Ideal transforms and local cohomology defined by a pair of ideals

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Abstract

In this paper, we introduce a generalization of the ordinary ideal transform, denoted by $D_{I,J}$, which is called the ideal transform with respect to a pair of ideals (I, J) and has an apparent algebraic structure. Then we study its various properties and explore the connection with the ordinary ideal transform. Also, we discuss the associated primes of local cohomology modules with respect to a pair of ideals. In particular, we give a characterization for the associated primes of the non-vanishing generalized local cohomology modules.

1 Introduction

Throughout this paper, without particular explanation, R is always a Noetherian commutative ring with non-zero identity, I, J are ideals of R and M is an R-module.

Now let R be a commutative unitary domain and K its quotient field. The ideal transform of R with respect to I was defined by Nagata [Na1]:

$$T_I(R) := \bigcup_{n\geq 1}(R :_K I^n).$$

[*]Work partially supported by Natural Science Foundation of China (11201326)

[†]Work partially supported by CNPq-Brazil - Grants 309316/2011-1, and FAPESP Grant 2012/20304-1.

[‡]Work partially supported by CAPES-Brazil 10056/12-2. *Key words*: Local Cohomology, Ideal Transforms, Associated primes

Ideal tranform turns out to be a significant tool in commutative algebra and algebraic geometry and is closely related to local cohomology modules of Grothendieck. A very important application is existing in the treatment of Hilbert's fourteenth problem (cf. [Na1], [Na2]). In order to study this algebra deeply, there are several methods for ideal transform (for example, [Br], [Ha]). In particular, a functorial description of ideal transform was given (see, for example, [Br]): an ideal transform $D_I(M)$ of an R-module M with respect to I defined as

$$D_I(M) := \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_R(I^n, M).$$

It can be verified that $T_I(R) = D_I(R)$. Moreover, let $Q(M)$ be the total module of quotients of M and let $T(M)$ denote the R-module:

$$\bigcup_{n\geq 1}(M :_{Q(M)} I^n).$$

Then $T_I(M) = D_I(M)$ if $\mathrm{grade}(I, M) > 0$.

In [TYY], Takahashi, Yoshino and Yoshizawa introduced the notion of generalized local cohomology modules, which is an extension of local cohomology modules to a pair of ideals (I, J), and they studied their various properties (another generalization may be seen in [FP]). More precisely, set $W(I, J) = \{\mathfrak{p} \in \mathrm{Spec}R \mid I^n \subseteq \mathfrak{p} + J \ \text{ for some integer } n\}$ and let $\widetilde{W}(I, J)$ denote the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a} + J$. Also, let the set

$$\Gamma_{I,J}(M) = \{x \in M \mid I^n x \subseteq Jx \quad \text{for} \ \ n \gg 1\}$$

of elements of M. The functor $\Gamma_{I,J}(-)$ is a left exact functor, additive and covariant, from the category of R-modules to itself, which is called (I, J)-torsion functor. For an integer i, the ith right derived functor of $\Gamma_{I,J}(-)$ is denoted by $H^i_{I,J}(-)$ and called the ith local cohomology functor with respect to (I, J). For an R-module M, we denote by $H^i_{I,J}(M)$, the ith local cohomology module of M with respect to (I, J), and $\Gamma_{I,J}(M)$ will be called the (I, J)-torsion part of M. When $J = 0$ or J is a nilpotent ideal, $H^i_{I,J}(-)$ coincides with the ordinary local cohomology functor $H^i_I(-)$ with the support in $V(I)$.

Naturally, in this paper we introduce a kind of ideal transforms as a generalization of the ordinary ideal transform, which will be called an ideal transform with respect to a pair of ideals (I, J). We recall that $D_I(R)$ is a commutative ring with an identity; the generalized ideal transform introduced here will have a structure as the quotient of a direct sum of some

$\{D_I(R)\}_{I\subseteq R}$ (see Remark 2.1), and therefore, it will have a structure of ring, as well. We study its various properties and discuss its connection with the ordinary ideal transform. Also, we give some results about the associated primes of the local cohomology module $H^i_{I,J}(M)$, which improves some known results.

The organization of the paper is as follows. In Section 2, we give the definition of $D_{I,J}(M)$, an ideal transform with respect to a pair of ideals (I, J), and give some elementary results of $D_{I,J}(M)$. In Section 3, we discuss the associated primes set of $H^i_{I,J}(M)$. In Section 4, we discuss the exactness property of an ideal transform module with respect to a pair of ideals (I, J).

2 Ideal transforms defined by a pair of ideals

In the following, we shall introduce a kind of idea transforms, as a generalization of an ideal transform $D_I(M)$ of an R-module M with respect to I.

Let R be a commutative Noetherian ring and let I and J be ideals of R. Consider the set $\tilde{W} := \tilde{W}(I, J)$ of all ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n. We define a partial order on $\tilde{W}$ by letting $\mathfrak{a} \leq \mathfrak{b}$ if $\mathfrak{b} \subseteq \mathfrak{a}$ for $\mathfrak{a}, \mathfrak{b} \in \tilde{W}$. Then the order relation on $\tilde{W}$ makes $\{D_{\mathfrak{a}}(M)\}_{\mathfrak{a}\in\tilde{W}}$ into a direct system of R-modules (see [BS, Exercise 2.2.20]). Now we define the covariant R-linear functors from the category of R-modules to itself:

$$D_{I,J} := \varinjlim_{\mathfrak{a}\in\tilde{W}} D_{\mathfrak{a}}(-), \quad \text{where } D_{\mathfrak{a}} = \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_R(\mathfrak{a}^n, -)$$

We shall refer to $D_{I,J}$ as the (I, J)-*transform functor*. As $D_{\mathfrak{a}}$ is a left exact functor, and exact sequences are preserved by taking a direct limit, we conclude that the functor $D_{I,J}$ is also left exact.

For an R-module M, we call $D_{I,J}(M) = \varinjlim_{\mathfrak{a}\in\tilde{W}} D_{\mathfrak{a}}(M)$ the *ideal transform of* M *with respect to* (I, J), or, alternatively, the (I, J)-*transform* of M. For $i \in \mathbb{N}_0$, we use $\mathfrak{R}^i D_{I,J}$ to denote the ith right derived functor of $D_{I,J}$ and call $\mathfrak{R}^i D_{I,J}(M)$ the ith right derived module of $D_{I,J}$.

Remark 2.1. (i) (I, J)-transform functor $D_{I,J}$ is a natural generalization for the notion of I-transform functor D_I, just like from the definition of $H^i_I(M)$ to that of $H^i_{I,J}(M)$:

It is known that

$$H^i_I(M) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/I^n, M).$$

In fact, there is another isomorphism:

$$H_I^i(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{V}(I)} \operatorname{Ext}_R^i(R/\mathfrak{a}, M).$$

Here, $\tilde{V}(I)$ denotes the set of all ideals $\mathfrak{a}$ of R such that $I \subseteq \sqrt{\mathfrak{a}}$.
By Theorem 2.2 (below),

$$H_{I,J}^i(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}} \operatorname{Ext}_R^i(R/\mathfrak{a}, M).$$

While, for the ideal transforms, by Theorem 2.2,

$$D_I(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{V}(I)} \operatorname{Hom}_R(\mathfrak{a}, M);$$

$$D_{I,J}(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}} \operatorname{Hom}_R(\mathfrak{a}, M).$$

(ii) Now we let $Q(M)$ be the total module of quotients of M,

$$T_{I,J}(M) := \bigcup_{\mathfrak{a}\in\tilde{W}} (M :_{Q(M)} \mathfrak{a}),$$

and

$$\beta_{\mathfrak{a}} : (M :_{Q(M)} \mathfrak{a}) \to \operatorname{Hom}_R(\mathfrak{a}, M) \ \ (\mathfrak{a} \in \tilde{W}),$$

defined by $\beta_{\mathfrak{a}}(y)(x) = xy$ $(x \in \mathfrak{a}, y \in (M :_{Q(M)} \mathfrak{a}))$, be a canonical homomorphism. Taking the direct limit $\varinjlim_{\mathfrak{a}\in\tilde{W}}$, there is a homomorphism

$$\beta_{\tilde{W}} : T_{I,J}(M) \to D_{I,J}(M).$$

Let $\operatorname{depth}_R(I, J, M) = \inf\{\operatorname{grade}(\mathfrak{a}, M) \mid \mathfrak{a} \in \tilde{W}(I, J)\}$. One can then deduce that $\operatorname{depth}_R(I, J, M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in W(I, J)\}$. Then, by [TYY, Theorem 4.1], $\operatorname{depth}_R(I, J, M) = \inf\{i \mid H_{I,J}^i(M) \neq 0\}$. If $\operatorname{depth}_R(I, J, M) > 0$ (that is, $\Gamma_{I,J}(M) = 0$), then $T_{I,J}(M) \cong D_{I,J}(M)$. It follows similarly as [Br, Lemma 2.8].

(iii) From [BS, Exercise 2.2.3 (iv)], $D_I(R)$ is a commutative ring with identity, and $D_I(M)$ can be viewed as $D_I(R)$-module. Therefore,

$$D_{I,J}(-) = \varinjlim_{\mathfrak{a}\in\tilde{W}} D_{\mathfrak{a}}(-)$$

can be viewed as the quotient of a direct sum of $\{D_{\mathfrak{a}}(-)\}_{\mathfrak{a}\in\tilde{W}}$. Thus, $D_{I,J}(R)$ is a commutative ring with identity and $D_{I,J}(M)$ can be viewed as $D_{I,J}(R)$-module. So all the $\mathfrak{R}^i D_{I,J}$ $(i \in \mathbb{N}_0)$ can be viewed as additive functors from the category of R-modules to the category of $D_{I,J}(R)$-modules.

(iv) Similarly as the proof of [BS, Remark 1.3.7], and since taking the direct limits is an exact functor, we have that

$$\left(\varinjlim_{\mathfrak{a}\in\tilde{W}} \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(\mathfrak{a}^n, -)\right)_{i\in\mathbb{N}_0}$$

is a negative strongly connected sequence of functors from R-modules. Now, by [BS, Theorem 1.3.5] we show that there is a unique isomorphism of connected sequence of functors of R-modules

$$\left(\mathfrak{R}^i D_{I,J}(-)\right)_{i\in\mathbb{N}_0} \cong \left(\varinjlim_{\mathfrak{a}\in\tilde{W}} \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(\mathfrak{a}^n, -)\right)_{i\in\mathbb{N}_0},$$

which extends the identity natural equivalence from $D_{I,J}$ to itself.

Theorem 2.2 (i) below is a special case of [BS, Remark 1.3.7], while item (ii) is a special case of [BS, Definition 2.2.3].

Theorem 2.2. *Let M be an R-module and let I, J be ideals of R. Then, there are some natural isomorphisms*:

(i) $H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}} \operatorname{Ext}^i_R(R/\mathfrak{a}, M)$, $\forall i \geq 0$;

(ii) $D_{I,J}(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}} \operatorname{Hom}_R(\mathfrak{a}, M)$;

(iii) $D_I(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{V}(I)} \operatorname{Hom}_R(\mathfrak{a}, M)$, *where* $\tilde{V}(I) := \tilde{W}(I, 0)$.

Proof. (i) Firstly, we prove that $\Gamma_{I,J}(M) = \bigcup_{\mathfrak{a}\in\tilde{W}}(0 :_M \mathfrak{a})$. For an element $x \in \Gamma_{I,J}(M)$, there exists an integer $n \geq 0$ such that $I^n x \subseteq Jx$, that is, $I^n \subseteq \operatorname{Ann}(x) + J$. Let $\mathfrak{b} = \operatorname{Ann}(x)$. Thus,

$$x \in (0 :_M \mathfrak{b}) \subseteq \bigcup_{\mathfrak{a}\in\tilde{W}} (0 :_M \mathfrak{a}).$$

On the other hand, if $x \in \bigcup_{\mathfrak{a}\in\tilde{W}}(0 :_M \mathfrak{a})$, then there exists $\mathfrak{b} \in \tilde{W}$ with $x \in (0 :_M \mathfrak{b})$. Since, for the ideal $\mathfrak{b}$, there exists m such that $I^m \subseteq \mathfrak{b} + J$, this concludes that $I^m x \subseteq Jx$. So $x \in \Gamma_{I,J}(M)$.

Clearly, $\{H^i_{I,J}(-)\}_{i\geq 0}$ and $\{\varinjlim_{\mathfrak{a}\in\tilde{W}}\mathrm{Ext}^i_R(R/\mathfrak{a},-)\}_{i\geq 0}$ are both strongly connected. Moreover, $H^i_{I,J}(E)=\varinjlim_{\mathfrak{a}\in\tilde{W}}\mathrm{Ext}^i_R(R/\mathfrak{a},E)=0$ for all $i>0$ and all injective R-modules E. Hence,

$$H^i_{I,J}(M)\cong\varinjlim_{\mathfrak{a}\in\tilde{W}}\mathrm{Ext}^i_R(R/\mathfrak{a},M),\quad \forall i\geq 0.$$

(ii) Let $\mathfrak{a}$ be an ideal of R. From the short exact sequence

$$0\to\mathfrak{a}\to R\to R/\mathfrak{a}\to 0,$$

we have the following commutative graph (the two horizontal lines are both exact sequences):

$$\begin{array}{ccccccccc}
0\to & \mathrm{Hom}(R/\mathfrak{a},M) & \to & M & \to & \mathrm{Hom}(\mathfrak{a},M) & \to & \mathrm{Ext}^1_R(R/\mathfrak{a},M) & \to\ 0\\
 & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow & & \downarrow & \\
0\to & \Gamma_{\mathfrak{a}}(M) & \to & M & \to & D_{\mathfrak{a}}(M) & \to & H^1_{\mathfrak{a}}(M) & \to\ 0
\end{array}$$

The four modules $\Gamma_{\mathfrak{a}}(M), M, D_{\mathfrak{a}}(M)$ and $H^1_{\mathfrak{a}}(M)$ in the below line are viewed as the direct limits of the four direct systems. The maps f and g are given by the inclusion. Next, we take the direct limits $\varinjlim_{\mathfrak{a}\in\tilde{W}}$ in the above commutative graph, and by (i) and [TYY, Theorem 3.2], we have the following commutative graph:

$$\begin{array}{ccccccccc}
0\to & \Gamma_{I,J}(M) & \to & M & \to & \varinjlim_{\mathfrak{a}\in\tilde{W}}\mathrm{Hom}(\mathfrak{a},M) & \to & H^1_{I,J}(M) & \to\ 0\\
 & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle \cong}\downarrow & \\
0\to & \Gamma_{I,J}(M) & \to & M & \to & D_{I,J}(M) & \to & H^1_{I,J}(M) & \to\ 0
\end{array}$$

Hence, $D_{I,J}(M)\cong\varinjlim_{\mathfrak{a}\in\tilde{W}}\mathrm{Hom}_R(\mathfrak{a},M)$.

(iii) Let $\mathfrak{a}\in\tilde{V}(I)$. Then there exists $n>0$ such that $I^n\subseteq\mathfrak{a}$. Then we have the following commutative graph (the three horizontal lines are all exact sequences):

$$\begin{array}{ccccccccc}
0\to & \mathrm{Hom}(R/\mathfrak{a},M) & \to & M & \to & \mathrm{Hom}(\mathfrak{a},M) & \to & \mathrm{Ext}^1_R(R/\mathfrak{a},M) & \to 0\\
 & {\scriptstyle f_1}\downarrow & & {\scriptstyle g_1}\downarrow & & \downarrow & & \downarrow & \\
0\to & \mathrm{Hom}(R/I^n,M) & \to & M & \to & \mathrm{Hom}(I^n,M) & \to & \mathrm{Ext}^1_R(R/I^n,M) & \to 0\\
 & {\scriptstyle f_2}\downarrow & & {\scriptstyle g_2}\downarrow & & \downarrow & & \downarrow & \\
0\to & \Gamma_I(M) & \to & M & \to & D_I(M) & \to & H^1_I(M) & \to 0
\end{array}$$

The maps f_1, f_2, g_1 and g_2 are given by the inclusion. It follows that the following graph is commutative (the two horizontal lines are both exact sequences):

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}(R/\mathfrak{a}, M) & \to & M & \to & \mathrm{Hom}(\mathfrak{a}, M) & \to & \mathrm{Ext}^1_R(R/\mathfrak{a}, M) & \to & 0 \\ & {}^{f}\downarrow & & {}^{g}\downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \Gamma_I(M) & \to & M & \to & D_I(M) & \to & H^1_I(M) & \to & 0 \end{array}$$

The maps $f = f_2 \circ f_1, g = g_2 \circ g_1$ are given by the inclusion. Then, we take the direct limits $\varinjlim_{\mathfrak{a}\in\tilde{V}(I)}$, and by (i), we have the following commutative graph (note that the four modules in the second line have nothing to do with taking the direct limits):

$$\begin{array}{ccccccccc} 0 \to & \Gamma_I(M) & \to & M & \to & \varinjlim_{\mathfrak{a}\in\tilde{V}(I)}\mathrm{Hom}(\mathfrak{a}, M) & \to & H^1_I(M) & \to & 0 \\ & {}^{=}\downarrow & & {}^{=}\downarrow & & \downarrow & & {}^{\cong}\downarrow & & \\ 0 \to & \Gamma_I(M) & \to & M & \to & D_I(M) & \to & H^1_I(M) & \to & 0 \end{array}$$

Hence, we have the isomorphism: $D_I(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{V}(I)}\mathrm{Hom}_R(\mathfrak{a}, M)$.

□

The following corollary is an immediate result of Theorem 2.2.

Corollary 2.3. *Let M be an R-module and let I be an ideal of R. Then $D_{I,0}(M) \cong D_I(M)$. In Particular,*

$$D_{\mathfrak{a}}(M) \cong D_{\mathfrak{b}}(M)$$

for two ideals $\mathfrak{a}, \mathfrak{b}$ in R such that $V(\mathfrak{a}) = V(\mathfrak{b})$.

In [BS, Theorem 2.2.1], there are very important connections between the $\mathfrak{a}$-transform functor and the local cohomology functors. Similarly, we will obtain some connections between the (I, J)-transform functor and the local cohomology functors with respect to a pair of ideals.

Theorem 2.4. *Let M be an R-module and let I, J be ideals of R.*

(i) *There exists an exact sequence*

$$0 \to \Gamma_{I,J}(M) \to M \xrightarrow{\eta_M} D_{I,J}(M) \to H^1_{I,J}(M) \to 0;$$

(ii) *For each $i \in \mathbb{N}$, $\mathfrak{R}^i D_{I,J}(M) \cong H^{i+1}_{I,J}(M)$.*

Proof. (i) For each $\mathfrak{a} \in \tilde{W}(I, J)$, by [BS, Theorem 2.2.4 (i)], we obtain the exact sequence $0 \to \Gamma_{\mathfrak{a}}(M) \to M \to D_{\mathfrak{a}}(M) \to H^1_{\mathfrak{a}}(M) \to 0$. Taking direct limits on this sequence, by the above definition and [TYY, Theorem 3.2], one obtains the result.
(ii) For each $\mathfrak{a} \in \tilde{W}(I, J)$, by [BS, Theorem 2.2.4 (i)], one has $\mathfrak{R}^i D_{\mathfrak{a}}(M) \cong H^{i+1}_{\mathfrak{a}}(M)$. Then, by passing to direct limits and by [TYY, Theorem 3.2], it can be derived that

$$\varinjlim_{\mathfrak{a}\in\tilde{W}} \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(\mathfrak{a}^n, M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}} \mathfrak{R}^i D_{\mathfrak{a}}(M) \cong H^{i+1}_{I,J}(M).$$

□

Remark 2.5. Let M be an R-module and let I, J be ideals of R. From Theorem 2.4(i) we obtain the following:

(i) M and $D_{I,J}(M)$ are isomorphic if and only if $\Gamma_{I,J}(M) = H^1_{I,J}(M) = 0$.

(ii) There is an exact sequence

$$0 \to M/\Gamma_{I,J}(M) \to D_{I,J}(M) \to H^1_{I,J}(M) \to 0.$$

Lemma 2.6. *Let M be an R-module and let I, J be two ideals of R. Then the followings hold:*

(i) $D_{I,J}(\Gamma_{I,J}(M)) = 0$*;*

(ii) $D_{I,J}(M) \cong D_{I,J}(M/\Gamma_{I,J}(M))$*;*

(iii) $D_{I,J}(M) \cong D_{I,J}(D_{I,J}(M))$*;*

(iv) $\Gamma_{I,J}(D_{I,J}(M)) = 0 = H^1_{I,J}(D_{I,J}(M))$;

(v) $H^i_{I,J}(M) \cong H^i_{I,J}(D_{I,J}(M))$ *for all* $i > 1$.

Proof. The proof is straightforward, it follows by using Theorem 2.4, [TYY, Corollary 1.13] and Remark 2.5(i). □

Proposition 2.7. *Let $\phi : M \to M'$ be an R-homomorphism such that* $\mathrm{Ker}\phi$ *and* $\mathrm{Coker}\phi$ *are both (I, J)-torsion R-modules. Then*

(i) *The map $D_{I,J}(\phi) : D_{I,J}(M) \to D_{I,J}(M')$ is an isomorphism;*

(ii) *There exists an unique R-homomorphism $\psi' : M' \to D_{I,J}(M)$ such that the diagram*

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "\eta_M"'] & M' \arrow[d, "\psi'"] \\ & D_{I,J}(M) \end{tikzcd}$$

commutes. In fact, $\psi' = D_{I,J}(\phi)^{-1} \circ \eta_{M'}$.

(iii) *The map ψ' above is an isomorphism if and only if $\eta_{M'}$ is an isomorphism if and only if $\Gamma_{I,J}(M') = H^1_{I,J}(M') = 0$.*

Proof. To prove (i) we make use of Theorem 2.4, [TYY, Corollary 1.13] and Lemma 2.6(i). For item (ii), make use of Lemma 2.6(iii). Finally, item (iii) results from Theorem 2.4(i).

□

In [TYY, Corollary 2.5], it is shown that if M is a J-torsion then $H^i_{I,J}(M) \cong H^i_I(M)$. Here, we provide a more direct proof for this fact. Further, next result is a similar result for ideal transforms.

Proposition 2.8. *Let I, J be ideals of R. If M is a J-torsion R-module, then*

$$D_{I,J}(M) \cong D_I(M) \cong D_I(D_{I,J}(M)) \cong D_{I,J}(D_I(M)).$$

Proof. Since M is a J-torsion R-module, then we can take an injective resolution $E^\bullet$ of M such that in which each term is a J-torsion R-module by [TYY, Proposition 1.12]. Note that for any J-torsion R-module N, $\Gamma_{I,J}(N) = \Gamma_I(N)$. Then the diagram

$$\begin{tikzcd} \cdots \arrow[r] & \Gamma_{I,J}(E^t(M)) \arrow[r] \arrow[d, "="'] & \Gamma_{I,J}(E^{t+1}(M)) \arrow[r] \arrow[d, "="'] & \cdots \\ \cdots \arrow[r] & \Gamma_I(E^t(M)) \arrow[r] & \Gamma_I(E^{t+1}(M)) \arrow[r] & \cdots \end{tikzcd}$$

commutes. It implies that

$$H^i_{I,J}(M) = H^i(\Gamma_{I,J}(E^\bullet)) = H^i(\Gamma_I(E^\bullet)) = H^i_I(M)$$

for any $i \geq 0$.

Let $\mathfrak{a} \in \tilde{V}(I) \subseteq \tilde{W}(I, J)$. We have the following commutative graph:

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}(R/\mathfrak{a}, M) & \to & M & \to & \mathrm{Hom}(\mathfrak{a}, M) & \to & \mathrm{Ext}^1_R(R/\mathfrak{a}, M) & \to & 0 \\ & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \Gamma_{I,J}(M) & \to & M & \to & D_{I,J}(M) & \to & H^1_{I,J}(M) & \to & 0. \end{array}$$

By taking limits $\varinjlim_{\mathfrak{a} \in \tilde{V}(I)}$ and by Theorem 2.2, we obtain the commutative graph below (the two horizontal lines are both exact sequence):

$$\begin{array}{ccccccccc} 0 \to & \Gamma_I(M) & \to & M & \to & D_I(M) & \to & H^1_I(M) & \to & 0 \\ & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle \cong}\downarrow & & \\ 0 \to & \Gamma_{I,J}(M) & \to & M & \to & D_{I,J}(M) & \to & H^1_{I,J}(M) & \to & 0. \end{array}$$

It follows that $D_{I,J}(M) \cong D_I(M)$. Since any subquotient module of a J-torsion R-module is still a J-torsion and $H^i_{I,J}(M)$ is a J-torsion for $i \geq 0$, it follows from Remark 2.5(ii) that $D_{I,J}(M)$ is a J-torsion. For a similar reason, $D_I(M)$ is a J-torsion. Hence,

$$D_{I,J}(M) \cong D_{I,J}(D_{I,J}(M)) \cong D_I(D_{I,J}(M))$$

and

$$D_I(M) \cong D_I(D_I(M)) \cong D_{I,J}(D_I(M))$$

by Lemma 2.6(iii). □

The following theorem is a generalization of the base ring independence theorem concerning ideal transforms (see [BS, Theorem 2.2.21]).

Theorem 2.9. *Let I,J be ideals of R and $\varphi : R \to R'$ a ring homomorphism, and let M' be an R'-module. Assume that $\varphi(J) = JR'$. Then there exists an isomorphism $D_{I,J}(M') \cong D_{IR',JR'}(M')$ of R'-modules.*

Proof. From the proof of [BS, Theorem 2.2.21], one observes there exists an isomorphism $D_{\mathfrak{a}}(M') \cong D_{\mathfrak{a}R'}(M')$ as R and R'-modules. Thus, in passing to limits, we obtain an isomorphism

$$D_{I,J}(M') = \varinjlim_{\mathfrak{a} \in \tilde{W}(I,J)} D_{\mathfrak{a}}(M') \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(I,J)} D_{\mathfrak{a}R'}(M').$$

as R and R'-modules, as well. Also, if $\varphi(J) = JR'$ then

$$\tilde{W}(IR', JR') = \{\mathfrak{b} \subseteq R' \mid \mathfrak{b} \text{ is an ideal of } R' \text{ and } \varphi^{-1}(\mathfrak{b}) \in \tilde{W}(I, J)\}.$$

In fact, let $\mathfrak{b}$ be an ideal of R'. So $\mathfrak{b} \in \tilde{W}(IR', JR')$ iff $\varphi(I^n) \subseteq \mathfrak{b} + \varphi(J)$ for some integer n iff $I^n \subseteq \varphi^{-1}(\mathfrak{b}) + J$ for some n. Besides,

$$\{\mathfrak{a}R' \mid \mathfrak{a} \in \tilde{W}(I, J)\} \subseteq \tilde{W}(IR', JR').$$

As for any $\mathfrak{b} \in \tilde{W}(IR', JR')$, $\mathfrak{b} \supseteq \varphi^{-1}(\mathfrak{b})R'$, the set $\{\mathfrak{a}R' \mid \mathfrak{a} \in \tilde{W}(I, J)\}$ becomes a cofinal subset of $\tilde{W}(IR', JR')$ so that by [R, Exercise 2.43], we obtain

$$\varinjlim_{\mathfrak{a}\in\tilde{W}(I,J)} D_{\mathfrak{a}R'}(M') \cong \varinjlim_{\mathfrak{b}\in\tilde{W}(IR',JR')} D_{\mathfrak{b}}(M') = D_{IR',JR'}(M').$$

□

Proposition 2.10. *Let R be a Noetherian ring and let M be an R-module. Then* $\operatorname{Ass}(D_{I,J}(M)) = \operatorname{Ass}(M/\Gamma_{I,J}(M))$.

Proof. Firstly, note that if $\mathfrak{p} \in \operatorname{Ass}(D_{I,J}(M))$, then there exists an injective map $R/\mathfrak{p} \hookrightarrow D_{I,J}(M)$, and since $\Gamma_{I,J}(D_{I,J}(M)) = 0$ (by Lemma 2.6(iv)), one obtains $\Gamma_{I,J}(R/\mathfrak{p}) = 0$. Thus, by [TYY, Proposition 1.10], $\mathfrak{p} \notin W(I, J)$.

The short exact sequence

$$0 \to M/\Gamma_{I,J}(M) \to D_{I,J}(M) \to H^1_{I,J}(M) \to 0$$

implies that

$$\operatorname{Ass}(D_{I,J}(M)) \subseteq \operatorname{Ass}(M/\Gamma_{I,J}(M)) \cup \operatorname{Ass}(H^1_{I,J}(M)),$$

Since $\operatorname{Ass}(H^1_{I,J}(M)) \subseteq W(I, J)$ (see [TYY, Proposition 1.7]), one has

$$\operatorname{Ass}(D_{I,J}(M)) \subseteq \operatorname{Ass}(M/\Gamma_{I,J}(M)).$$

On the other hand, by virtue of the above short exact sequence,

$$\operatorname{Ass}(M/\Gamma_{I,J}(M)) \subseteq \operatorname{Ass}(D_{I,J}(M)).$$

□

The following is an immediate result of Corollary 2.3 and Proposition 2.9.

Corollary 2.11. $\operatorname{Ass}(D_I(M)) = \operatorname{Ass}(M/\Gamma_I(M))$.

It is well-known that

$$\mathrm{Ass}(\Gamma_I(M)) = \mathrm{Ass}(M) \cap V(I)$$

and

$$\mathrm{Ass}(M/\Gamma_I(M)) = \mathrm{Ass}(M) \backslash V(I).$$

The authors of [TYY] showed that

$$\mathrm{Ass}(\Gamma_{I,J}(M)) = \mathrm{Ass}(M) \cap W(I,J)$$

as a corresponding result of one of the above results for (I, J)-torsion module $\Gamma_{I,J}(M)$ in [TYY, Proposition 1.10]. Now we make a supplement for it by showing the remaining part.

Lemma 2.12. [S, Proposition 4.5 and Exercise 3.13] *Let $\{M_i\}_{i\in A}$ be a direct system of R-modules. If $\mathfrak{p} \in \mathrm{Ass}_R(\varinjlim M_i)$ then $\mathfrak{p} \in \bigcup_{i\in A} \mathrm{Ass}_R M_i$.*

Proposition 2.13. *Let R be a Noetherian ring and M be an R-module. Then, $\mathrm{Ass}(M/\Gamma_{I,J}(M)) = \mathrm{Ass}(M)\backslash W(I,J)$. Thus, $\mathrm{Ass}(M) = \mathrm{Ass}(\Gamma_{I,J}(M)) \cup \mathrm{Ass}(M/\Gamma_{I,J}(M))$, as a disjunct union.*

Proof. From the short exact sequence

$$0 \to \Gamma_{I,J}(M) \to M \to M/\Gamma_{I,J}(M) \to 0,$$

we know that

$$\mathrm{Ass}(M) \subseteq \mathrm{Ass}(\Gamma_{I,J}(M)) \cup \mathrm{Ass}(M/\Gamma_{I,J}(M)).$$

By [TYY, Proposition 1.10], $\mathrm{Ass}(\Gamma_{I,J}(M)) \subseteq W(I,J)$. Then,

$$\mathrm{Ass}(M)\backslash W(I,J) \subseteq \mathrm{Ass}(M/\Gamma_{I,J}(M)).$$

Now we prove the other inclusion. By Proposition 2.9, it is enough to prove that $\mathrm{Ass}(D_{I,J}(M)) \subseteq \mathrm{Ass}(M)$. Note that

$$D_{I,J}(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}} \mathrm{Hom}_R(\mathfrak{a}, M) \quad \text{(by Theorem 2.2(ii))}.$$

By Lemma 2.12, we have that,

$$\mathrm{Ass}(D_{I,J}(M)) \subseteq \bigcup_{\mathfrak{a}\in\tilde{W}} \mathrm{Ass}(\mathrm{Hom}_R(\mathfrak{a}, M)).$$

While, for each $\mathfrak{a} \in \tilde{W}$, by a well-known fact in [B],

$$\mathrm{Ass}(\mathrm{Hom}_R(\mathfrak{a}, M)) = \mathrm{Ass}(M) \cap \mathrm{Supp}(\mathfrak{a}) \subseteq \mathrm{Ass}(M).$$

Thus, $\mathrm{Ass}(D_{I,J}(M)) \subseteq \mathrm{Ass}(M)$ and

$$\mathrm{Ass}(M) = \mathrm{Ass}(\Gamma_{I,J}(M)) \cup \mathrm{Ass}(M/\Gamma_{I,J}(M)).$$

This completes the proof. □

By Proposition 2.10, we obtain the following.

Corollary 2.14. $\mathrm{Ass}(D_{I,J}(M)) = \mathrm{Ass}(M) \setminus W(I, J)$.

Proposition 2.15. *Let M be a finite R-module. Then*

$$\mathrm{Supp}(D_{I,J}(M)) = \bigcap_{\mathfrak{a} \in \tilde{W}(I,J)} \mathrm{Supp}(D_{\mathfrak{a}}(M)).$$

In particular, if $\tilde{W}(I, J) \subseteq \tilde{W}(I_1, J_1)$, then $\mathrm{Supp}(D_{I_1,J_1}(M)) \subseteq \mathrm{Supp}(D_{I,J}(M))$.

Proof. By Proposition 2.10, $\mathrm{Supp}(D_{I,J}(M)) = \mathrm{Supp}(M/\Gamma_{I,J}(M))$. So we only need to prove that

$$\mathrm{Supp}(M/\Gamma_{I,J}(M)) = \bigcap_{\mathfrak{a} \in \tilde{W}(I,J)} \mathrm{Supp}(M/\Gamma_{\mathfrak{a}}(M)).$$

Since for $\mathfrak{a} \in \tilde{W}(I, J)$, $\Gamma_{\mathfrak{a}}(M)$ is a submodule of $\Gamma_{I,J}(M)$, $M/\Gamma_{I,J}(M)$ can be viewed as a quotient of $M/\Gamma_{\mathfrak{a}}(M)$ and

$$\mathrm{Supp}(M/\Gamma_{I,J}(M)) \subseteq \bigcap_{\mathfrak{a} \in \tilde{W}(I,J)} \mathrm{Supp}(M/\Gamma_{\mathfrak{a}}(M)).$$

Let $\mathfrak{p} \notin \mathrm{Supp}(M/\Gamma_{I,J}(M))$. Then,

$$M_{\mathfrak{p}}/(\Gamma_{I,J}(M))_{\mathfrak{p}} = (M/\Gamma_{I,J}(M))_{\mathfrak{p}} = 0.$$

So $M_{\mathfrak{p}} = (\Gamma_{I,J}(M))_{\mathfrak{p}}$. If $\{m_1, m_2, \cdots, m_k\}$ is a generator set for the R-module M, then there exists $s_i \in R\setminus\mathfrak{p}, 1 \leq i \leq k$, such that $s_i m_i \in \Gamma_{I,J}(M)$. Then, there exist $n_i, 1 \leq i \leq k$, such that $I^{n_i} \subseteq J + (0 :_R s_i m_i)$. Denote

$$\mathfrak{b} = \bigcap_{i \in \{1,2,\cdots,k\}} (0 :_R s_i m_i).$$

Clearly, $I^{n_1+n_2+\cdots+n_k} \subseteq J+\mathfrak{b}$, and $\mathfrak{b} \in \tilde{W}(I,J)$. Moreover, for $1 \leq i \leq k$, $s_i m_i \in \Gamma_{\mathfrak{b}}(M)$. This shows that $\frac{m_i}{1} \in (\Gamma_{\mathfrak{b}}(M))_{\mathfrak{p}}$, and $M_{\mathfrak{p}} = (\Gamma_{\mathfrak{b}}(M))_{\mathfrak{p}}$. Therefore, $\mathfrak{p} \notin \mathrm{Supp}(M/\Gamma_{\mathfrak{b}}(M))$. The proof is completed.

□

3 Associated primes of $H^i_{I,J}(M)$ and $\mathfrak{R}^i D_{I,J}(M)$.

Let M be a non-zero R-module. The Krull dimension $\dim_R M$ of M is the supremum of lengths of chains of prime ideals in $\mathrm{Supp}_R M$ if this supremum exists, and ∞ otherwise. In the case when M is finitely generated, this is equal to $\dim_R R/(0:M)$. If $M=0$, we set $\dim_R M = -1$.

Lemma 3.1. *Let M be a non-zero finite R-module. Let k, t be two integers. The followings are equivalent:*

(i) $\dim_R \Gamma_{I,J}(M) \leq k$;

(ii) $\dim_R \mathrm{Hom}_R(R/\mathfrak{a}, M) \leq k, \ \forall \mathfrak{a} \in \tilde{W}(I,J)$;

(iii) $\dim_R \Gamma_{\mathfrak{a}}(M) \leq k, \ \forall \mathfrak{a} \in \tilde{W}(I,J)$.

Proof. Since

$$\mathrm{Hom}_R(R/\mathfrak{a}, M) \subseteq \Gamma_{\mathfrak{a}}(M) \subseteq \Gamma_{I,J}(M), \ \ \forall \mathfrak{a} \in \tilde{W}(I,J),$$

we have

$$\mathrm{Supp}_R(\mathrm{Hom}_R(R/\mathfrak{a}, M)) \subseteq \mathrm{Supp}_R(\Gamma_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}_R(\Gamma_{I,J}(M)), \ \forall \mathfrak{a} \in \tilde{W}(I,J).$$

So the proof of (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) completes.

On the other hand, for any chain of prime ideals having the maximal length in $\mathrm{Supp}_R(\Gamma_{I,J}(M))$, there exists $\mathfrak{a} \in \tilde{W}(I,J)$ such that this chain is included in $\mathrm{Supp}_R(\mathrm{Hom}_R(R/\mathfrak{a}, M))$. This completes the proof of (ii) $\Rightarrow$ (i).

□

Theorem 3.2. *Let R be a Noetherian ring and M a non-zero finite R-module. Let $E^\bullet(M)$ a minimal injective resolution of M. Let k, t be two integers. The followings are equivalent:*

(i) $\dim_R H^i_{I,J}(M) \leq k$, $\forall i < t$;

(ii) $\dim_R \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \leq k$, $\forall i < t$, $\forall \mathfrak{a} \in \tilde{W}(I,J)$;

(iii) $\dim_R H^i_{\mathfrak{a}}(M) \leq k$, $\forall i < t$, $\forall \mathfrak{a} \in \tilde{W}(I,J)$;

(iv) $\dim_R \Gamma_{I,J}(E^i(M)) \leq k$, $\forall i < t$;

(v) $\dim_R \mathrm{Hom}_R(R/\mathfrak{a}, E^i(M)) \leq k$, $\forall i < t$, $\forall \mathfrak{a} \in \tilde{W}(I,J)$;

(vi) $\dim_R \Gamma_{\mathfrak{a}}(E^i(M)) \leq k$, $\forall i < t$, $\forall \mathfrak{a} \in \tilde{W}(I,J)$.

Moreover, if $k = -1$, and one of these equivalent conditions is satisfied, then there is an isomorphism:

$$\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M)) \cong \mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)) \cong \mathrm{Ext}^t_R(R/\mathfrak{a}, M), \ \forall \mathfrak{a} \in \tilde{W}(I,J).$$

Proof. Firstly we prove the equivalence of (i) and (iv).

Note that the following is a commutative graph:

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\Gamma_{I,J}(d^{r-1})} & \Gamma_{I,J}(E^r(M)) & \xrightarrow{\Gamma_{I,J}(d^r)} & \Gamma_{I,J}(E^{r+1}(M)) & \xrightarrow{\Gamma_{I,J}(d^{r+1})} \cdots \\
& \downarrow & & \downarrow & \\
\cdots \xrightarrow{d^{r-1}} & E^r(M) & \xrightarrow{d^r} & E^{r+1}(M) & \xrightarrow{d^{r+1}} \cdots
\end{array}$$

Since $\mathrm{Ker} d^r \subseteq E^r(M)$ is an essential extension, then $\mathrm{Ker}\Gamma_{I,J}(d^r) = \mathrm{Ker} d^r \cap \Gamma_{I,J}(E^r(M)) \subseteq \Gamma_{I,J}(E^r(M))$ is an essential extension. Note that if $K \subseteq L$ is an essential extension, then $\mathrm{Ass}_R K = \mathrm{Ass}_R L$, so that $\dim_R K \leq k$ if and only if $\dim_R L \leq k$, for some integer k. Therefore,

$$(a) \qquad \dim_R \mathrm{Ker}\Gamma_{I,J}(d^r) \leq k \Longleftrightarrow \dim_R \Gamma_{I,J}(E^r(M)) \leq k$$

for some integer k. On the other hand, it is clear that

$$(b) \qquad \dim_R \mathrm{Im}\Gamma_{I,J}(d^r) \leq k \text{ if } \dim_R \Gamma_{I,J}(E^r(M)) \leq k.$$

By using the following exact sequence

$$0 \longrightarrow \mathrm{Im}\Gamma_{I,J}(d^{r-1}) \longrightarrow \mathrm{Ker}\Gamma_{I,J}(d^r) \longrightarrow H^r_{I,J}(M) \longrightarrow 0$$

for $r = 1, 2, \cdots, t-1$ and $\mathrm{Ker}\Gamma_{I,J}(d^0) \cong H^0_{I,J}(M)$, it follows from the results (a) and (b) that

$\dim_R H^i_{I,J}(M) \leq k$, $\forall i < t \Longleftrightarrow \dim_R \Gamma_{I,J}(E^i(M)) \leq k$, $\forall i < t$.

Using the above method, for each $\mathfrak{a} \in \tilde{W}(I,J)$, we can also prove that

$\dim_R \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \leq k$, $\forall i < t \Longleftrightarrow \dim_R \mathrm{Hom}_R(R/\mathfrak{a}, E^i(M)) \leq k$, $\forall i < t$, and

$\dim_R H^i_{\mathfrak{a}}(M) \leq k$, $\forall i < t \Longleftrightarrow \dim_R \Gamma_{\mathfrak{a}}(E^i(M)) \leq k$, $\forall i < t$.

Finally, it follows from Lemma 3.1 the equivalence of these conditions.

Let $k = -1$. Assume that $\Gamma_{I,J}(E^i(M)) = 0$ for $\forall i < t$. Let $\mathfrak{a} \in \tilde{W}(I,J)$. Set $T(-) = \Gamma_{I,J}(-)$, $\Gamma_{\mathfrak{a}}(-)$ or $\mathrm{Hom}_R(R/\mathfrak{a}, -)$. From the following commutative graph:

$$\begin{tikzcd}
\cdots \arrow[r, "T(d^{t-1})"] & T(E^t(M)) \arrow[r, "T(d^t)"] \arrow[d] & T(E^{t+1}(M)) \arrow[r, "T(d^{t+1})"] \arrow[d] & \cdots \\
\cdots \arrow[r, "d^{t-1}"] & E^t(M) \arrow[r, "d^t"] & E^{t+1}(M) \arrow[r, "d^{t+1}"] & \cdots ,
\end{tikzcd}$$

it follows that, for $\forall \mathfrak{a} \in \tilde{W}(I,J)$,

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Ker}\Gamma_{I,J}(d^t)) \cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Ker} d^t) \cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Ker}\Gamma_{\mathfrak{a}}(d^t)).$$

This shows that, for $\forall \mathfrak{a} \in \tilde{W}(I,J)$,

$$\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M)) \cong \mathrm{Ext}^t_R(R/\mathfrak{a}, M) \cong \mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)).$$

□

By virtue of the above theorem for the cases that $k = -1, 0$, the following corollaries may be obtained in another way.

Corollary 3.3. ([TYY, Theorem 4.1]) *For a finite R-module M, there is an equality*

$$\inf\{i \mid H^i_{I,J}(M) \neq 0\} = \inf\{\mathrm{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in W(I,J)\}.$$

Proof. Note the well-known fact that

$$\inf\{i \mid \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\} = \inf\{\mathrm{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in V(\mathfrak{a})\}.$$

It is clear that $\mathfrak{p} \in W(I,J)$ if and only if $\exists\, \mathfrak{a} \in \tilde{W}(I,J)$, $\mathfrak{p} \in V(\mathfrak{a})$. Then, by Theorem 3.2,

$$\inf\{i \mid H^i_{I,J}(M) \neq 0\} = \inf_{\mathfrak{a} \in \tilde{W}(I,J)} \inf\{i \mid \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\}$$

$$= \inf_{\mathfrak{a}\in\tilde{W}(I,J)} \inf\{\mathrm{depth} M_{\mathfrak{p}} \mid \mathfrak{p}\in V(\mathfrak{a})\}$$
$$= \inf\{\mathrm{depth} M_{\mathfrak{p}} \mid \mathfrak{p}\in W(I,J)\,\}.$$

$\square$

Corollary 3.4. ([CW, Theorem 2.4]) *Let M be a finite module on a local ring $(R,\mathfrak{m})$. Then there is an equality*

$$\inf\{i \mid H^i_{I,J}(M) \text{ is not Artinian}\} = \inf\{\mathrm{depth} M_{\mathfrak{p}} \mid \mathfrak{p}\in W(I,J)\backslash\{\mathfrak{m}\}\,\}.$$

Proof. From the proof of Theorem 3.2, it is clear that

$$\begin{aligned}\dim_R H^i_{I,J}(M) \leq 0 \text{ for all } i<t &\iff \dim_R \Gamma_{I,J}(E^i(M)) \leq 0 \text{ for all } i<t\\ &\iff \Gamma_{I,J}(E^i(M)) = \Gamma_{\mathfrak{m}}(E^i(M)) \text{ for all } i<t\\ &\iff \Gamma_{I,J}(E^i(M)) \text{ is Artinian for all } i<t\\ &\iff H^i_{I,J}(M) \text{ is Artinian for all } i<t.\end{aligned}$$

Note the well-known fact that

$$\inf\{i \mid H^i_{\mathfrak{a}}(M) \text{ is not Artinian}\} = \inf\{\mathrm{depth} M_{\mathfrak{p}} \mid \mathfrak{p}\in V(\mathfrak{a})\backslash\{\mathfrak{m}\}\,\}.$$

By using the similar method to the above corollary, we get this result. $\square$

The following proposition improves one of the main results in [TT] (See [TT, Theorem 3.6]).

Proposition 3.5. *Let M be a finite R-module and $t = \inf\{i \mid H^i_{I,J}(M) \neq 0\}$. Then, there are some equalities*

$$\begin{aligned}\mathrm{Ass}_R H^t_{I,J}(M) &= \bigcup_{\substack{\mathfrak{a}\in\tilde{W}(I,J)\\ \mathrm{grade}(\mathfrak{a},M)=t}} \mathrm{Ass}_R H^t_{\mathfrak{a}}(M) = \bigcup_{\substack{\mathfrak{a}\in\tilde{W}(I,J)\\ \mathrm{grade}(\mathfrak{a},M)=t}} \mathrm{Ass}_R \mathrm{Ext}^t_R(R/\mathfrak{a},M).\\ &= \bigcup_{\substack{\mathfrak{p}\in W(I,J)\\ \mathrm{grade}(\mathfrak{p},M)=t}} \mathrm{Ass}_R H^t_{\mathfrak{p}}(M) = \bigcup_{\substack{\mathfrak{p}\in W(I,J)\\ \mathrm{grade}(\mathfrak{p},M)=t}} \mathrm{Ass}_R \mathrm{Ext}^t_R(R/\mathfrak{p},M).\end{aligned}$$

Proof. We recall the well-known fact that

$$\mathrm{Ass}_R H^r_I(M) = \mathrm{Ass}_R \mathrm{Ext}^r_R(R/I,M)$$

for $r = \operatorname{grade}(I, M)$. So it is enough to prove that

$$\begin{aligned}
\operatorname{Ass}_R H^t_{I,J}(M) &\subseteq \bigcup_{\substack{\mathfrak{p} \in W(I,J) \\ \operatorname{grade}(\mathfrak{p}, M) = t}} \operatorname{Ass}_R H^t_{\mathfrak{p}}(M) \\
&\subseteq \bigcup_{\substack{\mathfrak{a} \in \tilde{W}(I,J) \\ \operatorname{grade}(\mathfrak{a}, M) = t}} \operatorname{Ass}_R H^t_{\mathfrak{a}}(M) \\
&\subseteq \operatorname{Ass}_R H^t_{I,J}(M).
\end{aligned}$$

The second inclusion is clear. We will prove the first and the third inclusions.

Let $\mathfrak{p} \in \operatorname{Ass}_R H^t_{I,J}(M)$. By [TYY, Proposition 1.7] and [TYY, Corollary 1.13], $\mathfrak{p} \in W(I, J)$. Note the well-known fact that, for a finitely generated $R-$module K and an arbitrary $R-$module L, $\operatorname{Ass}_R \operatorname{Hom}_R(K, L) = \{\mathfrak{p} \in \operatorname{Ass}_R L \mid \exists\, \mathfrak{q} \in \operatorname{Ass}_R K, \mathfrak{q} \subseteq \mathfrak{p}\}$ (see [B]). Then, by Theorem 3.2,

$$\mathfrak{p} \in \operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{p}, H^t_{I,J}(M)) = \operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{p}, H^t_{\mathfrak{p}}(M)) = \operatorname{Ass}_R H^t_{\mathfrak{p}}(M).$$

Then $H^t_{\mathfrak{p}}(M) \neq 0$, and $\operatorname{grade}(\mathfrak{p}, M) \leq t$. On the other hand, $t = \inf\{\operatorname{depth} M_{\mathfrak{q}} \mid \mathfrak{q} \in W(I, J)\} \leq \inf\{\operatorname{depth} M_{\mathfrak{q}} \mid \mathfrak{q} \in V(\mathfrak{p})\} = \operatorname{grade}(\mathfrak{p}, M)$. So $t = \operatorname{grade}(\mathfrak{p}, M)$, and the proof of the first inclusion completes.

Let $\mathfrak{a} \in \tilde{W}(I, J), \operatorname{grade}(\mathfrak{a}, M) = t$ and $\mathfrak{p} \in \operatorname{Ass}_R H^t_{\mathfrak{a}}(M)$. Then by Theorem 3.2,

$$\mathfrak{p} \in \operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)) = \operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M)) \subseteq \operatorname{Ass}_R H^t_{I,J}(M).$$

This completes the proof of the third inclusion. □

In the following, a characterization for the associated primes of the non-vanishing local cohomology with respect to a pair ideals is given. It generalizes [TYY, Proposition 5.6] (see Remark 3.8) and [TT, Theorem 3.6].

Theorem 3.6. *Let M be a finite R-module and $t = \inf\{i \mid H^i_{I,J}(M) \neq 0\}$. Then, there is an equality*

$$\operatorname{Ass}_R H^t_{I,J}(M) = \{\mathfrak{p} \in W(I, J) \mid \operatorname{depth} M_{\mathfrak{p}} = t\}.$$

Proof. We take $\mathfrak{p} \in W(I, J)$ such that $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = t$. Then $(H^t_\mathfrak{p}(M))_\mathfrak{p} \cong H^t_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. Therefore, $H^t_\mathfrak{p}(M) \neq 0$. So $\mathrm{grade}(\mathfrak{p}, M) \leq t$. Note that

$$t = \inf\{\mathrm{depth} M_\mathfrak{q} \mid \mathfrak{q} \in W(I, J)\} \leq \inf\{\mathrm{depth} M_\mathfrak{q} \mid \mathfrak{q} \in V(\mathfrak{p})\} = \mathrm{grade}(\mathfrak{p}, M).$$

Therefore, $\mathrm{grade}(\mathfrak{p}, M) = t$. From the fact that

$$\mathrm{Ass}_{R_\mathfrak{p}} \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, H^t_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) = \mathrm{Ass}_{R_\mathfrak{p}} H^t_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq \emptyset,$$

we know that $\mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, H^t_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) \neq 0$. This implies that

$$\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_\mathfrak{p}} H^t_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{Ass}_{R_\mathfrak{p}} (H^t_\mathfrak{p}(M))_\mathfrak{p}.$$

Then $\mathfrak{p} \in \mathrm{Ass}_R H^t_\mathfrak{p}(M)$. By Proposition 3.5, we get that

$$\{\mathfrak{p} \in W(I, J) \mid \mathrm{depth} M_\mathfrak{p} = t\} \subseteq \mathrm{Ass}_R H^t_{I,J}(M).$$

On the other hand, let $\mathfrak{p} \in \mathrm{Ass}_R H^t_{I,J}(M)$. Then $\mathfrak{p} \in W(I, J)$. Then, by Theorem 3.2,

$$\mathfrak{p} \in \mathrm{Ass}_R \mathrm{Hom}_R(R/\mathfrak{p}, H^t_{I,J}(M)) = \mathrm{Ass}_R \mathrm{Hom}_R(R/\mathfrak{p}, H^t_\mathfrak{p}(M)) = \mathrm{Ass}_R H^t_\mathfrak{p}(M).$$

Since $H^t_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) = (H^t_\mathfrak{p}(M))_\mathfrak{p} \neq 0$, we get $\mathrm{depth} M_\mathfrak{p} \leq t$. Also,

$$t = \inf\{\mathrm{depth} M_\mathfrak{q} \mid \mathfrak{q} \in W(I, J)\} \leq \mathrm{depth} M_\mathfrak{p}.$$

Therefore, $\mathrm{depth} M_\mathfrak{p} = t$, and

$$\mathrm{Ass}_R H^t_{I,J}(M) \subseteq \{\mathfrak{p} \in W(I, J) \mid \mathrm{depth} M_\mathfrak{p} = t\}.$$

This completes the proof.

$\square$

Corollary 3.7. *Let M be a finite $R-$module and $t = \inf\{i \mid H^i_I(M) \neq 0\}$(= $\mathrm{grade}(I, M)$). Then, there is an equality*

$$\mathrm{Ass}_R H^t_I(M) = \{\mathfrak{p} \in V(I) \mid \mathrm{depth} M_\mathfrak{p} = t\}.$$

Hence, if $x_1, x_2, \cdots, x_t$ is a maximal $M-$sequence in I, then

$$\mathrm{Ass}_R H^t_I(M) = V(I) \cap \mathrm{Ass}_R M/(x_1, x_2, \cdots, x_t)M.$$

Proof. In Theorem 3.6, we take $J = 0$ and get that

$$\mathrm{Ass}_R H_I^t(M) = \{\mathfrak{p} \in V(I) \mid \mathrm{depth} M_{\mathfrak{p}} = t\}.$$

Now we shall prove that

$$\mathrm{Ass}_R H_I^t(M) = V(I) \cap \mathrm{Ass}_R M/(x_1, x_2, \cdots, x_t)M.$$

Let $\mathfrak{p} \in \mathrm{Ass}_R H_I^t(M)$. Since $\mathrm{depth} M_{\mathfrak{p}} = t$, it follows that $x_1/1, x_2/1, \cdots, x_t/1$ is a maximal $M_{\mathfrak{p}}-$sequence in $pR_{\mathfrak{p}}$, and so $pR_{\mathfrak{p}} \in \mathrm{Ass}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}/(x_1, x_2, \cdots, x_t)M_{\mathfrak{p}}$. Hence, $\mathfrak{p} \in \mathrm{Ass}_R M/(x_1, x_2, \cdots, x_t)M$. On the other hand, we let $\mathfrak{p} \in \mathrm{Ass}_R M/(x_1, x_2, \cdots, x_t)M$. Then

$$pR_{\mathfrak{p}} \in \mathrm{Ass}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}/(x_1, x_2, \cdots, x_t)M_{\mathfrak{p}}.$$

This means that $\mathrm{depth} M_{\mathfrak{p}} = t$. This completes the proof. □

Remark 3.8. Takahashi, Yoshino and Yoshizawa gave the following result [TYY, Proposition 5.6]*: Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* d *with canonical module* $K_R = \mathrm{Hom}(H_{\mathfrak{m}}^d(R), E(R/\mathfrak{m}))$*. Assume that* J *is a perfect ideal of grade* r*. Then there is the following equality*

$$\mathrm{Ass}_R H_{\mathfrak{m},J}^{d-r}(K_R) = \{\mathfrak{p} \in W(\mathfrak{m}, J) \mid \mathrm{ht}\mathfrak{p} = d - r\}.$$

Since K_R *is a Cohen-Macaulay module of dimension* d*, by* [TYY, Proposition 5.6]*, it follows that, for* $\mathfrak{p} \in W(\mathfrak{m}, J)$*,*

$$\mathrm{depth}(K_R)_{\mathfrak{p}} = \dim(K_R)_{\mathfrak{p}} = \dim K_{R_{\mathfrak{p}}} = \dim R_{\mathfrak{p}} = d - r.$$

Therefore, Theorem 3.6 *generalizes* [TYY, Proposition 5.6].

Proposition 3.9. *Let* R *be a Noetherian ring,* M *a finite* R*-module and* $t = \inf\{i \mid H_{I,J}^i(M) \neq 0\}$*. Then*

(i) $\mathrm{Ass}(H_{I,J}^t(M)) = \bigcup_{\mathfrak{a} \in \tilde{W}(I,J)} \mathrm{Ass}(H_{\mathfrak{a}}^t(M))$.

(ii) $\mathrm{Ass}(H_{I,J}^i(M)) \subseteq \bigcup_{\mathfrak{a} \in \tilde{W}(I,J)} \mathrm{Ass}(H_{\mathfrak{a}}^i(M))$ *for each* $i \geq 0$.

(iii) $\mathrm{Ass}(\mathfrak{R}^i D_{I,J}(M)) \subseteq \bigcup_{\mathfrak{a} \in \tilde{W}(I,J)} \mathrm{Ass}(\mathfrak{R}^i D_{\mathfrak{a}}(M))$ *for each* $i \geq 0$.

Proof. Item (i) can be deduced as same as Proposition 3.5 (we do not consider the grade(depth) in the proof of Proposition 3.5). Further, since

$$H_{I,J}^i(M) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}} H_{\mathfrak{a}}^i(M),$$

and

$$\mathfrak{R}^i D_{I,J}(M) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}} \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}_R^i(\mathfrak{a}^n, M) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}} \mathfrak{R}^i D_{\mathfrak{a}}(M),$$

items (ii) and (iii) follow by Lemma 2.12. □

4 Exactness of ideal transforms

In this section, we discuss about exactness of ideal transforms with respect to a pair of ideals (I, J). In fact, in this sense the behavior of the functor $D_{I,J}$ is very similar to the ordinary case for one ideal.

Remark 4.1. *Using the notation of Theorem* 2.4*, it can be shown that* $D_{I,J}(\eta_M) = \eta_{D_{I,J(M)}}$*, and further,* $\eta_R : R \to D_{I,J}(R)$ *is a ring homomorphism. We recall that* $D_{I,J}(R)$ *carries a structure of a commutative ring since it is given by limits of commutative rings.*

Proposition below may be seen for the usual case in [BS, Proposition 2.2.17].

Lemma 4.2. *Let* $\phi : R \to R'$ *a ring homomorphism such that* $\mathrm{Ker}\phi$ *and* $\mathrm{Coker}\phi$ *are both* (I, J)*-torsion, where* R' *is regarded as an* R*-module through* ϕ*. Then the unique* R*-homomorphism* $\psi' : R \to D_{I,J}(R)$ *(obtained from Proposition* 2.7*) such that the diagram*

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[dr, "\eta_R"'] & R' \arrow[d, "\psi'"] \\ & D_{I,J}(R) \end{tikzcd}$$

commutes is a ring homomorphism.

Proof. Firstly, note that $\psi'(1_{R'}) = \psi'(\phi(1_R)) = \eta_R(1_R) = 1_{D_{I,J}(R)}$. Now let $r'_1, r'_2 \in R'$ and let $\overline{r'_1}, \overline{r'_2}$ be their natural images in $\mathrm{Coker}\phi$. So, by hypothesis, there exists n such that $I^n \subseteq \mathrm{ann}(\overline{r'_i}) + J$ for $i = 1, 2$. Let $x := \psi'(r'_1)\psi'(r'_2) - \psi'(r'_1 r'_2) \in D_{I,J}(R)$. We claim $I^{2n} \subseteq \mathrm{ann}(x) + J$ so that $x \in \Gamma_{I,J}(D_{I,J}(R)) = 0$. In fact, let $b_1, b_2 \in I^n$. Thus $b_i = a_i + j_i$ where $a_i \in \mathrm{ann}(\overline{r'_i})$ and $j_i \in J$ for $i = 1, 2$. In this way, $a_i r'_i = \phi(r_i)$ for some $r_i \in R$ and $b_1 b_2 = a_1 a_2 + j$ with $j \in J$. In conclusion, see that

$$\begin{aligned} a_1 a_2 \psi'(r'_1)\psi'(r'_2) = \psi'(a_1 r'_1)\psi'(a_2 r'_2) &= \psi'(\phi(r_1))\psi'(\phi(r_2)) \\ &= \eta_R(r_1)\eta_R(r_2) = \eta_R(r_1 r_2) = \psi'(\phi(r_1 r_2)) \\ &= \psi'(\phi(r_1)\phi(r_2)) = \psi'(a_1 r'_1 a_2 r'_2) = a_1 a_2 \psi'(r'_1 r'_2), \end{aligned}$$

so that $a_1 a_2 \in \mathrm{ann}(x)$. □

Next result is a special case of [BS, Corollary 2.2.18].

Corollary 4.3. *Let M be an R-module and S a multiplicatively closed subset of R formed entirely of non-zerodivisors on M such that $S \cap \mathfrak{a} \neq \emptyset$ for all $\mathfrak{a} \in \tilde{W}(I, J)$. Then there exists an R-isomorphism*

$$\psi'_M : \bigcup_{\mathfrak{a} \in \tilde{W}(I,J)} (M :_{S^{-1}M} \mathfrak{a}) \to D_{I,J}(M)$$

for which the diagram

$$\begin{tikzcd} M \arrow[r, "\subseteq"] \arrow[dr, "\eta_M"'] & \bigcup_{\mathfrak{a} \in \tilde{W}(I,J)} (M :_{S^{-1}M} \mathfrak{a}) \arrow[d, "\psi'"] \\ & D_{I,J}(M) \end{tikzcd}$$

commutes.

Proof. By the ordinary case (see [BS, Corollary 2.2.15]), we know that there exists a R-isomorphism

$$\begin{tikzcd} M \arrow[r, "\subseteq"] \arrow[dr, "\eta_{\mathfrak{a}}"'] & \bigcup_{n \in \mathbb{N}} (M :_{S^{-1}M} \mathfrak{a}^n) \arrow[d, "\psi'_{\mathfrak{a}}"] \\ & D_{\mathfrak{a}}(M) \end{tikzcd}$$

By passing to limits, we obtain

$$\begin{tikzcd} M \arrow[r, "\subseteq"] \arrow[dr, "\eta_M"'] & \bigcup_{\mathfrak{a} \in \tilde{W}(I,J)} \bigcup_{n \in \mathbb{N}} (M :_{S^{-1}M} \mathfrak{a}^n) \arrow[d, "\psi'_M"] \\ & D_{I,J}(M) \end{tikzcd}$$

Note that $\bigcup_{\mathfrak{a} \in \tilde{W}(I,J)} \bigcup_{n \in \mathbb{N}} (M :_{S^{-1}M} \mathfrak{a}^n) = \bigcup_{\mathfrak{a} \in \tilde{W}(I,J)} (M :_{S^{-1}M} \mathfrak{a})$. $\square$

The following is about exactness of ideal transforms which is an extension of [BS, Lemma 6.3.1] and the proof is very similar (we make use of Theorem 2.4(ii)), so we omit it.

Proposition 4.4. *Let M be an R-module. Then the following conditions are equivalent:*

(i) $D_{I,J}$ *is an exact functor;*

(ii) $H^i_{I,J}(R) = 0$ *for all* $i > 1$*;*

(iii) $H^2_{I,J}(M) = 0$ *for each finite* R*-module* M*;*

(iv) $H^2_{I,J}(M) = 0$ *for each* R*-module* M*.*

We will see the hypothesis from next result implies the exactness of $D_{I,J}$. The usual case for next proposition would be [BS, Proposition 6.3.4].

Proposition 4.5. *Set* $\mathfrak{c} := \bigcap_{\mathfrak{a}\in\tilde{W}(I,J)}\mathfrak{a}$ *and assume that* $\mathfrak{c}D_{I,J}(R) = D_{I,J}(R)$*. Then* $D_{I,J}(R)$ *is a finitely generated* R*-algebra.*

Proof. The ring R is an (I,J)-torsion if and only if $D_{I,J}(R) = 0$ by Remark 2.5. So, we assume $\Gamma_{I,J}(R) \neq R$ and set $\overline{R} := R/\Gamma_{I,J}(R)$.

By Remark 2.5, there is an injective homomorphism (induced by η_R) $\phi_R : \overline{R} \to D_{I,J}(R)$ and $\operatorname{Coker}(\eta_R) \cong H^1_{I,J}(R)$, and consequently, $\operatorname{Coker}(\phi_R)$ is $(\overline{I},\overline{J})$-torsion, where $\overline{I} := I\overline{R}$ and $\overline{J} := J\overline{R}$. In conclusion, the $\overline{R}$-homomorphism $\psi : D_{I,J}(R) \to D_{\overline{I},\overline{J}}(\overline{R})$ (obtained from Lemma 4.2) such that the diagram

$$\begin{tikzcd} \overline{R} \arrow[r, "\phi_R"] \arrow[dr, "\eta_{\overline{R}}"'] & D_{I,J}(R) \arrow[d, "\psi"] \\ & D_{\overline{I},\overline{J}}(R) \end{tikzcd}$$

commutes is a ring homomorphism. Besides, by [TYY, Theorem 2.7] and Lemma 2.6(iv), $H^i_{\overline{I},\overline{J}}(D_{I,J}(R)) \cong H^i_{I,J}(D_{I,J}(R)) = 0$ for $i = 0, 1$. Because of Proposition 2.7(iii), the map ψ is an isomorphism, so that $\overline{\mathfrak{c}}D_{\overline{I},\overline{J}}(\overline{R}) = D_{\overline{I},\overline{J}}(\overline{R})$. A fortiori, $(\bigcap_{\mathfrak{a}\in\tilde{W}(I,J)}\overline{\mathfrak{a}})D_{\overline{I},\overline{J}}(\overline{R}) = D_{\overline{I},\overline{J}}(\overline{R})$. As $\tilde{W}(\overline{I},\overline{J}) = \{\overline{\mathfrak{a}} \mid \mathfrak{a} \in \tilde{W}(I,J)\}$ and $\Gamma_{\overline{I},\overline{J}}(\overline{R}) = 0$, we may assume $\Gamma_{I,J}(R) = 0$. It means that for each $\mathfrak{a} \in \tilde{W}(I,J)$ there exists an element $s_{\mathfrak{a}} \in \mathfrak{a}$ which is a non-zerodivisor on R. Now define $S := \{s^{n_1}_{\mathfrak{a}_1}\cdots s^{n_r}_{\mathfrak{a}_r} \mid s_{\mathfrak{a}_i} \in \mathfrak{a}_i,\ n_i \in \mathbb{N}\}$. By Corollary 4.3, the ring $D := \bigcup_{\mathfrak{a}\in\tilde{W}(I,J)}(R :_{S^{-1}R} \mathfrak{a})$ is such that $\mathfrak{c}D = D$. Thus, if $c_1, ..., c_t$ generate $\mathfrak{c}$, then there exist $d_1, ..., d_t \in D$ such that $1 = \sum_{i=1}^t c_id_i$. We claim $D = R[d_1, ..., d_t]$. Let $z \in (R :_{S^{-1}R} \mathfrak{a})$, so that $c_iz \in (R :_{S^{-1}R} R) \subseteq R[d_1, ..., d_t]$ for $i = 1, ..., t$. Hence, $z = \sum_{i=1}^t c_id_iz = \sum_{i=1}^t (c_iz)d_i \in R[d_1, ..., d_t]$. $\square$

Next result complements Proposition 4.5. See [BS, Proposition 6.3.5] for the ordinary case.

Proposition 4.6. *Set* $\mathfrak{c} := \bigcap_{\mathfrak{a}\in\tilde{W}(I,J)}\mathfrak{a}$. *The followings are equivalent.*

(i) *The functor* $D_{I,J}$ *is exact and* $\mathfrak{c} \in \tilde{W}(I,J)$;

(ii) $\mathfrak{c}D_{I,J}(R) = D_{I,J}(R)$.

Proof. (i) $\Rightarrow$ (ii) By [BS, Exercise 6.1.8], one has

$$D_{I,J}(R/\mathfrak{c}) \cong D_{I,J}(R)/\mathfrak{c}D_{I,J}(R). \tag{4.1}$$

Now see that by Lemma 2.6(i), $\mathfrak{c} \in \tilde{W}(I,J)$ implies $D_{I,J}(R/\mathfrak{c}) = 0$.

(ii) $\Rightarrow$ (i) We have $\mathfrak{a}D_{I,J}(R) = D_{I,J}(R)$ for all $\mathfrak{a} \in \tilde{W}(I,J)$. By Proposition 4.5, $D_{I,J}(R)$ is Noetherian, so by the usual Independence Theorem,

$$H^i_{\mathfrak{a}}(D_{I,J}(R)) \cong H^i_{\mathfrak{a}D_{I,J}(R)}(D_{I,J}(R)) = 0 \ \text{ for all } i \geq 0.$$

Then by [TYY, Theorem 3.2], one has $H^i_{I,J}(D_{I,J}(R)) = 0$. Lemma 2.6(v) says $H^i_{I,J}(R) = H^i_{I,J}(D_{I,J}(R))$ for $i > 1$. The exactness of $D_{I,J}$ follows by Proposition 4.4. On the other hand, by the isormorphism above (4.1), one gets $D_{I,J}(R/\mathfrak{c}) = 0$, so that $R/\mathfrak{c} = \Gamma_{I,J}(R/\mathfrak{c})$ (see Remark 2.5(ii)). In particular, $I^n \subseteq \operatorname{ann}_R(\overline{1}) + J$ for some $n > 0$. As $\operatorname{ann}_R(\overline{1}) = \mathfrak{c}$, we have $\mathfrak{c} \in \tilde{W}(I,J)$. $\square$

Acknowlegment. *The authors thank the referee for the revision and suggestions. This paper was finished when the first author visited Institute of Mathematics and Computer Science, ICMC, University of São Paulo. He is deeply grateful for the warm hospitality. The author Pedro Lima is deeply grateful to Sathya Sai Baba for the guidance.*

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