International Journal of Control
Vol. 00, No. 00, Month 200x, 1-14

RESEARCH ARTICLE

Lyapunov stability of 2D finite-dimensional behaviors

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(Received 00 Month 200x; final version received 00 Month 200x)

In this paper we investigate a Lyapunov approach to the stability of finite-dimensional 2D systems. We use the behavioral framework and consider a notion of stability following the ideas in (15, 18, 19). We characterize stability in terms of the existence of a (quadratic) Lyapunov function and provide a constructive algorithm for the computation of all such Lyapunov functions.

Keywords: 2-D system; Lyapunov function; quadratic difference form; behavioral approach;

1 Introduction

The stability of two dimensional (2D) systems has been the subject of extensive investigation in the past decades; among these research efforts, some have also been focused on the computation of Lyapunov functions. Past research has predominantly been concerned with systems whose set of trajectories is infinite-dimensional, and almost exclusively has concerned specific class of models, for example Fornasini-Marchesini or Roesser models (see (3, 10)). Moreover, in those investigations a specific (usually nonnegative quarter-plane) notion of causality has been assumed.

In this paper we follow the behavioral approach: we study the stability of 2D systems described by higher-order difference equations without reference to special representations; the central object of interest in our investigation is the set of all admissible trajectories of the system, the behavior, rather than any of its specific representations. Following the pioneering approach of (19), stability is accordingly defined at the level of trajectories, although we will be using a different but ultimately equivalent definition to that proposed in (19). We also adopt the eminently reasonable position proposed in (19) to let the system dynamics themselves dictate what notion of causality is most appropriate for the case at hand.

A Lyapunov analysis of stability of infinite-dimensional square 2D behaviors has been presented in (8); in this paper we concentrate our attention on the case of finite-dimensional 2D discrete behaviors, i.e. finite-dimensional subspaces of the set of trajectories from \( \mathbb{Z}^2 \) to \( \mathbb{R}^n \). We define a finite-dimensional 2D system stable if all trajectories of the behavior go to zero along every “discrete line” in a cone, which without loss of generality we take to be the first orthant of the lattice \( \mathbb{Z}^2 \); this is a different but equivalent definition from that adopted in section 3, Def. 3.1 of (19), and follows the approach of (12, 15, 18). The main result of the paper is a characterization of stability for finite-dimensional 2D behaviors in term of the existence of a Lyapunov function, defined as a quadratic function of the system variables and their 2D shifts which is positive along

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each discrete line in the first orthant, and whose increment along each such line is negative. In
this paper we also give necessary and sufficient conditions for a quadratic function of the system
variables and their 2D shifts to be a Lyapunov function; and we illustrate an algorithm to
compute a Lyapunov function for a given finite-dimensional 2D behavior.

In the following we make extensive use of the concepts and calculus of 2D quadratic difference
forms (see (9)), and their association with four-variable polynomial matrices. We will also use
extensively the concepts and terminology of the behavioral approach to 2D systems. In order
to make the paper self-contained we have included some background material in section 2; the
reader interested in a more thorough introduction to 2D behavioral system theory is referred to
(16, 17, 19). The main result of the paper is illustrated in section 3. In section 4 we outline an
algorithm for the construction of Lyapunov functions.

2 Preliminaries

We consider sets \( \mathcal{B} \) of trajectories defined over \( \mathbb{Z}^2 \) that can be described by a set of linear partial
difference equations, i.e.,
\[
\mathcal{B} = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) \subseteq (\mathbb{R}^w)^{\mathbb{Z}^2},
\]
where \( \sigma_i \)'s are the 2D shift operators defined by
\[
\sigma_i w(k_1, k_2) = w((k_1, k_2) + e_i),
\]
for \( (k_1, k_2) \in \mathbb{Z}^2 \) and \( e_i \) the \( i \)th element of the canonical basis of \( \mathbb{R}^2 \), \( i = 1, 2 \);
and \( R(s_1, s_1^{-1}, s_2, s_2^{-1}) \) is a 2D \( (p \times w) \)-dimensional Laurent-polynomial matrix. We call (1) a kernel representation
of the behavior \( \mathcal{B} \).

We now introduce finite-dimensional behaviors, and briefly discuss their representation by
means of state equations.

Given a full column rank Laurent-polynomial matrix \( R \in \mathbb{R}^{p \times w}[s_1, s_1^{-1}, s_2, s_2^{-1}] \), we define its
Laurent variety (or simply variety) as
\[
\mathcal{V}(R) := \{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid \text{rank}(R(\lambda_1, \lambda_2)) < \text{rank}(R) \},
\]
where \( \text{rank}(R(\lambda_1, \lambda_2)) \) is the rank of the complex matrix \( R(\lambda_1, \lambda_2) \), while \( \text{rank}(R) \) is the rank
of the Laurent-polynomial matrix \( R \). It can be shown (21) that any two different representations
of \( \mathcal{B} \) share the same Laurent variety; consequently in the following we refer to the variety of \( \mathcal{B} \),
denoted by \( \mathcal{V}(\mathcal{B}) \), as the Laurent variety of any of its kernel representations. It is well known
(see (2, 17)) that a behavior \( \mathcal{B} \) is finite dimensional (when considered as a subspace of the vector
space over \( \mathbb{R} \) consisting of all trajectories from \( \mathbb{Z}^2 \) to \( \mathbb{R}^w \)) if and only if \( \mathcal{V}(\mathcal{B}) \) consists of a finite
number of points, or equivalently if \( \mathcal{B} \) admits a left factor prime representation (see (5) for a
definition).

It was shown in (4) that for every finite dimensional behavior \( \mathcal{B} \) there exist a hybrid representa-
tion of first order, i.e., there exist matrices \( A_1 \in \mathbb{R}^{n \times p} \), \( A_2 \in \mathbb{R}^{n \times n} \) and \( C \in \mathbb{R}^{w \times n} \) such that \( \mathcal{B} \)
consists of all trajectories \( w \) for which there exists a trajectory \( x : \mathbb{Z}^2 \to \mathbb{R}^n \) such that
\[
\begin{align*}
\sigma_1 x &= A_1 x \\
\sigma_2 x &= A_2 x \\
w &= C x,
\end{align*}
\]
holds, where also the matrices \( A_1 \) and \( A_2 \) commute: \( A_1 A_2 = A_2 A_1 \). In particular, \( A_1, A_2 \) and \( C \)
can be chosen so that the state variable $x$ is observable from $w$, i.e.,

$$[(w, x) \text{ satisfy (2) and } w = 0] \implies [x = 0].$$

There are several different characterization of observability in terms of the algebraic properties of the representation; the first one is

$$\ker \begin{bmatrix} \sigma_1 I - A_1 \\ \sigma_2 I - A_2 \\ C \end{bmatrix} = \{0\}.$$ 

This condition is equivalent with the extended observability matrix, defined as the column block matrix

$$O(A_1, A_2, C) := \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_1 A_2 \\ CA_2^2 \\ \vdots \\ CA_2^{n-1} \end{bmatrix},$$

having rank equal to $n$. This implies that there exists a matrix $E$ such that $E : O(A_1, A_2, C) = I_n$, where $I_n$ is the $n \times n$ identity matrix. Thus we obtain that $x = X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w$ where

$$X(s_1, s_1^{-1}, s_2, s_2^{-1}) := E \begin{bmatrix} C \\ CS_1 \\ CS_2 \\ C s_1 s_2 \\ C s_1^2 s_2 \\ \vdots \\ C s_1^{n-1} \end{bmatrix} \in \mathbb{R}^{|s_1, s_1^{-1}, s_2, s_2^{-1}| \times w}.$$ 

In this case the state map $X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ is minimal, i.e., the dimension of the state variable $x$ is minimal among all possible representations (2).

In the following, discrete lines in the lattice $\mathbb{Z}^2$ will play an important role; we now introduce the basic notation and discuss behaviors restricted to lines.

The set of lines in the first orthant of $\mathbb{Z}^2$ is defined by

$$L := \{ \ell \subset \mathbb{N}^2 \mid \ell = \{(a, b) \in \mathbb{N}^2 \mid \alpha \in \mathbb{N} \}, a, b \in \mathbb{N} \text{ are coprime} \}.$$ 

The lines in the first orthant corresponding to the vertical, respectively horizontal, axes, will be denoted in the following with $\ell_1$ and $\ell_2$ respectively. Given a 2D behavior $\mathcal{B}$ and a line $\ell \in L$, we define the restriction of $\mathcal{B}$ to $\ell$ as

$$\mathcal{B} | \ell := \{ w_\ell : \mathbb{Z} \to \mathbb{R}^w \mid \text{there exists } w \in \mathcal{B} \text{ such that } w|_\ell = w_\ell \},$$

where $w|_\ell$ denotes the restriction of the trajectory $w$ to the domain $\ell$. Note that $w_\ell$ is a 1D trajectory, while $w|_\ell$ is a trajectory depending on two indices. It has been shown in (11, Th.6
and Th. 7) that a 2D behavior restricted to a line is a 1D behavior; that is, $B$ is the kernel of some polynomial operator in the 1D shift defined by a Laurent-polynomial matrix $R(s) \in \mathbb{R}[s, s^{-1}]$. It is easy to see that if $B$ is described by (2) and if $B_1$ and $B_2$ denote the restrictions of $B$ to the axes, then $B_i$ is described in the state-space form as

$$\begin{align*}
s_i x &= A_i x \\
w &= C x,
\end{align*}$$

(5)
i = 1, 2. Note that these state representations may be non-minimal even if (2) is minimal.

In many modeling and control problems it is necessary to study quadratic functionals of the system variables and their derivatives; for example, in linear quadratic optimal control, dissipativity theory, etc. Following the seminal work of (20), successively extended to the 2D case in (7, 8, 16), we will use polynomial matrices in 4 variables as a tool to express quadratic functionals of functions of 2 independent variables and their shifts. We next review the definitions regarding these functionals which are most relevant to the problems treated in this paper.

In the following, we use the multi-indices $k = (k_1, k_2)$ and $l = (l_1, l_2)$, and indeterminates by $\zeta = (\zeta_1, \zeta_2)$ and $\eta = (\eta_1, \eta_2)$. We also denote $\zeta^k = \zeta_1^{k_1} \zeta_2^{k_2}$ and $\eta^k = \eta_1^{k_1} \eta_2^{k_2}$. We denote with $\mathbb{R}^{w \times w}[\zeta, \eta]$ the set of real polynomial $w \times w$ matrices in the 4 indeterminates $\zeta$ and $\eta$ that is, an element of $\mathbb{R}^{w \times w}[\zeta, \eta]$ is of the form

$$\Phi(\zeta, \eta) = \sum_{k,l} \Phi_{k,l} \zeta^k \eta^l$$

(6)

where $\Phi_{k,l} \in \mathbb{R}^{w \times w}$; the sum ranges over a finite set of multi-indices $k, l \in \mathbb{N}^2$. The 4-variable polynomial matrix $\Phi(\zeta, \eta)$ is called symmetric if $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$, equivalently if $\Phi_{k,l} = \Phi_{l,k}^\top$ for all $l$ and $k$. In this paper we restrict our attention to the symmetric elements in $\mathbb{R}^{w \times w}[\zeta, \eta]$, and denote this subset by $\mathbb{R}_s^{w \times w}[\zeta, \eta]$. Any symmetric $\Phi$ induces a quadratic functional

$$Q_{\Phi} : (\mathbb{R}^w)^{2 \times 2} \times (\mathbb{R}^w)^{2 \times 2} \to (\mathbb{R})^{2 \times 2}$$

$$Q_{\Phi}(w) = \sum_{k,l} (\sigma^k w)^\top \Phi_{k,l} \sigma^l (w)$$

where the $k$-th shift operator $\sigma^k$ is defined as $\sigma^k = \sigma^{k_1} \sigma^{k_2}$ (similarly for $\sigma^l$). We call $Q_{\Phi}$ the quadratic difference form (in the following abbreviated with QDF) associated with $\Phi$. Given two QDFs $Q_{\Phi_1}, Q_{\Phi_2}$ we say that $Q_{\Phi_1}$ is equivalent to $Q_{\Phi_2}$ on $B$, denoted by $Q_{\Phi_1} \overset{B}{=} Q_{\Phi_2}$, if

$$Q_{\Phi_1}(w) = Q_{\Phi_2}(w) \text{ for all } w \in B.$$ We call a QDF $Q_{\Phi}$ nonnegative along $B$, denoted $Q_{\Phi} \overset{B}{\geq} 0$, if $Q_{\Phi}(w) \geq 0$ for all $w \in B$. We call $Q_{\Phi}$ positive along $B$, denoted $Q_{\Phi} \overset{B}{>} 0$, if $Q_{\Phi} \overset{B}{\geq} 0$, and moreover $\forall w \in B \ [Q_{\Phi}(w) = 0] \implies [w = 0]$.

In the following we will be also using operations on QDFs. Given a QDF $Q_{\Phi}$ and a line $\ell = \{\alpha(a, b) \mid \alpha \in \mathbb{N}\} \in I$, we define the increment of $Q_{\Phi}$ along the line $\ell$, denoted $\nabla_\ell(Q_{\Phi})$, as

$$\nabla_\ell Q_{\Phi}(w)(\alpha(a, b)) := Q_{\Phi}(w)((\alpha + 1)(a, b)) - Q_{\Phi}(w)(\alpha(a, b)).$$

The increment along the vertical, respectively horizontal, line will be denoted by $\nabla_{\ell_1}, \nabla_{\ell_2}$ respectively.
3 2D stability and Lyapunov functions

In this paper we will consider stability as defined in (18), i.e. with respect to a cone $S$. A set $S \subset \mathbb{R} \times \mathbb{R}$ is called a cone if $\alpha S \subset S$ for all $\alpha \geq 0$. A cone $S$ is solid if it contains an open ball in $\mathbb{R} \times \mathbb{R}$, and pointed if $S \cap (-S) = \{(0,0)\}$. A cone is proper if it is closed, pointed, solid, and convex. Since an appropriate change of independent variables transforms any proper cone $S$ into the first orthant, in the following we assume, without loss of generality, that $S$ is the first orthant in $\mathbb{Z}^2$. For the sake of brevity, in the following we will use the expression “stable” instead of “stable with respect to the first orthant”.

The definition of asymptotic stability that we shall use in the rest of this paper is the following; note that it is the discrete counterpart of that considered in the continuous-time case in (15) for $nD$ behaviors.

**Definition 3.1:** Let $B$ be a 2D behavior. $B$ is asymptotically stable if

$$[w \in B] \implies [\forall (a,b) \in \mathbb{N}^2 \lim_{\alpha \to \infty} w(\alpha(a,b)) = 0].$$

It is straightforward to see that $B$ is asymptotically stable if and only if $\forall \ell \in L$ it holds that $w_\ell$, the 1D trajectory associated with the restriction $w_\ell$, goes to zero as the independent variable goes to infinity. This definition of stability is equivalent to the definition considered in (19) for (finite dimensional) 2D behaviors. It was shown in (18) that in the discrete case all stable behaviors according to Definition 3.1 are finite dimensional; note that in the continuous case this is not necessarily true.

Having defined stability as in Definition 3.1, we now define Lyapunov functions as follows.

**Definition 3.2:** A functional $F : (\mathbb{R}^w)^{\mathbb{Z}^2} \to (\mathbb{R})^{\mathbb{Z}^2}$ is a Lyapunov function for a 2D behavior $B$ if for all $\ell \in L$ it holds that for all $w \in B$,

$$F(w)|_{\ell} > 0 \quad \text{and} \quad \nabla_{\ell} F(w) < 0.$$

If $F$ is a quadratic functional of $w \in B$ and its shifts, we call it a quadratic Lyapunov function (QLF) for $B$.

In order to state the main result of this section, a characterization of asymptotic stability in terms of Lyapunov functions, we need some preliminary concepts and results. The first one is the notion of quadratic functionals of the state. Let $B$ be a finite-dimensional 2D behavior, and let (2) be a minimal state representation of $B$. We say that a quadratic functional $Q_\Phi(w)$ of $w \in B$ and its shifts is a quadratic function of the state of $B$ if there exists a symmetric constant matrix $P$ such that for all trajectories $(w,x)$ satisfying (2) it holds that $Q_\Phi(w) = x^TPx$. As the following result shows, any quadratic functional of the system variables and their shifts is a quadratic functional of the state.

**Proposition 3.3:** Let $B$ be a 2D behavior, and let (2) be a state representation of $B$. Let $Q_\Phi : (\mathbb{R}^w)^{\mathbb{Z}^2} \to (\mathbb{R})^{\mathbb{Z}^2}$ be a QDF. Then there exists a symmetric matrix $P \in \mathbb{R}^{n \times n}$ such that for all $w \in B$ with associated state trajectory $x$ it holds

$$Q_\Phi(w) = x^TPx.$$
Proof: Write

$$Q_{\Phi}(w) = \sum_{i_1,i_2,k_1,k_2=0}^N (\sigma_1^{i_1} \sigma_2^{i_2} w)^\top \Phi_{i_1,i_2,k_1,k_2}(\sigma_1^{k_1} \sigma_2^{k_2} w)$$

$$= (w^\top \sigma_1 w^\top \sigma_2 w^\top \ldots) \begin{pmatrix} \Phi_{0000} & \Phi_{0010} & \Phi_{0001} & \ldots \\ \Phi_{1000} & \Phi_{1010} & \Phi_{0011} & \ldots \\ \Phi_{0100} & \Phi_{0110} & \Phi_{0010} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} w \\ \sigma_1 w \\ \sigma_2 w \\ \vdots \end{pmatrix}.$$ 

Now observe that $\sigma_1^{i_1} \sigma_2^{i_2} w = CA_1^{i_1} A_2^{i_2} x$; consequently the last expression can be rewritten as

$$( (Cx)^\top (CA_1 x)^\top (CA_2 x)^\top \ldots) \begin{pmatrix} \Phi_{0000} & \Phi_{0010} & \Phi_{0001} & \ldots \\ \Phi_{1000} & \Phi_{1010} & \Phi_{0011} & \ldots \\ \Phi_{0100} & \Phi_{0110} & \Phi_{0010} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Cx \\ CA_1 x \\ CA_2 x \\ \vdots \end{pmatrix}.$$ 

Now define

$$P := O(A_1,A_2,C)^\top \begin{pmatrix} \Phi_{0000} & \Phi_{0010} & \Phi_{0001} & \ldots \\ \Phi_{1000} & \Phi_{1010} & \Phi_{0011} & \ldots \\ \Phi_{0100} & \Phi_{0110} & \Phi_{0010} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} O(A_1,A_2,C);$$

the claim follows. $\square$

It follows from the proof of Proposition 3.3 that every QDF $Q_{\Phi}$ can be written as $Q_{\Phi}(w) = (X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1}) w)^\top P(X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1}) w) = x^\top Px$ for $w \in \mathcal{B}$, where $X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$ is a polynomial operator in the shift that induces the state variable $x$ when acting on the trajectory $w$, and whose expression is derived in a straightforward manner from the extended observability matrix (3). Consequently, every QDF $Q_{\Phi}$ is equivalent on $\mathcal{B}$ to a QDF $Q_{\Phi'}$ induced by a 4-variable polynomial matrix of the form $\tilde{\Phi}(\zeta_1,\zeta_2,\eta_1,\eta_2) = X(\zeta_1,\zeta_2)^\top PX(\eta_1,\eta_2)$, with $P$ symmetric and $X$ inducing a state variable. We call such a QDF $Q_{\Phi'}$ a canonical representative of $Q_{\Phi}$. Note that if $\tilde{x}$ is another minimal state variable for $\mathcal{B}$, then it is easy to see that $\tilde{x} = Tx$. Consequently, $Q_{\Phi}(w) = \tilde{x}^\top \tilde{P} \tilde{x}$ where $\tilde{P} = (T^\top)^{-1} PT^{-1}$. If a well-ordering (see (1)) has been fixed in the space of polynomials $\mathbb{R}^{w\times w}[\zeta_1,\zeta_2,\eta_1,\eta_2]$, then a unique canonical representative can be defined, see (7).

We now give an example of the computation of a canonical representative of a QDF.

Example 3.4 Let $\mathcal{B} = \ker R(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$ be a 2D behavior where

$$R(s_1,s_1^{-1},s_2,s_2^{-1}) = \begin{pmatrix} (s_2 - \frac{1}{2})(s_1 - \frac{1}{2}) & 0 \\ (s_1 - \frac{1}{2})(s_2 - \frac{1}{2}) & 0 \\ 0 & s_1 - \frac{1}{2} \\ 0 & s_2 - \frac{1}{2} \end{pmatrix},$$

and $\Phi(\eta_1,\eta_2,\zeta_1,\zeta_2)$ a 4-variable polynomial matrix given by

$$\Phi(\eta_1,\eta_2,\zeta_1,\zeta_2) = \begin{pmatrix} 1 + \eta_1 \zeta_1 & \zeta_1 \\ \eta_1 & \eta_2 \zeta_2 \end{pmatrix}.$$
It is a matter of straightforward verification to check that the variety of $B$ consists of the points

$$\mathcal{V}(B) = \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{4}), (\frac{1}{3}, \frac{1}{5})\} ;$$

since $\mathcal{V}(B)$ is finite, it follows that $B$ is finite-dimensional.

Following the procedure illustrated in (4), a minimal state-space realization of $B$ as in (2) is given by the matrices

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  

It is easy to compute that $x = X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w$ is the state variable corresponding to the matrices $A_1, A_2$ and $C$, where

$$X(s_1, s_1^{-1}, s_2, s_2^{-1}) = \begin{pmatrix} -4(s_1 - \frac{1}{2}) & 0 \\ 0 & 1 \\ 4(s_1 - \frac{1}{4}) & 0 \end{pmatrix}.$$  

We now compute a canonical representative of $Q\Phi$. It is easy to check that $Q\Phi(w) = x^TPx$, where

$$P = \mathcal{O}(A_1, A_2, C) \begin{pmatrix} \Phi_{0000} & \Phi_{0010} & \Phi_{0001} & \ldots \\ \Phi_{1000} & \Phi_{1010} & \Phi_{0110} & \ldots \\ \Phi_{0100} & \Phi_{0101} & \Phi_{0101} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mathcal{O}(A_1, A_2, C)$$

$$= \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}.$$  

Hence, a canonical representative for $Q\Phi$ is

$$\Phi'(\zeta_1, \zeta_2, \eta_1, \eta_2) = X(\zeta_1, \zeta_2)^TPX(\eta_1, \eta_2).$$

The notion of canonical representative of a QDF is important for the proof of the following theorem, that constitutes the main result of this section. It relates the 2D stability of the 2D behavior $B$ with the 1D stability of the 1D behaviors resulting from the restriction of $B$ to the axes and with the existence of a quadratic Lyapunov functional.

**Theorem 3.5:** Let $B$ be a finite dimensional 2D behavior and denote with $B_1, B_2$ the restrictions of $B$ to the vertical, respectively horizontal, axis. The following statements are equivalent:

1. $B$ is stable.
(2) $B_1$ and $B_2$ are stable 1D behaviors;
(3) There exist three 4 variable polynomial matrices $\Phi$, $\Delta_1$ and $\Delta_2$ such that

$$Q_\Phi^B > 0, \quad Q_{\Delta_i}^B > 0 \quad \text{and} \quad \nabla_\ell Q_\Phi^B = -Q_{\Delta_i}, \quad i = 1, 2.$$  

(4) There exists a QLF for $B$.

Proof: We begin with some general considerations about finite-dimensional behaviors which will make the proof of the result easier. Since $B$ is finite dimensional, $\mathcal{V}(B)$ is finite and every trajectory of $B$ is a linear combination of polynomial exponentials of the form

$$w_{\lambda_1, \lambda_2}(k_1, k_2) = p_{\lambda_1, \lambda_2}(k_1, k_2)\lambda_1^{k_1}\lambda_2^{k_2},$$  

for some suitable nonzero w-vector polynomial function $p_{\lambda_1, \lambda_2}$, i.e.

$$p_{\lambda_1, \lambda_2}(k_1, k_2) = \sum_{ij \in I} \alpha_{ij} k_1^i k_2^j,$$

where $I \subset \mathbb{N}^2$ is a finite bi-index set and $\alpha_{ij} \in \mathbb{R}^2$. This implies that the trajectories of $B_1$ are linear combinations of trajectories of the form

$$p_{\lambda_1, \lambda_2}(k_i e_i)\lambda_{\alpha}^{k_i}, \quad i = 1, 2.$$  

Furthermore, it follows from (18, Th. 8) that $B$ is stable if and only if $\mathcal{V}(B)$ is finite and for all $(\lambda_1, \lambda_2) \in \mathcal{V}(B)$, it holds that $|\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}| < 1$ for all $(\alpha_1, \alpha_2) \in \mathbb{N}^2$.

1) $\Rightarrow$ 2): Let $\hat{\lambda}_1 \in \mathcal{V}(B_1)$. By assumption $B$ is finite dimensional and the trajectories of $w \in B$ are of the form described in (7). Also, the trajectories of $B_1$ are of the form described in (8) with $i = 1$. This implies that there exists $\hat{\lambda}_2$ such that $p_{\hat{\lambda}_1, \hat{\lambda}_2}(k_1, k_2)\lambda_1^{k_1}\lambda_2^{k_2} \in B$. Since $B$ is stable it follows that $|\lambda_1^{\alpha_1}| < 1$, for all $(\alpha_1, \alpha_2) \in \mathbb{N}^2$. In particular, if $\alpha_2 = 0$ we obtain that we have that $|\lambda_1^{\alpha_1}| < 1$ for all $\alpha_1 \in \mathbb{N}$, which amounts to saying that $B_1$ is stable. The same argument shows that $B_2$ is stable.

2) $\Rightarrow$ 3): Consider a representation of $B$ as in (2). Then $B_i$ is described by (5); by assumption we have that $A_1, A_2$ are Schur matrices, i.e. all their eigenvalues have modulus less than one. Since these matrices commute, there exists (see (13)) a matrix $P > 0, \Delta_1 > 0$ and $\Delta_2 > 0$ of suitable sizes such that

$$A_i^T P A_i - P = -\tilde{\Delta}_i, \quad \text{for } i = 1, 2.$$  

Define

$$Q_\Phi(w) := (X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w)^T P X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w = x^T P x,$$

$$Q_{\Delta_i}(w) := (X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w)^T \tilde{\Delta}_i X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w = x^T \tilde{\Delta}_i x,$$

$i = 1, 2$, where $X$ is the polynomial operator in the shift inducing the state variable $x$. Now since $\nabla_\ell Q_\Phi(w) = x^T (A_i^T P A_i - P)x$, $\forall w \in B$, $i = 1, 2$, it is easy to verify that $Q_\Phi, Q_{\Delta_i}$ and
\( Q_{\Delta_k} \) satisfy the conditions of statement 3).

3) \( \Rightarrow \) 4): We show that \( Q_{\Phi} \) is a QLF for \( B \). It follows from the Proposition 3.3 that there exists a symmetric polynomial matrix \( P \) such that \( Q_{\Phi}(w) = x^TPx \) with \( x = X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w \). Observe that the 1D dynamics along a line \( \ell = \{\alpha(i, j) \mid \alpha \in \mathbb{N}\} \in \mathcal{L} \) are described by \( \sigma_\ell x = A_1^\ell A_2^\ell x, \ w = CX \), for some \( i, j \) fixed but otherwise arbitrary. Thus, it is enough to prove that \( P \) satisfies a matrix Lyapunov equation for \( A_1^\ell A_2^\ell \), i.e. \( (A_1^\ell A_2^\ell)^TP(A_1^\ell A_2^\ell) - P < 0 \). Observe first that

\[
(A_1^\ell A_2^\ell)^TP(A_1^\ell A_2^\ell) - P = (A_2^\ell)^TP(A_1^\ell)A_1^\ell) - P < 0,
\]

where we have used the fact that

\[
[A_1^\ell^TPA_1 - P < 0] \Rightarrow [(A_1^\ell)^TP(A_1^\ell) - P < 0]
\]

as can be readily proved by induction. From the same argument it follows that \( (A_2^\ell)^TP(A_2^\ell) - P < 0 \) and consequently \( (A_1^\ell A_2^\ell)^TP(A_1^\ell A_2^\ell) - P < 0 \). Now define \( Q_{\Phi}(w) := x^TPx \), and conclude that along the line \( \ell \) it holds that \( \nabla_{\ell} Q_{\Phi} < 0 \), as was to be proved.

4) \( \Rightarrow \) 2) It is a matter of straightforward verification to check that \( x|_{\ell_i} \) is a state trajectory for the 1D behavior \( B_i, \ i = 1, 2 \). Moreover, if \( P \) is a matrix corresponding to a canonical representative of the Lyapunov function \( Q_{\Phi} \), then \( (x|_{\ell_i})^TPx|_{\ell_i} \) is a Lyapunov function for \( B_i \). This implies that for \( i = 1, 2 \) it holds that \( w_i \) goes to zero along the line \( \ell_i, \ i = 1, 2, \) i.e., \( B_i \) is stable. \( \square \)

We are now in the position of stating a characterization of Lyapunov functions in terms of canonical representatives.

**Definition 3.6:** Let \( A_1, A_2 \in \mathbb{R}^{n \times n} \). A matrix \( P > 0 \) is said to be a common Lyapunov solution (CLS) for \( A_1 \) and \( A_2 \) if

\[
\begin{align*}
A_1^TPA_1 - P < 0 \\
A_2^TPA_2 - P < 0.
\end{align*}
\]

(9)

Using this definition, we can give yet another equivalent statement to those of Theorem 3.5.

**Proposition 3.7:** Let \( B \) be a 2D behavior and \( \Phi \) be a QDF. \( \Phi \) is a QLF for \( B \) if and only if given any minimal state-space representation \( (A_1, A_2, C) \) of \( B \) together with a state map \( X \in \mathbb{R}^{n\times w}[\xi_1, \xi_2] \) inducing the state variable \( x \) for the representation, the following two conditions hold:

1) There exists a symmetric matrix \( P > 0 \) which is a CLS for \( A_1 \) and \( A_2 \);

2) the canonical representative of \( Q_{\Phi} \) is equal to \( X^TPX \).

**Proof:** The claim follows easily using the same arguments of the proof 4) \( \Rightarrow \) 2), 2) \( \Rightarrow \) 3) and 3) \( \Rightarrow \) 4) of Theorem 3.5. \( \square \)

An interesting question is the construction of all Lyapunov functions for a given 2D behavior \( B_i \); this is addressed in the next section.

4 Construction of Lyapunov functions

The result of Proposition 3.7 shows that the problem of finding a Lyapunov function can be reduced to that of finding a common solution to a pair of discrete-time 1D Lyapunov equations.
In this section, we provide a constructive algorithm for the construction for Lyapunov functions for a given stable 2D behavior.

We present some preliminary results which will be instrumental to this end. The first one introduces two useful maps. In the following, we denote with $\mathbb{R}^{n \times n}$ the $n \times n$ symmetric matrices.

**Definition 4.1:** Let $A_1, A_2 \in \mathbb{R}^{n \times n}$. The Lyapunov map associated with $A_i$, $i = 1, 2$ are defined as

$$\mathcal{L}_i : \mathbb{R}_s^{n \times n} \to \mathbb{R}^{n \times n}$$

$$P \mapsto A_i^\top P A_i - P. \quad (10)$$

The name Lyapunov map is adopted following (14).

**Lemma 4.2:** If the matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$ in (10) commute then the Lyapunov maps $\mathcal{L}_1$ and $\mathcal{L}_2$ associated with $A_1, A_2$ commute.

**Proof:** $(\mathcal{L}_1 \mathcal{L}_2)(P) = A_1^\top [A_2^\top P A_2 - P] A_1 - [A_2^\top P A_2 - P]$

$$= A_1^\top A_2^\top P A_2 A_1 - A_1^\top P A_1 - A_2^\top P A_2 + P$$

$$= A_2^\top A_1^\top P A_1 A_2 - A_2^\top P A_2 - A_1^\top P A_1 + P$$

$$= A_2^\top [A_1^\top P A_1 - P] A_2 - [A_1^\top P A_1 - P]$$

$$= (\mathcal{L}_2 \mathcal{L}_1)(P).$$

\[ \square \]

The next result states a necessary and sufficient condition for the existence of a common Lyapunov solution for $A_1$ and $A_2$, given in terms of the Lyapunov maps $\mathcal{L}_i$.

**Theorem 4.3:** Let the matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$ in (10) be Schur commuting matrices. Then the associated Lyapunov maps $\mathcal{L}_i$ are invertible. Moreover, a matrix $P > 0$ is a CLS for $A_1$ and $A_2$ if and only if there exists a matrix $S > 0$ such that

$$P = (\mathcal{L}_2^{-1} \mathcal{L}_1^{-1})(S). \quad (11)$$

**Proof:** The fact that the maps $\mathcal{L}_i$ are invertible follows from standard knowledge regarding stability of 1D systems. We prove the second part of the claim.

$(\Rightarrow)$ This part of the claim can be proved using the same argument of the proof of statement ii) of Theorem 1 of (13). By assumption we have that $\mathcal{L}_2(P) = \mathcal{L}_1^{-1}(S) < 0$ because of the linearity of maps $\mathcal{L}_i$, as $\mathcal{L}_1^{-1}(-S) > 0$. Thus, $P$ is a Lyapunov solution for $A_2$. In the same way we prove that $P$ is a Lyapunov solution for $A_1$. Thus $P$ is a CLS for $A_1$ and $A_2$.

$(\Leftarrow)$: Let $P$ be a CLS for $A_1$ and $A_2$, and define $Q_i = \mathcal{L}_i(P)$, $i = 1, 2$. Note that $Q_i < 0$, $i = 1, 2$. Define $S = \mathcal{L}_1(Q_2) = \mathcal{L}_1(\mathcal{L}_2(P)) = \mathcal{L}_2(\mathcal{L}_1(P)) = \mathcal{L}_2(Q_1)$. Then, $(\mathcal{L}_2^{-1} \mathcal{L}_1^{-1})(S) = \mathcal{L}_2^{-1}(Q_2) = P$. Note that $S > 0$.

The result of Theorem 4.3 characterizes the common Lyapunov solutions for a pair of Schur commuting matrices; it also constitutes a generalization of Theorem 1 of (13), since it shows that the condition (11) is not only sufficient, but also necessary for the existence of a common Lyapunov function. Moreover, Theorem 4.3 also suggests an algorithm to compute a CLS by inversion of the maps $\mathcal{L}_i$.

We now proceed to investigate further the properties of these maps; a similar approach has been adopted for one polynomial Lyapunov equation in (14). The assumption that the matrices $A_i$, $i = 1, 2$, have a basis of common eigenvectors is crucial in our approach; consequently, we concentrate on the case in which both $A_1$ and $A_2$ are diagonalizable. Note that this is the generic case.
**Lemma 4.4:** Let $A_1$ and $A_2$ be two $n \times n$ diagonalizable commuting matrices and $\mathcal{L}_i$ the Lyapunov maps associated with $A_1, A_2$. Then the set

$$V = \{(v, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} | \begin{bmatrix} I\lambda - A_1 \\ I\mu - A_2 \end{bmatrix} v = 0\}$$

has cardinality $n$; denote its $i$-th element with $(v_i, \lambda_i, \mu_i)$, $i = 1, \ldots, n$. Moreover, the set

$$\hat{V} = \{\hat{v}_{ij} = v_i v_j^\top + v_j v_i^\top | 1 \leq i \leq j \leq n\}$$

forms a basis of common eigenvectors of $\mathcal{L}_1$ and $\mathcal{L}_2$ of dimension $\frac{n(n+1)}{2}$, each associated with the eigenvalues

$$\lambda_i \lambda_j - 1$$

$$\mu_i \mu_j - 1,$$

for $1 \leq i \leq j \leq n$.

**Proof:** That the set $V$ has cardinality $n$ follows from the well-known fact that commuting matrices are diagonalizable if and only if they have a basis of common eigenvectors. The claim that the matrices $\hat{v}_{ij}$ are eigenvectors of $\mathcal{L}_i$ follows easily after verifying that $\mathcal{L}_1(\hat{v}_{ij}) = (\lambda_i \lambda_j - 1) \hat{v}_{ij}$ and $\mathcal{L}_2(\hat{v}_{ij}) = (\mu_i \mu_j - 1) \hat{v}_{ij}$; note that these formulas also show what the eigenvalues associated with each eigenvector are. In order to prove that these matrices form a basis for the set of $n \times n$ symmetric matrices, observe that any linear combination of the $\hat{v}_{ij}$ can be written down as

$$\begin{pmatrix} v_1^\top \\ v_2^\top \\ \vdots \\ v_n^\top \end{pmatrix} K \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

with $K$ a nonsingular symmetric matrix. The linear independence of the $\hat{v}_{ij}$ then follows from the linear independence of the vectors $v_i$, $i = 1, \ldots, n$. \[\square\]

The result of Lemma 4.4 shows that if the matrices $A_i$, $i = 1, 2$ are diagonalizable, a basis of common eigenvectors of $\mathcal{L}_i$, $i = 1, 2$, can be computed in a straightforward way from a basis of common eigenvectors of $A_1$ and $A_2$. Consequently, the inversion necessary to compute the matrix $P$ as in Theorem 4.3 is a straightforward matter. These considerations lead us to stating the following algorithm to compute Lyapunov functions for a given stable behavior $\mathcal{B}$, a Lyapunov function for $\mathcal{B}$.

**Algorithm**

**Input:** A stable, finite-dimensional behavior $\mathcal{B}$;

**Output:** $\Phi \in \mathbb{R}^{w \times w}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ inducing a Lyapunov function for $\mathcal{B}$.

**Step 1:** Compute a representation of $\mathcal{B}$ as in (2), together with a state map $X \in \mathbb{R}^{n \times w}[\xi_1, \xi_2]$ inducing the state variable $x$ for the representation.

**Step 2:** Using the matrices $A_1$ and $A_2$ from Step 1 construct $V$ and $\hat{V}$ as described in Lemma 4.4.
Step 3: Select $\alpha_{ij}$, $1 \leq i \leq j \leq n$ such that

$$S = \sum_{1 \leq i \leq j \leq n} \alpha_{ij} \hat{v}_{ij} > 0.$$ 

Step 4: Output

$$\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) := (X(\zeta_1, \zeta_2))^T \sum_{1 \leq i \leq j \leq n} \frac{\alpha_{ij}}{(\lambda_i \lambda_j - 1)(\mu_i \mu_j - 1)} \hat{v}_{ij} X(\eta_1, \eta_2).$$

Some remarks are in order.

Remark 1: Note that since $\{\hat{v}_{ij}\}_{1 \leq i \leq j \leq n}$ forms a basis for the space of $n \times n$ symmetric matrices, the construction of the matrix $S$ in the step 3 generates all negative definite matrices; consequently the algorithm produces all possible Lyapunov functions for a given stable 2D behavior. In particular, a selection of $S$ can be performed for example by taking $S = -\sum_{1 \leq i \leq j \leq n} v_i v_i^T$.

Remark 2: The increments $\nabla \ell_i; i = 1, 2$ of the Lyapunov function $Q_\Phi$ computed in Step 4 are easily seen to be equal respectively to

$$\Delta_1(\zeta_1, \zeta_2, \eta_1, \eta_2) = X(\zeta_1, \zeta_2)^T \left( \sum_{1 \leq i \leq j \leq n} \frac{1}{\lambda_i \lambda_j - 1} \alpha_{ij} \hat{v}_{ij} \right) X(\eta_1, \eta_2),$$

$$\Delta_2(\zeta_1, \zeta_2, \eta_1, \eta_2) = X(\zeta_1, \zeta_2)^T \left( \sum_{1 \leq i \leq j \leq n} \frac{1}{\mu_i \mu_j - 1} \alpha_{ij} \hat{v}_{ij} \right) X(\eta_1, \eta_2).$$

Remark 3: It follows from Theorem 9.1.1 of (6) that there exist non-diagonalizable matrices which do not have a basis of common generalized eigenvectors. The problem of finding efficient algorithms to compute a common Lyapunov solution in this non-generic case is a matter of further investigation.

Example 4.5 Consider $(A_1, A_2, X)$ as in the example 3.4. In step 3, take, for instance $S = I_3$, the $3 \times 3$ identity matrix. Step 4 produces

$$\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) = \begin{pmatrix}
9(\zeta_1 - \frac{1}{2})(\eta_1 - \frac{1}{2}) + \frac{225}{16}(\zeta_1 - \frac{1}{4})(\eta_1 - \frac{1}{4}) & 0 & \frac{64}{75} \\
0 & 0 & \frac{64}{75}
\end{pmatrix}. $$

5 Conclusions

In this paper we have illustrated a Lyapunov approach to the stability of finite-dimensional 2D systems. We have adopted as definition of stability the one given in Def. 3.1, namely the asymptotic stability along all lines in the first orthant. The main results are Theorem 3.5, which characterizes stability in terms of the existence of a Lyapunov function, defined as a quadratic functional of the system variables which is positive along all lines, and whose increments are negative along all lines; and the algorithm given in section 4 for the computation of Lyapunov functions for a stable 2D behavior $\mathcal{B}$. 
Acknowledgement

The second author gratefully acknowledges the financial support of The Royal Society and of the Engineering and Physical Sciences Research Council for financially supporting the visit to the Faculty of Electrical Engineering and Computer Science of the University of Oporto (Portugal) during which the results presented in this paper were obtained. The research of the other two authors has been financed by the Fundação para a Ciência e a Tecnologia, through the R&D Unit Centro de Investigação e Desenvolvimento em Matemática e Aplicações.

Referências

REFERÊNCIAS

