Representations of monomiality principle with Sheffer-type polynomials
and boson normal ordering

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Received 30 August 2005; received in revised form 18 November 2005; accepted 21 November 2005
Available online 28 November 2005
Communicated by P.R. Holland

Abstract

We construct explicit representations of the Heisenberg–Weyl algebra \([ P, M ] = 1\) in terms of ladder operators acting in the space of Sheffer-type polynomials. Thus we establish a link between the monomiality principle and the umbral calculus. We use certain operator identities which allow one to evaluate explicitly special boson matrix elements between the coherent states. This yields a general demonstration of boson normal ordering of operator functions linear in either creation or annihilation operators. We indicate possible applications of these methods in other fields.

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PACS: 03.65.Fd; 02.30.Vv; 02.10.Ox
Keywords: Monomiality principle; Sheffer-type polynomials; Boson normal ordering

1. Introduction

The Heisenberg–Weyl algebra, i.e., the algebra of \( P, M \) and 1 satisfying the commutation relations

\[
[ P, M ] = 1, \quad [ P, 1 ] = [ M, 1 ] = 0,
\]

has attracted attention of physicists and mathematicians from a long time. In quantum mechanics the canonical commutators of momentum and position operators provide the most famous example of Eq. (1). Also, in the second quantization method the boson creation \( a^\dagger \) and annihilation \( a \) operators acting in the Fock space satisfy \([a, a^\dagger] = 1\).

Obviously, just mentioned representations of Heisenberg–Weyl algebra do not exhaust its applications. One should point out canonical transformations leading from the standard Fock space ladder operators to the so-called quasi-particle operators, i.e., the Bogolubov–Valatin transformation, successfully applied in condensed matter physics and quantum optics from many years. More recently, realizations of Eq. (1) by ordinary differential operators acting in the space of polynomials were found to play important role in various branches of mathematics and mathematical physics. The crucial step was to establish connection between differential operators having polynomial eigenfunctions and finite-dimensional irreducible representations of the algebra \( sl_2 \) of the first-order differential operators. In fact, this result allowed to single out which operators belonging to the universal enveloping algebra of Eq. (1) admit polynomial eigenfunctions [1] and linked the problem under consideration to representations of Lie algebras by higher-order differential operators [2]. Further consequence was to develop a method called Lie-algebraic discretization of differential equations [3,4] and to adopt the latter, using representations of Eq. (1) given in terms of finite-difference operators, in order to analyze isospectral problems for differential and finite-difference operators having the same functional shape.
Such an approach provides us with a scheme of discretization of continuous problems which obey physically well justified properties—continuous and discrete problems both have the same eigenvalues while eigenfunctions are modified in a simple and controllable way. Consequently, both models with discretized space–time and systems with discrete spaces of internal degrees of freedom were studied. As examples of problems belonging to the first group we can mention construction of various finite difference replacements for the derivative $D$ and its conjugate $X$ satisfying either the standard commutation rule $[D, X] = 1$, see [5–7], or its one parameter deformations [7]. Concerning the second group of problems alternative realizations of Eq. (1) emerged in string theory where, for some discrete models of two-dimensional theories, Eq. (1) appears as the string equation whose solutions describe properties of the model under consideration [8–10].

Our particular interest in the problem of non-standard realizations of Eq. (1) comes from the observation that solutions to the Eq. (1) given by differential operators acting on suitably chosen polynomials lead to ladder structures being identical with those spanned by usual multiplication and derivative operators acting on monomials—the property named afterwards the monomiality principle. Moreover, this property enables one to construct new types of functions related to classical (orthogonal) polynomials and to investigate effectively their properties [11,12].

In this Letter we shall exploit the link between alternative realizations of the Heisenberg–Weyl algebra, the monomiality principle and the umbral calculus [13–15]. Next, we connect special solutions of the monomiality principle given by the so-called Sheffer-type polynomials, the latter explicitly constructed by the umbral calculus methods, to the problem of boson normal ordering [16,17]. In such a way we achieve an alternative derivation of some results of the boson normal ordering problem initiated in [18]. In addition, this setting permits one a rather straightforward calculation of a number of special boson matrix elements in the coherent state representation.

2. Realizations of the Heisenberg–Weyl algebra and quasi-monomials

The simplest representations of Eq. (1) is that given in terms of $X$, the multiplication by $x$ and the derivative $D \equiv \frac{d}{dx}$, as they satisfy

$$[D, X] = 1.$$  

Evidently, $X$ and $D$ acting on the monomials $x^n$ give

$$X x^n = x^{n+1}, \quad D x^n = n x^{n-1}. \quad (3)$$

We shall be concerned with constructing and studying the representations of Eq. (1) in terms of the operators $M(X, D)$ and $P(D)$ such that the action of $M$ and $P$ on certain polynomials $s_n(x)$ is analogous to the action of $X$ and $D$ on monomials. More specifically, we shall search for $M(X, D)$ and $P(D)$ and their associated polynomials $s_n(x)$ (of degree $n$, $n = 0, 1, 2, \ldots$) that satisfy

$$M s_n(x) = s_{n+1}(x), \quad P s_n(x) = n s_{n-1}(x). \quad (4)$$

The polynomials $s_n(x)$ are then called quasi-monomials with respect to $M$ and $P$. These operators can be immediately recognized as raising and lowering operators acting on the $s_n(x)$’s. Obviously, $M$ and $P$ satisfy Eq. (1). Further consequence of Eq. (4) is the eigenproperty of $M P$

$$M P s_n(x) = n s_n(x). \quad (5)$$

The polynomials $s_n(x)$ are obtained through the action of $M^n$ on $s_0(x)$

$$s_n(x) = M^n s_0(x) \quad (6)$$

(in the following we shall always set $s_0(x) = 1$), and consequently the exponential generating function (egf) of $s_n(x)$ is

$$G(\lambda, x) \equiv \sum_{n=0}^{\infty} s_n(x) \frac{\lambda^n}{n!} = e^{\lambda M} 1. \quad (7)$$

Also, if we write the quasimonomial $s_n(x)$ explicitly as

$$s_n(x) = \sum_{k=0}^{n} s_{n,k} x^k, \quad \quad (8)$$

then

$$s_n(x) = \left[ \sum_{k=0}^{n} s_{n,k} x^k \right] 1. \quad (9)$$

Several types of such polynomial sequences were studied recently using this monomiality principle embodied in Eqs. (4)–(7). If $s_n(x)$ are of the Sheffer-type, i.e., of the type of polynomials related to exponential generating functions of a very specific form, then it is always possible to find explicit representations of $M$ and $P$. Conversely, if $M = M(X, D)$ and $P = P(D)$ then $s_n(x)$ of Eq. (4) are necessarily of Sheffer-type.

3. Sheffer-type polynomials and monomiality

As we mentioned earlier the properties of Sheffer-type polynomials are naturally handled within the so-called umbral calculus [14,15]. Let us recall some relevant facts about them with special emphasis on their ladder structure. Suppose we have a polynomial sequence $s_n(x)$, $n = 0, 1, 2, \ldots$ (in the latter being a polynomial of degree $n$). It is called of a Sheffer A-type zero [13] (which we shall call here Sheffer-type) if there exists a function $f(x)$ such that

$$f(D)s_n(x) = n s_{n-1}(x), \quad \quad \quad (10)$$

which is the lowering operator. This characterization is not unique, i.e., there are a lot of Sheffer-type sequences $s_n(x)$ satisfying Eq. (10) for a given $f(x)$. We can further classify them by postulating the existence of the associated raising operator. A general theorem [14,15] states that a polynomial sequence $s_n(x)$ satisfying the monomiality principle Eq. (4) with an operator $P$ given as a function of the derivative operator only, is uniquely determined by two (formal) power series

$$f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} \quad \text{such that} \quad f_0 = 0,$$
\( f_1 \neq 0 \) and \( g_0 \neq 0 \). The egf of \( s_n(x) \) is then equal to
\[
\sum_{n=0}^{\infty} s_n(x) \frac{\lambda^n}{n!} = \frac{1}{g(f^{-1}(\lambda))} e^{x f^{-1}(\lambda)}, \quad (11)
\]
and their associated raising and lowering operators of Eq. (4) are given by [15]
\[
P = f(D), \quad M = \left[ X - \frac{g'(D)}{g(D)} \right] \frac{1}{f'(D)}. \quad (12)
\]
Observe the important fact that \( X \) enters \( M \) only linearly. Also, the order of \( X \) and \( D \) in \( M \) matters. The above holds true also for \( f(x) \) and \( g(x) \) which are formal power series. Any pair \( M, P \) from Eq. (12) automatically satisfies Eq. (1). Among the polynomials encountered in quantum mechanics, Hermite and Laguerre polynomials are of Sheffer-type, whereas Legendre, Jacobi and Gegenbauer polynomials are not. Here are some examples of so obtained representations of the Heisenberg–Weyl algebra:

(a) 
\[
M(X, D) = 2X - D, \quad P(D) = \frac{1}{2} D, \quad s_n(x) = H_n(x) \text{ Hermite polynomials:} \quad G(\lambda, x) = e^{2\lambda x - \lambda^2}.
\]

(b) 
\[
M(X, D) = -XD^2 + (2X - 1) D - X + 1, \quad P(D) = -\sum_{n=1}^{\infty} D^n, \quad s_n(x) = n! L_n(x), \text{ where } L_n(x) \text{ are Laguerre polynomials:} \quad G(\lambda, x) = \frac{1}{1 - \lambda} e^{\lambda x}. \quad (13)
\]

(c) 
\[
M(X, D) = X \frac{1}{1 - D}, \quad P(D) = -\frac{1}{2} D^2 + D, \quad s_n(x) = P_n(x) \text{ Bessel polynomials [23]:} \quad G(\lambda, x) = e^{-\lambda(1 - \sqrt{1 - 2\lambda})}. \quad (14)
\]

(d) 
\[
M(X, D) = X(1 + D), \quad P(D) = \ln(1 + D), \quad s_n(x) = B_n(x) \text{ the exponential (Bell) polynomials:} \quad G(\lambda, x) = e^{\lambda(e^\lambda - 1)}. \quad (15)
\]

(e) 
\[
M(X, D) = X e^{-D} , \quad P(D) = e^D - 1, \quad s_n(x) = \frac{\Gamma(x + 1)}{\Gamma(x + 1 - n)} \text{ the lower factorial polynomials [3]:} \quad G(\lambda, x) = e^{\lambda \ln(1 + \lambda)}. \quad (16)
\]

(f) 
\[
M(X, D) = (X - \tan D) \cos^2 D, \quad P(D) = \tan D, \quad s_n(x) = R_n(x) \text{ Hahn polynomials [24]:} \quad G(\lambda, x) = \frac{1}{\sqrt{1 + \lambda^2}} e^{\lambda \arctan \lambda}.
\]

(g) 
\[
M(X, D) = X \frac{1 + W_L(D)}{W_L(D)} D, \quad P(D) = W_L(D), \quad \text{where } W_L(x) \text{ is the Lambert W function [25],} \quad s_n(x) = I_n(x) \text{ the idempotent polynomials [26]:} \quad G(\lambda, x) = e^{\lambda e^{\lambda x}}.
\]

(h) The monomiality principle has also been applied to the polynomial sequences of more than one variable [11,12,19–22]. A case in point are Hermite–Kampé de Fériet polynomials of index \( m = 1, 2, \ldots \), of two variables \( x, y \), defined as:
\[
H_n^{(m)}(x, y) = n! \sum_{r=0}^{[n/m]} \frac{x^{n-mr} y^r}{(n-mr)!}! \quad (17,27),
\]
for which (here \( D_x = \frac{d}{d x} \))
\[
P = D_x, \quad M = x + m y D_x^{m-1}. \quad (18)
\]
Their egf is a two-variable generalization of \( G(\lambda, x) \) of example (a) and it reads
\[
G(\lambda, x, y) = \sum_{n=0}^{\infty} H_n^{(m)}(x, y) \frac{\lambda^n}{n!} = e^{\lambda x + \lambda y}. \quad (19)
\]

For other examples see, e.g., [19].

### 4. Boson normal order and monomiality

Now we give an application of the above formalism exploiting the analogy with the second quantization approach. We shall consider boson creation \( a^\dagger \) and annihilation \( a \) operators satisfying \([a, a^\dagger] = 1\). There are two sets of states of great importance in that representation. First, there are number states \(|n\rangle, (n = 0, 1, 2, \ldots, \) which are eigenstates of the number operator, \(a^\dagger a|n\rangle = n|n\rangle, \langle n|n'\rangle = \delta_{n,n'}\). The action of boson operators in the number states basis is \(a^\dagger|n\rangle = \sqrt{n + 1}|n+1\rangle\) and \(a|n\rangle = \sqrt{n}|n-1\rangle\). The second set are coherent states \(|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle\) [17,27]. Here we will show how to use the above formulas to calculate some special matrix elements of operator functions, which are linear in either \(a\) or \(a^\dagger\). As a byproduct we shall produce relations for the normally ordered form of these operator functions.

To make the analogy with the monomiality property Eqs. (3) and (4), it is convenient to redefine the number states
\[
|\tilde{n}\rangle = \sqrt{n!}|n\rangle \quad (16)
\]
Recalling Eqs. (6), (8) and (9) we get

\[ 0 \text{ and } g(x) \]

\[ \text{Equation (16)} \]

we arrive at

\[ \text{The result of Equation (23)} \]

\[ \text{can be further extended to a general} \]

\[ \text{matrix element} \]

\[ \langle z|e^{M(a^\dagger,a)}|z' \rangle \]

\[ \text{To this end recall [27] that} \]

\[ \text{if for an arbitrary operator} \]

\[ F(a^\dagger,a) \text{ without} \]

\[ \text{taking into account the commutation relations.} \]

\[ \text{Using the latter} \]

\[ \text{operation the normal ordering problem is solved for} \]

\[ F(a^\dagger,a) \text{ if we are able to find} \]

\[ \text{an operator} \]

\[ G(a^\dagger,a) \text{ for which} \]

\[ F(a^\dagger,a) = G(a^\dagger,a) : \text{satisfied}. \]

\[ \langle z|F(a^\dagger,a)|z' \rangle = \langle z|z' \rangle G(z^\ast,z') \]

\[ \text{then the normally ordered form of} \]

\[ F(a^\dagger,a) \text{ is given by} \]

\[ \mathcal{N}[F(a^\dagger,a)] = :G(a^\dagger,a): \]

\[ \text{Eqs. (24) and (26) then provide the central result} \]

\[ \mathcal{N}[e^{M(a^\dagger,a)}] = e^{a\dagger[f^{-1}(\lambda + f(x)) - a]} \frac{g(a)}{g(f^{-1}(\lambda + f(x)))} : \]

\[ \text{Let us point out again that} \]

\[ a^\dagger \text{ appears linearly in} \]

\[ M(a^\dagger,a) \text{, see} \]

\[ \text{Eq. (20).} \]

\[ \text{By hermitian conjugation of Eq. (27) we obtain the} \]

\[ \text{expression for the normal form of} \]

\[ e^{M(a^\dagger,a)} \text{, where} \]

\[ M(a^\dagger,a) \text{ is linear in} \]

\[ a. \]

\[ \text{In this context we refer to our previous work [18]} \]

\[ \text{where the results equivalent to Eqs. (24) and (27) were obtained using the substitution group approach [22].} \]

\[ \text{We shall conclude by enumerating some examples of evaluation of coherent state matrix elements of Eqs. (22) and (24).} \]

\[ \langle z|(-a + 2a^\dagger)^n|0 \rangle = H_n(z^\ast)|z|0\rangle, \]

\[ \langle z|e^{\lambda M(a^\dagger,a)}|z' \rangle = e^{\lambda(2z^\ast - z^2)} \langle z|z' \rangle. \]

\[ \langle z| [-a^2 + (2a^\dagger - 1)a - a^\dagger + 1]^n|0 \rangle \]

\[ = n!L_{n-1}(z^\ast)|z|0\rangle, \]

\[ \langle z|e^{\lambda[-a^\dagger a + (2a^\dagger - 1)a - a^\dagger + 1]}|z' \rangle \]

\[ = \frac{\langle z' |z|0 \rangle}{(1 - z^\ast)^2} e^{\lambda z^2 - \lambda z^\ast + 1}. \]
(c) \[
\langle z \bigg| \left( a^\dagger \frac{1}{1-a} \right)^n \bigg| 0 \rangle = P_n(z) \langle z | 0 \rangle,
\]
\[
\langle z \bigg| e^{\lambda (a^\dagger + a)} \bigg| z' \rangle = e^{\lambda [1 - \sqrt{1 - 2(z-z' - \frac{z^2}{2})^2}]} \langle z | z' \rangle.
\]

(d) \[
\langle z \big| (a^\dagger a + a^\dagger a)^n \big| 0 \rangle = B_n(z) \langle z | 0 \rangle,
\]
\[
\langle z \bigg| e^{\lambda (a^\dagger + a)} \bigg| z' \rangle = e^{\lambda (z' + (e^\lambda - 1))} \langle z | z' \rangle.
\]

(e) \[
\langle z \bigg| (a^\dagger e^{-a})^n \bigg| 0 \rangle = \frac{\Gamma(z+n+1)}{\Gamma(z+n+1-n)} \langle z | 0 \rangle,
\]
\[
\langle z \bigg| e^{\lambda (e^{-a} - a)} \bigg| z' \rangle = e^{\lambda [\text{ln}(e^{z' + \lambda}) - z]} \langle z | z' \rangle.
\]

(f) \[
\langle z \bigg| \left[ (a^\dagger \cos(a) + a) \cos^2(a) \right]^n \bigg| 0 \rangle = R_n(z) \langle z | 0 \rangle,
\]
\[
\langle z \bigg| e^{\lambda (a^\dagger + a)^2} \bigg| z' \rangle = \frac{\cos[\text{arctan}(\lambda + \text{tan}(z'))]}{\cos(z')} \langle z | z' \rangle.
\]

(g) \[
\langle z \bigg| \left[ a^\dagger \frac{1 + W_L(a)}{W_L(a)} \right]^n \bigg| 0 \rangle = I_n(z) \langle z | 0 \rangle,
\]
\[
\langle z \bigg| e^{\lambda (a^\dagger + W_L(a))} \bigg| z' \rangle = e^{\lambda [\lambda e^{z' + \lambda} + (e^\lambda - 1)]} \langle z | z' \rangle.
\]

5. Conclusions and outlook

In this work we have used the prescription from the umbral calculus to find explicit forms of the raising and lowering operators acting in the space of Sheffer-type polynomials. Their specific form given by Eq. (12) permitted us to calculate a large number of coherent state matrix elements. Those in turn have given directly, see Eq. (27), an explicit expression for the normally ordered form of operator functions of $M(a^\dagger, a)$ since $M$ is linear in either $a^\dagger$ or $a$. A legitimate question is if other types of sequences would be quasimonomial with respect to some operators $M$ and $P$. The affirmative answer was given in [28] where it has been demonstrated that for any sequence such operators can be found. For a number of polynomial sequences the operators $M$ and $P$ can actually be written down explicitly. However, only in the Sheffer case $P = P(D)$ and $M(X, D)$ is linear in $X$. In any other case $P = P(X, D)$ and it appears that this circumstance renders the calculation of coherent state matrix elements rather difficult. This is without doubt another demonstration of an intrinsic difficulty to perform a normal ordering of an arbitrary operator function.

Let us mention that the operator methods based on monomiality principle developed above have interesting, and as yet non-explored, ramifications to at least two other fields [29]: non-local evolution equations and generalized heat equations. We shall briefly sketch these points here, leaving the details for the forthcoming publication.

Let us consider here the case (c) above for which $s_n(x)$ are the Bessel polynomials with $M(X, D) = X \frac{1}{1-D}$ and $P(D) = -\frac{1}{2} D^2 + D$. Then it is possible to look for the solution of the following operator evolution equation:

$$
\frac{\partial}{\partial y} F(X, y) = \left[ P(D) + M(X, D) \right] F(X, y),
$$

with the ‘initial’ condition $F(X, 0) = q(X)$. Using $A^{-1} = \int_0^\infty e^{-sA} d s$ we find $M = X \int_0^\infty e^{-s} e^{sD} d s$, and then with $Mq(X) = X \int_0^\infty e^{-s} q(X + s) d s$, Eq. (28) takes the following non-local form

$$
\frac{\partial}{\partial y} F(X, y) = \left[ -\frac{1}{2} D^2 + D \right] F(X, y)
$$

$$
+ X \int_0^\infty e^{-s} F(X + s, y) d s,
$$

whose formal solution can be written as

$$
F(X, y) = e^{x(P + M)} q(X) = e^{-\frac{1}{2} y^2} e^{yP} e^{yM} q(X).
$$

For given $q(X)$, $e^{yM} q(X)$ can be evaluated recursively through

$$
e^{yM} q(X) = \sum_{n=0}^{\infty} \pi_n(X) \frac{y^n}{n!}
$$

with

$$
\pi_n(X) = X \int_0^\infty e^{-s} \pi_{n-1} (X + s) d s.
$$

With additional standard transformations a closed form of $F(X, y)$ can be obtained.

Another perspective will be offered by further two variable [30] extension of Eq. (14). Define

$$
\Pi = f(D_x),
$$

$$
\Theta = M_x + 2M_y f(D_y),
$$

with

$$
D_y = \frac{\partial}{\partial \eta}, \quad M_y = \left[ \eta - \frac{g'(D_x)}{g(D_y)} \right] \frac{1}{f'(D_x)},
$$

for $\eta = x, y$. Evidently, $[\Pi, \Theta] = 1$. This particular two-variable generalization of monomiality operators allows one the following extension of Hermite–Kampé de Fériet polynomials for $m = 2$ of Eq. (13):

$$
S_n(x, y) = n! \sum_{r=0}^{[n/2]} \binom{\lambda - 2r}{r} \frac{y^n}{(n - 2r)!} r!
$$

which are umbral extensions [30] of the ordinary case, where $s_n(\eta)$ are Sheffer polynomials associated with $f$ and $g$. It is worth noting that $S_n(x, y)$ are solutions of generalized heat equation

$$
f(D_x) S_n(x, y) = \left[ \frac{f(D_y)}{f(D_x)} \right]^2 S_n(x, y).
$$

These and related topics will be developed in subsequent publications.
Acknowledgements

We thank A.Z. Gorski, L. Haddad and J. Katriel for important discussions and constructive comments.

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