Quantum Measurement and Entropy Production

Paolo Grigolini1,2,3, Marco G. Pala1, Luigi Palatella1
1 Dipartimento di Fisica dell’Università di Pisa and INFM, Piazza Torricelli 2, 56127 Pisa, Italy
2 Istituto di Biofisica CNR, Area della Ricerca di Pisa, Via Alfieri 1, San Cataldo 56010 Ghezzano-Pisa, Italy
3 Center for Nonlinear Science, University of North Texas, P.O. Box 5368, Denton, Texas 76201

We study the time evolution of a quantum system without classical counterpart, undergoing a process of entropy increase due to the environment influence. We show that if the environment-induced decoherence is interpreted in terms of wave-function collapses, a symbolic sequence can be generated. We prove that the Kolmogorov-Sinai entropy of this sequence coincides with rate of von Neumann entropy increase.

According to Landau and Lifshitz [1], the foundation of the second law might lie in the processes of quantum measurement. This point of view, still under the form of a plausible conjecture, has been recently reformulated by Srivastava, Vitiello and Widom [3]. The authors of this interesting paper note that the von Neumann entropy, which is kept constant by the unitary transformation of quantum mechanics, increases as a consequence of the von Neumann projective measurement, and consequently as an effect of the occurrence of a quantum measurement. Their approach makes manifest the concept of heat and work during the measurement process. In summary, they prove that the von Neumann entropy expressed in terms of the von Neumann projected density matrix, as a consequence of quantum measurement, becomes indistinguishable from the physical entropy.

The present letter focuses on a different aspect of the same fundamental issue. This has to do with the relation between physical entropy and Kolmogorov-Sinai (KS) entropy [4]. This latter form of entropy is actually a property of a classical trajectory [4]. The classical phase space is divided into cells, the cells are labelled with symbols, and a trajectory running in this phase space creates a symbolic sequence. Finally, using the KS entropy prescription this trajectory is assigned the value $h_{KS}$ that can be interpreted as a rate of entropy increase. In the recent past many papers have been devoted to the discussion of the possible connection between the KS entropy and the physical entropies [5–7]. Of special relevance for the discussion of this paper is the work of Latora and Baranger [17]. These authors study the time evolution of the physical entropy moving from an out of equilibrium initial condition and prove that three distinct time regimes exist: An initial regime of transition, an intermediate regime of linear increase and a saturation regime. The rate of entropy increase in the intermediate regime is proved by them to coincide with the KS entropy. Results of the same kind have been derived by Pattanyak [12] along lines that essentially adopt a perspective originally advocated by Zurek and Paz [8]. In a sense, the perspective of Zurek and Paz is the same as the perspective of Refs. [6–8] if we interpret the influence of the environment as a nature-made form of measurement [14,15]. However, the main limitation of all these papers is given by the fact that the adopted approach works only when the system studied has a classical counterpart. Thus, the correspondence between the original conjecture of Landau and Lifshiz [1] and physical entropy [8], on one hand, and physical entropy and KS entropy [14,15] on the other hand, is only partially established.

The main purpose of this letter is to fill this gap and to show that the KS entropy shows up even throughout the environment-done measurement process of systems which do not have a classical analog. Let us consider the quantum mechanical Hamiltonian:

$$ H = (|1⟩⟨1| + |2⟩⟨2|)V. $$

We imagine this system as consisting of two distinct sites, for instance two wells of a lattice coupled to one another by a tunneling process with rate $\hbar/2V$. The solution corresponding to the initial condition $|\psi(0)⟩ = |1⟩$ reads

$$ |\psi(t)⟩ = \cos\left(\frac{Vt}{\hbar}\right)|1⟩ - i \sin\left(\frac{Vt}{\hbar}\right)|2⟩. $$

In the presence of the interaction with an environment causing decoherence, with the rate $2\sigma/\hbar$, it is convenient to use the statistical density matrix. This physical condition is thus expressed by

$$ \dot{ρ}_{11}(t) = -i \frac{V}{\hbar}(ρ_{21} - ρ_{12}) $$
$$ \dot{ρ}_{12}(t) = -i \frac{V}{\hbar}(ρ_{22} - ρ_{11}) - 2\frac{\sigma}{\hbar}ρ_{12}(t) $$
$$ \dot{ρ}_{22}(t) = -i \frac{V}{\hbar}(ρ_{12} - ρ_{21}) $$
$$ \dot{ρ}_{21}(t) = -i \frac{V}{\hbar}(ρ_{11} - ρ_{22}) - 2\frac{\sigma}{\hbar}ρ_{21}(t). $$

In this letter we limit our attention to the case:

$$ V << \sigma. $$

Note that the condition of Eq.(4) makes it possible for us to define the time region where the intermediate regime
of KS type is expected to be located. The time \( \tau \equiv \hbar/\sigma \) is evidently the time at which entropy starts increasing. The time evolution of \( \delta(t) \equiv \rho_{11}(t) - \rho_{22}(t) \) is easily proven \cite{21} to be equivalent to that of a damped harmonic oscillator with frequency \( 2V/\hbar \) and damping \( 2\sigma/\hbar \). The condition of Eq. (3) sets the overdamping condition, and consequently establishes the time scale \( T_{\text{relax}} \equiv (\hbar\sigma)/2V^2 \) as the time necessary for the system to reach equilibrium. Consequently, the KS regime is expected to be located in the time region: \( \tau < t < T_{\text{relax}} \). Note that, due to the condition of Eq. (3), \( T_{\text{relax}} \equiv 1/(2(\hbar\sigma)/V^2) = 1/2(\hbar/\sigma)(\sigma/V)^2 \gg \hbar/\sigma \).

The exact system dynamics are described by Eqs. (3), which means 4 coupled differential equations. Using either a perturbation approach or a projection method \cite{22}, both resting on the basic condition of Eq. (3), we reduce the number of coupled differential equations from 4 to 2. The solution is obtained by diagonalizing a two-dimensional matrix, whose eigenvalues are:

\[
\Lambda_{\pm} = \frac{1}{2} \pm \frac{1}{2} \exp(-\frac{2V^2t}{\sigma\hbar}). \tag{5}
\]

The time evolution of the von Neumann entropy

\[
S(t) = -Tr\{\rho(t) \log \rho(t)\}, \tag{6}
\]
corresponding to the earlier approximations reads

\[
S(t) = -\left[ \frac{1}{2} + \frac{1}{2} \exp(-\frac{2V^2t}{\sigma\hbar}) \right] \log \left[ \frac{1}{2} + \frac{1}{2} \exp(-\frac{2V^2t}{\sigma\hbar}) \right] \tag{7}
\]

\[
-\left[ \frac{1}{2} - \frac{1}{2} \exp(-\frac{2V^2t}{\sigma\hbar}) \right] \log \left[ \frac{1}{2} - \frac{1}{2} \exp(-\frac{2V^2t}{\sigma\hbar}) \right].
\]

Note that this time evolution has the same initial condition as that of Eq. (3), namely, the system at the initial time is in the state \( |1\rangle \).

The approximated time evolution of \( S(t) \) given by Eq. (3) corresponds to setting equal to 0 the finite size of the transition region. To establish the slope of the regime of increasing linearly in time, we have to consider a time scale of the order of \( \tau \), namely \( V^2t/\langle \sigma \hbar \rangle = (V/\sigma) \ll 1 \) thereby making it possible for us to replace Eq. (3) with the following approximated expression

\[
S(t) \approx \frac{V^2t}{\sigma\hbar} + \frac{V^2t}{\sigma\hbar} \log(\frac{\sigma^2}{V^2}). \tag{8}
\]

Consequently, we obtain that a good candidate for the quantum KS entropy is given by

\[
\dot{S}(t) = \frac{V^2}{\sigma\hbar} [1 + \log(\frac{\sigma^2}{V^2})]. \tag{9}
\]

To prove that this is really a KS entropy we decide to adopt the following perspective. Following the authors of Refs. \cite{13,21} we interpret the decoherence process as a true wave-function collapse occurring with the rate \( \sigma/\hbar \). We can adopt the following prescription. We imagine that a measurement process takes place at regular intervals of time: \( \tau, 2\tau, \ldots, n\tau \ldots \). where, in accordance with the earlier definition, \( \tau = \hbar/\sigma \). At the moment of the first wave-function collapse, the system, which in the absence of the measurement process would follow Eq. (3), jumps to the state \( |2\rangle \) with probability \( x = \sin(\sqrt{V\tau/\hbar})^2 \) and to the state \( |1\rangle \) with probability \( 1 - x = \cos(\sqrt{V\tau/\hbar})^2 \). Due to the fact that we adopt the condition of Eq. (3) we set

\[
x = (\frac{V}{\sigma})^2. \tag{10}
\]

Since \( x \ll 1 \), at the first step the system will jump with a very large probability into the state \( |1\rangle \). At the next step the probability of jumping into the same state is the same as at the first step, and so on. Consequently, the system will collapse for a large number of times into the state \( |1\rangle \). However, since the probability of collapsing into the state \( |2\rangle \) is small but not vanishing, sooner or later the system will collapse into that state, and, afterward it will keep collapsing into that state for a large number of times. In literature this effect is called Zeno effect \cite{23}. If we associate the state \( |1\rangle \) with 1 and the state \( |2\rangle \) with 0, this will have the effect of creating a symbolic sequence of 1’s and 0’s.

Let us calculate the KS entropy of this symbolic sequence. According to the literature prescriptions to evaluate the KS entropy \cite{6,7}, we have to proceed as follows: We fix a window of size \( N \), namely, containing \( N \) symbols of this sequence, and we move the window along the sequence, recording the combinations of symbols \( \omega_0, \omega_1, \ldots, \omega_{N-1} \), with \( \omega_i = 0, 1 \), corresponding to any window position. Since the sequence is assumed to be infinite, we can evaluate the frequency of occurrence of any given string and we can identify this frequency with the probability \( p(\omega_0, \omega_1, \ldots, \omega_{N-1}) \). The KS entropy is defined as

\[
h_{KS} \equiv \lim_{N \to \infty} \frac{H(N)}{N}, \tag{11}
\]

where \( H(N) \) is the Shannon entropy:

\[
H(N) \equiv - \sum_{\omega_0, \ldots, \omega_{N-1}} p(\omega_0, \ldots, \omega_{N-1}) \log p(\omega_0, \ldots, \omega_{N-1}). \tag{12}
\]

To make our calculation easier, let us imagine a case where the probability of getting \( |1\rangle \), \( 1-x \), and the probability of getting \( |2\rangle \), are fixed. This produces a sequence with long strings of 1’s interspersed with very few 0’s. This is the former of the two kinds of sequences that we want to discuss here. The Zeno effect results in a different kind of sequence, the latter of the two here under discussion. In fact, if a collapse into \( |2\rangle \) occurs, it is easily seen that since that moment the quantity \( x \) becomes
the probability for the wave function to collapse into $|1\rangle$ and $1 - x$ becomes the probability for the wave function to collapse into $|2\rangle$. In other words, the latter sequence, corresponding to the realization of the Zeno effect, is obtained from the former by changing the 1's of the second, fourth, sixth string, and so on, into 0's, and by ignoring the 0's at the border between a given long string and the next long string.

We prove that the KS entropy of the former sequence is the same as the KS entropy of the latter. The evaluation of the KS entropy of the former sequence is done noticing that at the first step there is a probability $x$ of getting 0 and a probability $(1 - x)$ of getting 1. At the second step there are four possibilities, the string 00, with probability $x^2$, the string 11, with probability $(1 - x)^2$, and the strings 01 and 10, both with the same probability $x(1 - x)$. In general at the $N$th step there will be $2^N$ different strings, with probability

$$p_k^{(N)} = x^k(1 - x)^{N-k} \quad k = 0, \ldots, N.$$  

As a consequence, we can write:

$$H(N) = - \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} x^k(1 - x)^{N-k} \log x^k(1 - x)^{N-k}.$$  

After some algebra, Eq. (14) is proven to be proportional to $N$ through such a factor that the adoption of Eq. (11) yields

$$h_{KS} = x \log \left( \frac{1}{x} \right) + (1 - x) \log \left( \frac{1}{1 - x} \right).$$  

It is interesting to notice that this result coincides with that of Dorfman, Ernst and Jacobs [24] who studied the asymmetric Bernoulli map given by

$$y_{n+1} = y_n/(1 - x) \quad \text{for} \quad 0 \leq y_n < x,$$

$$y_{n+1} = (y_n - x)/x \quad \text{for} \quad x \leq y_n < 1.$$  

These authors evaluated the Lyapunov coefficient of this map and found it to be equal to the KS entropy of Eq. (15). In fact, we know from the the Pesin theorem [24] that the KS entropy of a chaotic map is its Lyapunov coefficient.

To prove that the theoretical prediction of Eq. (15) is really equivalent to the KS entropy of a Zeno effect, we create a symbolic sequence by randomly drawing a number of the interval $[0,1]$. The drawing is done at the integer times: $n=1,2,\ldots$. If this number is smaller than the number $x << 1$, we select 0, if it is larger we select 1. The probability of selecting 1 is $1-x$, and consequently it is much larger than the probability of selecting 0. As small as the probability of 0 is, if we get 0, we reverse the procedure, and we associate the numbers larger than $x$ with 0, and those smaller than $x$ with 1. Proceeding along these lines we find the sequence illustrated in Fig. 1. The KS entropy of this sequence is then evaluated numerically using the prescription of Eq. (12). For windows of size $N < 20$ this can be easily done. The result is illustrated in Fig. 2, for $x = 0.05$. We see that the agreement between the theoretical prediction of Eq. (15) and the numerical result is excellent. We can thus conclude that the asymmetric map of Dorfman, Ernst and Jacobs [24] is a dynamic system statistically equivalent to the Zeno effect [23].

Note that to make complete the connection with the quantum measurement process, we have to move from the unit time picture adopted to derive Eq. (15) to a picture expressed in terms of physical time. To do that we have to notice that in the physical process under study the elementary time step is, as we have seen, $\tau = h/\sigma$. We replace $x = (V/\sigma)^2$ into Eq. (15), make the approximation $\log(1-x) \approx -x$ and multiply the result by $\sigma/h$. The result is:

$$h_{KS} = \frac{V^2}{\sigma h} \left[ 1 + \log \left( \frac{\sigma^2}{V^2} \right) \right],$$  

which coincides with Eq. (6), which is the rate of the physical entropy in the time regime that the earlier theoretical arguments have identified with the quantum KS regime.

In addition to the benefit, of merely conceptual interest, of extending the correspondence between the physical entropy and the KS entropy also to cases with no classical counterpart, the results of this letter, as simple as they are, makes it possible to derive the results about fluorescence spectra in a single-atom double-resonance experiment [26] using methods borrowed from the literature on intermittent processes [27]. In fact, when the condition $x << 1$ applies, the left hand part of the asymmetric map of Eq. (16) becomes equivalent to the continuous time equation

$$\dot{y} = (x/(1-x))y(t)$$  

ranging from $y = 0$ to $y = 1$. Note that Eq. (13) applies to the whole interval [0,1] but the point $y = 1$. Thus the point $y = 1$ and the remainder part of the interval [0,1] play the same role as the chaotic region and the laminar region of the intermittent map of Ref. [27], respectively. As done in Ref. [27], we assume that the injection of the trajectory back into the laminar region is uniform. Thus, using Eq. (13) and adopting the same approach as that of Ref. [27], we evaluate the distribution of sojourn times, $\psi(t)$. This is:

$$\psi(t) = \exp(-x/(1-x)t).$$  

On the other hand, we know [28] that the second-order time derivative of the correlation function of the telegraphic noise $C(t)$ is proportional to the waiting time distribution $\psi(t)$. From Eq. (19) we immediately derive, in a full agreement with the theory of Ref. [27], that also this function is exponential [29].

This paper rests on a perspective which might depart from that of the KS entropy. Here we are imagining that
a genuine random process is acting at regular intervals of time, $n$. The "time" $n$ is a sort of internal time, and the physical entropy is expected to increase linearly with it. The observation time $T$ is easily related to the internal time through $n = T/\langle t \rangle$, where $\langle t \rangle$ denotes the mean time of sojourn in the laminar region. This perspective might lead to a departure from the KS entropy in the cases where $\psi(t)$ is the inverse power law:

$$\lim_{t \to \infty} \psi(t) = \frac{\text{const}}{t^\mu}, \text{ with } \mu > 1. \quad (20)$$

We know that this inverse power law asymptotic behavior can be obtained with a slow modulation of the rate $x$ [30]. In the case where $2 < \mu < 3$ one would expect that an ordinary KS entropy emerges [31]. However, the adoption of the KS perspective seems to imply an average of the fluctuating probability $x/(1-x)$. This paper suggests that the rate of increase of the physical entropy is proportional to $T/\langle t \rangle$. This implies that we have to make the average of $(1-x)/x$ rather than that of $x/(1-x)$. We think that all this might raise interesting new questions on the meaning of the KS entropy.

[29] Note that the authors of Ref. [26] considered a case where the waiting time distribution in one state can be different from that of the other state. Thus, we have to compare our results to the theoretical predictions of Ref. [26] in the symmetric case.
FIG. 1. A sample of the sequence generated using the method of random drawing of the numbers of the interval [0, 1]. The sequence refers to the case \( x = 0.05 \). Time is expressed in terms of the ”integer time” \( n \).

FIG. 2. The entropy \( H(N) \) as a function of \( N \). The dots denote the results of the numerical analysis made using a moving window of size \( N \) to evaluate the frequency \( p(\omega_0, \omega_1, ..., \omega_{N-1}) \). This probability is then adopted to define \( H(N) \) using the prescription of Eq. (12). The dashed line denotes a straight line whose slope is given by the theoretical prediction of Eq. (15). The abscissa denotes the window size \( N \).