Nonlinear Model for the Design of Stable MEMS Resonators

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Summary: A double-ended tuning fork resonator is shown to demonstrate nonlinear resonance. The geometric nonlinearity of the structure is modeled using finite element analysis. To validate the model the measured frequency response is curve-fit using Duffing’s equation for nonlinear oscillators. The results validate the model, and show how a resonator may be designed for linear operation.

Keywords: Nonlinear, Resonator, Modeling

1 Introduction

Resonant MEMS devices have been widely reported and exploited in a number of commercial applications. Due to the excellent mechanical properties of single-crystal silicon, devices with high quality factors in excess of 30,000 are routinely fabricated\textsuperscript{1}. However as vibration amplitude increases geometric non-linearity may cause the resonator to operate in an unstable regime. As device amplitude increases the resonator effectively becomes stiffer, as such its resonant frequency increases. This leads to the classic non-linear frequency response, described by Duffing’s equation\textsuperscript{2}, see figure 1.

![Graph showing the frequency response of a Duffing oscillator](image)

\textit{Fig. 1. Nonlinear frequency response curve of a Duffing oscillator.}

As critical amplitude is reached the response curve is sufficiently skewed that the resonator becomes unstable. From this point onwards the resonator may snap to lower amplitude. For example, from point A to B on figure 1 for increasing frequency or point C to D for decreasing frequency. To prevent this, the resonator must be driven with a lower drive force resulting in lower response amplitude.

Determining the maximum allowable amplitude is therefore a critical step in designing MEMS resonators, as this must be known before a suitable detection method is designed. This paper shows how the non-linear behavior of any mode may be predicted using finite element analysis (FEA). Experimental data collected from a double-ended tuning fork (DETF) is curve-fitted, with a Duffing oscillator response. The determined stiffness non-linearity is shown to be in agreement with FEA predictions. Based on this the maximum safe vibration amplitude may be determined.

2 The Resonator

A double-ended tuning fork resonator was fabricated using deep reactive ion etching (DRIE) and silicon on insulator (SOI) wafers. The device was patterned using a direct write electron beam writer. The beams are approximately 2.5\textmu m wide, and 1000\textmu m long. The device depth is approximately 18\textmu m. The device was etched and released using a one-step dry release process\textsuperscript{3}. This resonator is designed to be electro-statically driven with capacitive sensing, by the two electrodes either side of the resonator as shown by figure 2.

![Image of a double-ended tuning fork resonator](image)

\textit{Fig. 2. Double-ended tuning fork resonator.}

3 Finite Element Analysis

The Resonator is designed to be electrically well coupled to the first and second modes. A modal analysis using Abaqus FEA software was carried out. The mode shapes and corresponding frequencies are shown in figure 3. Previous work has shown it is possible to estimate the degree of nonlinearity using FEA to generate a force extension curve, which may be used to calculate the device stiffness\textsuperscript{4}. Here simple force extension may not be used to model the equivalent stiffness of each mode shape, therefore a different approach was taken. The structure was constrained to deformations equivalent to each mode shape, at a range of amplitudes. By relating the total strain energy or internal work for each amplitude an equivalent stiffness could be determined by applying Castigliano’s theorem. This theorem relates the loads on a structure to the partial derivative of strain energy with
respect to displacement\(^2\). This is done in terms of one degree of freedom. As the resonance is sensed from one electrode only this degree of freedom was taken to be the movement of one tine relative to the sense electrode.

![Mode 1: 14.4 KHz, Mode 2: 16.8 KHz](image)

**Fig. 3.** Mode shapes and frequencies of first two modes.

### 4 Resonator Characterisation

The frequency response for the DETF resonator was measured using a Soletron 1250 frequency response analyser. The response curves for a range of drive force amplitudes were collected for both the first and second modes. Figure 4 shows response curves for mode 1. As amplitude increases the response becomes progressively more asymmetric. The unstable ‘snapping’ phenomenon is also clearly visible for the higher drive amplitudes.

![Frequency Response Curves with Increasing Drive Amplitudes](image)

**Fig.4.** Measured Response Curves for mode 1.

### 5 The Duffing Equation

The Duffing equation states that the equation of motion for a class of non-linear oscillators may be written as (Equation 1). Where \( F \) is the forcing amplitude, \( m \) is the resonator mass, \( c \) is the damping coefficient, \( k \) is the standard linear stiffness coefficient, and \( \mu \) is the non-linear cubic stiffness coefficient.

\[
m\ddot{x} + c\dot{x} + kx + \mu x^3 = F \sin \omega t
\]

(Equation 1)

Duffing shows how this may be solved analytically\(^2\), to yield a relation between drive amplitude \( F \), response amplitude \( A \), frequency \( \omega \), and natural frequency \( \omega_n \), shown by equation 2.

\[
F^2 = (\omega_n^2 - \omega^2)A + \frac{3}{4}A^3 + [\omega_0A]^2
\]

(Equation 2)

By solving equation 2 numerically using Matlab the nonlinear response of a resonator may be plotted in terms of the drive force amplitude, damping, resonant frequency, and cubic stiffness. By fitting the Duffing equation response to the measured response these parameters including the cubic stiffness for the resonator may be determined.

### 6 Results & Discussion

Table 1 gives a comparison between the cubic stiffness of each mode determined by curve fitting the Duffing equation, and those predicted using FEA. The results show that the absolute values of cubic stiffness only agree to within an order of magnitude. Given the many assumptions made to convert the measured amplitude from a voltage to a real displacement this is deemed to be an acceptable error. The model may be validated however by comparison of the first and second modes as all assumptions are the same for each mode. The percentage increase in cubic stiffness from the first to the second mode is approximately 160\% for both the predicted and measured values.

<table>
<thead>
<tr>
<th>Cubic Stiffness (N/m(^3))</th>
<th>Predicted</th>
<th>Measured</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>2.05 \times 10^{11}</td>
<td>1.06 \times 10^{12}</td>
</tr>
<tr>
<td>Mode 2</td>
<td>5.31 \times 10^{11}</td>
<td>2.80 \times 10^{12}</td>
</tr>
<tr>
<td>Increase</td>
<td>159%</td>
<td>162%</td>
</tr>
</tbody>
</table>

### 7 Conclusion

A fabricated MEMS resonator has been characterised and shown to exhibit a nonlinear resonant behaviour. It is shown that this may be well modelled using the Duffing equation for nonlinear oscillators. By using simple FEA static analysis the cubic stiffness term may be calculated for any mode shape. This may then be used to plot the frequency response of a device based on the Duffing equation. For stable resonance a response curve that is not prone to ‘snapping’ is required, so a suitable maximum amplitude may be determined. The designer may then design a suitable detection method, or redesign the resonator to give a more linear response.

### References