New and Improved Spanning Ratios for Yao Graphs

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Abstract

For a set of points in the plane and a fixed integer $k > 0$, the Yao graph $Y_k$ partitions the space around each point into $k$ equiangular cones, and connects each point to a nearest neighbor in each cone. With the exception of $Y_5$, it is known for all Yao graphs whether or not they are geometric spanners. In this paper we resolve this gap and show that $Y_5$ is a $t$-spanner with spanning ratio $t$ less than 10.9. We also improve the known spanning ratio of all the Yao graphs for odd $k > 5$. Finally, we revisit the $Y_6$ graph, which plays a particularly important role as the transition between the graphs ($k > 6$) for which simple inductive proofs are known, and the graphs ($k ≤ 6$) whose best spanning ratios are established by complex arguments. Here we reduce the known spanning ratio of $Y_6$ from 17.6 to 5.8.

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1 Introduction

The complete Euclidean graph defined on a planar point set $S$ is the graph with vertex set $S$ and edges connecting each pair of points in $S$, where each edge has as weight the Euclidean distance between its endpoints. Although this graph is useful in many different contexts, its main disadvantage is that it has a quadratic number of edges. As such, much effort has gone into the development of various methods for constructing graphs that approximate the complete Euclidean graph. What does it mean to approximate this graph? One standard approach is to construct a spanning subgraph with fewer edges (typically linear) with the additional property that every edge of the complete Euclidean graph is approximated by a path in the subgraph whose weight is not much more than that of the edge. This gives rise to the notion of a $t$-spanner. A $t$-spanner of the complete Euclidean graph is a spanning subgraph with the property that for all pairs of vertices, the weight of the shortest path in the subgraph between two vertices $x$ and $y$ is at most $t \geq 1$ times $|xy|$. The spanning ratio is the smallest $t$ for which the subgraph is a $t$-spanner. Spanners find many applications, such as approximating shortest paths or minimum spanning trees. For a comprehensive overview of geometric spanners and their applications, we refer the reader to the book by Narasimhan and Smid [1].

One of the simplest ways of constructing a $t$-spanner is to first partition the plane around each vertex into a fixed number of cones and then add an edge between the vertex and a closest vertex in each cone. Intuition suggests that this would yield a graph whose spanning ratio depends on the number of cones. Indeed, this is one of the first approximations of the complete Euclidean graph, referred to as Yao graphs in the literature, introduced independently by Flinchbaugh and Jones [2] and Yao [3]. We will denote the Yao-graph by $Y_k$ where $k$ is the number of cones each having angle $\theta = 2\pi/k$. Yao used them to simplify computation of the Euclidean minimum spanning tree. Flinchbaugh and Jones studied their graph theoretic properties. Neither of them actually proved that they are $t$-spanners.

To the best of our knowledge, the first proof that Yao-graphs are spanners was given by Althöfer et al. [4]. They showed that for every $t > 1$, there exists a $k$ such that $Y_k$ is a $t$-spanner. It appears that some form of this result was known earlier, as Clarkson [5] already remarked in 1987 that $Y_{12}$ is a $1 + \sqrt{3}$-spanner, albeit without providing a proof or reference. Bose et al. [6] provided a more specific bound on the spanning ratio, by showing that for $k > 8$, $Y_k$ is a geometric spanner with spanning ratio at most $1/(\cos \theta - \sin \theta)$, where $\theta = 2\pi/k$. This was later strengthened to show that for $k > 6$, $Y_k$ is a $(1 + \sqrt{2 - 2 \cos \theta})/(2 \cos \theta - 1)$-spanner [7]. Bose et al. [8] showed that $Y_4$ is a constant spanner. For $k < 4$, Molla [9] showed that there is no constant $t$ such that $Y_k$ is a $t$-spanner. This leaves open the question of whether $Y_5$ is a constant spanner.

In this paper we close this gap by showing that $Y_5$ is a $t$-spanner with spanning ratio $t$ less than 10.9. Our approach to solving this open question in fact improves the known spanning ratio of all Yao graphs for odd $k > 5$. Finally, we revisit the $Y_6$ graph, which plays a particularly important role as the transition between the graphs ($k > 6$) for which simple inductive proofs are known, and the graphs ($k \leq 6$) whose best spanning ratios are established by complex arguments. We reduce the known spanning ratio of $Y_6$ from 17.6 to 5.8.

Due to space constraints, several proofs have been deferred to the Appendix.

\footnote{the orientation of the cones is the same for all vertices}
2 Spanning ratio of $Y_k$, for odd $k \geq 5$

In this section we study the spanning properties of the Yao graphs defined by an odd number of cones $k \geq 5$. For $k = 5$ in particular, this is the first result showing that $Y_5$ is a constant spanner. For odd values $k > 5$ we improve the currently known bound on the spanning ratio of $Y_k$.

We start with a few definitions. For a fixed $k$, let $Q_i(a)$ be the half-open cone of angle $360^\circ/k$ with apex at $a$, including the angle range $[i, i+1)360^\circ/k$, $i = 0, \ldots, k-1$, where angles are measured counterclockwise from the $+x$ axis. The directed graph $Y_k$ includes exactly one directed edge from $a$ to a closest point in $Q_i(a)$, for each $i = 0, \ldots, k-1$. If there are several equally-closest points within $Q_i(a)$, then ties are broken arbitrarily. The graph $Y_k$ is the undirected version of $Y_k$. For any two points $a, b \in S$, let $p(a,b)$ be the length of a shortest path in $Y_k$ from $a$ to $b$.

![Figure 1: If $\alpha$ is small, there is a close relation between $|ac|$ and $|bc|$.](image)

Lemma 1 Given three points $a$, $b$, and $c$, such that $|ac| \leq |ab|$ and $\angle bac \leq \alpha < 180^\circ$, then

$$|bc| \leq |ab| - (1 - 2 \sin(\alpha/2)) \cdot |ac|.$$  

Proof. Let $c'$ be the point on $ab$ such that $|ac| = |ac'|$ (see Figure 1). Since $acc'$ forms an isosceles triangle, $|cc'| = 2 \sin(\angle bac/2) \cdot |ac| \leq 2 \sin(\alpha/2) \cdot |ac|$. Now, by the triangle inequality, $|bc| \leq |cc'| + |c'b| \leq 2 \sin(\alpha/2) \cdot |ac| + |ab| - |ac'| = |ab| - (1 - 2 \sin(\alpha/2)) \cdot |ac|$. □

Theorem 2 For any odd integer $k \geq 5$, the $Y_k$-graph defined on a point set $S$ has spanning ratio at most $t = 1/(1 - 2 \sin(3 \cdot \theta/8))$, where $\theta = 360^\circ/k$.

Proof. Let $a, b \in S$ be an arbitrary pair of points. We show that there is a path in $Y_k$ from $a$ to $b$ no longer than $t|ab|$. For simplicity, let $Q(a)$ denote the cone with apex $a$ that contains $b$, and let $Q(b)$ denote the cone with apex $b$ that contains $a$. Rotate the point set $S$ such that the bisector of $Q(a)$ is in the direction of the positive $y$-axis, as depicted in Fig. 2. Assume without loss of generality that $b$ lies to the right of this bisector; the case when $b$ lies to the left of this bisector is symmetric.

Let $\alpha$ be the angle formed by the segment $ab$ with the bisector of $Q(a)$, and let $\beta$ be the angle formed by $ab$ with the bisector of $Q(b)$. Since $k$ is odd, the bisector of $Q(a)$ is parallel to the right boundary of $Q(b)$. Hence, we have that $\alpha = \theta/2 - \beta$. Assume without loss of generality that $\alpha$ is the smaller of these two angles (if not, we exchange the roles of $a$ and $b$). It follows that $\alpha \leq \theta/4$.

Our proof is by induction on the distance $|ab|$. In the base case $|ab|$ is minimal among all distances between pairs of points, which implies that there is no point $c \in Q(a)$ that is strictly closer to $a$ than $b$. Therefore either $ab \in Y_k$, case in which our proof for the base case is finished, or there is a point $c \in Q(a)$ such that $|ab| = |ac|$ and $ac \in Y_k$. In this latter case, since $\alpha \leq \theta/4$ and
Since opposite cones are not symmetric, either $\alpha$ or $\beta$ is small.

$k \geq 5$, the angle between $ab$ and $ac$ is at most $\theta/2 + \alpha \leq 3 \cdot \theta/4 \leq 3/4 \cdot 72^\circ = 54^\circ$. This is lower than $60^\circ$, which implies that $|bc| < |ab|$. This contradicts our assumption that $|ab|$ is minimal. It follows that $\overrightarrow{ab} \in \overrightarrow{Y}_k$ and the base case holds.

For the inductive step, let $c \in Q(a)$ be such that $\overrightarrow{ac} \in \overrightarrow{Y}_k$. If $c$ coincides with $b$, then $p(a, b) = |ab|$ and the proof is finished. So assume that $c \neq b$. Because $c$ is closer to $a$ than $b$, and because $\angle cab \leq \theta/2 + \alpha \leq 3 \cdot \theta/4$, we can use Lem. 1 to derive $|cb| \leq |ab| - (1 - 2 \sin(\frac{\theta}{8})) \cdot |ac| = |ab| - |ac|/t$, which is strictly lower than $|ab|$. Thus we can apply the inductive hypothesis to $cb$ to determine a path between $a$ and $b$ of length

$$p(a, b) \leq |ac| + t \cdot |cb| \leq |ac| + t \cdot \left( |ab| - \frac{|ac|}{t} \right) = t \cdot |ab|$$

This concludes the proof.

Applying this result to $Y_5$ yields the following spanning ratio.

**Corollary 3** The $Y_5$-graph has spanning ratio at most $1/(1 - 2 \sin(27^\circ)) \approx 10.868$.

## 3 Spanning ratio of $Y_6$

In this section we fix $k = 6$ and show that, for any pair of points $a, b \in S$, $p(a, b) \leq 5.8|ab|$. Our proof is inductive and it relies on an elementary triangle lemma, which we introduce next.

### 3.1 Triangle Lemma

Let $a, b \in S$ and let $\overrightarrow{ab} \in \overrightarrow{Y}_6$ be the edge from $a$ within the $60^\circ$-sector that includes $b$. The triangle lemma below will be relevant in the context where we seek to bound the length of a path from $a$ to $b$ by applying the induction hypothesis to the path from $c$ to $b$. The basic geometry is illustrated in Fig. 3. Next we present two key lemmas.
Lemma 4 [Triangle] Let \( \triangle abc \) be labeled as in Fig. 3 with \(|ac| \leq |ab|\), \(|bc| < |ab|\), \(x = |ab| - |bc|\) and \(s = |ac|\). The ratio \(s/x\) is equal to some function \(t\) depending on \(\alpha\) and \(\beta\):

\[
\frac{s}{x} = t(\alpha, \beta) = \frac{\cos(\beta/2)}{\cos(\alpha + \beta/2)}.
\]

The following lemma derives an upper bound on the function \(t(\alpha, \beta)\) from Lem. 4, which will be used in Thm. 6 to derive an optimal value for \(\delta\).

Lemma 5 Let \(a, b, c \in S\) satisfy the conditions of Lem. 4 and let \(t(\alpha, \beta)\) be as defined in (1). Let \(\delta \in (0^\circ, 60^\circ)\) be a fixed positive angle. If \(\alpha \leq 60^\circ - \delta\), or \(\beta \leq 60^\circ - \delta\), then

\[
t(\alpha, \beta) \leq t(60^\circ, 60^\circ - \delta) = \frac{\cos(30^\circ - \delta/2)}{\sin(\delta/2)}.
\]

We defer the proofs of Lems. 4, 5 to Section A of the Appendix.

3.2 Main Result

We prove the following result.

Theorem 6 The \(Y_6\)-graph on a point set \(S\) has spanning ratio at most 5.8.

This result follows from the following lemma, with the variable \(\delta\) substituted by the quantity \(\delta_0 = 18.564^\circ\) that minimizes \(t(\delta)\). (It can be easily verified that \(t(\delta) \geq 5.8 > t(18.564)\).)

Lemma 7 Let \(\delta \in (0^\circ, 20^\circ)\) be a strictly positive real value. The \(Y_6\)-graph on a point set \(S\) has spanning ratio bounded above by

\[
t = t(\delta) = \max \left\{ \frac{\cos(30^\circ - \delta/2)}{\sin(\delta/2)}, \frac{2}{1 - \sin(2\delta) \csc(30^\circ + 2\delta)} \right\}.
\]

Proof. The proof is by induction on the pairwise distance between pairs of points \(a, b \in S\). We imagine sorting the \(\binom{n}{2}\) distances determined by points in \(S\). At any stage in the induction proof for the pair of points \((a, b)\), we have established the theorem for all distances strictly smaller than \(|ab|\), and we seek to establish that \(p(a, b) \leq t|ab|\). Without loss of generality let \(b \in Q_0(a)\).
**Base Case.** We show that, if \((a, b)\) is a closest pair of points, then \(\overrightarrow{ab} \in \overrightarrow{Y}_6\) and so \(p(a, b) = |ab|\). If \(\overrightarrow{ab} \in \overrightarrow{Y}_6\), then the lemma has been established. So assume that \(\overrightarrow{ab} \notin \overrightarrow{Y}_6\); we will derive a contradiction. Because \(\overrightarrow{ab} \notin \overrightarrow{Y}_6\), there must be another point \(c \in Q_0(a)\) such that \(\overrightarrow{ac} \in \overrightarrow{Y}_6\). Because \((a, b)\) is a closest pair, it must be that \(|ac| = |ab|\). Let \(\alpha_1\) and \(\alpha_2\) be the angles that \(ab\) and \(ac\) make with the horizontal respectively. Because both \(\alpha_1, \alpha_2 \in [0, 60^\circ]\), necessarily \(|\alpha_1 - \alpha_2| < 60^\circ\). Thus \(|bc| < |ab| = |ac|\), contradicting the assumption that \((a, b)\) is a closest pair. So in fact it must be that \(\overrightarrow{ab} \notin \overrightarrow{Y}_6\), and the lemma is established.

**Main Idea of the Inductive Step.** It has already been established [7] that \(Y_7\) is a spanner; the sector angles for \(Y_7\) are 51.4°. The main idea of our inductive proof is to partition the 60°-sectors of \(Y_6\) into peripheral cones of angle \(\delta\), for some fixed \(\delta \in (0, 20^\circ)\), leaving a central sector of angle \(60^\circ - 2\delta\). (The \(\delta\)-cones are the shaded regions in Fig. [4]).

When a \(Y_6\) edge falls inside the central sector, induction will apply, because an edge within the central sector makes definite progress toward the goal in that sector (as it does in \(Y_7\)), ensuring that the remaining distance to be covered is strictly smaller than the original. This idea is captured by the flowing lemma.

**Lemma 8** [Induction Step] Let \(a, b, c \in S\) such that \(b, c\) lie in the same 60°-sector with apex \(a\), and \(\overrightarrow{ac} \in Y_6\). Let \(\alpha = \angle cab\) and \(\beta = \angle cba\). If either \(\alpha < 60^\circ - \delta\) or if \(\beta < 60^\circ - \delta\), then we may use induction on \(p(c, b)\) to conclude that \(p(a, b) \leq t|ab|\).

**Proof.** This configuration is depicted in Fig. [3]. Because \(\overrightarrow{ac} \in \overrightarrow{Y}_6\) and \(c, b\) lie in the same 60°-sector with apex \(a\), we have that \(|ac| \leq |ab|\). Because at least one of \(\alpha\) or \(\beta\) is strictly smaller than 60°, we have that \(|bc| < |ab|\). Thus the conditions of Lem. [4] are satisfied, so we can use it to bound \(|ac|\) in terms of \(|ab| - |ac|\); since \(|ac|/x < t\), \(|ac| < tx\). Because \(|cb| < |ab|\), we may apply induction to bound \(p(c, b)\): \(p(c, b) \leq t|cb|\). Hence

\[
p(a, b) \leq |ac| + p(c, b) \leq tx + t|cb| = t(x + |cb|) = t|ab|
\]

We will henceforth use the symbol \(\text{Induct}\) as shorthand for applying Lem. [8] to a triangle equivalent to that in Fig. [3].

Lem. [8] leaves out \(Y_6\) edges falling within the \(\delta\)-cones, that could conceivably not make progress toward the goal. For example, following one edge of an equilateral triangle leaves one exactly as far away from the other corner as at the start. However, we will see that when all relevant edges of \(Y_6\) fall with the \(\delta\)-cones near 60°, the restricted geometric structure ensures that progress toward the goal is indeed made, and again induction applies.

**Inductive Step.** The inductive step proof first handles the cases where \(Y_6\) edges from \(a\) or from \(b\) fall in the central portion of the relevant sectors, and so satisfy Lem. [5], and so Lem. [8] applies.

Recall that \(b \in Q_0(a)\) by our assumption. If \(\overrightarrow{ab} \in \overrightarrow{Y}_6\), then \(p(a, b) = |ab|\) and we are finished. Assuming otherwise, there must be a point \(c \in Q_0(a)\) such that \(\overrightarrow{ac} \in \overrightarrow{Y}_6\) and \(|ac| \leq |ab|\). For the remainder of the proof, we are in this situation, with \(ac \in Y_6\) and \(|ac| \leq |ab|\). The proof now partitions into three parts: (1) when only \(Q_0(a)\) is relevant and leads to \(\text{Induct}\); (2) when \(Q_2(b)\) leads to \(\text{Induct}\); (3) when we fall into a special situation, for which induction also applies, but for different reasons.
(1) The $Q_0(a)$ Sector. Consider $\triangle abc$ as previously illustrated in Fig. [3]. If either $b$ or $c$ is not in one of the $\delta$-cones of $Q_0(a)$, then $\alpha = \angle bac < 60^\circ - \delta$: [Induct.]

So now assume that both $b$ and $c$ lie in $\delta$-cones of $Q_0(a)$. If they both lie within the same $\delta$-cone (Fig. 4a), then again $\alpha$ is small: [Induct.] So without loss of generality let $b$ lie in the lower $\delta$-cone, and $c$ in the upper $\delta$-cone of $Q_0(a)$; see Fig. 4b. We cannot apply induction in this situation because the ratio $s/x$ in Lem. 4 has no upper bound.

(2) The $Q_2(b)$ Sector. Now we consider $Q_2(b)$, the sector with apex at $b$ aiming to the left of $b$, and assume that $c \in Q_2(b)$. Refer to Fig. [5]. The case $c \notin Q_2(b)$ will be discussed later (special situation).

Because $b$ may subtend an angle as large as $\delta$ at $a$ with the horizontal, the "upper 2$\delta$-cone" of $Q_2(b)$ becomes the relevant region. If $c$ is not in the upper 2$\delta$-cone of $Q_2(b)$ (as depicted in Fig. [5]), then $\triangle abc$ satisfies Lem. 5 with $\beta < 60^\circ - \delta$: [Induct.] Note that this conclusion follows even if $c$ is in the small region outside of and below of $Q_2(b)$: the angle $\beta$ at $b$ is then very small.

Assume now that $c$ is in the upper 2$\delta$-cone of $Q_2(b)$. Let $d \in Q_2(b)$ be the point such that $\overrightarrow{bd} \in \overrightarrow{Y_6}$. We now consider possible locations for $d$. If $d = c$, then $p(a,b) \leq |ac| + |cb| \leq 2|ab|$, and we are finished. So assume henceforth that $d$ is distinct from $c$.

If $d$ is not in the upper $\delta$-cone of $Q_0(a)$ (Fig. [6a]), then $\triangle abd$ satisfies Lem. 5 with the roles of $a$ and $b$ reversed: $bd$ takes a step toward $a$, with the angle at $a$ satisfying $\angle bad < 60^\circ - \delta$: [Induct.]

If $d$ is not in the upper 2$\delta$-cone of $Q_2(b)$ (Fig. [6b]), then $\triangle abd$ satisfies Lem. 5 again with the roles of $a$ and $b$ reversed and this time the angle at $b$ bounded away from $60^\circ$, $\angle abd < 60^\circ - \delta$: [Induct.]

Assume now that $d$ is in the intersection region between the upper $\delta$-cone of $Q_0(a)$ and the upper 2$\delta$-cone of $Q_2(b)$. Recall that we are in the situation where $c$ lies in the same region, so it is close to $d$. See Fig. [7]. This suggests the strategy of following $ac$ and $db$, connected by $p(c,d)$. We show that in fact $|cd| < |ab|$, so the inductive hypothesis can be applied to $p(c,d)$. More precisely, we show the following result.
Lemma 9 Let \( a, b, c, d \in S \) be as in Fig. 10a, with \( \overrightarrow{bd} \in Y_6 \), \( b, c \in Q_0(a) \) and \( c, d \in Q_2(b) \). If both \( c \) and \( d \) lie above the lower rays bounding the upper \( 2\delta \)-cones of \( Q_0(a) \) and \( Q_2(b) \), then for any \( 0 \leq \delta \leq 20^\circ \),
\[
|cd| \leq \frac{\sin(2\delta)}{\sin(30^\circ + 2\delta)}|ab|
\] (3)

Note that \( c \) lies in the intersection region between the upper \( 2\delta \)-cones of \( Q_0(a) \) and \( Q_2(b) \), because \( c \in Q_0(a) \cap Q_2(b) \) (by the statement of the lemma). However, Lem. 9 does not restrict the location of \( d \) to the same region. Indeed, \( d \) may lie either below or above the upper ray bounding \( Q_0(a) \), as long as it satisfies the condition \( |bd| \leq |bc| \). (This condition must hold because \( c, d \) are in the same sector \( Q_2(b) \), and \( \overrightarrow{bd} \in Y_6 \).) To keep the flow of our main proof uninterrupted, we defer a proof of Lem. 9 to Section 13 of the Appendix.

By Lem. 9 we have \( |cd| < |ab| \). Thus we can use the induction hypothesis to show that \( p(c, d) \leq t|cd| \). We know that \( |ac| \leq |ab| \) because both \( c \) and \( b \) are in \( Q_0(a) \) and \( \overrightarrow{ac} \in Y_6 \). We also know that \( |bd| \leq |bc| \) because both \( c \) and \( d \) are in \( Q_2(b) \) and \( \overrightarrow{bd} \in Y_6 \). Let \( u \) and \( i \) be the upper and lower intersection points between the rays bounding \( Q_2(b) \) and the upper ray of \( Q_0(a) \), as in Fig. 7. Note that \( \triangle bui \) is equilateral, and because \( c \) lies in this triangle, we have \( |bc| \leq |bu| = |bi| \leq |ab| \). It follows that \( |bd| \leq |ab| \). So in this situation (illustrated in Fig. 7), we have
\[
p(a, b) \leq |ac| + p(c, d) + |bd| \leq 2|ab| + p(c, d)
\]
\[
\leq 2|ab| + t|cd| \leq 2|ab| + t\frac{\sin(2\delta)}{\sin(30^\circ + 2\delta)}|ab| \leq t|ab|
\]
Here we have applied Lem. 9 to bound \( |cd| \). Note that the latter inequality above is true for the value of \( t \) from (2).

(3) Special Situation. The only case left to discuss is the one in which \( c \) lies in the upper \( \delta \)-cone of \( Q_0(a) \) and to the right of the upper ray of \( Q_2(b) \). This situation is depicted in Fig. 8.
Figure 6: (a) $d$ not in the upper $\delta$-cone of $Q_0(a)$: $\angle bad$ is small. (b) $d$ not in the upper $2\delta$-cone of $Q_2(b)$: $\angle abd$ is small.

Figure 7: Lem. |$cd| <$ |$ab$|.

consider $Q_4(c)$. Because $b \in Q_4(c)$, there exists $\overrightarrow{cz} \in Y_6$, with $z \in Q_4(c)$ and $|cz| \leq |cb|$. Clearly $z \in Q_0(a) \cup Q_5(a)$. Note that the disk sector $D_0(a, |ac|) \subset Q_0(a)$ with center $a$ and radius $|ac|$ must be empty, because $\overrightarrow{ac} \in Y_6$.

**Case 3(a).** If $z \in Q_0(a)$, then $z$ lies in the lower $\delta$-cone of $Q_0(a)$ and to the right of $D_0(a, |ac|)$, close to $b$. See Fig. 8a. In this case we show that the quantity on the right side of inequality 3 is a loose upper bound on $|bz|$, and that similar inductive arguments hold here as well. Let the circumference of $D_0(a, |ac|)$ intersect the right ray of $Q_4(c)$ and the lower ray of $Q_0(a)$ at points $z' \neq c$ and $b'$, respectively. Refer to Fig. 8b. Let $\gamma \leq \delta$ be the angle formed by $ac$ with the upper ray of $Q_0(a)$. Then $\angle z'ab' = \gamma$ and $\angle z'cb' = \gamma/2$. This implies that both $b'$ and $z'$ lie in the intersection region between the lower $\delta$-cone of $Q_0(a)$ and the right $\delta/2$-cone of $Q_4(c)$. Thus $a, b, c, z \in S$ satisfy
the conditions of Lem. 9 with the roles of $b$ and $c$ reversed:

$$|bz| \leq \frac{\sin(2\delta)}{\sin(30^\circ + 2\delta)} |ac|$$

Arguments similar to the ones used for the case depicted in Fig. 10 show that $|cz| \leq |ac|$. This along with $|ac| \leq |ab|$ (because $\overrightarrow{ac} \in Y_6$) and the above inequality imply

$$p(a, b) \leq |ac| + |cz| + p(z, b) \leq 2|ab| + t|bz| \leq t|ab|$$

for any $t$ satisfying the conditions stated by this lemma.

**Case 3(b).** Assume now that $z \notin Q_0(a)$. Then $z \in Q_5(a)$, as depicted in Fig. 9. In this case $z$ lies in the disk sector $D_5(c, |cb|)$ (because $|cz| \leq |cb|$) and below the horizontal through $a$ (because $D_0(a, |ac|)$ is empty). This implies that there exists $\overrightarrow{ae} \in Y_6$, with $e \in Q_5(a)$ and $|ae| \leq |az|$. Similarly, there exists $\overrightarrow{bf} \in Y_6$, with $f \in Q_3(b)$ and $|bf| \leq |bz|$. If $e$ lies above the lower $2\delta$-cone of $Q_5(a)$, then $\angle bae \leq 60^\circ - \delta$, which leads to $\text{Induct}$ and settles this case. Similarly, if $f$ lies above the lower $\delta$-cone of $Q_3(b)$, then $\angle abf \leq 60^\circ - \delta$, which again leads to $\text{Induct}$. Otherwise, we show that the following lemma holds.

**Lemma 10** Let $a, b, c, z \in S$ be in the configuration depicted in Fig. 9 with $\overrightarrow{ae}, \overrightarrow{cf} \in Y_6$. Let $\overrightarrow{ae}, \overrightarrow{bf} \in Y_6$, with $e$ in the lower $2\delta$-cone of $Q_5(a)$ and $f$ in the lower $\delta$-cone of $Q_3(b)$. Then at least one of the following is true: (a) $e \in Q_3(b)$, or (b) $f \in Q_5(a)$.

We defer a proof of Lem. 10 to Section C of the Appendix.

Lem. 10 guarantees that, if condition (a) holds, then $ae$ may not cross the lower ray bounding $Q_3(b)$. This case reduces to one of the cases depicted in Figs. 6 and 10 with $e$ playing the role of $c$ and the path passing under $ab$ rather than above. Because $ae$ does not cross the lower ray bounding $Q_3(b)$, the special situation depicted in Fig. 8 with $e$ playing the role of $c$ may not occur.
Figure 9: Case $c \notin Q_2(b)$ and $z \in Q_5(a)$ (a) $z$ left of $uv$ (b) $z$ right of $uv$.

in this case. Similarly, condition (b) from Lem. 10 reduces to one of the cases depicted in Figs. 6 and 10 with the roles of $a$ and $b$ reversed and with $f$ playing the role of $c$; the special situation depicted in Fig. 8 (with $bf$ playing the role of $ac$) may not occur in this case. Having exhausted all cases, we conclude the proof.

4 Conclusions

By exploiting the asymmetry of Yao graphs with an odd number of cones, we proved the first constant upper bound on the spanning ratio of $Y_5$ and improved the bounds for all other odd Yao graphs. The improvement is significant for low values of $k$. For instance, the upper bound on the spanning ratio of $Y_7$ was reduced by more than half, from 7.562 to 2.946.

We also improved the upper bound on the spanning ratio of $Y_6$ from 17.64 to 5.8. Instead of proving that the graph spans the edges of another spanner, our proof uses a direct argument. This results in a cleaner proof and a better spanning ratio.

Of course, there is still work to be done. Thus far, none of these bounds have matching lower bound examples. The best known lower bound for $Y_6$ has a spanning ratio of 2 and consists of just four points. The closely related $\theta_6$-graph was recently shown to have a spanning ratio of exactly 2 [10], the first graph in this family to have matching upper and lower bounds. We conjecture that the spanning ratio of $Y_6$ is also 2.
References


A Proofs of Lemmas 4, 5

Lemma 4 Let \( \triangle abc \) be labeled as in Fig. 3, with \(|ac| \leq |ab|, |bc| < |ab|\), \(x = |ab| - |bc|\) and \(s = |ac|\). The ratio \(s/x\) is equal to some function \(t\) depending on \(\alpha\) and \(\beta\):

\[
\frac{s}{x} = t(\alpha, \beta) = \frac{\cos(\beta/2)}{\cos(\alpha + \beta/2)}
\]

Proof. Normalize the triangle so that \(|ab| = 1\); this does not alter the quantity we seek to compute, \(s/x\). Let \(|bc| = r\) to simplify notation. Then \(x = 1 - r\) and \(x \geq 0\) because \(r = |bc| \leq |ab| = 1\). Note that each of the angles \(\angle cab\) and \(\angle cba\) is strictly less than 90°, because \(|ac| \leq |ab|\) and \(|bc| \leq |ab|\). Thus the projection of \(c\) onto \(ab\) is interior to the segment \(ab\). Computing the altitude \(h\) of \(\triangle abc\) in two ways yields

\[
s \sin \alpha = r \sin \beta
\]

Also projections onto \(ab\) yield

\[
s \cos \alpha + r \cos \beta = 1
\]

Solving these two equations simultaneously yields expressions for \(r\) and \(s\) as functions of \(\alpha\) and \(\beta\):

\[
\begin{align*}
\quad r &= \frac{\sin \alpha}{\sin \alpha \cos \beta + \cos \alpha \sin \beta}, \\
\quad s &= \frac{\sin \beta}{\sin \alpha \cos \beta + \cos \alpha \sin \beta}
\end{align*}
\]

Now we can compute \(s/x = s/(1 - r)\) as a function of \(\alpha\) and \(\beta\). This simplifies to

\[
\frac{s}{x} = \frac{\cos(\beta/2)}{\cos(\alpha + \beta/2)}
\]

as claimed. \(\square\)

Lemma 5 Let \(a, b, c \in S\) satisfy the conditions of Lem. 4, and let \(t(\alpha, \beta)\) be as defined in 4). Let \(\delta \in (0^\circ, 60^\circ)\) be a fixed positive angle. If \(\alpha \leq 60^\circ - \delta\), or \(\beta \leq 60^\circ - \delta\), then

\[
t(\alpha, \beta) \leq t(60^\circ, 60^\circ - \delta) = \frac{\cos(30^\circ - \delta/2)}{\sin(\delta/2)}.
\]

Proof. The derivative of \(t(\alpha, \beta)\) with respect to \(\alpha\) is

\[
\frac{\partial t}{\partial \alpha} = \frac{\sin \alpha + \sin(\alpha + \beta)}{1 + \cos(2\alpha + \beta)} > 0
\]

This means that, for a fixed \(\beta\) value, \(t(\alpha, \beta)\) reaches its maximum when \(\alpha\) is maximum. Similarly, the derivative of \(t(\alpha, \beta)\) with respect to \(\beta\) is

\[
\frac{\partial t}{\partial \beta} = \frac{\sin \alpha}{2 \cos(\alpha + \beta/2)^2} > 0
\]

So for a fixed value \(\alpha\) value, \(t(\alpha, \beta)\) reaches its maximum when \(\beta\) is maximum. Because \(|ac| \leq |ab|\), \(\beta \leq \angle acb\). The sum of these two angles is \(180^\circ - \alpha\), therefore \(\beta \leq 90^\circ - \alpha/2\). This along with
the derivations above implies that, for a fixed value $\alpha \leq 60^\circ - \delta$, $t(\alpha, \beta) \leq t(\alpha, 90^\circ - \alpha/2) \leq t(60^\circ - \delta, 60^\circ + \delta/2)$ (we substituted $\alpha = 60^\circ - \delta$ in this latter inequality). Next we evaluate

$$\frac{t(60^\circ, 60^\circ - \delta)}{t(60^\circ - \delta, 60^\circ + \delta/2)} = \frac{\cos(30^\circ - \delta/2) \sin(3\delta/2)}{\cos(30^\circ + \delta/2) \sin(\delta/2)} > 1$$

It follows that $t(60^\circ, 60^\circ - \delta)$ is maximal, as claimed by the lemma. \hfill \Box

**B Proof of Lemma 9**

**Lemma 9** Let $a, b, c, d \in S$ be as in Fig. 10, with $bd \in Y_6$, $b, c \in Q_0(a)$ and $c, d \in Q_2(b)$. If both $c$ and $d$ lie above the lower rays bounding the upper $2\delta$-cones of $Q_0(a)$ and $Q_2(b)$, then for any $0 \leq \delta \leq 20^\circ$,

$$|cd| \leq \frac{\sin(2\delta)}{\sin(30^\circ + 2\delta)}|ab|$$

**Proof.** Let $u$ and $v$ be the top and bottom points of the intersection quadrilateral $R$ between the upper $2\delta$-cones of $Q_0(a)$ and $Q_2(b)$. See Fig. 10. Then $c \in R$. For any $\delta \leq 20^\circ$, the angles opposite to the diagonal $uv$ of $R$ are bounded below by $100^\circ$, therefore $uv$ is the diameter of $R$.

![Figure 10: Lem. 9](image)

Assume first that $d \in R$ as well. In this case, the quantity $|cd|$ is bounded above by the length $|uv|$ of the diameter of $R$. Let $\gamma$ be the angle formed by $ab$ with the horizontal. We show that $|uv|$ is maximized when $\gamma = 0$. Set a coordinate system with the origin at $a$. Scale the point set $S$ so that $|ab| = 1$. Then $b$'s coordinates are $(\cos \gamma, \sin \gamma)$. The point $u$ is at the intersection of the two lines passing through $a$ and $b$ with slopes $\tan 60^\circ$ and $-\tan 60^\circ$ respectively, given by $y = \sqrt{3}x$ and $y = -\sqrt{3}(x - \cos \gamma) + \sin \gamma$. Solving for $x$ and $y$ gives the coordinates of $u$

$$x_u = \frac{\sqrt{3}\cos \gamma + \sin \gamma}{2\sqrt{3}}, \quad y_u = \frac{\sqrt{3}\cos \gamma + \sin \gamma}{2}$$

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Similarly, the point $v$ is at the intersection of two lines given by $y = \tan(60^\circ - 2\delta)x$ and $y = -\tan(60^\circ - 2\delta)(x - \cos \gamma) + \sin \gamma$. Solving for $x$ and $y$ gives $v'$ coordinates

$$x_v = \frac{\tan(60^\circ - 2\delta) \cos \gamma + \sin \gamma}{2 \tan(60^\circ - 2\delta)}, \quad y_v = \frac{\tan(60^\circ - 2\delta) \cos \gamma + \sin \gamma}{2}$$

We can now compute $|uv| = \sqrt{(x_u - x_v)^2 + (y_u - y_v)^2}$ as a function of $\gamma$ and $\delta$. The derivative of this function with respect to $\gamma$ is represented as a graph in Fig. 11 for $\gamma, \delta \in [0, 20^\circ]$. Note that this function is negative on the given interval, therefore $|uv|$ increases as $\gamma$ decreases. Thus $|uv|$ is maximum when $\gamma = 0$. We now set $\gamma = 0$ and compute $|uv| \leq \sqrt{3}/2 - \cot(2\delta + 30^\circ)/2 = \sin(2\delta)/\sin(2\delta + 30^\circ)$ as claimed.

Assume now that $d \notin R$, so $d$ lies above the upper ray bounding $Q_0(a)$. Let $i$ be the intersection point between the upper ray bounding $Q_0(a)$ and the lower ray bounding the upper $2\delta$-cone of $Q_2(b)$. Then $c$ must lie outside the disk $D_2(b, |bi|)$, because $d$ lies outside this disk (by assumption) and $|bd| \leq |bc|$ (because $\overline{bd} \in Y_6$). Refer to Fig. 10b. Let $j$ be the intersection point between the lower ray bounding the upper $2\delta$-cone of $Q_2(b)$ and the circumference of $D_2(b, |bu|)$. Then both $c$ and $d$ lie in the strip delimited by $D_2(b, |bi|)$, $D_2(b, |bj|)$ and the two rays bounding the upper $2\delta$-cone of $Q_2(b)$. Thus $cd$ is no greater than the diameter of this strip, which we show to be no greater than the diameter of $R$. For this, it suffices to show that max\{|ui|, |uj|, |ij|\} $\leq |uv|.

Because $ui$ is an edge of $R$, $|ui|$ is clearly no greater than the diameter $|uv|$ of $R$. Next we show that $|uj| \leq |uv|$. From the isosceles triangle $\triangle buj$ we derive $\angle uju = 90^\circ - \delta$. Angle $\angle uv j$ is exterior to $\triangle uv b$, therefore $\angle uv j = \angle vub + 2\delta \leq 30^\circ + 2\delta$ (note that $\angle vub = 30^\circ$ when $ab$ is horizontal, otherwise $\angle vub < 30^\circ$). It follows that $\angle uju \leq \angle uv j$ for any $\delta \leq 20^\circ$. This along with the Law of Sines applied to $\triangle uju$ yields $|uj| \leq |uv|.

It remains to show that $|ij| < |uv|$. We will in fact show that $|ij| < |uj|$, which along with the conclusion above that $|uj| \leq |uv|$, yields $|ij| < |uv|$. Angle $\angle uij$ is exterior to $\triangle uib$, therefore $60^\circ \leq \angle uij \leq 60^\circ + 2\delta$. Earlier we showed that $\angle uju = 90^\circ - \delta \geq 70^\circ$, for any $\delta \leq 20^\circ$. It follows that $\angle uij \leq 180^\circ - (70^\circ + 60^\circ) = 50^\circ$ is the smallest angle of $\triangle uij$, therefore $|ij| < |uv|$. This completes the proof. 

\[\square\]
C Proof of Lemma 10

Lemma 10 Let \( a, b, c, d \in S \) be in the configuration depicted in Fig. 3 with \( \overline{ae}, \overline{cd} \in Y_6 \). Let \( \overline{af}, \overline{bf} \in Y_6 \), with \( e \) in the lower \( 2\delta \)-cone of \( Q_5(a) \) and \( f \) in the lower \( \delta \)-cone of \( Q_3(b) \). Then at least one of the following is true:

(a) \( e \in Q_3(b) \)
(b) \( f \in Q_5(a) \)

Proof. We define four intersection points \( u, v, i \) and \( j \) as follows: \( u \) is at the intersection between the top rays of \( Q_3(a) \) and \( Q_2(b) \); \( v \) is at the intersection between the bisector of \( \angle uab \) and the boundary of the disk sector \( D_1(u, |ub|) \); \( i \) is the foot of the perpendicular from \( a \) on the lower ray of \( Q_3(b) \); and \( j \) is the foot of the perpendicular from \( b \) on the lower ray of \( Q_5(a) \). Refer to Fig. 3.

Note that \( |ae| \leq |ai| \) implies condition (a), and \( |bf| \leq |bj| \) implies condition (b). We show that the first holds if \( d \) lies left of \( uv \), and the latter holds if \( d \) lies right of \( uv \) (and so at least one of the two conditions holds). We first show that \( d \in D_4(u, |ub|) \). This follows immediately from the inequality \( |ud| + |cb| < |cd| + |ab| \) (which can be derived using the triangle inequality twice on the triangles induced by the diagonals of \( ucbd \)), and the fact that \( |cd| \leq |cb| \) (because \( \overline{cd} \in Y_6 \)). It follows that \( |ud| < |ub| \), therefore \( d \in D_4(u, |ub|) \).

Condition (a). Assume that \( d \) is to the left of \( uv \) (as in Fig. 3). Because \( d \in D_4(u, |ub|) \) is below the horizontal through \( a \), \( \angle adv \) is obtuse and therefore \( |ad| \leq |av| \) (equality holds when \( d \) coincides with \( v \)). Also \( |ae| \leq |ad| \), because \( d \) and \( e \) are in the same sector \( Q_5(a) \) and \( \overline{ae} \in Y_6 \). It follows that \( |ae| \leq |av| \). We now show that \( |av| \leq |ai| \), which implies \( |ae| \leq |ai| \), thus settling this case.

Let \( \gamma \in [0, \delta] \) be the angle formed by \( ab \) with the horizontal through \( a \). Then \( \angle ab = 60^\circ - \gamma \) and \( |ai| = |ab| \sin(60^\circ - \gamma) \). The Law of Sines applied to \( \triangle uv \) tells us that

\[
\frac{|av|}{\sin 30^\circ} = \frac{|ua|}{\sin \angle uwa} = \frac{|uv|}{\sin \angle uav}.
\]

Note that \( |uv| = |ub| \leq |ua| \), because \( v \) lies on the circumference of \( D(u, |ub|) \) and \( a \) lies outside of this disk. This along with the latter equality above yields \( \angle uwa \leq \angle uva \). The sum of these two angles is \( 150^\circ \) (recall that \( uv \) is the bisector of \( \angle uab \)), therefore \( \angle uwa \geq 75^\circ \). Also note that \( \angle uwa < 90^\circ \), because \( v \) lies strictly below the horizontal through \( a \) (otherwise \( d \) may not exist). It follows that \( \sin \angle uwa \geq \sin 75^\circ \). Substituting this in the equality above yields \( |av| \leq |ua| \sin 30^\circ / \sin 75^\circ \). The Law of Sines applied to triangle \( \triangle abu \) yields \( |au| = |ab| \sin(60^\circ + \gamma) / \sin 60^\circ \), which substituted in the previous equality yields \( |av| \leq |ab| \sin(60^\circ + \gamma) \sin 30^\circ / (\sin 60^\circ \sin 75^\circ) \). Thus the inequality \( |av| \leq |ai| \) holds for any \( \gamma \) satisfying

\[
\frac{\sin(60^\circ + \gamma) \sin 30^\circ}{\sin 60^\circ \sin 75^\circ} \leq \sin(60^\circ - \gamma).
\]

It can be easily verified that this inequality holds for any \( \gamma \leq \delta \leq 23^\circ \), and in particular for the \( \delta \) values restricted by Lem. 7.
Condition (b). Assume now that \( d \) is to the right of \( uv \) (as in Fig. 9b). In this case \( |bf| \leq |bd| \leq |bv| \). We now show that \( |bv| \leq |bj| \), which implies \( |bf| \leq |bj| \), thus settling this case. From the right triangle \( \triangle baj \) with angle \( \angle baj = 60^\circ + \gamma \) we derive \( |bj| = |ab| \sin(60^\circ + \gamma) \). Next we derive an upper bound on \( |bv| \). From the isosceles triangle \( \triangle vub \), having angle \( \angle vub = 30^\circ \), we derive \( |bv| = 2|bu| \sin 15^\circ \). The Law of Sines applied to triangle \( \triangle uab \) gives us \( |ub| = |ab| \sin(60^\circ - \gamma) / \sin 60^\circ \), which substituted in the previous equality yields \( |bv| = 2|ab| \sin(60^\circ - \gamma) \sin 15^\circ / \sin 60^\circ \). Thus the inequality \( |bv| \leq |bj| \) holds for any \( \gamma \) value satisfying

\[
\frac{2 \sin(60^\circ - \gamma) \sin 15^\circ}{\sin 60^\circ} \leq \sin(60^\circ + \gamma).
\]

It can be verified that this inequality holds for any \( \gamma \leq \delta \leq 60^\circ \), and in particular for the \( \delta \) values restricted by Lem. 7. \( \square \)