theta(t)u(t) + \xi(t + 1);$$  \(1c\)

$$\text{cov}(\xi(t), \xi(t - u)) = \delta(u) \Psi$$  \(1d\)

$$\theta(t + 1) = \theta(t) + \xi(t + 1);$$  \(1e\)

$$\text{cov}(\xi(t), \xi(t - u)) = \delta(u) \text{diag} - \Phi.$$  \(1f\)

The first pair of Eqs. \((1a)\) and \((1b)\) describes the way in which the $p$-variate process $y(t)$ is related to the latent $q$-variate state process $x(t)$. The $(p, q)$-dimensional matrix $A\theta(t)$ is akin to a matrix of time-varying factor loadings. It describes the linear dependence of $y(t)$ on $x(t)$. The argument $\theta(t)$ will be specified shortly. The $p$-variate time series $v(t)$ represents a Gaussian measurement error process. Eq. \((1b)\) specifies that the measurement error consists of $p$ components which are mutually independent univariate white noise processes.

The second pair of Eqs. \((1c)\) and \((1d)\) describes the evolution of the state process $x(t)$ in time. The $(q \times r)$-dimensional matrix $B\theta(t)$ and the $(q \times r)$-dimensional matrix $\Gamma\theta(t)$ are matrices of time-varying regression coefficients. $u(t)$ denotes a given (fixed) $r$-variate time-varying covariable. Notice that $u(t)$ can include a constant term or deterministic trend terms associated with time-varying mean functions. The $q$-variate time series $\xi(t)$ represents Gaussian process noise, which according to \((1d)\) is a white noise process.

It is noted that all parameter matrices in this linear state-space model depend upon $\theta(t)$, which is an $s$-variate time-varying vector containing all unknown parameters. The third pair of Eqs. \((1e)\) and \((1f)\) describes the evolution in time of $\theta(t)$ as a first-order random walk. The $s$-variate time series $\xi(t)$ represents the Gaussian process noise in this random walk which, according to \((1f)\), is composed of $s$ components which are mutually independent univariate white noise processes.

### 2.1. Aspects of the client model

The following standard aspects have to be addressed with respect to the client model \((1)\): stability and (structural) identifiability. Moreover, at the close of this section several possible generalizations of \((1)\) will be indicated.

To address stability of \((1)\), the exogenous variable $u(t)$ can be treated along the lines as explained in Durbin and Koopman (2001, p. 122). That is, the state vector is extended by adding the exogenous variables. Let $I$ denote the $(r, r)$-dimensional identity matrix and $0_{r\times q}$ the $(a, b)$-dimensional zero matrix. Define $y'(t) = [y'(t), u'(t)]$, $A^\theta(t) = [A\theta(t), I]$, $x'(t) = [x'(t), u'(t)]$, $v'(t) = [v'(t), 0_{q\times q}]$, $\zeta'(t) = [\zeta'(t), 0_{1\times q}]$, and the $(q + r)$-dimensional matrix $B^\theta(t)$ consisting of four blocks: $B^\theta(t)_{11} = B\theta(t)$, $B^\theta(t)_{12} = \Gamma\theta(t)$, $B^\theta(t)_{21} = 0_{q\times q}$, and $B^\theta(t)_{22} = I$. Then the Durbin–Koopman representation is:

$$y'(t) = A^\theta(t)x'(t) + v'(t),$$

$$x'(t + 1) = B^\theta(t)x'(t) + \zeta'(t).$$

In general, exponential stability of a linear system with time-varying uncertain parameters like \((1)\) can be established by finding a quadratic Lyapunov function $v'(t)\Psi(t)v'(t)$ in which, for $\forall t \in N_0$, $P(t)$ is positively bounded and obeys the linear matrix inequality $B^\theta(t)P(t) + P(t)B^\theta(t) - P(t) < -CL_2\Phi$ a positive scalar. The random walk \((1e)\), however, implies that parameters in the client model are slowly varying in time (cf. Priestley (1988, p. 149) for a mathematical definition of slowly time-varying functions). Because parameters are slowly varying in time, i.e., there exists a sufficiently small scalar $\epsilon > 0$ such that $\|B^\theta(t + 1) - B^\theta(t)\|_2 < \epsilon \|\| denotes the $L_2$ norm; cf. Grigoriu (2002) section 2.6), the following easy alternative test applies: (a) is exponentially stable if, for $\forall t \in N_0$, $B^\theta(t)\Psi(t)$ is bounded; (b) the eigenvalues of $B^\theta(t)$ are uniformly located in the unit disk (cf. Amato, 2006, Theorem 2.15, p. 27). In case \((1)\) lacks exponential stability, an attempt can be made to stabilize via feedback. This requires solving a pair of linear matrix inequalities depending upon two matrix functions like $P(t)$, the solution of which (if any) allows for the construction of a dynamical stabilizing controller (cf. Amato, 2006, Theorem 6.7, p. 167). In psychotherapeutic process analysis, the assumption that $B^\theta(t)\Psi(t)$ is bounded will always be met given that measurement scales have bounded range (e.g. Meier, 2008; Molenaar, 1987). Hence under the assumptions of client model \((1)\), the critical test for exponential stability is whether the eigenvalues of $B^\theta(t)$ are uniformly located in the unit disk. If this is not the case, a dynamical stabilizing controller should be used.

Structural identifiability of the stationary state-space model is discussed in, e.g., Hannan and Deister (1988, p. 20–21). Testing for structural identifiability in state-space models with time-varying parameters requires alternative techniques which are developed for nonlinear state-space models. For instance, algebraic generating power series (gps) expansions of nonlinear state-space models can be used. The gps defines a generalization of the Laplace transform for nonlinear state-space models. Identifiability is tested in the usual way by ascertaining whether $\text{gps}(\theta) = \text{gps}(\theta^*)$ implies that $\theta = \theta^*$. This approach is described in Lecourtier, Lamnabhi-Lagarrigue, and Walter (1987), who provide an explicit algorithm for the generating power series expansion of systems with time-varying parameters (cf. also Fliess, Lamnabhi, and Lamnabhi-Lagarrigue (1983)).

Although model \((1)\) is a vector autoregression with external input which can accommodate a wide range of sequentially dependent observations, it can be generalized in several ways. For instance, $v(t)$ in \((1a)\) and $\xi(t)$ in \((1c)\) can be correlated: $\text{cov}(v(t), \xi(t)) \neq 0$. In addition, $v(t)$ and/or $\xi(t)$ can have serial correlation (the so-called colored noise): $\text{cov}(v(t), v(t - u)) \neq 0$ and/or $\text{cov}(\xi(t), \xi(t - u)) \neq 0$ for $u \neq 0$. Simon (2006, chapter 7) presents a complete discussion of these generalizations. In any application of the client model \((1)\) to empirical data \((y(t), u(t)) = 1, \ldots, T\) the appropriate assumptions regarding $v(t)$ and $\xi(t)$ can be determined in a posteriori model validation (cf. Ljung, 1999, section 16.5).

### 2.2. Parameter estimation in client model

To fit \((1)\) to an observed time series $y(t)$, a $(q + s)$-dimensional extended state process $z(t)$ is defined by adding the parameter process $\theta(t)$ to the original state process $x(t) = [x(t), \theta(t)]$. Accordingly, the original linear state-space model \((1)\) is rewritten as a nonlinear analogue in terms of the extended state process $z(t)$:

$$y(t) = h(z(t), t) + v(t)$$  \(2a\)

$$\text{cov}(v(t), v(t - u)) = \delta(u)\mathbf{V}$$  \(2b\)

$$z(t + 1) = f(z(t), t) + w(t + 1)$$  \(2c\)

$$\text{cov}(w(t), w(t - u)) = \delta(u)\mathbf{W}$$  \(2d\)

where $w(t) = [\xi(t), \xi'(t)]$. The $(q + s, q + s)$-dimensional covariance matrix $\mathbf{W}$ is block-diagonal with $\Psi$ at $NW$ and $\Phi$ at $SE$ position. Notice that $h(z(t), t)$ and $f(z(t), t)$ are vector-valued nonlinear functions of $z(t)$. More specifically, each entry of $h(z(t), t)$ and $f(z(t), t)$ is a linear combination of constants, elements of $z(t)$ and products of pairs of elements of $z(t)$.

To fit \((2)\) involves (a) recursive estimation of the state $z(t)$ by means of a second-order extended Kalman filter, and (b) estimation of the covariance matrices of $v(t)$ and $w(t)$. 

2.3. Second-order extended Kalman filter

Suppose that all parameters in a linear state-space model are known. Then the recursive estimator of the state process is called the Kalman filter. An easy derivation of the Kalman filter based on conditional expectations for Gaussian variables is given in Shumway and Stoffer (2006, section 6.2). The extended Kalman filter involves application of the Kalman filter to nonlinear state-space models which have been linearized around the conditional expectation of the state process. A derivation of the extended Kalman filter using the conditional Fokker–Planck equation is given in Sage and Melsa (1971, section 9.2).

Let \( \mathbf{Y}^k := [y(1), \ldots, y(k)] \) and denote the conditional expectation of \( z(t) \) given \( \mathbf{Y}^k \) by \( E[z(t|\mathbf{Y}^k)] = \mathbf{z}(t|k) \). Let \( E[\mathbf{z}(t) - \mathbf{z}(t)] = \mathbf{P}(t|k) \). The following derivatives of \( \mathbf{h}(\mathbf{z}(t), t) \) and \( \mathbf{f}(\mathbf{z}(t), t) \) in (2) are needed: (a) the first-order derivative of \( \mathbf{h}(\mathbf{z}(t), t) \) with respect to \( \mathbf{z}(t) \) at \( z(t|t-1) \), denoted by \( \mathbf{h}_t \); (b) the second-order derivative of the \( k \)th component of \( \mathbf{h}(\mathbf{z}(t), t) \) with respect to \( \mathbf{z}(t) \) at \( z(t|t-1) \), denoted by \( \mathbf{h}_{tt}^{(k)} \); (c) the first-order derivative of \( \mathbf{f}(\mathbf{z}(t), t) \) with respect to \( \mathbf{z}(t) \) at \( z(t|t) \), denoted by \( \mathbf{f}_t \); and (d) the second-order derivative of the \( k \)th component of \( \mathbf{f}(\mathbf{z}(t), t) \) with respect to \( \mathbf{z}(t) \) at \( z(t|t) \), denoted by \( \mathbf{f}_{tt}^{(k)} \).

Given \( \mathbf{Y}^k \), \( \mathbf{V} \) and \( \mathbf{W} \), the second-order Kalman filter for estimation of the state process \( \mathbf{z}(t) = [x(t), \theta(t)] \) in (2) consists of the following recursion (Tr denotes the trace):

Initial condition: \( \mathbf{z}(0|0), \mathbf{P}(0|0), \text{cov}[\mathbf{z}(0|0), \mathbf{v}(t)] = 0 \).

For \( t = 1, 2, \ldots, T \):

\[
\begin{align*}
\mathbf{z}(t|t-1) &= \mathbf{f}(z(t-1|t-1), t-1) + \mathbf{w}(t) \\
\mathbf{P}(t|t-1) &= \mathbf{f}(z(t-1|t-1), t-1) + \mathbf{W}(t) \\
\mathbf{K}(t) &= \mathbf{P}(t|t-1)\mathbf{H}(t)\mathbf{P}(t|t-1) + \mathbf{V}^{-1} \\
\pi(t) &= 0.5\mathbf{K}(t)\sum_{k=1,1}^{k=1,2} \{\mathbf{k}(t)\mathbf{P}(t|t-1)\} \\
\mathbf{z}(t|t) &= \mathbf{z}(t|t-1) + \mathbf{K}(t)(\mathbf{y}(t) - \mathbf{h}(\mathbf{z}(t|t-1), t)) + \pi(t) \\
\mathbf{P}(t|t) &= \mathbf{I}_n - \mathbf{K}(t)\mathbf{H}(t)\mathbf{P}(t|t-1) \\
\end{align*}
\]

Because elements of \( \mathbf{h}(\mathbf{z}(t), t) \) and \( \mathbf{f}(\mathbf{z}(t), t) \) are polynomials in \( z(t) \) of degree strictly smaller than 3, the linearization of (2) underlying the second-order extended Kalman filter has zero remainder.

2.4. Maximum likelihood estimation of variance terms

Given \( \mathbf{V} \) and \( \mathbf{W} \), recursive estimation of \( \theta(t) = [x(t), \theta(t)] \) according to (3) yields parameter estimates \( \hat{\theta}(t|t), t = 1, 2, \ldots, T \). In general, however, \( \mathbf{V} \) and \( \mathbf{W} \) will be unknown. This holds in particular for the covariance matrix \( \mathbf{V}(\theta(t)) \) in the random walk model (1e): \( \theta(t) = \theta(t) + \xi(t) \). It can be shown (cf. Young, 2000, p. 79) that (1e) implies that the kth recursive parameter estimate \( \hat{\theta}_k(t|t) \) only depends on local data in the neighborhood of \( t \), where the span of this neighborhood is an inverse monotonic function of \( \text{var}[\xi(t)] \). If \( \text{var}[\xi(t)] \to 0, k \in [1, \ldots, s] \), then the kth parameter estimate approaches constancy in time: \( \hat{\theta}_k(t|t) \to \hat{\theta}_k, t = 1, 2, \ldots, T \).

Maximum likelihood estimates of \( \mathbf{V} \) and \( \mathbf{W} \) are obtained by the following algorithm:

Initial values are \( \mathbf{V}^{(0)} \) and \( \mathbf{W}^{(0)} \).

For some small positive constant \( \text{eps} \) and \( n = 0, 1, \ldots \)

Apply (3) to obtain \( \mathbf{z}^{(n)}(t|t) \) and \( \mathbf{P}^{(n)}(t|t-1) \), \( t = 1, 2, \ldots, T \).

Minimize \( -\ln L^{(n)} \), yielding \( \mathbf{V}^{(n+1)} \) and \( \mathbf{W}^{(n+1)} \).

If \( -\ln L^{(n)} + \ln L^{(n-1)} > \text{eps} \) then \( n \to n + 1 \)

and go to (4b).

2.5. Illustration of non-stationary time series modeling

A bivariate non-stationary time series \( \mathbf{y}(t), t = 1, 2, \ldots, 100, \) is generated according to the following model. \( \mathbf{A}([\theta(t)) = \mathbf{I}_2, \mathbf{x}(t) = \{x_1(t), x_2(t)\}, \) and in (1b) let \( \mathbf{V} = \mathbf{V} \) [1...1]. Parameter values in \( \mathbf{B}(\theta(t)) \) in (1c) are: \( \beta_1(t) = \beta_1(t) = 0.5; \beta_2(t) = \beta_2(t) = 0; \beta_2(t) = \beta_3(t) = 0.5; \beta_2(t) = \beta_4(t) = 0.5. \) Hence all 4 parameters in \( \mathbf{B}(\theta(t)) \) are constant. \( u(t) \) is univariate and generated as: \( u(t) = 0.5u(t-1) + \alpha(t), \) where \( \alpha(t) \) is Gaussian white noise with variance equal to 0.0. The entries of \( \mathbf{G}(\theta(t)) \) are chosen as: \( \gamma_1(t) = \gamma_2(t) = -3; \gamma_3(t) = \gamma_4(t) = 0.5 - 0.5t, t = 1, 2, \ldots, 100. \) Hence \( \gamma_1(t) \) is constant, but \( \gamma_2(t) \) is time-varying according to a linear trend: \( \gamma_2(10) = 0.5 \) and \( \gamma_2(100) = 0.5. \) In (1d) \( \text{diag} - \mathbf{V} = [1, 1, 0, 0]. \)

Application of (4) with initial values (in (3)) \( \mathbf{z}(0|0) = 0, \mathbf{P}(0|0) = \mathbf{I}_p, \) and (in (4)) \( \mathbf{V}^{(0)} = 0, \mathbf{W}^{(0)} = 0, \) yields the following constant parameter estimates \( \beta_1(t) = \beta_1(t) = 0.5, \beta_2(t) = \beta_2(t) = 0, \beta_3(t) = \beta_3(t) = 0.5, \beta_2(t) = \beta_4(t) = 0.5, \) and \( \gamma_1(t) = \gamma_2(t) = -22. \) Only \( \gamma_2(t) = \gamma_2(t) \) is correctly identified as time-varying (see Fig. 1).

3. Optimization of the psychotherapeutic process

Based on the client model (1), therapeutic manipulations at each session \( t \) can be determined in such a way to minimize future deviations of the client state process from desired levels while taking into account the costs of therapeutic intervention. To accomplish this, the model fitted to the data up to session...
t (i.e., fitted in real time) is used to predict the client state at the next session. This predicted value then is compared with the desired value for the client state and the difference (if any) is used to determine the optimal parameters of therapeutic manipulations to minimize the difference. The extent to which severity of deviations of client state from their desired levels is penalized and the cost of manipulations weighted is quantified by, respectively, the design matrices Q and R to be chosen by the therapist (or fixed at default values given by identity matrices of appropriate dimensions). This constitutes an entirely innovative approach in psychotherapeutic process analysis with which the effects of an ongoing individual psychotherapy can be optimized in flexible client-specific ways.

In what follows optimal feedback control will be derived for linear state-space systems and quadratic cost functions — the so-called LQ control, which is a standard feedback control strategy in the engineering sciences (e.g., Goodwin & Sin, 1984, section 10.4). Next, LQ control is generalized to Gaussian systems like (1) — the so-called LQG control.

3.1. LQ feedback control

To derive LQ control based on client model (1), we consider the systematic part of (1c): $x(t + 1) = B\theta(t)x(t) + G\theta(t)u(t)$. At each session $t$ the parameter estimates $\theta(t)$ is available. Substitution of $\theta(t)$ in (1e) yields the predicted parameter values $\theta(t + k) = \theta(t)$ for $k \geq 0$. (the random walk (1e) is a martingale). Consequently, derivation of LQ control at each session $t$ can be based on the model with constant parameters: $x(t + 1) = Bx(t) + Gu(t)$.

A quadratic cost function is minimized with respect to $u(t)$, starting at $t$ up to $t + h$, where $h > 1$ is called the control horizon. To ease notation, starting time $t$ is taken to be $t = 1$. The quadratic cost function is defined as:

$$J(x', u) = \sum_{r=1}^{h-1} \{ (x(r) - x')'Q(x(r) - x') + u(r)'Ru(r) \} + h_b \cdot \lambda_r \tag{5a}$$

$$h_b = (x(h) - x')'Q(x(h) - x'). \tag{5b}$$

The desired level of the client state process is denoted by the reference $x'$, which together with the positive-definite design matrices Q, Q, and R are chosen a priori. More general cost functions can be considered, including time-varying references for both $x(t)$ and $u(t)$ and risk-sensitive control (Whittle, 1990, p. 79–125) as well as asymmetric design matrices which only penalize deviations in one direction. Penalizing deviations in one direction also can be implemented by appropriate rescaling of $x(t)$ as well as the reference $x'$.

The following derivations using Pontryagin’s minimization principle are based on Kwon & Han (2005, p. 26–33). First a Hamiltonian is formed as follows:

$$H_r = \{ (x(r) - x')'Q(x(r) - x') + u(r)'Ru(r) \} + \lambda_r \cdot Bx(r) + Gu(r), \quad r \in [1, h - 1]$$

According to Pontryagin’s minimization principle (Kwon & Han, 2005, p. 20–24), the necessary conditions for minimization of (5) with respect to $u(t)$ are:

$$\partial H_r / \partial u(t) = 2Ru(t) + \Gamma \lambda_{r+1} = 0 \tag{6a}$$

$$\partial H_r / \partial x(r) = 2Q(x(r) - x') + B\lambda_r = \lambda_r \tag{6b}$$

$$\partial h_b / \partial x(h) = 2Q(x(h) - x') = \lambda_h. \tag{6c}$$

(6a) yields the optimal manipulation at $t$ as (a) $u* (t) = -0.5\Gamma^{-1} \lambda_{r+1}$. To solve for $\lambda_r$, assume that (b) $\lambda_r = 2L(\tau, x(t)) + 2g(\tau, h)$. Substitution of (b) in (6c) yields:

$$L(h, h) = Q_\tau \tag{7a}$$

$$g(h, h) = -Q_xx'. \tag{7b}$$

Substitution of (a) and $x(t + 1) = Bx(t) + Gu(t)$ in (b), and solving for $\lambda_{r+1}$ yields, after simplification:

$$\lambda_{r+1} = \left[ I + L_{r+1} + h(\Gamma R^{-1} \Gamma)^{-1} \right] L_{r+1} h + 2g(\tau + 1, h). \tag{7c}$$

Substitution of (7c) in (a) yields:

$$u*(\tau) = -R^{-1} \Gamma \left[ I + L_{r+1} + h(\Gamma R^{-1} \Gamma)^{-1} \right] L_{r+1} hBx(\tau) + g(\tau + 1, h). \tag{7d}$$

Finally, $L(\tau, h)$ and $g(\tau, h)$ are determined after substitution of (7c) and (6b), yielding:

$$L(\tau, h) = B[ L_{r+1} + h(\Gamma R^{-1} \Gamma)^{-1} L_{r+1} hB + Q \tag{7e}$$

$$g(\tau, h) = B[ L_{r+1} + h(\Gamma R^{-1} \Gamma)^{-1} g(\tau + 1, h) - Qx']. \tag{7f}$$

(7e) and (7f) are recursively solved backwards in time, starting at $\tau = h$ with initial conditions (7a) and (7b), until $\tau = 1$. Notice that if the reference $x'$ equals zero, then $g(\tau, h)$ vanishes.

3.2. Illustration of LQ control

LQ control is based on the client model: $x(t + 1) = Bx(t) + Gu(t)$. Consider the following simple bivariate instance of this model:

$$x_1(t + 1) = 0.8x_1(t) - 0.3x_2(t) + 0.7u(t) + p_1(t)$$

$$x_2(t + 1) = 0.5x_1(t) + 0.7x_2(t) + 0.4u(t) + p_2(t).$$

The control variable $u(t)$ is univariate; $p_1(t)$ and $p_2(t)$ denote deterministic perturbations. It is assumed that $x_1(t), x_2(t)$ and $u(t)$ have been centered with respect to their steady-state baselines (cf. Whittle, 1990, p. 37–38).

The process is started at the initial values $x_1(0) = 5$ and $x_2(0) is 5. Perturbations of $x_1(t)$ only occur at $t = 3$ and $t = 6$: $p_1(3) = 9$ and $p_1(6) = 9$. Perturbation of $x_2(t)$ only occur at $t = 4, 6,$ and $8$: $p_2(4) = 9 p_2(6) = 9 p_2(8) = 9$. The process thus generated under free conditions (no feedback; $u(t) = 0$ for all $t$) is depicted in Fig. 2a.

Fig. 2b shows the process with LQ feedback. Q, Q, and R in (5) are identity matrices of appropriate dimensions. The reference state is taken to be $x' = 0$, hence $g(\tau, h)$ defined by (7f) vanishes. The control horizon $h = 4$. At each time $t$, (7) is applied starting from $t + h$ backwards in time. Then only $u*(\tau) = u'(\tau = 1)$ is applied and the computations are repeated at the next time points.
control strategy is a simple instance of receding horizon control (cf. Kwon and Han (2005)).

3.3. LQG feedback control

The full client model (1) differs from the model underlying LQ control in some important respects. To start with, the state process \( x(t) \) in (1) is latent, whereas (7) has been derived for a manifest process model. The presence of Gaussian processes \( w(t) \) and \( \xi(t) \) in (1) defines the linear quadratic Gaussian (LQG) control problem, the solution of which is given by the certainty equivalence principle (e.g., Whittle, 1990, chapter 2). The essence of the certainty equivalence principle is that (7) still applies if: (a) the conditional expectation of the cost function (5) given \( Y \) is taken and (b) for \( x(t) \) in (7d) the conditional expectation \( E[x(t)|Y^t] = x(t|r) \) is substituted. Kwon and Han (2005, section 2.5.1) present a proof of the certainty equivalence principle (see also Goodwin & Sin, 1984, section 10.4.1).

Another important difference between the manifest process model underlying (7) and (1) is that the latter model has uncertain time-varying parameters \( \theta(t) \) whose evolution in time is described by the random walk (1e). In order to determine \( u^*(t) \) according to (7d) at each time \( t \), the future values of \( \theta(t) \) up to the control horizon \( t + h \) are needed. Because the random walk is a martingale, predicted values of \( \theta(t) \) are obtained as: \( \theta(t) = \hat{\theta}(t), \tau = t, \ldots, t + h \), hence are constant up to the control horizon.

Both (7) and its certainty equivalent analogue in the presence of Gaussian noise processes are based on the assumption that the system parameters are known. In psychotherapeutic process analysis, however, all model parameters in (1) will be uncertain. Consequently, two conflicting goals have to be pursued simultaneously: parameter estimation and exercising optimal control. This is called dual control (cf. Filatov and Unbehauen (2000)) which in most situations cannot be solved analytically. An approximate recursive scheme for dual control is presented in Tucci (2004, chapter 4), but its performance in a Monte Carlo experiment using a simple univariate state process with a univariate control variable and a single uncertain time-varying parameter is inferior to the certainty equivalence control described above if the control horizon \( h \) is larger than 2 (Tucci, 2004, p. 81–86). Another approximate solution based on the assumptions of constant uncertain parameters and allowing these parameters to have only values in a finite set is described in Li, Qian, and Fu (2008). It will be determined in future work whether this solution can be generalized to psychotherapeutic process analysis based on the client model (1).

4. Conclusion

A new computational paradigm is proposed to optimize individual psychotherapeutic processes. It is based on a dynamic client model of the sequential regularities induced in a client’s state as function of the therapeutic manipulations. The client’s state is assessed immediately before the beginning of each session. A state-of-the-art recursive estimator then updates the parameters in the client model, enabling determination of the optimal profile of therapeutic manipulations by means of a feedback control algorithm. This information is available to the therapist at the start of the session. At the close of each session the profile of therapeutic manipulations actually administered is assessed. The scales with which the state of a client and the therapeutic manipulations are measured as well as the cost functions in the feedback controller can be chosen in accordance with the specific goals of each psychotherapy.

The new paradigm involves at least two innovations: (i) use of a client model with time-varying parameters in order to accommodate the change processes induced by psychotherapeutic intervention, and (ii) application of feedback control theory to optimize the effects of interventions. As to (i), almost all analyses of multivariate psychological time series are based on models having constant parameters (e.g., Hamaker, Dolan, and Molenaar (2005)). The only exceptions of which the author is aware concern an application of extended Kalman filtering to fit a nonlinear model to rhythmic finger movements undergoing a bifurcation (Molenaar & Newell, 2003) and the fit of linear Gaussian state-space models in which the parameters are time-varying according to fixed polynomial trend functions (Molenaar, 1994).

The recursive estimation scheme (4) involves what appears to be a new combination of second-order extended Kalman filtering and maximum likelihood estimation of noise variances. The estimation scheme used in Young (2000) is similar to (4), but differs in detail. A large scale Monte Carlo study using client model (1) is currently being carried out to investigate the statistical properties of (4) (Molenaar, in preparation). Preliminary results thus obtained show excellent performance (e.g., almost 100% identification without a priori information of time-varying parameters in bivariate time series of length \( T = 200 \)). The approach followed by Tucci (2004) involves recursive estimation of both time-varying parameters and noise variances. However, without the invocation of nonlinear constraints positive definiteness of noise covariance matrices is not guaranteed in this approach.

Another innovation is the application of feedback control to optimize the effects of interventions in psychotherapeutic process analysis. An early application using suboptimal recursive Bayesian estimation and infinite horizon feedback control is reported in Molenaar (1987). The present proposal involves not only a more powerful estimation scheme, but also the use of a simple variant of receding horizon control. Receding horizon control provides for adaptive control algorithms with good performance characteristics and flexible handling of constraints (cf. Rossiter, 2003, chapter 11; Goodwin, Seron, & De Doná, 2005).

Several more complex variants of the proposed computational paradigm for psychotherapeutic process analysis and control can be developed along the lines presented here. In particular approaches in which parameters are not modeled as slowly varying in time, as in client model (1), but instead undergo sudden switches to a finite set of alternative regimes at random times. Recursive estimation and control theory for the latter models is presented in, e.g., Costa, Fragoso, and Marques (2005) and Elliott, Aggoun, and Moore (1995). It remains to be determined in future research whether these alternative approaches prove to be worthwhile in applications to psychotherapeutic process analysis and control.
References