Topological Properties of Silicate Networks

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Abstract

The silicates are the largest, the most interesting and the most complicated class of minerals by far. The basic chemical unit of silicates is the \((\text{SiO}_4)\) tetrahedron. A silicate sheet is a ring of tetrahedrons which are linked by shared oxygen nodes to other rings in a two dimensional plane that produces a sheet-like structure. We consider the silicate sheet as a fixed interconnection parallel architecture and call it a silicate network. We study its structure and properties from the perspective of computer science. We compare the performance of this architecture with the other similar mesh-like architectures. Moreover, we solve the minimum metric dimension problem. We then look at the compound from the perspectives of chemistry. We study the topological structure of silicates. We identify an equilateral property in silicates. We also demonstrate that the oxygen edges can be partitioned into edge disjoint cycles. We design a drawing algorithm that draws silicates aesthetically.

Keywords: silicate networks, topological and structural properties of interconnection networks, mesh-like architectures, embedding, minimum metric dimension, aesthetic drawing and drawing algorithm.

1. Introduction

A fixed interconnection parallel architecture is characterized by a graph, with vertices corresponding to processing nodes and edges representing communication links. Interconnection networks are notoriously hard to compare in abstract terms [14]; the relative merits of various networks change with hardware implementation technology, data routing scheme, computational workload, data/task distribution, and many other architectural, system, or application parameters. Researchers in parallel processing are thus motivated to propose new or improved interconnection networks, arguing the benefits and offering performance evaluations in different contexts [12].

![Figure 1: Designed Architectures](image)

Some interconnection network topologies are designed and some borrow from nature. For example grids, hypercubes, complete binary trees, butterfly and Benes networks are some of the designed architectures [12]. See Figure 1. Hexagonal, Honeycomb, and grid networks, for instance, bear resemblance to atomic or molecular lattice structures. We call them natural architectures. See Figure 2.
Honeycomb networks, built recursively using the hexagon tessellation [22, 23, 25], are widely used in computer graphics [13], cellular phone base station [9], image processing [5], and in chemistry as the representation of benzenoid hydrocarbons [24] and Carbon Hexagons of Carbon Nanotubes [11]. Hexagonal networks are based on triangular plane tessellation, or the partition of a plane into equilateral triangles [7, 15, 16, 25]. Hexagonal network represents a host cyclotriveratrylene with halogenated monocarbaborane anions [1] and Silicon Carbide [27]. Carbon nanotubes consist of shells of sp²-hybridized carbon atoms forming a hexagonal network, arranged helically within a tubular motif [1].

![4 x 4 Grid](image1)
![Hexagonal network](image2)
![Honeycomb](image3)

**Figure 2: Natural Architectures**

Silicates are obtained by fusing metal oxides or metal carbonates with sand. Essentially all the silicates contain SiO₄ tetrahedra. In chemistry, the corner vertices of SiO₄ tetrahedra represent oxygen ions and the center vertex represents the silicon ion. In graph theory, we call the corner vertices as oxygen nodes and the center vertex as silicon node. See Figure 3.

![SiO₄ tetrahedra](image4)

**Figure 3: SiO₄ tetrahedra where the corner vertices represent oxygen ions and the center vertex the silicon ion**

The minerals are obtained by successively fusing oxygen nodes of two tetrahedra of different silicates. The different types of silicate structure arise from the ways in which these tetrahedra are arranged: they may exist as separate unlinked entities, as linked finite arrays, as 1-dimensional chains, as 2-dimensional sheets or as 3-dimensional frameworks. Some of the structural units found in silicates are shown in Figures 4 and 5. They are termed orthosilicates, pyrosilicates, chain silicates, cyclic silicates and sheet silicates.

![Orthosilicates](image5)
![Pyrosilicates](image6)
![Chain Silicates](image7)

**Figure 4: Different kinds of silicates**

Simple orthosilicates contain discrete SiO₄ units. When two SiO₄ tetrahedra share an oxygen node, pyrosilicates are obtained. While tetrahedra are arranged linearly, chain silicates are obtained. See Figure 4.
In this paper, we study the topological properties of silicate networks as it has been studied for other interconnection networks [2, 3, 4, 7, 8, 10, 12, 14, 16, 18, 19, 20, 21, 23, 26]. We study its structure and properties from the perspective of computer science. In order to compare the computational powers of silicate networks with the other similar mesh-like architectures, the silicates are embedded into mesh-like architectures such as hexagons and honeycombs. We solve minimum metric dimension problem which is an NP-Complete problem for general graphs [17]. We also look at the compound from the perspectives of chemistry. We study the topological structure of silicates. We identify an equilateral property of silicates and we partition the oxygen edges into edge disjoint cycles. We propose an addressing scheme of silicate networks and map the nodes of silicate networks onto a Cartesian plane to draw silicates aesthetically.

2. Properties of Silicate Networks

We require the definition of a honeycomb network to construct a silicate network. A honeycomb network can be built from a hexagon in various ways. The honeycomb network \( HC(1) \) is a hexagon. The honeycomb network \( HC(2) \) is obtained by adding six hexagons to the boundary edges of \( HC(1) \). Inductively, honeycomb network \( HC(n) \) is obtained from \( HC(n – 1) \) by adding a layer of hexagons around the boundary of \( HC(n – 1) \). For instance, Figure 6 is \( HC(3) \). The parameter \( n \) of \( HC(n) \) is called the dimension of \( HC(n) \). The number of vertices and edges of \( HC(n) \) are \( 6n^2 \) and \( 9n^2 – 3n \) respectively. The diameter is \( 4n – 1 \) [22, 23, 24].

A silicate network can be constructed in the same way as a honeycomb is constructed. The silicate network \( SL(1) \) is a cyclic silicate with six \( SiO_4 \) tetrahedra units. The silicate network \( SL(2) \) is obtained by adding six units of \( SL(1) \) such that each outer \( SL(1) \) shares two consecutive tetrahedra of inner \( SL(1) \). Inductively, silicate network \( SL(n) \) is obtained from \( SL(n – 1) \) by adding a layer of \( SL(1) \) around the boundary of \( SL(n – 1) \). For instance,
the graph in Figure 7(b) is $SL(2)$. Notice that there are $6n$ number of $SL(1)$ in the outer layer of $SL(n)$. The parameter $n$ of $SL(n)$ is called the *dimension* of $SL(n)$.

A silicate network can be constructed in different ways. Here in this paper we describe the construction of a silicate network from a honeycomb network. Consider a honeycomb network $HC(n)$ of dimension $n$. Place silicon ions on all the vertices of $HC(n)$. Subdivide each edge of $HC(n)$ once. Place oxygen ions on the new vertices. Introduce $6n$ new pendant edges one each at the 2-degree silicon ions of $HC(n)$ and place oxygen ions at the pendent vertices. See Figure 7(a). With every silicon ion associate the three adjacent oxygen ions and form a tetrahedron as in Figure 7(b). The resulting network is a silicate network of dimension $n$, denoted $SL(n)$. The graph in Figure 7(b) is a silicate network of dimension two.

![Silicate Network Construction](image)

**Figure 7: Silicate Network Construction.**

The 3-degree oxygen nodes of silicates are called *boundary nodes*. In Figure 8, $c_1, c_2 \ldots c_{12}$ are boundary vertices.

![Boundary vertices of $SL(n)$](image)

**Figure 8: Boundary vertices of $SL(n)$**
Theorem 1: The number of nodes in $SL(n)$ is $15n^2 + 3n$. The number of edges of $SL(n)$ is $36n^2$.

Proof: Besides an oxygen node on each edge of $HC(n)$, there are $6n$ oxygen nodes in the boundary layer of $SL(n)$. The number of nodes of $SL(n) = \text{the number of nodes of } HC(n) + \text{the number of edges of } HC(n) + 6n = 6n^2 + (9n^2 - 3n) + 6n = 15n^2 + 3n$.

Now each silicon node of $SL(n)$ belongs to a unique $SiO_4$ tetrahedral unit and these tetrahedral units are edge-disjoint. Further each unit has 6 edges. Since the number of silicon nodes equals the number of nodes of $HC(n)$, the number of edges of $SL(n)$ is $6 \times 6n^2 = 36n^2$. □

When we delete all the silicon nodes from a silicate network we obtain a new network which we shall call as an Oxide Network. See Figure 9. An $n$-dimensional oxide network is denoted by $OX(n)$. Even though $HC(n)$ and $OX(n)$ are sub graphs of $SL(n)$, $OX(n)$ plays more important role in studying the properties of $SL(n)$. We note that the diameter of silicate network $SL(n)$ is equal to the diameter of the oxide network $OX(n)$.

3. Addressing the nodes of Silicate Networks

In order to study the properties of silicate networks, it is important to assign a unique identity $id$ (coordinate) to each node of silicate network. First we shall propose a coordinate system that can be used to assign an $id$ to each node of oxide network. Then we shall extend this coordinate system to silicate network.

We shall adapt the coordinate system that was proposed for a honeycomb network by Stojmenovic [23] or a hexagonal network by Nocetti et al. [15]. Three axes, $\alpha$, $\beta$ and $\gamma$ parallel to three edge directions and at mutual angle of 120 degrees between any two of them are introduced, as indicated in Figure 10. The three coordinate axes are $\alpha = 0$, $\beta = 0$, and $\gamma = 0$ respectively. We call lines parallel to the coordinate axes as $\alpha$-lines, $\beta$-lines and $\gamma$-lines. Here $\alpha = h$ and $\alpha = -k$ are $\alpha$-lines on either side of $\alpha$-axis. A node of $OX(n)$ is assigned a triple $(a, b, c)$ when the node is the intersection of lines $\alpha = a$, $\beta = b$, and $\gamma = c$.

Each silicon node is at the centroid of three oxygen nodes of a tetrahedral $SiO_4$. Thus it is enough to specify an addressing scheme for the oxygen nodes of $SL(n)$. For all practical purposes, we don’t need separate $id$ for silicon nodes since the silicate network is completely characterized by the oxide network. However for the sake of completeness, one can assign $ids$ to silicon nodes by applying the formula of centroid of an equilateral triangle.
4. Drawing Algorithm for Silicate Networks

Let us classify the edges of a tetrahedral SiO$_4$ as follows: An edge incident at a silicon node is called *silicon edge*. Otherwise it is called an *oxygen edge*. The drawing algorithm provides a method to draw $SL(n)$ in the Cartesian plane. That is, it provides a formula to map a node $(\alpha, \beta, \gamma)$ of $SL(n)$ into a point $(x, y)$ of Cartesian System. Our objective is to draw $SL(n)$ in Cartesian plane in such a way that all the drawn edges of $SL(n)$ are of equal length in the 2-dimnesional plane. However, it is not possible to draw a tetrahedral SiO$_4$ (a complete graph on 4 vertices) in the 2-dimensional plane such that all edges are of equal length. Therefore, we design a drawing algorithm such that all the silicon edges are of the same length and all the oxygen edges are of the same length. As it is mentioned earlier, it is enough to design a drawing algorithm for oxide network.

The oxygen edges of a tetrahedral SiO$_4$ form a triangle. In other words, these three oxygen edges should make an equilateral triangle in order to be of equal length. Trigonometrically, these three edges make an angle of $60^\circ$ with each other. Moreover, these
three oxygen edges are on $\alpha$-lines, $\beta$-lines, and $\gamma$-lines of $OX(n)$ respectively. In order to keep all the oxygen edges of equal length, $\alpha$-lines, $\beta$-lines, and $\gamma$-lines of $OX(n)$ are drawn as follows:

- All $\alpha$-lines are parallel to $X$-axis, $\beta$-lines make $60^\circ$ with $X$-axis and $\gamma$-lines make $120^\circ$ with $X$-axis.
- Successive $\alpha$-lines ($\beta$-lines, and $\gamma$-lines) are equally spaced in the Cartesian plane.

(1) Since the $\alpha$-lines are parallel to $X$-axis, the $\alpha$-line “$\alpha = 0$” is taken as the $x$-axis. Line $\alpha = (2h+1), h \in Z$ of $OX(n)$ is mapped to $y = (2h+1)$ in the Cartesian system.

(2) A $\beta$-line $\beta = (2k+1)$ makes an angle $60^\circ$ with $X$-axis and forms a $y$-intercept $2(2k+1)$ in the Cartesian system. See Figure 10. Thus line $\beta = (2k+1), k \in Z$ is mapped to $y = (\tan 60^\circ)x + 2(2k+1)$ which is $y = \sqrt{3}x + 2(2k+1)$.

(3) A $\gamma$-line $\gamma = (2\ell+1)$ makes an angle $120^\circ$ with $X$-axis and forms a $y$-intercept $2(2\ell+1)$ in the Cartesian system. See Figure 10. Thus line $\gamma = (2\ell+1), \ell \in Z$ is mapped to $y = (\tan 120^\circ)x + 2(2\ell+1)$ which is $y = -\sqrt{3}x + 2(2\ell+1)$.

From (1), we have

$$\alpha = (2h+1) \text{ and } y = (2h+1), \ h \in Z$$

Thus

$$y = \alpha. \quad (A)$$

From (2) and (3) we have

$$\beta = (2k+1) \text{ and } y = \sqrt{3}x + 2(2k+1).$$

$$\gamma = (2\ell+1) \text{ and } y = -\sqrt{3}x + 2(2\ell+1).$$

Solving the above two equations, we get

$$x = (\beta - \gamma)/\sqrt{3}. \quad (B)$$

Combining (A) and (B), we arrive at a function $f$ that maps a node $(\alpha, \beta, \gamma)$ of $OX(n)$ to a node of Cartesian System as follows:

$$f(\alpha, \beta, \gamma) = ((\beta - \gamma)/\sqrt{3}, \alpha). \quad (C)$$

This function $f$ provides an algorithm to draw $OX(n)$ in a Cartesian plane.

Once the oxide network is drawn in the Cartesian plane, placing silicon nodes is rather simple. As we know, a silicon node is at the centroid of three oxygen nodes of a tetrahedral SiO$_4$. If $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are the Cartesian coordinates of oxygen nodes of a tetrahedral SiO$_4$, then $((x_1+x_2+x_3)/3, (y_1+y_2+y_3)/3)$ is the Cartesian coordinate of the silicon node of the tetrahedron. Thus, it completes the drawing algorithm of $SL(n)$.

To complete the proof of correctness, it is enough to show that all the silicon edges of a tetrahedral SiO$_4$ are of equal length and all the oxygen edges of the tetrahedral are of equal length. Since $\beta$-lines and $\gamma$-lines make $60^\circ$ with $\alpha$-lines, all the triangles in $OX(n)$ are equilateral triangles. The algorithm is designed in such a way that the distance between any two successive $\beta$-lines and the distance between any two successive $\gamma$-lines are the same. Thus any two oxygen edges lying in an $\alpha$-line are equal. Hence any two equilateral triangles of three oxygen edges are identical. Hence the oxygen edges are of equal length. Since the silicon node is at the centroid of equilateral triangle formed by three oxygen nodes of the tetrahedral, the silicon edges are also of equal length. Thus we have proved the following result.
**Theorem 2:** A silicate network can be drawn in a two-dimensional Cartesian plane such that all the silicon edges are of equal length and all the oxygen edges are of equal length.

5. Equilateral Triangle Property of Silicate Network

Three vertices $u$, $v$, $w$ of a graph $G(V, E)$ are said to form an *equilateral triangle* if $d(u, v) = d(v, w) = d(w, u)$ where $d(x, y)$ denotes the distance between $x$ and $y$. There is an interesting equilateral triangular property of silicate networks. A rigorous mathematical statement and its proof are given below.

**Theorem 3:** Three vertices $A(x_1, x_2, x_3)$, $B(y_1, y_2, y_3)$ and $C(z_1, z_2, z_3)$ of $SL(n)$ form an equilateral triangle if $x_1 = y_1, y_2 = z_2$ and $z_3 = x_3$.

**Proof:** Since $x_1 = y_1$, $AB$ is an $α$-line. In the same way, $BC$ is a $β$-line and $CA$ is a $γ$-line. Let us recall the function $f$ defined in the section 4:

$$f(α, β, γ) = ((β - γ)/\sqrt{3}, α).$$

This function $f$ places the vertices of an $α$-line on a straight line in the Cartesian plane such that consecutive vertices of the $α$-line are equally spaced in the Cartesian plane. The nodes of $β$-line and $γ$-line are also similarly placed in the Cartesian plane. Thus geometrical proof is applicable to prove a graph theory result in our case.

Geometrically a triangle formed by an $α$-line, a $β$-line and a $γ$-line is equilateral, since any two of these lines make an angle of $60^0$ between themselves. □

Continuing the above theorem, we discuss a stronger result. Consider a triangle $ABC$ formed by some $α$-line, $β$-line and $γ$-line. By the above theorem, $ΔABC$ is equilateral. Let $a_1, a_2 \ldots a_r$ be the nodes on the $β$-line between $B$ and $C$. Let $b_1, b_2 \ldots b_r$ be the nodes on the $α$-line between $C$ and $A$. Let $c_1, c_2 \ldots c_r$ be the nodes on the $γ$-line between $A$ and $B$. See Figure 11. We know that $d(A, B) = d(A, C)$. The interesting observation is that $d(A, B) = d(A, a_i) = d(A, C)$ for $i = 1, 2 \ldots r$.

![Figure 11: Equilateral triangle property](image)

**Theorem 4:** Let $ΔABC$ denote a triangle of $SL(n)$ formed by three vertices $A(x_1, x_2, x_3)$, $B(y_1, y_2, y_3)$ and $C(z_1, z_2, z_3)$ such that $x_1 = y_1, y_2 = z_2$ and $z_3 = x_3$. Let $a$ be a node on the $β$-
line between B and C, let \( b \) be a node on the \( \alpha \)-line between C and A and let \( c \) be a node on the \( \gamma \)-line between A and B. Then
\[
d(A, B) = d(B, C) = d(C, A) = d(A, a) = d(B, b) = d(C, c).
\]

**Proof:** It is enough to prove that \( d(A, B) = d(A, a) = d(C, A) \) for some node \( a \) on the line \( \beta = k \) between B and C. We prove the result by induction on \( k \). Assuming that the result is true for \( \beta = k \), let us prove the result for \( \beta = k + 1 \). This is true since for every node \( u \) in \( \beta = k + 1 \), there is a node \( v \) in \( \beta = k \) such that \( d(u, v) = 1 \). □

6. The edge set of \( OX(n) \) is partitioned into edge disjoint cycles

**Definition:** An edge of \( OX(n) \) is called \( \alpha \)-edge if it is in some \( \alpha \)-line. A \( \beta \)-edge and a \( \gamma \)-edge are defined in the same way. A cycle of \( OX(n) \) is said to be a symmetric cycle if it is formed by an \( \alpha \)-edge, a \( \beta \)-edge and a \( \gamma \)-edge alternatively. Notice that the number of edges of any symmetric cycle of \( OX(n) \) is a multiple of 3. We demonstrate that the edge set of \( OX(n) \) is partitioned into edge-disjoint symmetric cycles.

**Theorem 5:** The edge set of \( OX(n) \) can be partitioned into edge-disjoint symmetric cycles.

![Figure 12: Edge set is partitioned into cycles](image)

**Proof:** As a strategy to partition the edge set of \( OX(n) \), we orient the edges of \( OX(n) \) as follows:

1. Orient the edges of lines \( \alpha = 1, 3 \ldots 2n + 1 \) in the positive direction.
2. Orient the edges of lines \( \alpha = -1, -3 \ldots -2n - 1 \) in the negative direction.
3. Orient the edges of lines \( \beta = 1, 3 \ldots 2n + 1 \) in the positive direction.
4. Orient the edges of lines \( \beta = -1, -3 \ldots -2n - 1 \) in the negative direction.
5. Orient the edges of lines \( \gamma = 1, 3 \ldots 2n + 1 \) in the positive direction.
6. Orient the edges of lines \( \gamma = -1, -3 \ldots -2n - 1 \) in the negative direction.

Let us observe the following facts in an oriented \( OX(n) \):

1. For each directed edge \( (a, b) \) of \( OX(n) \), there is exactly one incoming edge at node \( a \) which is qualified to form a symmetric cycle.
2. For each directed edge \((a,b)\) of \(OX(n)\), there is exactly one outgoing edge at node \(b\) which is qualified to form a symmetric cycle.

First we claim that an edge of \(OX(n)\) is in at most one symmetric cycle. Suppose a directed edge \((a,b)\) of \(OX(n)\) is in two different symmetric cycles \(C_1\) and \(C_2\). Between the two outgoing edges from the node \(b\), there is only one edge which will be qualified to form a symmetric cycle. Thus, that outgoing edge will be in both the symmetric cycles \(C_1\) and \(C_2\). In the same way, between the two incoming edges to the node \(a\), there is only one edge which will be qualified to form a symmetric cycle. Thus, that incoming edge will be in both the symmetric cycles \(C_1\) and \(C_2\). Inductively, both the cycles \(C_1\) and \(C_2\) will be the same which is a contradiction.

Now it is left to prove that every edge is in at least one symmetric cycle. Suppose a directed edge \((a,b)\) of \(OX(n)\) is not in any symmetric cycle. As before, there exists an incoming edge at \(a\) which is not in any symmetric cycle. Continuing this argument, this will lead to a cycle which will end up at \(b\) because it is a finite graph. Hence every edge is in at least one symmetric cycle.

Since symmetric cycles in the oriented \(OX(n)\) are unique, it is now easy to partition \(OX(n)\) into edge-disjoint symmetric cycles. The innermost symmetric cycle is a hexagon of the length 6. Then onwards, each successive symmetric cycle forms a layer on the previous one. See Figure 12. The length of each symmetric cycle is a multiple of 6. There are \(2n\) edge disjoint symmetric cycles of oriented \(OX(n)\). Hence the proof.

7. Embedding of Honeycomb and Hexagonal Networks in Silicate Networks.

Let \(G\) and \(H\) be finite graphs with \(n\) vertices. \(V(G)\) and \(V(H)\) denote the vertex sets of \(G\) and \(H\) respectively. \(E(G)\) and \(E(H)\) denote the edge sets of \(G\) and \(H\) respectively. An embedding \([6]\) \(f\) of \(G\) into \(H\) is defined as follows:

1. \(f\) is a bijective map from \(V(G)\rightarrow V(H)\)
2. \(f\) is a one-to-one map from \(E(G)\) to \(\{P_f(f(u),f(v)) : P_f(f(u),f(v))\text{ is a path in }H\text{ between }f(u)\text{ and }f(v), (u,v) \in E(G)\}\).

The dilation \(\hat{D}_f(G,H)\) of an embedding \(f\) of \(G\) into \(H\) is defined as
\[
\hat{D}_f(G,H) = \max_{(u,v) \in E(G)} |P_f(f(u),f(v))|
\]
where \(|P_f(f(u),f(v))|\) denotes the length of the path \(P_f(f(u),f(v))\).

Then, the dilation of \(G\) into \(H\) is defined as
\[
\hat{D}(G,H) = \min_f \hat{D}(G,H)
\]
where the minimum is taken over all embeddings \(f\) of \(G\) into \(H\). The dilation problem for a graph \(G\) into \(H\) is that of finding an embedding of \(G\) into \(H\) that induces the dilation \(\hat{D}(G,H)\).

We next state the results pertaining to embedding of honeycomb and hexagonal networks into silicate networks.
**Theorem 6:** The dilation of the embedding of a honeycomb network of dimension \( n \) into a silicate network of the same dimension is 2.

![Diagram](image)

*Figure 13: Dilation is 2*

**Proof:** From the structure of \( SL(n) \) described in Section 2, it follows that the subdivision of the honeycomb network of dimension \( n \) is a subgraph of \( SL(n) \). See Figure 13. Hence the dilation of the embedding of \( HC(n) \) into \( SL(n) \) is 2. \( \square \)

**Theorem 7:** The dilation of the embedding of a hexagonal network of dimension \( n \) into a silicate network of dimension \( n - 2 \) is at most 3.

![Diagram](image)

*Figure 14: The nodes of Hexagon \( HX(n) \) are labeled*
Figure 15: Hexagons of $HX(n)$ between successive $\alpha = 2k + 1$ and $\alpha = 2k + 3$ lines are mapped into the corresponding hexagons of $SL(n - 2)$.

Figure 16: Center node of the hexagon of $HX(n)$ is mapped into a node which sits on top of the corresponding hexagon of $SL(n - 2)$.
**Figure 17:** An embedding of $HX(n)$ into $SL(n - 2)$. The edge $(6, 11)$ of $HX(n)$ is dilated along $6, 1, 5, 11$ in $SL(n - 2)$. The dilation of $HX(n)$ into $SL(n - 2)$ is 3.

**Proof:** We provide an embedding of $HX(n)$ into $SL(n - 2)$ as follows:

Step 1: The nodes of $HX(n)$ are labeled as in Figure 14.

Step 2: Each hexagon of $HX(n)$ between successive $\alpha = 2k + 1$ and $\alpha = 2k + 3$ lines is mapped into the corresponding hexagon of $SL(n - 2)$. See Figure 15.

Step 3: The center node of a hexagon of $HX(n)$ is mapped into a node which sits on top of the corresponding hexagon of $SL(n - 2)$. See Figure 16.

The Figure 17 shows the embedding of $HX(n)$ into $SL(n - 2)$. It is easy to verify that the dilation of this embedding is 3. □

**8. Minimum Metric Dimension of Silicate Networks**

The problem of finding the minimum metric dimension is important and interesting from the point of view of Robotics [17]. A metric basis for a graph $G(V, E)$ is a subset of vertices $W \subseteq V$ such that for each pair of vertices $u$ and $v$ of $V \setminus W$, there is a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$.

![Definition of metric basis](image)

**Figure 18:** Definition of metric basis

A minimum metric basis is a metric basis of minimum cardinality. The cardinality of a minimum metric basis of $G$ is called minimum metric dimension and is denoted by $\beta(G)$; the members of a minimum metric basis are called landmarks. A minimum metric dimension (MMD) problem is to find a minimum metric basis. A minimum metric dimension problem is $NP$-Complete. The reader may refer to figure 8 to recall the definition of boundary nodes.
**Theorem 8:** $\beta(SL(n)) \geq 6n$.

**Proof:** Let $u$ be a boundary node of $SL(n)$. Let $v$ be the silicon node adjacent to $u$. Then for any node $w$ in $SL(n)$, we have
\[ d(u, w) = d(v, w) \]
Thus any metric basis will contain either $u$ or $v$. There are $6n$ boundary nodes in $SL(n)$. Hence any metric basis of $SL(n)$ should contain at least $6n$ nodes of $SL(n)$. □

**Theorem 9:** $\beta(SL(n)) = 6n$.

**Proof:** We claim that the set of boundary nodes is a metric basis. Let $u$ and $v$ be two oxygen nodes of $SL(n)$. Both the nodes $u$ and $v$ lie in some $\alpha$-line, $\beta$-line or $\gamma$-line. If not, both the nodes $u$ and $v$ will lie in two $\alpha$-lines, two $\beta$-lines or two $\gamma$-lines.

Suppose both the nodes $u$ and $v$ lie in some $\alpha$-line. Let $c$ be a boundary node of the $\alpha$-line. Then $d(u, c) \neq d(v, c)$. Suppose both the nodes $u$ and $v$ lie in two different $\alpha$-lines. Let $c$ be a boundary node of one of the two $\alpha$-lines. It is straightforward to observe that $d(u, c) \neq d(v, c)$.

When $u$ and $v$ are silicon nodes, similar arguments will apply. The other cases are also similar. Thus the set of boundary nodes is a metric basis and $\beta(SL(n)) \leq 6n$. □

8. Conclusion

In this paper we have considered a new interconnection network motivated by the molecular structure of certain chemical compounds. The different forms of silicates available in nature led to the introduction of the silicate networks. We have investigated various parameters related to this network. We have provided an addressing scheme for the nodes of the network and also an algorithm that enables us to draw the silicate network in the two dimensional plane. Certain properties of this network are brought out using classical geometry.

The embedding of the honeycomb and hexagonal networks into silicates is also considered in this paper. Finally the minimum metric dimension problem is also investigated.

This paper is an eye opener for researchers in the sense that different networks can be derived using the ores and compounds available in nature.

9. References


