DETECTION PERFORMANCE OF THE FRACTIONAL FOURIER TRANSFORM (CHIRP FFT) FOR FREQUENCY MODULATED SIGNALS

Jonathan D. Locke¹, Paul R White²
¹Maritime Data Analysis Group, Dstl Naval Systems, Farnborough GU14 0LX, U.K.
²Institute of Sound and Vibration Research, University of Southampton, Highfield, Hants, SO17 1BJ, UK.

Maritime Data Analysis Group, Dstl Naval Systems, Farnborough GU14 0LX, U.K., +44 (01252 451956), +44 (01252 451823), jdlocke@dstl.gov.uk

The Fractional Fourier transform (FrFT) is well suited to the processing of linear frequency modulated signals. Amongst the abilities it potentially offers are: the detection of signals at lower SNRs, better signal separation and the possibility of estimating the chirp rate of a signal. This work concentrates on using such a transform for detecting frequency modulated signals. For the detection problem the magnitude of the FrFT is sufficient and consequently the algorithm can be implemented efficiently by modifying a standard FFT, this implementation we shall refer to as a ‘Chirp-FFT’.

This study considers the performance of the Chirp-FFT routine compared to the FFT routine for the detection of quiet underwater signals of both biological and man-made origin whose chirp rates are unknown a priori. Critically, the performance analysis here includes the effect the algorithms have on the background noise – a factor overlooked in other similar studies. The comparison is made on the basis of real data. It is shown that, unsurprisingly, the detection advantage offered by the Chirp-FFT is very much dependent on the character of the signal, but for the examples considered the gain in performance is in the range 0 - 3 dB.

Keywords: Fractional Fourier transform, detection, sonar processing, spectrogram, frequency modulation, chirp transform.
1. THE CHIRP-FFT

The FFT exploits a basis set consisting of complex sinusoids, consequently it efficiently represents signals that are tonal or near tonal. For frequency modulated signals, the FFT is sub-optimal. This is because under such circumstances the energy of the signal is not represented in the FFT by a small number of complex sinusoidal basis functions, but by a combination of several basis functions.

An established alternative to the FFT is the Fractional Fourier Transform (FrFT) [1-3]. The FrFT creates a signal representation in which the basis functions are linearly chirped complex sinusoids and is tailored to the processing of signals containing linear frequency modulations. One can efficiently implement the FrFT in several ways [2]. In this work we shall only consider the problem of signal detection based on the (squared) magnitude of the transform and so we shall use an intuitively appealing and efficient implementation of the FrFT, referred to as the Chirp-FFT [4]. This consists of pre-multiplying the signal by a synthetic linear FM signal prior to performing an FFT; if the sweep rate of the synthetic signal is chosen appropriately then the results of this product is a signal that is nearly tonal and so suitable for analysis with the FFT. In essence, if the signal contains an up-chirp at a particular rate, then by multiplying by a down-chirp at the same rate, the combined signal has a chirp-rate which is zero, i.e. it is a sinusoid. It is important that mean frequency of the multiplying chirp is constructed to be zero to ensure that multiplication by the chirp does not affect the centre frequency of the signal. For this transformation to be effective the analytic form [5] of the input signal must be used.

Equation 1 below, defines the discrete form of the Chirp-FFT. The analytic form, \( \hat{x}(n) \), of the input signal, \( x(n) \), is obtained using the Hilbert transform. This is then multiplied by a chirp \( c(n;\alpha) \), where \( \alpha \) is the chirp-rate, before taking the FFT. Ideally the chirp-rate \( \alpha \) should be selected to coincide with the chirp-rate of the original signal \( x(n) \).

\[
X(k;\alpha) = \sum_{n=0}^{N-1} c(n;\alpha) \hat{x}(n) e^{-2\pi ink/N}
\]

In most cases any chirp-rates present in the signal are not known a priori, in which case one approach is to compute a set of Chirp-FFTs using a range of chirp-rates. The Chirp-FFT which yields the largest peak value is then selected as the output [5,6]. Fig 1. is a representation of the magnitude of the Chirp-FFTs computed using different chirp rates, where horizontal lines correspond to spectra from Chirp-FFTs with different chirp-rates. The Chirp-FFT which produces the largest peak in the plot corresponds to a chirp-rate of 60 Hz/s and a centre frequency 50 Hz. These are the chirp rate and centre frequency of the synthetic input signal. The FFT of the signal is represented in this plot as the line corresponding to a chirp rate of 0 Hz/s. The data in this region are not concentrated in a few frequency bins but spread (smeared) across several bins.
Fig. 1: Chirp-FFTs using different chirp rates for the signal $e^{2\pi(3t^2+20t)} \quad t \in [0,1)$

2. THE CHIRP-FFT SPECTROGRAM

The concept of a spectrogram can be extended by replacing the FFT with the Chirp-FFT [6]. This requires one to compute a Chirp-FFT for every time window, with the chirp-rate being recomputed independently for each window. Fig. 2 shows example spectrograms based on the FFT and Chirp-FFT for a synthetic chirp embedded in Gaussian white noise, the chirp has a quadratic instantaneous frequency law. The estimated chirp-rate derived from the Chirp-FFT for this signal is also depicted. From the figure it is evident that the FFT produces its highest output level when the signal is nearly tonal close to $t = 0.5$, in this region the signal energy is concentrated into a small number of frequency bins. As the signal’s chirp rate increases the signal energy occupies more frequency bins resulting in a reduction in the peak level. The Chirp-FFT is more robust to these phenomena; it is evident from the central frame in Fig. 2 that signal energy is well concentrated over the duration of the signal irrespective of chirp-rate. It is noteworthy that the Chirp-FFT spectrogram has altered the appearance of the background noise, introducing visual correlations coinciding with the direction of the chirping signal.

3. CHIRP-FFT NORMALISATION

One has to take considerable care when comparing the FFT against the Chirp-FFT using different window sizes. Such a comparison should be based on a metric which relates performance when presented with a signal in noise to performance in a noise only environment. When the input to the algorithms is noise (Gaussian and white) the magnitudes of both transforms are non-Gaussian. The statistics of the FFT in this case are well characterised, in particular, the squared magnitude of the FFT of white Gaussian noise is known to conform to an exponential distribution [7]. The statistical distribution of the Chirp-FFT for a single (known) chirp-rate is the same exponential distribution. However, the statistical character of the output of the Chirp-FFT, in the case where the chirp-rate is unknown and has to be estimated from the data, has not been studied. As we shall
demonstrate, the squared magnitude of the optimal Chirp-FFT (obtained by considering a set of chirp rates and selecting the best rate) has a statistical distribution which deviates significantly from an exponential distribution. The consequence is that when comparing the two methods extra care is required because of their different behaviours in noise only environments. This is important since the statistics’ of the algorithm output in the noise only case dictates the threshold levels used to generate a specified false alarm rate. One principled approach on which to base a comparison is to measure performance relative to a threshold, where that threshold is set to achieve the same (specified) false alarm rate for both algorithms. The underlying detection process consists of a simple threshold applied to the output of the transform, i.e., a detection is made whenever the algorithm output exceeds the threshold value. The thresholds are applied to each time frequency cell individually. In practice such detections are of limited operational use, since detections in isolated time-frequency cells do not provide sufficient information regarding the character of the signal. In a realistic system an alarm would probably not be triggered until a sequence of such detections had been made and identified as having appropriate characteristics. Consequently the false alarm rates used here may appear to be high when compared to operationally acceptable false alarm rates.

Fig. 2: Comparison of spectrogram (top), Chirp-FFT spectrogram (middle) and estimated chirp-rate (bottom) for a synthetic signal with a quadratic instantaneous frequency law

\[
e^{-2\pi(-2048t^2 + 3072t^2)}
\]

Fig. 3 has been obtained through Monté-Carlo simulation. The plots illustrate the probability of false alarm (the probability that a cell in the Chirp-FFT will exceed the given value) against spectral level. From these plots we can identify the spectral level associated with a specified false alarm rate. The difference between this level and the level associated with the FFT is the correction factor that needs to be applied to equalise the false alarm rates of the two methods and so render the comparison fair. Table 1 summarises these correction factors.

<table>
<thead>
<tr>
<th>N</th>
<th>256</th>
<th>512</th>
<th>1024</th>
<th>2048</th>
<th>4096</th>
<th>8192</th>
</tr>
</thead>
<tbody>
<tr>
<td>False Alarm Rate</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 in 10^4</td>
<td>1.1</td>
<td>1.4</td>
<td>1.8</td>
<td>2.0</td>
<td>2.2</td>
<td>2.3</td>
</tr>
<tr>
<td>1 in 10^5</td>
<td>1.0</td>
<td>1.2</td>
<td>1.5</td>
<td>1.7</td>
<td>1.9</td>
<td>2.0</td>
</tr>
</tbody>
</table>
4. RESULTS FOR THE BOTTLENOSE DOLPHIN SIGNAL

Fig. 4 and Fig. 5 illustrate the analysis on an example section of the whistle of a bottlenose dolphin (*Tursiops truncatus*) represented by four FFT spectrograms and four Chirp-FFT spectrograms respectively. The figures show the representations for different window sizes of 256 points to 2048 points; the overlap is fixed at 87.5%. As the noise statistics have been normalised, the peak levels from the Chirp FFT can be directly compared to that of the FFT to give an indication of signal-to-noise performance. The results in Table 2 show that for this signal the Chirp-FFT offers SNR gain over the FFT of 2.2 dB. The chirp-rate is estimated to be 54300 Hz/s.

This methodology has been applied to a wider range of narrow-band marine mammal vocalisations where it generally offers a similar level of SNR gain [8].

<table>
<thead>
<tr>
<th>$f_s$ = 148100 Hz</th>
<th>$N$</th>
<th>256</th>
<th>512</th>
<th>1024</th>
<th>2048</th>
<th>4096</th>
<th>8192</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated chirp-rate (Hz/s)</td>
<td>-1340000</td>
<td>-167000</td>
<td>-126000</td>
<td>-41800</td>
<td>-36600</td>
<td>-41200</td>
<td></td>
</tr>
<tr>
<td>FFT Level (dB)</td>
<td>0.6</td>
<td>2.4</td>
<td>3.0</td>
<td>3.2</td>
<td>2.8</td>
<td>2.8</td>
<td></td>
</tr>
<tr>
<td>Chirp-FFT Level (dB)</td>
<td>0.1</td>
<td>1.2</td>
<td>2</td>
<td>3.2</td>
<td>5.4</td>
<td>3.8</td>
<td></td>
</tr>
<tr>
<td>Chirp-FFT SNR gain (dB)</td>
<td></td>
<td></td>
<td></td>
<td>2.2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Bottlenose dolphin results
5. REFERENCES


