Nonexponential Evolution Equations and Operator Ordering

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Abstract—Nonexponential evolution equations can be treated using a formalism involving the evolution operator method, which, unlike the ordinary case, is not expressed in terms of exponential operators. The use of this technique requires particular care associated with the operator ordering. In this paper, we will present a first systematic approach to this type of problems. © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The equation,
\[ \frac{\partial}{\partial \tau} f(x, \tau) = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} f(x, \tau), \quad f(x, 0) = g(x), \] (1)
expressing a simple initial value problem (IVP), is recurring in different physical problems ranging from wave propagation to charged particle beam diffusion in storage rings [1].

The formal obvious solution of equation (1) is
\[ f(x, \tau) = \exp \left( -\tau \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \right) g(x). \] (2)

The operator on the left-hand side of equation (2) is usually called the evolution operator and is expressed in terms of an exponential because the exponential function is the eigenfunctions of the \( \frac{\partial}{\partial \tau} \) operator on the right-hand side of equation (1). Let us interchange the role of the operator appearing in equation (1), by considering the IVP,
\[ \frac{\partial}{\partial x} x \frac{\partial}{\partial x} f(x, \tau) = \frac{\partial}{\partial \tau} f(x, \tau), \quad f(0, \tau) = \gamma (\tau). \] (3)

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According to [2], the above problem can be solved using the evolution operator technique, by noting that the eigenfunction of the operator, \(-\frac{\partial}{\partial x} \frac{\partial}{\partial x}\), is provided by the function,

\[ C_O(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2}, \]  

which is a generalization of the exponential function and is known as the 0th-order Tricomi functions specified by [3],

\[ C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!} = x^{-n/2}J_n(2\sqrt{x}), \]

where \(J_n(x)\) is the ordinary cylindrical Bessel function of first kind [4].

The solution of equation (3), therefore, can be written as

\[ f(x,\tau) = C_O \left( x \frac{\partial}{\partial \tau} \right) \gamma(\tau), \]

where the operator \(C_O(x \frac{\partial}{\partial \tau})\) plays now the role of nonexponential evolution operator (NEEO).

As in the ordinary case, we have to specify how the evolution operator acts on the initial function. By noting, e.g., that the integral representation of the Tricomi function is [3],

\[ f(x) = \int_0^{2\pi} \exp(-in\theta)\exp(\frac{\partial}{\partial \tau}(e^{i\theta} - xe^{-i\theta})) \, d\theta, \]

we can write the solution of problem (3) using the following integral transform,

\[ f(x,\tau) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(e^{i\theta} - x \frac{\partial}{\partial \tau} e^{-i\theta}\right) \gamma(\tau) \, d\theta = \int_0^{2\pi} \exp(e^{i\theta}) \gamma(\tau - xe^{-i\theta}) \, d\theta. \]

The use of the above integral transform is a consequence of the integral representation (7) and is particularly convenient since it allows the use of the wealth of properties of the exponential operator.

One of the often encountered problems associated with exponential operators is that of the operatorial ordering.

In this paper, we will combine the properties of the Tricomi functions along with those of the exponential operators when NEEO of the type (6) involve operational ordering problems.

**2. EXAMPLES INVOLVING OPERATIONAL ORDERING PROBLEMS**

The first example, we consider is provided by the IVP,

\[ -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} f(x,\tau) = (\hat{A} + \hat{B}) f(x,\tau), \quad f|_{x=0} = f(0,\tau) = f_0, \]

where \(\hat{A}\) and \(\hat{B}\) are the noncommuting operators, satisfying the commutation relation [5],

\[ [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = k, \]

with \(k\) a complex number.

According to equation (8), the solution of our problem can be cast in the form,

\[ f(x,\tau) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(e^{i\theta} - x \left( \hat{A} + \hat{B} \right) e^{-i\theta}\right) f_0 \, d\theta, \]
and by recalling that the following decoupling rule holds [5],

\[ e^{(A+B)} = e^A e^B e^{-k/2}, \]

we find from equation (11),

\[ f(x, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \exp \left( e^{i\theta} - \frac{k}{2} x^2 e^{-2i\theta} - x^2 e^{-i\theta} \right) \exp \left( -x e^{-i\theta} \right) f_0 \, d\theta. \]  

This last equation provides an ordered form, obtained from the known properties of the exponential ordering. To better clarify how the method works, we consider the following realization of the operators \( A \) and \( B \),

\[ \hat{A} = \tau, \quad \hat{B} = \frac{\partial}{\partial \tau}, \]

so that \( k = 1 \), and we take \( f_0 = e^{-\lambda \tau} \), accordingly, we find from equation (13),

\[ f(x, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \exp \left( e^{i\theta} - x(\tau - \lambda) e^{-i\theta} + \frac{1}{2} x^2 e^{-2i\theta} \right) e^{-\lambda \tau} d\theta. \]  

The above integral representation can now be interpreted in terms of generalized forms of Tricomi functions. By recalling indeed that

\[ H_n(x, y) = \sum_{r=0}^{[n/2]} \frac{x^{n-2r} y^r}{r! (n-2r)!}, \]

being the two variable Hermite polynomials, we can cast equation (15) in the explicit form,

\[ f(x, \tau) = H C_{i\lambda} \left( x (\tau - \lambda), \frac{1}{2} \right) e^{-\lambda \tau}. \]  

It is evident that the above example shows the implications underlying NEEO. Let us now consider as a further possibility, problem (9) with

\[ [\hat{A}, \hat{B}] = m \hat{A}^{1/2}. \]  

In this case, the following decoupling rule holds for exponential operators [5],

\[ e^{(A+B)} = e^{m/12} e^{(m/2) \hat{A}^{1/2}} e^B. \]  

Therefore, we find from equation (11),

\[ f(x, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \exp \left( e^{i\theta} x e^{-i\theta} \hat{A} - \frac{m}{2} x^2 e^{-2i\theta} \hat{A}^{1/2} - \frac{m^2 x^3 e^{-3i\theta}}{12} \right) \times \exp \left( -x e^{-i\theta} \right) f_0 \, d\theta. \]  

In the case in which,

\[ \hat{A} = \tau^2, \quad \hat{B} = \frac{\partial}{\partial \tau}, \quad \text{and} \quad f_0 = e^{-\lambda \tau}, \]

we find

\[ f(x, \tau) = \frac{1}{2} \int_0^{2\pi} \exp \left( e^{i\theta} - (\tau^2 - \lambda) x e^{-i\theta} + x^2 \tau e^{-2i\theta} - \frac{2}{3} x^3 e^{-3i\theta} \right) e^{-\lambda \tau} d\theta. \]
and our problem can be written in terms of higher-order Tricomi functions, namely, the Hermite-Tricomi functions,

$$f(x, \tau) = \mathcal{H} C_0 \left( \left( \tau^2 - \lambda \right) x, x^2 \tau, \frac{x^3}{3} \right) e^{-\lambda \tau},$$

where

$$\mathcal{H} C_n(x, y, z) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r(x, y, z)}{r! (n+r)!}, \quad H_r(x, y, z) = \sum_{s=0}^{[r/3]} \frac{z^s H_{r-3s}(x, y)}{s! (r-3s)!}.$$ (24)

In the case in which

$$\hat{A} = \frac{\partial^2}{\partial \tau^2}, \quad \hat{B} = \tau, \quad \text{and} \quad f_0 = e^\tau,$$ (25)

we can write the solution using equation (20) in the form,

$$f(x, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \exp \left( e^{i\theta} - (1 + \tau) xe^{-i\theta} + x^2 e^{-2i\theta} - \frac{x^3}{3} e^{-3i\theta} \right) e^\tau d\theta,$$ (26)

which can be expressed again in terms of equation (24),

$$f(x, \tau) = \mathcal{H} C_0 \left( (1 + \tau) x, x^2, \frac{x^3}{3} \right) e^\tau.$$ (27)

Just to complete the analysis we will consider other two cases, namely,

$$[\hat{A}, \hat{B}] = m\hat{A} \quad \text{and} \quad [\hat{A}, \hat{B}] = m\hat{B}.$$ (28)

In the first case, the decoupling rule of the exponential writes [5],

$$e^{(\hat{A} + \hat{B})} = e^{((1-e^{-m})/m)\hat{A} e^{\hat{B}},}$$

thus, getting from equation (11),

$$f(x, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \exp \left( \frac{1 - e^{mxe^{-i\theta}}}{m} \hat{A} \right) \exp \left( -x \hat{B} e^{-i\theta} \right) f_0 d\theta.$$ (30)

We can now consider the following realization of the operators \( \hat{A} \) and \( \hat{B} \),

$$\hat{A} = -m\tau, \quad \hat{B} = -m\tau \frac{\partial}{\partial \tau},$$ (31)

so that, the solution of our evolution problem, after using the identity,

$$e^{az/\beta x} f(x) = f(e^a x),$$

can be written as

$$f(x, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \exp \left( e^{i\theta} - \left( 1 - e^{mxe^{-i\theta}} \right) \tau \right) \gamma \left( e^{mxe^{-i\theta}} \tau \right) d\theta.$$ (33)

In the second case, i.e., when \([\hat{A}, \hat{B}] = m\hat{A}^2\), we decouple the exponential by using the rule [5],

$$e^{(\hat{A} + \hat{B})} = \left( 1 + m\hat{A} \right)^{1/m} e^{\hat{B}},$$

(34)
and thus, we find from equation (11),

$$f (x, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \left(1 - m x \hat{A} e^{-i\theta} \right)^{1/m} \exp \left( e^{i\theta} - x \hat{B} e^{-i\theta} \right) f_0 \, d\theta. \quad (35)$$

Further, we consider the following realization of the operators $\hat{A}$ and $\hat{B}$,

$$\hat{A} = -\tau, \quad \hat{B} = m \tau^2 \frac{\partial}{\partial \tau}, \quad (36)$$

and thus, getting the solution of our evolution problem after making use of the identity,

$$e^{\lambda x^2 (\partial / \partial x)} f (x) = f \left( \frac{x}{1 - \lambda x} \right), \quad |x| < \frac{1}{\lambda}, \quad (37)$$

in the form,

$$f (x, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \left(1 + m x \tau e^{-i\theta} \right)^{1/m} \exp \left( \frac{\tau}{1 + m x \tau e^{-i\theta}} \right) f_0 \, d\theta, \quad |\tau x| < \frac{1}{m}. \quad (38)$$

The method which we have used to solve the IVP (9) for different possible relations between the operators $\hat{A}$ and $\hat{B}$ can be extended further to obtain the solutions of more general type of equations.

### 3. CONCLUDING REMARKS

We have shown that, apart from computational details, the technique associated with ordinary evolution equations can be easily extended to equations of the type (9).

To make the correspondence closer, we can use the operational rule [2,3],

$$C_0 (\alpha x) = \exp \left(-\alpha \hat{A}_x^{-1} \right), \quad (39)$$

where $\hat{A}_x^{-1}$ is the negative derivative operator discussed in [2,3].

According to the above definition, we can solve equation (9) in the pseudo exponential form,

$$f (x, \tau) = \exp \left[ -\hat{A}_x^{-1} \left( \hat{A} + \hat{B} \right) \right] f_o. \quad (40)$$

The exponential can be handled as in ordinary case. Assuming the condition (10), we find, e.g.,

$$f (x, \tau) = \exp \left[ -\frac{k}{2} \hat{A}_x^{-2} \right] \exp \left[ -\hat{A}_x^{-1} \hat{A} \right] \exp \left[ -\hat{A}_x^{-1} \hat{B} \right] f_0, \quad (41)$$

and then the explicit solution is obtained for a specific realization of $\hat{A}$ and $\hat{B}$ operators. In the case (14), and for $f_0$ = constant, we find, e.g.,

$$f (x, \tau) = \exp \left[ -\tau \hat{A}_x^{-1} + \frac{1}{2} \hat{A}_x^{-2} \right] f_0, \quad (42)$$

regarding the r.h.s. of the above equation as the generating function of the Hermite polynomials,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n (x, y) = \exp (xt + yt^2), \quad (43)$$

we find

$$f (x, \tau) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \Delta_x^{-r} H_r \left( \tau, \frac{1}{2} \right) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \Delta_x^{-r} H_r \left( \tau, \frac{1}{2} \right), \quad (44)$$
which, after using the fact $a^r H_r(x, y) = H_r(ax, a^2 y)$ and equation (16), give

$$f(x, \tau) = H C_0 \left( x \tau, \frac{1}{2} \tau^2 \right). \quad (45)$$

The use of this method can be efficient in some cases, but the integral representation method is more general and can be used even for higher-order NEEO.

We note indeed that in the hypothesis that we are dealing with the problem (9), in which we replace $-\frac{\partial^2}{\partial x^2} \frac{\partial}{\partial \tau}$ by

$$\hat{L} = -\frac{\partial}{\partial x} \frac{\partial}{\partial \tau} - m \frac{\partial}{\partial \tau}, \quad (46)$$

with $m$ being an integer, we can use the same procedure as before and being $C_m(x)$ and eigenfunction of the operator $\hat{L}$, we can write the solution in the form,

$$f(x, \tau) = \frac{m!}{2\pi} \int_0^{2\pi} \exp(-im\theta) \exp(ie^{i\theta} - (A + B) e^{-i\theta}) f_0 d\theta. \quad (47)$$

In this paper, we have shown that the significant progress can done in the theory of NEEO using operational methods, the extension of the theory will be discussed elsewhere.

REFERENCES