Necessary Conditions for Existence of Some Designs in Polynomial Metric Spaces

PETER BOYVALENKOV†, SILVIA BOUMOVA AND DANYO DANEV

In this paper we consider designs in polynomial metric spaces with relatively small cardinalities (near to the classical bounds). We obtain restrictions on the distributions of the inner products of points of such designs. These conditions turn out to be strong enough to ensure obtaining nonexistence results already for the first open cases.

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1. INTRODUCTION

Let $M$ be a metric space with a finite diameter $D$ and a finite normalized measure $\mu_M$. Let the Hilbert space $L_2(M)$ of complex-valued functions be decomposed into a countable (when $M$ is infinite) or finite (with $D + 1$ members when $M$ is finite) direct sum of mutually orthogonal subspaces

$$L_2(M) = V_0 \oplus V_1 \oplus \cdots$$

($V_0$ is the space of constant functions). We denote $N = D + 1$ if $M$ is finite and $N = \infty$ otherwise.

In definition and description of the notion of polynomial metric spaces we follow [26, 27].

**DEFINITION 1.1.** The space $M$ is called a polynomial metric space (PMS) if there exists an ordering of the spaces $V_i, i = 0, 1, \ldots, N$, and a system of real polynomials $(Q_i(t))_{i=0}^N$ (called zonal spherical functions (ZSF)) such that for all $y \in M$

$$Q_i(\sigma_M(d(x, y))) = \frac{1}{r_i} \sum_{j=1}^{r_i} v_{ij}(x)v_{ij}(y),$$

where $r_i = \dim(V_i)$, $\{v_{ij}(x) : 1 \leq j \leq r_i\}$ is an orthonormal basis of $V_i$ (so $Q_i(1) = 1$), and $t = \sigma_M(d)$ is a continuous decreasing real function (substitution) such that $\sigma_M(0) = 1$ and $\sigma_M(D) = -1$. For $x, y \in M$ we call the number $t = \sigma_M(d(x, y)) \in [-1, 1]$ their inner product.

Among the infinite PMS, we mention the compact symmetric spaces of rank 1 (called also two-point homogeneous spaces; see [13, 21, 23, 28, 33]). They are classified to be the euclidean spheres $S^{n-1}$ and the projective spaces $\mathbb{P}P^{n-1}$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ ($\mathbb{O}P^{n-1}$ exists for $n = 2, 3$ only). The finite PMS are represented by $(P$ and $Q)$-polynomial association schemes [7, 14]. Two of the most important examples are the Hamming and the Johnson spaces. However, the finite PMS have not yet been completely classified [7, 13, 14, 32].

The system $(Q_i(t))_{i=0}^N$ is orthogonal with respect to the measure $\nu(t) = 1 - \mu_M(\sigma_M^{-1}(t))$. The properties of this orthogonal system imply many important results in PMS. Together with the ZSF one considers their adjacent systems [26, Section 3] of polynomials $(Q^{a,b}_i(t))_{i=0}^N, (a, b \in [0, 1], N^{a,b} = N - 2 + \delta_{a,0} + \delta_{b,0})$ which are orthogonal with respect to the measure $(1 - t)^a(1 + t)^b\nu(t)$ and normalized for $Q^{a,b}_i(1) = 1$ (cf. Subsection 2.1).

† Author to whom all correspondence should be addressed.
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DEFINITION 1.2. A nonempty finite set (code) \( C \subset M \) is called a \( \tau \)-design if and only if the equality \( \sum_{x \in C} v(x) = 0 \) holds for any function \( v(x) \in V_1 \oplus V_2 \oplus \cdots \oplus V_\tau \).

The designs in polynomial metric spaces possess a number of regularity properties. Known as spherical designs, classical \( t \)-designs, orthogonal arrays, etc., they are investigated in the algebraic combinatorics, the classical combinatorics, coding theory, etc.

The problem for finding lower bounds on the minimum possible size of designs in PMS was considered by many authors (see e.g., Delsarte [14] for \( \tau = 2k \) and Dunkl [17] for \( \tau = 2k - 1 \) in finite PMS, also [30] for Hamming spaces, [31] for Johnson spaces, [16] for euclidean spheres, and [22] for projective spaces). The minimum possible cardinality \( B(M, \tau) \) among all \( \tau \)-designs in \( M \) is bounded from below by:

\[
B(M, \tau) \geq R(M, \tau) = \left\{ \begin{array}{ll}
1 - \frac{Q_{k-1}^{1,0}(-1)}{Q_k(-1)} & \text{if } \tau = 2k - 1, \\
\sum_{i=0}^{k-1} r_i & \text{if } \tau = 2k.
\end{array} \right.
\]

Although this bound was obtained in different cases by different authors (see above), we use the common name Delsarte bound for (1). Actually, the above presentation of the Delsarte bound is due to Levenshtein [26, 27].

A \( \tau \)-design in \( M \) is called tight if it attains the Delsarte bound. Tight designs were investigated by many authors (cf. [3–6, 8, 15, 16, 29]) and they seem to exist very rarely. The Delsarte bound is therefore improved by one in the cases where the nonexistence of tight designs was proved.

We make use of the following equivalent definition (see, for example, [18]).

DEFINITION 1.3. A code \( C \subset M \) is a \( \tau \)-design if and only if for any point \( y \in M \) the equality

\[
\sum_{x \in C} f(\sigma_M(d(x, y))) = |C| f_0
\]

holds for any real polynomial \( f(t) \) of degree at most \( \tau \), where \( f_0 = f(1) \) is the first coefficient in the expansion \( f(t) = \sum_{i=0}^{k} f_i Q_i(t) \) in terms of the ZSF. We denote \( \Omega(f) = f(1)/f_0 \).

DEFINITION 1.4. The number \( s(C) = \max\{\sigma_M(d(x, y)) : x, y \in C, x \neq y\} \) is called the maximal inner product of the code \( C \). The number \( d(C) = \min\{d(x, y) : x, y \in C, x \neq y\} \) is called the minimum distance of the code \( C \). It is clear that \( s(C) = \sigma_M(d(C)) \).

In this paper we firstly obtain bounds on inner products of the points of \( \tau \)-designs. This gives necessary conditions for the existence of designs. For both odd strengths and cardinalities these imply nonexistence results in many cases. Bounds on the maximal inner product and the minimum distance of \( \tau \)-designs are obtained as well.

In Section 2 we explain some known notations and properties of the polynomials used for obtaining universal linear programming bounds for codes and designs in PMS. The notion of antipodal PMS is given in Subsection 2.6. Our notation for the inner products is given in Subsection 2.7.

In Section 3 we obtain bounds on the least and greatest inner products of \((2k - 1)\)-designs. To do this we use suitable polynomials in (2) (for \( y \in C \)) and then use the properties of the Levenshtein polynomials (described in Subsections 2.3 and 2.4). The upper bound on
the least inner product implies a nonexistence result for designs with both odd strengths and cardinalities. This is expressed by the inequality

\[ p_0 |C| \geq 2 \]  

which must hold for all \((2k-1)\)-designs with odd cardinality \(|C|\).

In Section 4 we consider \((2k)\)-designs. Bounds for the inner products are obtained in a similar way as in Section 3. However, the results seem to have a somewhat different logic.

In Section 5 we consider self-complementary and antipodal designs. Necessary conditions for existence of self-complementary designs in finite and infinite spaces are obtained.

In Section 6 we show some applications of our results in the most interesting case – the euclidean spheres as the classical example of a PMS.

2. SOME PRELIMINARIES

2.1. ZSF and their adjacent systems. The ZSF of the compact symmetric spaces of rank 1 are Jacobi polynomials \(P_{n,\alpha,\beta}(t)\) \(\alpha, \beta \in \mathbb{R}\) \(n \geq 0\) [1, Chapter 22], where

\[ (\alpha, \beta) = \left( \frac{n-3}{2}, \frac{n-3}{2} \right), \quad \left( \frac{n-3}{2}, -\frac{1}{2} \right), \quad (n-2, 0), \quad (2n-3, 1) \]

for \(M = S^{n-1}, \mathbb{R}P^{n-1}, \mathbb{C}P^{n-1}, \mathbb{H}P^{n-1}\), respectively (see [22, 23, 26, 27]). The ZSF for the Hamming and Johnson spaces are Krawtchouk and Hahn polynomials respectively.

The adjacent (to the ZSF) polynomials \(Q_{a,b}^c(t)\) are defined to satisfy the orthogonality condition

\[ Q_{a,b}^c(t)Q_{\alpha,b}^c(t)(1-t)^\alpha (1+t)^\beta d\nu(t) = \delta_{i,j}, \]

where \(c_{a,b} \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta d\nu(t) = 1\) and \(Q_{k,b}^c(1) = 1\). Thus the adjacent polynomials for the infinite PMS are the Jacobi polynomials \(P_{\alpha,\beta,\gamma}^{(n)}(t)\). For more details see [26, 27].

2.2. The linear programming bound for designs in PMS. The following theorem is known as the ‘linear programming bound’ for designs in PMS (cf. [14, 17, 25, 26]).

**THEOREM 2.1 ([14]).** Let \(M\) be a PMS, \(\tau \geq 1\), and \(f(t)\) be a nonzero real polynomial such that

- **(B1)** \(f(t) \geq 0\) for \(-1 \leq t \leq 1\), and \(f(t)\) is a nonzero real polynomial

- **(B2)** The coefficients in the ZSF expansion \(f(t) = \sum_{i=0}^{k} f_i Q_i(t)\) satisfy \(f_{\tau+1} \leq 0, \ldots, f_k \leq 0\).

Then \(B(M, \tau) \geq f(1)/f_0\).

The Delsarte bound can be obtained by using the polynomial \((Q_{k}^{1,0}(t))^2\) for \(\tau = 2k\) [14], and by using the polynomial \((1+1)(Q_{k-1}^{1,1}(t))^2\) for \(\tau = 2k-1\) [17].

2.3. The linear programming bound for codes in PMS. The maximal possible cardinality among all codes in \(M\) with maximal inner product \(s\) is denoted by \(A(M, s)\). The following theorem is known as the ‘linear programming bound’ for codes in PMS (cf. [13, Chapter 9], [14, 25, 26]).
THEOREM 2.2. Let $\mathcal{M}$ be a PMS, $s \in [-1, 1)$, and $f(t)$ be a real polynomial such that

1. $f(t) \leq 0$ for $-1 \leq t \leq s$, and
2. The coefficients in the ZSF expansion $f(t) = \sum_{i=0}^{k} f_i Q_i(t)$ satisfy $f_0 > 0$, $f_i \geq 0$ for $i = 1, \ldots, k$.

Then $A(\mathcal{M}, s) \leq f(1)/f_0$.

For finite PMS, the condition (A1) is stronger than is really required. Indeed, then the polynomial $f(t)$ must be nonpositive in all possible inner products of points of $\mathcal{M}$ which belong to the interval $[-1, s]$. This fact was extensively used [14, 15, 28].

2.4. Levenshtein polynomials and Levenshtein bound. Set $T_{k}^{a,b}(u, v) = \sum_{i=0}^{k} t_{i}^{a,b} Q_{i}^{a,b}(u) Q_{i}^{a,b}(v)$. Levenshtein (cf. [24], see also [25–27]) obtained an universal upper bound on $A(\mathcal{M}, s)$ by using Theorem 2.2 with the polynomials

$$f^{(M,s)}_m(t) = \begin{cases} f_{2k-1}^{1,0}(t) = (t-s)f_{k-1}^{1,0}(t, s)^2 & \text{for } t_{k-1}^{1,0} \leq s \leq t_{k}^{1,0}, \\ f_{2k}^{1,1}(t) = (t+1)(t-s)f_{k-1}^{1,1}(t, s)^2 & \text{for } t_{k}^{1,0} \leq s \leq t_{k}^{1,1}, \end{cases}$$

where $t_{i}^{a,b}$ is the greatest zero of $Q_{i}^{a,b}(t)$. It is customary to call $f^{(M,s)}_m(t)$ the Levenshtein polynomials.

The real numbers $\{t_{i}^{1,1}\}_{i=0}^{k}$ (set $t_{0}^{1,1} = -1$) and $\{t_{i}^{1,0}\}_{i=1}^{k}$ divide the interval $[-1, 1]$ into consecutive closed nonoverlapping intervals $I_1$, $I_2$, ... . For each $m = 1, 2, \ldots$, and all $s \in I_m$ one has $A(\mathcal{M}, s) \leq L_m(\mathcal{M}, s)$, where

$$L_m(\mathcal{M}, s) = \begin{cases} L_{2k-1}(\mathcal{M}, s) = \left(1 - \frac{Q_{k-1}^{1,0}(s)}{Q_{k}(s)}\right) \sum_{i=0}^{k-1} r_i & \text{for } t_{k-1}^{1,1} \leq s \leq t_{k}^{1,0}, \\ L_{2k}(\mathcal{M}, s) = \left(1 - \frac{Q_{k}^{1,0}(s)}{Q_{k}^{0,1}(s)}\right) \sum_{i=0}^{k} r_i & \text{for } t_{k}^{1,0} \leq s \leq t_{k}^{1,1}. \end{cases}$$

In the common boundary points of $I_m$ and $I_{m+1}$ we have [26]

$$L_{2k-1}(\mathcal{M}, t_{k}^{1,0}) = L_{2k}(\mathcal{M}, t_{k}^{1,0}) = \sum_{i=0}^{k} r_i = R(\mathcal{M}, 2k)$$

and

$$L_{2k}(\mathcal{M}, t_{k}^{1,1}) = L_{2k+1}(\mathcal{M}, t_{k}^{1,1}) = \left(1 - \frac{Q_{k}^{1,0}(-1)}{Q_{k+1}^{0,1}(-1)}\right) \sum_{i=0}^{k} r_i = R(\mathcal{M}, 2k + 1).$$

These equalities shed light on the ‘duality’ on the bounds (1) and (4) (compare also Theorems 2.1 and 2.2).

Note also that the ‘even’ bounds $L_{2k}(\mathcal{M}, s)$ are proved to be valid when $f^{(M,s)}_m(t)$ is proved to be expanded with nonnegative ZSF coefficients. This is true in the most important examples, such as the compact symmetric spaces of rank 1, the Hamming spaces, etc. For more information see [26, 27].
2.5. Some properties of the Levenshtein polynomials. The polynomial $f^{(\lambda_1,s)}_{2k-1}(t)$ (resp. $f^{(\lambda_2,s)}_{2k}(t)$) has exactly $k$ (resp. $k+1$) different zeros $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} = s$ (resp. $-1 = \beta_0 < \beta_1 < \cdots < \beta_k = s$) in the interval $[-1,1]$. Furthermore, there exist positive weights $\rho_i$, $i = 0, 1, \ldots, k-1$ (resp. $y_1 = 0, 1, \ldots, k$; in fact $y_0 = 0$ for $s = t_k^{1,0}$ and $y_0 > 0$ for $s > t_k^{1,0}$) and a number $\rho_k$ (resp. $y_{k+1}$), positive for $s < t_k^{0,0}$ (resp. for $s < t_k^{0,1}$), such that the equality

$$f_0 = \rho_k f(1) + \sum_{i=0}^{k-1} \rho_i f(\alpha_i) \quad \text{(resp. } f_0 = y_{k+1} f(1) + \sum_{i=0}^{k} y_i f(\beta_i))$$

holds for any polynomial of degree at most $2k-1$ (resp. $2k$).

**Theorem 2.3** ([18, 26]). We have $\rho_k = 1/L_{2k-1}(\mathcal{M},s)$ (resp. $y_{k+1} = 1/L_{2k}(\mathcal{M},s)$) for $t_k^{1,1} \leq s \leq t_k^{0,0}$ (resp. for $t_k^{0,0} \leq s \leq t_k^{1,1}$).

**Proof.** Set the polynomial $f^{(\lambda_1,s)}_{2k-1}(t)$ (resp. $f^{(\lambda_2,s)}_{2k}(t)$) in (5).

As the functions $L_{2k}(\mathcal{M},s)$ are continuous and strictly increasing in $s$, it follows that for any integer (cardinality) $M \in [L_{2k-1}(\mathcal{M},t_k^{1,1}), \infty) \equiv [R(\mathcal{M},2k-1), \infty)$ (resp. $M \in [L_{2k}(\mathcal{M},t_k^{1,0}), \infty) \equiv [R(\mathcal{M},2k), \infty)$) there exists a unique $s \in [t_k^{1,1}, t_k^{0,0}]$ (resp. $s \in [t_k^{1,0}, t_k^{0,1}]$) such that $M = \Omega(f^{(\lambda_1,s)}_{2k-1})$ (resp. $M = \Omega(f^{(\lambda_2,s)}_{2k})$). In what follows we always associate any $\tau$-design $C \subseteq \mathcal{M}$ with the unique $s \in [t_k^{1,1}, t_k^{0,0}]$ (resp. $s \in [t_k^{1,0}, t_k^{0,1}]$) for $\tau = 2k-1$ or $s \in [t_k^{0,1}, t_k^{0,0}]$ for $\tau = 2k$ such that $|C| = \Omega(f^{(\lambda_{\tau},s)}_{k})$. Then all parameters from this subsection come with this $s$.

2.6. Antipodal PMS. A PMS $\mathcal{M}$ is called antipodal if for every point $x \in \mathcal{M}$ there exists a point $y \in \mathcal{M}$ such that for any point $y \in \mathcal{M}$ we have

$$\sigma_{\mathcal{M}}(d(x,y)) + \sigma_{\mathcal{M}}(d(y,x)) = 0.$$

The point $y$ is uniquely determined by $d(x,y) = D$, i.e., $d(x,y) = -1$. In antipodal PMS, the ZSF form a symmetric system, i.e., $Q_i(t) = (-1)^i Q_i(-t)$ for all $i$ and $t$. The above description of antipodal PMS is due to Levenshtein [27].

2.7. Some notations. For a $\tau$-design $C \subseteq \mathcal{M}$ and $y \in C$ we denote $I(y) = |\sigma_{\mathcal{M}}(d(x, y)) : x \in C, x \neq y|$ counting with the multiplicities. Thus we may assume that $I(y) = \{t_1, t_2, \ldots, t_{|C|-1}\}$ where $-1 \leq t_1 \leq t_2 \leq \cdots \leq t_{|C|-1} < 1$. Then (2) becomes

$$\sum_{i=1}^{\lfloor C \rfloor - 1} f(t_i) = |C| f_0 - f(1)$$

and we use it in this form.

3. Necessary Conditions for Existence of $(2k-1)$-Designs

Let $C \subseteq \mathcal{M}$ be a $(2k-1)$-design, $y \in C$, and let $s \in [t_k^{1,1}, t_k^{0,0}]$ be such that $|C| = \Omega(f^{(\lambda_1,s)}_{2k-1})$. We firstly derive an upper bound on the inner products $t_1$ and a lower bound on $t_{|C|-1}$.
THEOREM 3.1. We have \( t_1 \leq \alpha_0 \) and \( t_{|C|^{-1}} \geq s = \alpha_{k-1} \). If equality holds in one of these two cases then \( I(y) \subseteq \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\} \).

PROOF. We set in (6) the polynomial

\[
f(t) = (t - t_1)(t - s) f^{(M,s)}_{2k-1}(t)/(t - \alpha_0)^2 = (t - t_1) \prod_{i=1}^{k-1} (t - \alpha_i)^2.
\]

Then the LHS is nonnegative, and the RHS by (5) is \( f(t_0) - f(t_1) = \rho_0 f(\alpha_0)|C| \). Therefore, we have \( f(\alpha_0) \geq 0 \) which immediately implies \( t_1 \leq \alpha_0 \). Analogously, by

\[
f(t) = (t - t_{|C|^{-1}}) f^{(M,s)}_{2k-1}(t)/(t - \alpha_0)^2 = (t - t_{|C|^{-1}}) \prod_{i=0}^{k-2} (t - \alpha_i)^2
\]

we obtain \( t_{|C|^{-1}} \geq s = \alpha_{k-1} \).

\[\square\]

Let \( C \subset M \) be a \((2k-1)\)-design with \(|C|\) odd. Theorem 3.1 gives \( t_1 \leq \alpha_0 \) for any point \( y \in C \). We conclude that the same inequality must be satisfied by \( t_2 \) for some \( y \in C \) and otherwise the design could not exist.

THEOREM 3.2. If \(|C|\) is odd and \( t_2 > \alpha_0 \) for all \( y \in C \) then \( C \) does not exist.

PROOF. The inequalities \( t_1 \leq \alpha_0 < t_2 \) mean that for the point \( y \) there exists a unique point \( x \) such that \((y, x) \in [-1, \alpha_0] \). Conversely, \( y \) uniquely corresponds to \( x \) in this situation. Therefore, the points of \( C \) must be divided into disjoint pairs (every point together with its furthest) which is impossible when \(|C|\) is odd.

\[\square\]

THEOREM 3.3. Let \( C \subset M \) be a \((2k-1)\)-design with \(|C|\) odd. Then \( \rho_0|C| \geq 2 \). If equality holds then \( I(y) \subseteq \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\} \) for all \( y \in C \).

PROOF. By Theorem 3.2, there exists \( y \in C \) such that \( t_2 \leq \alpha_0 \). We set

\[
f(t) = f^{(M,s)}_{2k-1}(t)/(t - \alpha_0)^2 \prod_{i=1}^{k-1} (t - \alpha_i)^2
\]

in (6) and obtain

\[
2 f(\alpha_0) \leq 2 f(t_2) \leq f(t_1) + f(t_2) \leq \sum_{i=1}^{\frac{|C| - 1}{2}} f(t_i)
\]

\[
= f_0|C| - f(1) = |C| \sum_{i=0}^{k-1} \rho_i f(\alpha_i) = |C| \rho_0 f(\alpha_0)
\]

(because \( f(t) \) is decreasing in \([-1, \alpha_0] \), \( f_0 = \rho_0 f(\alpha_0) + \rho_k f(1) \) by (5), and \(|C| = 1/\rho_k \) which implies our inequality. If equality holds then \( t_1 = \alpha_0 \) and Theorem 3.1 is applied.

To study the existence condition \( \rho_0|C| \geq 2 \), one needs expressions for the parameter \( \rho_0 \).

LEMMA 3.4. (a) [26, Eqn 4.3] \( \rho_0 = \frac{1}{e^{1.0}(1 - \alpha_0)T_{k-1,0}(\alpha_0, \alpha_0)} \).

(b) [10, Theorem 3.8] If \( M \) is antipodal then

\[
\rho_0 = \frac{(1 - \alpha_0^2)(1 - \alpha_1^2) \cdots (1 - \alpha_{k-1}^2)}{\alpha_0 L_{2k-1}(M, s)(\alpha_0^2 - \alpha_1^2)(\alpha_0^2 - \alpha_2^2) \cdots (\alpha_0^2 - \alpha_{k-1}^2)}.
\]
**Corollary 3.5.** If $\mathcal{M}$ is antipodal and $C \subset \mathcal{M}$ is a $(2k - 1)$-design with $|C|$ odd, then

$$
\frac{1}{2a_0} \prod_{i=1}^{k-1} \frac{1}{\alpha_0^2 - \alpha_i^2} \leq -1.
$$

If equality holds then $I(y) \subseteq \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$ for all $y \in C$.

**Remark 3.6.** In the case $\rho_0|C| = 2$, the inclusion $I(y) \subseteq \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$ is valid for all $y \in C$. This implies very strong restrictions on the code $C$. Indeed, it follows that $C$ attains the Levenshtein bound $L_{2k-1}(\mathcal{M}, s)$ if $s \leq t_k^{1,0}$. Thus its distance distribution can be computed in terms of $|C|$, $s$, and the ZSF (cf. [10, Section 3]; see also [27, Remark 5.58]). Note also that equality is impossible when $s > t_k^{1,0}$.

In what follows in this section we assume that $\rho_0|C| < 2$, i.e., $|C|$ is even. For such designs we obtain bounds on their minimum distance and maximal inner product.

**Lemma 3.7.** Let $\delta_1$ and $\mu_1$ be the smallest and the greatest root, respectively, of the equation

$$
f(t) = f(\alpha_0)(\rho_0|C| - 1),
$$

where $f(t) = \prod_{i=1}^{k-1} (t - \alpha_i)^2$. Then $t_2 \geq \delta_1$ and $t_{|C|-1} \leq \mu_1$.

**Proof.** By $f(t) = \prod_{i=1}^{k-1} (t - \alpha_i)^2$ in (6) we obtain

$$
\sum_{i=2}^{|C|-1} f(t_i) = |C|\rho_0 f(\alpha_0) - f(t_1)
$$

(7)

(use (5) and Theorem 3.1). The estimation $t_2 \geq \delta_1$ follows because $f(t_2) \leq f(\alpha_0)(\rho_0|C| - 1)$ and $f(t)$ is decreasing in $(-\infty, \alpha_1]$, and the estimation $t_{|C|-1} \leq \mu_1$ follows because $f(t_{|C|-1}) \leq f(\alpha_0)(\rho_0|C| - 1)$ and $f(t)$ is increasing in $[s, \infty)$.

The bound $t_2 \geq \delta_1$ allows us to improve the bound $t_1 \leq \alpha_0$.

**Lemma 3.8.** Let $\lambda_1$ be the smallest root of the equation

$$
f(t) = f(\alpha_0)\rho_0|C|,
$$

where $f(t) = (t - \delta_1) \prod_{i=1}^{k-1} (t - \alpha_i)^2$. Then $t_1 \leq \lambda_1 < \alpha_0$.

**Proof.** We use $f(t) = (t - \delta_1) \prod_{i=1}^{k-1} (t - \alpha_i)^2$ in (6). The LHS is at least $f(t_1)$ and the RHS equals $f(\alpha_0)\rho_0|C|$ and we are done because $f(t)$ is increasing in $(-\infty, \delta_1]$. The inequality $\lambda_1 < \alpha_0$ follows from $\rho_0|C| > 1$.

The better bound $t_1 \leq \lambda_1$ gives an improvement of the above bounds. Indeed, we now can use $t_1 \leq \lambda_1$ in Lemma 3.7. We replace $f(t_1)$ by $f(\lambda_1)$ instead of $f(\alpha_0)$ in (7). This implies bounds $t_2 \geq \delta_2$ and $t_{|C|-1} \leq \mu_2$, where $\delta_2$ and $\mu_2$ are the smallest and the greatest root, respectively, of the equation

$$
f(t) = |C|\rho_0 f(\alpha_0) - f(\lambda_1)
$$
and \( f(t) = \prod_{i=1}^{k} (t - \alpha_i)^2 \). As \(|C| \rho_0 f(\alpha_0) - f(\alpha_0) < |C| \rho_0 f(\alpha_0) - f(\lambda_1)\), we have \( \delta_2 > \delta_1 \) and \( \mu_2 < \mu_1 \). Now the better bound \( t_2 \geq \delta_2 \) can be used in an analog of Lemma 3.8 for obtaining a further better bound \( t_1 \leq \lambda_2 < \lambda_1 \).

It is clear that the above procedure can be used infinitely many times, i.e., we are able to obtain bounds \( t_2 \geq \delta_k \), \( \delta_{k-1} \), \( \ldots \), \( \delta_1 \), \( t_{|C|-1} \leq \mu_k < \mu_{k-1} < \ldots < \mu_1 \), and \( t_1 \leq \lambda_k < \lambda_{k-1} < \ldots < \lambda_1 \) for any positive integer \( k \). Of course, it is not difficult to prove that the sequences \( \langle \delta_k \rangle_{k=1}^\infty \), \( \langle \mu_k \rangle_{k=1}^\infty \), and \( \langle \lambda_k \rangle_{k=1}^\infty \) are convergent. Therefore, the following is true.

**Theorem 3.9.** We have \( t_2 \geq \delta = \lim_{k \to \infty} \delta_k \), \( t_{|C|-1} \leq \mu = \lim_{k \to \infty} \mu_k \), and \( t_1 \leq \lambda = \lim_{k \to \infty} \lambda_k \).

**Corollary 3.10.** For any \((2k-1)\)-design \( C \subset \mathcal{M} \) we have

\[
 s \leq s(C) \leq \mu \quad \text{and} \quad \sigma_{\mathcal{M}}^{-1}(\mu) \leq d(C) \leq \sigma_{\mathcal{M}}^{-1}(s).
\]

**Proof.** Combine Theorem 3.1 and Theorem 3.9. \( \square \)

Actually, the bounds \( s \leq s(C) \) and \( d(C) \leq \sigma_{\mathcal{M}}^{-1}(s) \) are due to Levenshtein [25] (see also [18, 27]).

4. **Necessary Conditions for Existence of \((2k)\)-Designs**

Let \( C \subset \mathcal{M} \) be a \((2k)\)-design, \( y \in C \), and \( s \geq t_{1,0} \) be such that \(|C| = L_{2k}(\mathcal{M}, s)\). We derive a lower bound on \( t_1 \) and an upper bound on \( t_{|C|-1} \).

**Lemma 4.1.** Let \( \xi_1 \) and \( \eta_1 \) be the least and the greatest roots, respectively, of the equation

\[
 f(t) = \gamma_0 f(-1)|C|,
\]

where \( f(t) = \prod_{i=1}^{k} (t - \beta_i)^2 \). Then for every point \( y \in C \) we have \( t_1 \geq \xi_1 \) and \( t_{|C|-1} \leq \eta_1 \) (i.e., \( f(y) \subset [\xi_1, \eta_1] \)).

**Proof.** We use the polynomial \( f(t) \) in (6). The LHS is nonnegative, while the RHS equals \( \gamma_0 f(-1)|C| \). It is clear that outside the interval \([\xi_1, \eta_1]\) we have \( f(t) > \gamma_0 f(-1)|C| \) and this implies \( f(y) \subset [\xi_1, \eta_1] \). \( \square \)

In the next lemma we obtain bounds on the inner products \( t_2 \) and \( t_{|C|-2} \).

**Lemma 4.2.** Let \( \xi_2 \) and \( \eta_2 \) be the least and the greatest roots, respectively, of the equation

\[
 f(t) = \frac{\gamma_0 f(-1)|C|}{2},
\]

where \( f(t) = \prod_{i=1}^{k} (t - \beta_i)^2 \). Then for every \( y \in C \) we have \( t_2 \geq \xi_2 \) and \( t_{|C|-2} \leq \eta_2 \) (i.e., \( [t_2, t_3, \ldots, t_{|C|-2}] \subset [\xi_2, \eta_2] \)).

**Proof.** Let us assume that \( t_2 < \xi_2 \). We use the polynomial \( f(t) \) in (6). The RHS is \( \gamma_0 f(-1)|C| \), while the LHS is at least \( f(t_1) + f(t_2) \geq 2 f(t_2) > 2 f(\xi_2) = \gamma_0 f(-1)|C| \), a contradiction. Therefore \( t_2 \geq \xi_2 \). Analogously we obtain \( t_{|C|-2} \leq \eta_2 \). \( \square \)

For odd cardinalities \(|C|\), we prove stronger restrictions for at least one point \( y \in C \).
LEMMA 4.3. If \(|C|\) is odd then there exists a point \(y \in C\) such that \(t_1 \geq \xi_2\) and \(t_{|C|-1} \leq \eta_2\) (i.e., \(I(y) \subset [\xi_2, \eta_2]\) for this point).

PROOF. Let us assume that for all points \(y \in C\) we have \(t_1 < \xi_2\). Then the points of \(C\) can be divided into disjoint pairs as in Theorem 3.2 which is impossible. Therefore we have \(t_1 \geq \xi_2\) for some (at least one) point \(y \in C\). Similarly, there exists a point \(y \in C\) such that \(t_{|C|-1} \leq \eta_2\). Let \(A = \{x \in C : t_1 \geq \xi_2\}\) and \(B = \{x \in C : t_{|C|-1} \leq \eta_2\}\). We have to prove that \(A \cap B \neq \emptyset\). Let us assume that \(A \cap B = \emptyset\) and consider the sets \(C \setminus A\) and \(C \setminus B\). Again as in Theorem 3.2 we see that the points in these two sets can be divided into disjoint pairs. Hence the cardinalities \(|C \setminus A|\) and \(|C \setminus B|\) are even. As \(|C|\) is odd, this shows that \(|A|\) and \(|B|\) are odd as well. Then \(A \cup B = C\) is impossible (because \(|A \cap B| = 0\)) and we conclude that there exists \(y \in C\) which does not belong to \(A\) and \(B\). This means that \(t_1 < \xi_2\) and \(t_{|C|-1} > \eta_2\) for \(I(y)\). We apply (6) for \(y\) and the polynomial \(f(t)\) from Lemma 4.1. The RHS is \(\gamma_0 f(-1)|C|\), while the LHS is at least \(f(t_1) + f(t_{|C|-1}) > f(\xi_2) + f(\eta_2) = \gamma_0 f(-1)|C|\), a contradiction that completes the proof.

For arbitrary \(|C|\), Lemma 4.1 can be extended in the following way.

LEMMA 4.4. For every point \(y \in C\) we have \(t_1 \geq \xi_2\) or \(t_{|C|-1} \leq \eta_2\) (i.e., \(I(y) \subset [\xi_2, \eta_2]\) or \(I(y) \subset [\xi_1, \eta_2]\)).

PROOF. Let us suppose that \(t_1 < \xi_2\) and \(t_{|C|-1} > \eta_2\) for some point \(y \in C\). Using (6) for this point and for the polynomial \(f(t) = \prod_{i=1}^{k} (t - \beta_i)^2\) we reach a contradiction as in the end of the proof of Lemma 4.3.

The analog of Theorem 3.1 follows.

THEOREM 4.5. We have \(t_1 \leq \beta_1\) and \(t_{|C|-1} \geq s = \beta_k\). If equality holds in one of these two cases then \(I(y) \subset \{\beta_0, \beta_1, \ldots, \beta_k\}\).

PROOF. As in Theorem 3.1, with the polynomials \((t - t_1)(t - s)f_{2k}^{(M,s)}(t)/(t - \beta_1)^2\) and \((t - t_{|C|-1})f_{2k}^{(M,s)}(t)/(t - s)\) respectively.

By using Lemma 4.1, Theorem 4.5 can be slightly improved as follows.

LEMMA 4.6. We have
\[
t_1 \leq v_1 = \beta_1 - \frac{\gamma_0(1 + \eta_1)(1 + \beta_1)\prod_{i=2}^{k}(1 + \beta_i)^2}{\gamma_0(1 + \eta_1)\prod_{i=2}^{k}(1 + \beta_i)^2 - \gamma_1(\beta_1 - \eta_1)\prod_{i=2}^{k}(\beta_1 - \beta_i)^2}
\]
and
\[
t_{|C|-1} \geq v_2 = s + \frac{\gamma_0(1 - s)(1 + \xi_1)\prod_{i=1}^{k-1}(1 + \beta_i)^2}{\gamma_k(s - \xi_1)\prod_{i=1}^{k-1}(s - \beta_i)^2 - \gamma_0(1 + \xi_1)\prod_{i=1}^{k-1}(1 + \beta_i)^2}
\]

PROOF. We use the polynomials \(f(t) = (t - \eta_1)(t - t_1)\prod_{i=2}^{k}(t - \beta_i)^2\) (for the first estimation) and \(f(t) = (t - \xi_1)(t - t_{|C|-1})\prod_{i=2}^{k-1}(t - \beta_i)^2\) (for the second). The RHS is nonpositive, and the LHS equals \(\gamma_0 f(-1) + \gamma_1 f(\beta_1)\) or \(\gamma_0 f(-1) + \gamma_k f(s)\). It is easy to check that \(v_1 < \beta_1\) and \(v_2 > s\).

COROLLARY 4.7. For any \((2k)\)-design \(C \subset M\) we have
\[
v_2 \leq s(C) \leq \eta_1 \quad \text{and} \quad \sigma_{M}^{-1}(\eta_1) \leq d(C) \leq \sigma_{M}^{-1}(v_2).
\]
5. SELF-COMPLEMENTARY AND ANTIPODAL DESIGNS

**Definition 5.1.** The code $C \subseteq M$ is said to be self-complementary if for any point $y \in C$ there exists $x \in C$ such that $\sigma_M(d(x, y)) = -1$. When the point $x$ is uniquely determined by $y$ (by definition, this is the case in antipodal spaces) the code is called antipodal.

**Theorem 5.2.** Let $C \subseteq M$ be a self-complementary $(2k - 1)$-design, $y \in C$, and $m_y = |\{x \in C : \sigma(d(x, y)) = -1\}|$. Then

$$m_y \leq \rho_0|C| \prod_{i=1}^{k-1} \left( \frac{\alpha_0 - \alpha_i}{1 + \alpha_i} \right)^2 < \rho_0|C|.$$  

**Proof.** Use the polynomial

$$f(t) = (t-s)f^{(M,x)}(t)/(t-\alpha_0)^2 = \prod_{i=1}^{k-1} (t-\alpha_i)^2$$

in (6). The LHS is at least $m_y f(-1)$ while the RHS equals $\rho_0|C| f(\alpha_0)$. Therefore $\rho_0|C| f(\alpha_0) \geq m_y f(-1)$ which is equivalent to (8). The last inequality holds because $|(\alpha_0 - \alpha_i)/(1 + \alpha_i)| < 1$ for $i = 1, \ldots, k$. \hfill \Box

**Corollary 5.3.** Let $C \subseteq M$ be a self-complementary $(2k - 1)$-design and

$$\rho_0|C| \prod_{i=1}^{k-1} \left( \frac{\alpha_0 - \alpha_i}{1 + \alpha_i} \right)^2 < 2.$$  

Then $C$ is antipodal and, in particular, $|C|$ is even.

In a finite PMS, all possible inner products form a discrete set, say $\{s_0 = -1 < s_1 < \cdots < s_{N-1} < s_N = 1\}$. The next theorem gives a sufficient condition for a $(2k - 1)$-design in a finite PMS to be self-complementary. In fact, we see that the $(2k - 1)$-designs with relatively small sizes must be self-complementary.

**Theorem 5.4.** Let $M$ be finite, $C \subseteq M$ be a $(2k - 1)$-design and $\alpha_0 < s_1$. Then $C$ is self-complementary.

**Proof.** As $t_1 \leq \alpha_0$ by Theorem 3.1, it follows that $t_1 = -1$ for any point $x \in C$. \hfill \Box

The results concerning self-complementary $(2k)$-designs are mainly negative. We use some notations from Section 4.

**Theorem 5.5.** Let $C \subseteq M$ be a $(2k)$-design.

(i) If $\xi_1 > -1$, then $C$ is not self-complementary.

(ii) If $\xi_2 > -1$ and $|C|$ is odd, then $C$ is not self-complementary.

(iii) If $C$ possesses a pair of antipodal points, then $\gamma_0|C| \geq 1$.

(iv) If $M$ is antipodal and $C$ possesses a pair of antipodal points, then $|C| \geq R(M, 2k+1)$.

**Proof.** The assertions (i) and (ii) are obvious. For (iii), let $\{x, -x\} \subseteq C$. We use (6) for the point $x$ and the polynomial $f(t) = \prod_{i=1}^{k} (t - \beta_i)^2$. The LHS is at least $f(-1)$ and the RHS equals $f(-1)\gamma_0|C|$. Therefore $\gamma_0|C| \geq 1$. For (iv), we use a result from [10, Section 4] showing that in antipodal spaces $\gamma_0|C| \geq 1$ is equivalent to $s \geq t_k\gamma_{k+1}$. Therefore $|C| \geq L_{2k}(M, t_k\gamma_{k+1}) = R(M, 2k+1)$. \hfill \Box

**Remark 5.6.** In the case $M = S^{n-1}$, Theorem 5.5(iv) was already proved by Godsil [19] (this is Lemma 6.1 in Chapter 16).
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6. SOME APPLICATIONS: SPHERICAL DESIGNS

In the euclidean case, the bound (1) (proved by Delsarte, Goethals and Seidel [16]) states

\[ B(S^{n-1}, \tau) \geq R(S^{n-1}, \tau) = \left\{ \begin{array}{ll}
2 \binom{n+k-2}{n-1} & \text{if } \tau = 2k-1, \\
\binom{n+k-1}{n-1} + \binom{n+k-2}{n-1} & \text{if } \tau = 2k.
\end{array} \right. \]

(9)

For \( n \geq 3 \), tight spherical designs are known for \( \tau = 1, 2, 3 \) in all dimensions, and for \( \tau \geq 4 \) in only eight cases (cf. [3, 4, 16], [13, Chapter 14]).

The investigations on the structure of spherical \((2k-1)\)-designs of odd cardinalities were started in [12]. There the cases \( \tau = 3 \) and \( \tau = 5 \) are considered in detail. In particular, the first open cases were ruled out. Namely, on \( S^2 \) there exist no 3-designs with 7 points and 5-designs with 13 points.

In the general case \( \tau = 2k-1 \), the bound

\[ |C| \geq \frac{1 + 2^{1/\tau}}{(k-1)!} n^{k-1} \quad \text{as } n \to \infty, \]

(10)

for \( |C| \) odd, was obtained. In this case (1) gives \( |C| \geq 2n^{k-1}/(k-1)! \) as \( n \to \infty \).

The condition (3) works in small dimensions as well. Some new bounds for the minimum possible odd cardinalities of spherical \((2k-1)\)-designs are shown in Table 1. Further numerical consequences of (3) are available upon request [11].

We have collected many numerical results concerning the results in Section 4 (for spherical \((2k)\)-designs). Details are described in [9]. In particular, it is shown there that 4-designs on \( S^2 \) with 10 points do not exist, which rules out the first open case.
Theorem 5.5(iv) (see Remark 5.6) shows that the minimum possible size of a spherical $(2k)$-design $C \subset S^{n-1}$ which possesses a pair of antipodal points satisfies

$$|C| \geq R(S^{n-1}, 2k + 1) = 2 \binom{n + k - 1}{n - 1}.$$ 

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P. Boyvalenkov, S. Boumova
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
8 G. Bonchev street,
1113 Sofia,
Bulgaria
E-mail: peter,silvi@moi2.math.bas.bg

D. Danev
Department of Electrical Engineering,
Linköping University,
S-581 83 Linköping,
Sweden
E-mail: danyo@isy.liu.se