Minimum wirelength of hypercubes into n-dimensional grid networks

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Abstract. In the paper [Exact wirelength of hypercube on a grid, Discrete Applied Mathematics, 157 (2009), no. 7, 1486 - 1495], the minimum wirelength of an r-dimensional hypercube into a $2^{[r/2]} \times 2^{[r/2]}$ grid has been obtained. In this paper, we obtain the same when the $2^{[r/2]} \times 2^{[r/2]}$ rectangular grid is replaced by a generalized grid of size $2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}$ where $r_1 + r_2 + \cdots + r_n = r$, $r_1 \leq r_2 \leq \cdots \leq r_n$.

Keywords: Embedding; congestion; wirelength; parallel processing.

1 Introduction

The hypercube is one of the most popular versatile and efficient topological structures of interconnection networks. The hypercube has many excellent features and thus becomes the first choice of topological structure of parallel processing and computing systems. The main focus of this paper is to find the minimum wirelength of hypercubes into n-dimensional grid networks.

An important feature of an interconnection network is its ability to efficiently simulate programs written for other architectures. Such a simulation problem can be mathematically formulated as graph embedding. An embedding of a guest graph $G$ into a host graph $H$ is defined by a bijective mapping $f : V(G) \to V(H)$ together with a mapping $P_f$ which assigns to each edge $(u, v)$ of $G$ a path between $f(u)$ and $f(v)$ in $H$. The edge congestion of an embedding $f$ of $G$ into $H$ is the maximum number of edges of the graph $G$ that are embedded on any single edge of $H$. Let $EC_f(e)$ denote the number of edges $(u, v)$ of $G$ such that $e$ is in the path $P_f((u, v))$ between $f(u)$ and $f(v)$ in $H$.

The wirelength $[3, 4]$ of an embedding $f$ of $G$ into $H$ is given by

$$WL_f(G, H) = \sum_{(u, v) \in E(G)} d_H(f(u), f(v)) = \sum_{e \in E(H)} EC_f(e)$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f((u, v))$ in $H$.

The wirelength of $G$ into $H$ is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings of $f$ of $G$ into $H$. The wirelength problem of a graph $G$ into $H$ is to find an embedding of $G$ into $H$ that induces the minimum wirelength $WL(G, H)$.

For $r \geq 1$, let $Q^r$ denote the graph of $r$-dimensional hypercube. The vertex set of $Q^r$ is formed by the collection of all $r$-dimensional binary representations. Two vertices $x, y \in V(Q^r)$ are adjacent if and only if the corresponding binary representations differ exactly in one bit. Equivalently if $|V(Q^r)| = 2^r$ then the vertices of $Q^r$ can also be identified with integers $0, 1, \ldots, 2^r - 1$ so that if a pair of vertices $i$ and $j$ are adjacent then $i - j = \pm 2^p$ for some $p \geq 0$.

A set of $m$ vertices of a graph $G$ is said to be a composite set if the number of edges of the subgraph induced by these $m$ vertices is not less than the number of edges of a subgraph induced by any other set of $m$ vertices of $G$.

Theorem 1 [2, 3] Let $Q^r$ be an r-dimensional hypercube. For $1 \leq i \leq 2^r$, $L_i = \{0, 1, \ldots, i - 1\}$ is a composite set. $\square$

The $n$-dimensional grid network, denoted by $M[d_1 \times d_2 \times \cdots \times d_n]$, is defined as $P_{d_1} \times P_{d_2} \times \cdots \times P_{d_n}$, where $d_i \geq 2$ is an integer for each $i = 1, 2, \ldots, n$. The vertex set of $M[d_1 \times d_2 \times \cdots \times d_n]$ is the set $V = \{[x_1, x_2, \ldots, x_n] : 0 \leq x_i \leq d_i - 1, i = 1, 2, \ldots, n\}$ and two vertices $x = [x_1, x_2, \ldots, x_n]$ and $y = [y_1, y_2, \ldots, y_n]$ are linked by an edge if $\sum_{i=1}^{n} x_i - y_i = 1$. 

1
2 Lexicographic Embedding

The lexicographic embedding \([1]\) of \(Q^r\) with the labeling 0 to \(2^r - 1\) into \(M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}]\), \(r_1 + r_2 + \cdots + r_n = r\), \(r_1 \leq r_2 \leq \cdots \leq r_n\), is an assignment of label to the vertex \([x_1, x_2, \ldots, x_n]\) of \(M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}]\) as \(x_1 + x_2 \cdot 2^{r_1} + x_3 \cdot 2^{r_1+r_2} + \cdots + x_n \cdot 2^{r_1+r_2+\cdots+r_{n-1}}\), \(0 \leq x_i \leq 2^{r_i} - 1\), \(1 \leq i \leq n\). This lexicographic embedding is denoted by \(\text{lex}\).

Lemma 1 For \(1 \leq j \leq n\) and \(i = 1, 2, \ldots, 2^{r_j} - 1\)

\[
\text{Lex}_j^i = \{ x_1 + x_2 \cdot 2^{r_1} + x_3 \cdot 2^{r_1+r_2} + \cdots + x_j \cdot 2^{r_1+r_2+\cdots+r_{j-1}} + \cdots + x_n \cdot 2^{r_1+r_2+\cdots+r_{n-1}} : 0 \leq x_k \leq 2^{r_k} - 1, 1 \leq k \leq n, k \neq j ; 0 \leq x_j \leq i - 1 \}
\]
is a composite set on \(i \times 2^{r_j-1}\) vertices in \(Q^r\) where \(r_1 + r_2 + \cdots + r_n = r\), \(r_1 \leq r_2 \leq \cdots \leq r_n\).

Proof. Define \(\varphi : \text{Lex}_j^i \to L_i \times 2^{r_j-1}\) by

\[
\varphi(x_1 + x_2 \cdot 2^{r_1} + x_3 \cdot 2^{r_1+r_2} + \cdots + x_j \cdot 2^{r_1+r_2+\cdots+r_{j-1}} + \cdots + x_n \cdot 2^{r_1+r_2+\cdots+r_{n-1}}) = \begin{cases} 
(x_1 + x_2 \cdot 2^{r_1} + \cdots + x_j \cdot 2^{r_1+r_2+\cdots+r_{j-1}}) \cdot 2^{r_1+r_2+\cdots+r_{j-1}} + \cdots + x_n \cdot 2^{r_1+r_2+\cdots+r_{n-1}} : j < n \\
(x_1 + x_2 \cdot 2^{r_1} + \cdots + x_j \cdot 2^{r_1+r_2+\cdots+r_{j-1}}) \cdot 2^{r_1+r_2+\cdots+r_{j-1}} : j = n
\end{cases}
\]

Case 1 \((j = n)\): If the binary representation of

\[
x = x_1 + x_2 \cdot 2^{r_1} + \cdots + x_j \cdot 2^{r_1+r_2+\cdots+r_{j-1}} + \cdots + x_n \cdot 2^{r_1+r_2+\cdots+r_{n-1}}
\]
is

\[
\alpha_1 \alpha_2 \cdots \alpha_n \beta_1 \beta_2 \cdots \beta_{r_{n-1}} \cdots \gamma_1 \gamma_2 \cdots \gamma_{r_{j+1}} \delta_1 \delta_2 \cdots \delta_{r_j} \mu_1 \mu_2 \cdots \mu_{r_{j-1}} \cdots \eta_1 \eta_2 \cdots \eta_{r_{j}}
\]
then the binary representation of

\[
\varphi(x) = (x_1 + x_2 \cdot 2^{r_1} + \cdots + x_j \cdot 2^{r_1+r_2+\cdots+r_{j-1}}) \cdot 2^{r_1+r_2+\cdots+r_{j-1}}
\]
is

\[
\alpha_1 \alpha_2 \cdots \alpha_n \beta_1 \beta_2 \cdots \beta_{r_{n-1}} \cdots \gamma_1 \gamma_2 \cdots \gamma_{r_{j+1}} \delta_1 \delta_2 \cdots \delta_{r_j} \mu_1 \mu_2 \cdots \mu_{r_{j-1}} \cdots \eta_1 \eta_2 \cdots \eta_{r_{j}}
\]

Case 2 \((j < n)\): If the binary representation of

\[
x = x_1 + x_2 \cdot 2^{r_1} + \cdots + x_j \cdot 2^{r_1+r_2+\cdots+r_{j-1}} + \cdots + x_n \cdot 2^{r_1+r_2+\cdots+r_{n-1}}
\]
is

\[
\alpha_1 \alpha_2 \cdots \alpha_n \beta_1 \beta_2 \cdots \beta_{r_{n-1}} \cdots \gamma_1 \gamma_2 \cdots \gamma_{r_{j+1}} \delta_1 \delta_2 \cdots \delta_{r_j} \mu_1 \mu_2 \cdots \mu_{r_{j-1}} \cdots \eta_1 \eta_2 \cdots \eta_{r_{j}}
\]
then we prove that the binary representation of

\[
\varphi(x) = (x_1 + x_2 \cdot 2^{r_1} + \cdots + x_j \cdot 2^{r_1+r_2+\cdots+r_{j-1}}) \cdot 2^{r_1+r_2+\cdots+r_{j-1}} + x_{j+1} \cdot 2^{r_1+r_2+\cdots+r_{j-1}} + \cdots + x_{n-1} \cdot 2^{r_n} + x_{n}
\]
is

\[
\delta_1 \delta_2 \cdots \delta_{r_j} \mu_1 \mu_2 \cdots \mu_{r_{j-1}} \cdots \eta_1 \eta_2 \cdots \eta_{r_{j}}
\]
Suppose \(x_k = 0\), \(j < k \leq n\), in \(x\). Then \(x = x_1 + x_2 \cdot 2^{r_1} + x_3 \cdot 2^{r_1+r_2} + \cdots + x_j \cdot 2^{r_1+r_2+\cdots+r_{j-1}}\). Since \(0 \leq x_k \leq 2^{r_k} - 1\), \(1 \leq k \leq j\) and \(0 \leq x_j \leq i - 1\), \(i = 1, 2, \ldots, 2^{r_j} - 1\), the binary representation of \(x\) is

\[
\underbrace{000 \ldots 000}_{r_{j+1}+r_{j+2}+\cdots+r_{n} \text{ times}} \delta_1 \delta_2 \cdots \delta_{r_j} \mu_1 \mu_2 \cdots \mu_{r_{j-1}} \cdots \eta_1 \eta_2 \cdots \eta_{r_j}.
\]

2
This implies that the binary representations of \( \varphi(x) = (x_1 + x_2 \cdot 2^{r_1} + x_3 \cdot 2^{r_1+r_2} + \ldots + x_j \cdot 2^{r_1+r_2+\ldots+r_{j-1}}) \cdot 2^r \) is
\[
\delta_1 \delta_2 \cdots \delta_{r_j} \mu_1 \mu_2 \cdots \mu_{r_{j-1}} \cdots \eta_2 \eta_3 \cdots \eta_{r_1} \underbrace{000 \ldots 000}_{r_{j+1}+r_{j+2}+\ldots+r_n \text{ times}}.
\]
Suppose \( x_k = 0, \ 1 \leq k \leq j, \) in \( x. \) Then \( x = x_{j+1} \cdot 2^{r_1+r_2+\ldots+r_{j+1}} + \ldots + x_n \cdot 2^{r_1+r_2+\ldots+r_{n-1}}. \) Since \( 0 \leq x_k \leq 2^{r_k} - 1, \) \( j+1 \leq k \leq n, \) the binary representation of \( x \) is
\[
\alpha_1 \alpha_2 \cdots \alpha_{r_n} \beta_1 \beta_2 \cdots \beta_{r_{n-1}} \cdots \gamma_{r_j} \gamma_{r_{j+1}} \cdots \gamma_{r_{j+1}} \underbrace{000 \ldots 000}_{r_{j+1}+r_{j+2}+\ldots+r_1 \text{ times}}.
\]
This implies that the binary representations of \( \varphi(x) = x_{j+1} \cdot 2^{r_1+r_2+\ldots+r_{j+1}} + \ldots + x_{n-1} \cdot 2^{r_{n-1}} + x_n \) is
\[
\underbrace{000 \ldots 000}_{r_{j+1}+r_{j+2}+\ldots+r_1 \text{ times}} \gamma_{r_{j+1}} \cdots \beta_1 \beta_2 \cdots \beta_{r_{n-1}} \alpha_1 \alpha_2 \cdots \alpha_{r_n}.
\]
Similarly we can prove all other cases.

Let the binary representations of two numbers \( x \) and \( y \) be respectively

\[
\underbrace{0 \text{ (resp. 1)}}_{r_n + r_{j+2} + r_{j+1} \text{ bits}} \underbrace{0 \text{ (resp. 1)}}_{r_j + r_{j-1} + \ldots + r_1 \text{ bits}}
\]

and

\[
\underbrace{1 \text{ (resp. 0)}}_{r_n + r_{j+2} + r_{j+1} \text{ bits}} \underbrace{0 \text{ (resp. 1)}}_{r_j + r_{j-1} + \ldots + r_1 \text{ bits}}
\]

Then the binary representations of numbers \( \varphi(x) \) and \( \varphi(y) \) are respectively

\[
\underbrace{0 \text{ (resp. 1)}}_{r_{j+1} + r_{j+2} + \ldots + r_n \text{ bits}} \underbrace{0 \text{ (resp. 1)}}_{r_{j+1} + r_{j+2} + \ldots + r_n \text{ bits}}
\]

and

\[
\underbrace{1 \text{ (resp. 0)}}_{r_{j+1} + r_{j+2} + \ldots + r_n \text{ bits}} \underbrace{1 \text{ (resp. 0)}}_{r_{j+1} + r_{j+2} + \ldots + r_n \text{ bits}}
\]

Hence the binary representations of two numbers \( x \) and \( y \) differ in exactly one bit if and only if the binary representations of \( \varphi(x) \) and \( \varphi(y) \) differ in exactly one bit. Therefore (\( x,y \)) is an edge in \( \text{Lex}^1_j \) if and only if \( (\varphi(x),\varphi(y)) \) is an edge in \( L_{i \times 2^{r_j}}. \) Hence \( \text{Lex}_j^1 \) and \( L_{i \times 2^{r_j}} \) are isomorphic. By Theorem 1, \( \text{Lex}_j^1 \) is a composite set in \( Q^r. \) \( \square \)

We need the following lemmas to compute the minimum wirelength of hypercubes into \( n \)-dimensional grid networks.

**Lemma 2 (Congestion Lemma)** \([4]\) Let \( G \) be an \( r \)-regular graph and \( f \) be an embedding of \( G \) into \( H. \) Let \( S \) be an edge cut of \( H \) such that the removal of edges of \( S \) leaves \( H \) into 2 components \( H_1 \) and \( H_2 \) and let \( G_1 = f^{-1}(H_1) \) and \( G_2 = f^{-1}(H_2). \) Also \( S \) satisfies the following conditions:

(i) For every edge \((a, b) \in G_i, \ i = 1, 2, P_f((a, b)) \) has no edges in \( S. \)

(ii) For every edge \((a, b) \in G \) with \( a \in G_1 \) and \( b \in G_2, \) \( P_f((a, b)) \) has exactly one edge in \( S. \)

(iii) \( G_1 \) is a composite set on \( k \) vertices where \( k = |V(G_1)|. \)

Then \( EC_f(S) \) is minimum and \( EC_f(S) = rk - 2|E(G_1)|. \) \( \square \)
Lemma 3 (Partition Lemma) [4] Let \( f : G \rightarrow H \) be an embedding. Let \( \{S_1, S_2, ..., S_p\} \) be a partition of \( E(H) \) such that each \( S_i \) is an edge cut of \( H \). Then

\[
WL_f(G, H) = \sum_{i=1}^{p} EC_f(S_i). \quad \square
\]

Lemma 4 The lexicographic embedding \( \text{lex} \) of \( Q^r \) into \( M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}] \), \( r_1 + r_2 + \cdots + r_n = r \), \( r_1 \leq r_2 \leq \cdots \leq r_n \) induces a minimum wirelength \( WL(Q^r, M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}]) \).

Proof. For \( 1 \leq j \leq n \) and \( i = 1, 2, ..., 2^{j-1} \), let \( S_j^i \) be an edge cut of \( M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}] \) such that \( S_j^i \) disconnects \( M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}] \) into two components \( X_j^i \) and \( \overline{X}_j^i \) where \( V(X_j^i) \) is \( \text{Lex}^i \). Let \( G_j^i \) and \( \overline{G}_j^i \) be the inverse images of \( X_j^i \) and \( \overline{X}_j^i \) under \( \text{lex} \) respectively. The edge cuts \( S_j^i \) satisfy conditions (i) and (ii) of the Congestion Lemma. Since \( G_j^i \) is a subgraph induced by the vertices of \( \text{Lex}^i \), by Lemma 1, \( G_j^i \) is a composite set satisfying condition (iii) of the Congestion Lemma. Thus by the Congestion Lemma, \( EC_{\text{lex}}(S_j^i) \) is minimum for \( 1 \leq j \leq n \) and \( i = 1, 2, ..., 2^{j-1} \). By Partition Lemma, \( WL_{\text{lex}}(Q^r, M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}]) \) is minimum. \( \square \)

Theorem 2 The lexicographic embedding \( \text{lex} \) of \( Q^r \) into \( M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}] \), \( r_1 + r_2 + \cdots + r_n = r \), \( r_1 \leq r_2 \leq \cdots \leq r_n \) induces a minimum wirelength

\[
WL(Q^r, M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}]) = \sum_{i=1}^{n} 2^{r_i-1} (2^{2r_i-1} - 2^{r_i-1}).
\]

Proof. By Lemma 4, \( WL(Q^r, M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}]) = WL_{\text{lex}}(Q^r, M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}]) \). But we have \( WL_{\text{lex}}(Q^r, P_{2^r}) = 2^{2r-1} - 2^{r-1} \) [3, 4] and by the symmetric property of the lexicographic embedding, the edges of \( Q^r \) are stretched in the grid \( M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}] \) either vertical path or horizontal path. Therefore each edge of the edge cut \( S_j^i \) has the same edge congestion. Hence the sum of the edge congestions in each vertical path (resp. horizontal path) is the same. Therefore

\[
WL(Q^r, M[2^{r_1} \times 2^{r_2} \times \cdots \times 2^{r_n}]) = 2^{r_1+r_2+\cdots+r_n-1} (2^{2r_n-1} - 2^{r_n-1}) + 2^{r_1+r_2+\cdots+r_{n-2}} 2^{r_n} (2^{2r_{n-1}-1} - 2^{r_{n-1}-1}) + 2^{r_1+r_2+\cdots+r_{n-3}} (2^{r_{n-1}}+r_n) (2^{2r_{n-2}-1} - 2^{r_{n-2}-1}) + \cdots + 2^{r_1+r_2+\cdots+r_{j-1}} (2^{r_{j+1}}+\cdots+r_n) (2^{2r_j-1} - 2^{r_j-1}) + \cdots + 2^{r_1+r_2+\cdots+r_{n}} (2^{r_1} - 2^{r_1-1}) = \sum_{i=1}^{n} 2^{r_i-1} (2^{2r_i-1} - 2^{r_i-1}). \quad \square
\]

3 Concluding Remark

Following the techniques used in this paper, we raise a question: Is it possible to compute the minimum wirelength of hypercubes into \( n \)-dimensional tori networks?

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References


