Note

Sparse graph certificates for mixed connectivity

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Abstract

We give a short proof for the Mixed Connectivity Certificate Theorem of Even, Itkis and Rajsbaum and provide an upper bound on the edge number of a certificate of local $T$-mixed connectivity up to $k$.

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1. Introduction

1.1. Basic concepts

Let $k$ be a positive integer, and $G = (V, E)$ an undirected graph with $n$ nodes and $m$ edges. For two distinct nodes $x$ and $y$ of $G$ let $\lambda(x, y; G)$ (respectively $\kappa(x, y; G)$) denote the local edge connectivity (local node connectivity) of $x$ and $y$ in $G$, that is, the maximum number of edge-disjoint (internally node-disjoint) paths between $x$ and $y$ in $G$.

A sparse certificate of local node connectivity up to $k$ (local edge connectivity up to $k$) for $G$ is a subgraph $G' = (V, E')$ of $G$, $(E' \subseteq E)$, such that $G'$ has $O(kn)$ edges and $\kappa(x, y; G') \geq \min\{k, \kappa(x, y; G)\}$, $\lambda(x, y; G') \geq \min\{k, \lambda(x, y; G)\}$ for every two nodes $x$ and $y$.

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These notions have been unified in [4] as follows. For a subset $T$ of nodes, $G$ is called $T$-simple if it does not contain loops and all sets of parallel edges connect nodes in $V - T$; thus nodes from $T$ cannot be incident to parallel edges. All graphs in this note are assumed to be $T$-simple, and so we omit this condition from the statement of each result.

We say that a family of paths connecting two nodes $x$ and $y$ is $T$-independent if they are edge-disjoint and every element of $T$ is contained in at most one path as an inner node.

Let $\lambda_T(x, y; G)$ denote the maximum number of $T$-independent paths connecting $x$ and $y$ in $G$. Two nodes $x$ and $y$ are said to be $T$-mixed $k$-connected if $\lambda_T(x, y; G) \geq k$.

Observe that $\lambda_T = \lambda$ when $T = \emptyset$ and $\lambda_T = \kappa$, when $T = V$. The global $T$-mixed connectivity of graph $G$ is $\lambda_T(G) := \min_{x,y \in V} \lambda_T(x, y; G)$. $G$ is $T$-mixed $k$-connected if $\lambda_T(G) \geq k$.

A sparse certificate of local $T$-mixed connectivity up to $k$ for $G$ is a subgraph $G' = (V, E')$, $E' \subseteq E$, such that $G'$ has $O(kn)$ edges and for any two nodes $x$ and $y$, $\lambda_T(x, y; G') \geq \min[k, \lambda_T(x, y; G)]$. Similarly, $G'$ is a sparse certificate of global $T$-mixed connectivity up to $k$ for $G$ if it has $O(kn)$ edges and $\lambda_T(G') \geq \min[k, \lambda_T(G)]$. Note that certificates of local connectivity up to $k$ are also certificates of global connectivity up to $k$; the converse is generally not true.

Let $\{Z, A, B\}$ be a partition of $V$ such that $\emptyset \subseteq Z \subseteq T$, $A \neq \emptyset$ and $B \neq \emptyset$, $\Delta_E(A, B)$ denotes the set of all edges from $E$ between $A$ and $B$ and $d_E(A, B) := |\Delta_E(A, B)|$. We say that a pair $C := (Z, \Delta_E(A, B))$ is a $T$-mixed cut. $C$ separates two nodes $x$ and $y$ if one of them belongs to $A$ and the other belongs to $B$. The cardinality of $C$ is defined to be $|Z| + d_E(A, B)$. Menger's theorem implies that $\lambda_T(x, y; G)$ is equal to the minimum cardinality of a $T$-mixed cut separating $x$ and $y$.

1.2. Previous work

Nishizeki and Poljak [9] showed that the union of $k$ forests $F_1, F_2, \ldots, F_k$ of the graph, where each forest $F_i$ is maximal in the remaining graph $G - (F_1 \cup \cdots \cup F_{i-1})$ is a local edge connectivity certificate up to $k$ of size $< k|V|$.

Nagamochi and Ibaraki [8] constructed sparse certificates of local edge connectivity up to $k$ in linear time in case of simple graphs.

We need some notations and a definition before formulating their results: Let $G = (V, E)$ be an undirected graph with no loops. For an ordering $v_1, \ldots, v_n$ of the nodes, $V_i$ denotes the set of the first $i$ elements. If for two subsets $X, Y$ of nodes $d(X, Y)$ denotes the number of edges connecting $X - Y$ and $Y - X$, and using $d(X) := d(X, V - X)$, we say that an ordering $v_1, \ldots, v_n$ of the nodes is legal if $d(v_i, V_{i-1}) \geq d(v_j, V_{i-1})$ for every pair $i, j$ $(2 \leq i < j \leq n)$. With the help of an appropriate data structure, a legal ordering may be constructed in $O(|E|)$ time.

Each legal ordering $v_1, \ldots, v_n$ of the nodes determines a forest $F$ in the following way (see [8]): for each node $v_j$, $j \geq 2$ of the legal ordering, we take the smallest index $1 \leq i < j$ such that $\{v_i, v_j\}$ is an edge of the graph and consider all these edges. (In case of parallel edges, we take an arbitrary edge. In this sense, the forest $F$ is not unique.) The union of these edges forms a maximal forest. Note that a legal ordering of $G$ remains legal for the graph obtained by deleting the edges of a forest determined by the legal ordering.

For a legal ordering $v_1, \ldots, v_n$ of the nodes and $i = 2, \ldots, k$, let $F_i$ be the forest determined by the legal ordering of the graph $G - (F_1 \cup \cdots \cup F_{i-1})$. If $G$ is simple,
then the union of these forests is a local node connectivity certificate up to \( k \) for \( G \) (see [3, 4]).

In [4], Frank et al. show that for all \( T \subseteq V \) the algorithm of [8] applied to \( T \)-simple graphs produces certificates of local \( T \)-mixed connectivity up to \( k \).

In [3], we can find the generalization of the main theorem of [4, 2], named the Mixed Connectivity Certificate Theorem.

We give a short proof for this theorem and give an upper bound on the edge number of a small \( T \)-mixed \( k \)-connected subgraph of a \( T \)-mixed \( k \)-connected \( T \)-simple graph and show the existence of a node with degree \( k \) in minimal \( T \)-mixed \( k \)-connected \( T \)-simple graphs.

2. Sparse undirected graph certificates

In [3], the authors define a \( T \)-greedy forest \( F \) in a \( T \)-simple graph as a maximal forest obtained by the following algorithm: initially \( F = \emptyset \). During the procedure, each node is visited at least once. Edges added to \( F \) are incident to the visited node. The first visited node can be arbitrary. Whenever a visit of a node terminates, the next node to be visited can be chosen to be any other node of \( V(F) \), or any node of a component of \( G \) which has no nodes in \( V(F) \) yet. During the first visit of a node \( v \in T \), for every neighbour \( x \) of \( v \) such that \( x \notin V(F) \), add the edge \( \{v, x\} \) to \( F \). When visiting \( v \notin T \) if \( \{v, x\} \) is an edge of the graph and if \( x \notin V(F) \), one is allowed to add \( \{v, x\} \) to \( F \). If an edge is added to \( F \), then its parallel edges, if there are any, may never be added to \( F \). Edges incident to \( v \notin T \) may be added during several visits of the node \( v \). The forests produced by such a procedure are called \( T \)-greedy. In [3], Even et al. prove that any forest determined by a legal ordering of the nodes is a \( T \)-greedy forest, but the converse does not hold.

**Theorem 1** (Mixed Connectivity Certificate Theorem of Even, Itkis and Rajsbaum [3]). Let \( G = (V, E) \) be a graph with \( n \) nodes. Let \( k \) be a positive integer and \( F_1 \) a \( T \)-greedy forest of \( G \). For \( i = 2, \ldots, k \), let \( F_i \) be a \( T \)-greedy forest of the graph \( G - (F_1 \cup \cdots \cup F_{i-1}) \), and denote \( E_k := F_1 \cup \cdots \cup F_k \), \( G_k = (V, E_k) \). Then \( \lambda_T(x, y; G_k) \geq \min(k, \lambda_T(x, y; G)) \) for all \( x, y \in V \) (i.e. \( F_1 \cup \cdots \cup F_k \) is a certificate of local \( T \)-mixed connectivity up to \( k \) for \( G \)) and this certificate has at most \( k(n - 1) \) edges.

**Lemma 2.** If \( x \) and \( y \) are two nodes of the same component of the forest \( F_k \), then \( \lambda_T(x, y; G_k) \geq k \).

**Proof.** By induction on \( k \), the claim is trivial for \( k = 1 \). Let \( k \geq 2 \) and \( (Z, \Delta_E(A, B)) \) a \( T \)-mixed cut such that \( x \in A \), \( y \in B \). Since each \( F_i \) is maximal, we must have \( x \) and \( y \) in the same component of \( F_i \), \( i \leq k \). Consider \( G' := G - F_1 \) and \( J := F_2 \cup \cdots \cup F_k \). If \( F_1 \) contains an edge from \( \Delta_E(A, B) \), then we use the inductive hypothesis for \( G' \) and forests \( F_2, \ldots, F_k \).

\[
|Z| + d_{E_k}(A, B) \geq |Z| + d_J(A, B) + 1 \geq (k - 1) + 1 = k,
\]

so \( \lambda_T(x, y, G_k) \geq k \). Assume now that \( F_1 \) does not contain any edge from \( \Delta_E(A, B) \). Let \( H_1 \) be the component of \( x \) and \( y \) in \( F_1 \). We can assume that the first visited node \( r \) of \( F_1 \)
does not belong to $B$. Let $z_1$ be the first visited node of $Z \cap V(H_1)$ in the construction of $F_1$. Since $z_1$ belongs to $T$, on its first visit all edges which connect it to new vertices join $F_1$. Thus, in $J$, there are no edges between $z_1$ and $B$, i.e. $A' := A + z_1$, where $A' := A + z_1$. If $Z' := Z - z_1$ the inductive hypothesis for $G'$ and $F_2, \ldots, F_k$ gives us

$$|Z| + d_{E_k}(A, B) = 1 + |Z'| + d_{J}(A, B) = 1 + |Z'| + d_{J}(A', B) \geq 1 + (k - 1) = k.$$  

**Lemma 3.** If $F_1 \cup \cdots \cup F_k$ does not contain each edge of a $T$-mixed cut $(Z, A_E(A, B))$, then the cardinality of the $T$-mixed cut $(Z, A_{E_k}(A, B))$ is at least $k$.

**Proof.** Consider $e = \{x, y\} \in A_E(A, B) - (F_1 \cup \cdots \cup F_k)$. $F_k$ is maximal, and so $x$ and $y$ belong to the same component of $F_k$. Hence Lemma 2 can be used here. □

**Proof of Theorem 1.** It is enough to prove the theorem for $k \leq \lambda_T(x, y; G)$. If indirectly $\lambda_T(x, y; G_k) < \min(k, \lambda_T(x, y; G))$ (i.e. $\lambda_T(x, y; G_k) < k$), then there is a $T$-mixed cut $(Z, A_{E_k}(A, B))$ which separates $x$ and $y$ in $G_k$ with cardinality at most $k - 1$, but no such $T$-mixed cut may exist in $G$, and so the cardinality of the $T$-mixed cut $(Z, A_{E_k}(A, B))$ separating $x$ and $y$ in $G$ is at least $k$. This means that at least one edge $e \in A_{E_k}(A, B)$ has been left out, i.e. $F_1 \cup \cdots \cup F_k$ does not contain each edge of $A_{E_k}(A, B)$; hence Lemma 3 can be used here and we obtain $|Z| + d_{E_k}(A, B) \geq k$, a contradiction. □

Note, that if $x$ and $y$ are two nodes of the same component of $F_k$, then the $k$ paths between $x$ and $y$ determined by the forests $F_1$, $F_2$, \ldots, $F_k$ are not necessarily $T$-independent. For example, let $T = V$, $k = 2$ and let us consider the next graph with 9 nodes and 13 edges with $x := v_3$ and $y := v_8$ (see Fig. 1, where solid lines are used for $F_1$ and dashed lines for $F_2$). As the authors observed in [3], for $T = V$ in Theorem 1, we obtain the theorem of Cheriyan et al. [2] for sparse certificates for $k$-node connectivity.

Using Theorem 1, we get the next sparse graph certificate for $T$-mixed connectivity up to $k$.

**Theorem 4.** Each $T$-mixed $k$-connected graph $G = (V, E)$ with $n$ nodes contains a $T$-mixed $k$-connected subgraph $G_k = (V, E_k)$, such that $|E_k| \leq |T|(|T| + 1)/2 + (n - |T| - 1)k$, if $|T| < k$ and $|E_k| \leq k(k - 1)/2 + (n - k)k$, if $|T| \geq k$. 

![Fig. 1. Example with 9 nodes and 13 edges.](image-url)
Proof. $G_k = (V, F_1 \cup \cdots \cup F_k)$ is a certificate of local $T$-mixed connectivity up to $k$ by Theorem 1. Let $s_i$ denote the first visited node of the forest $F_i$ which is not isolated in $F_i$. Take $s_i$ from $T$, if it is possible. If $|T| < k$

$$|E| \leq (n - 1) + (n - 2) + \cdots + (n - |T|) = n|T| - \frac{|T|(|T| + 1)}{2}.$$  

We obtain

$$|E_k| \leq n|T| - \frac{|T|(|T| + 1)}{2} + (k - |T|)(n - |T| - 1),$$

because for $|T| + 1 \leq j \leq k$, each $F_j$ has at most $n - |T| - 1$ edges, and thus

$$|E_k| \leq \frac{|T|(|T| + 1)}{2} + (n - |T| - 1)k.$$  

If $|T| \geq k$

$$|E_k| \leq (n - 1) + (n - 2) + \cdots + (n - k) = \frac{k(k - 1)}{2} + (n - k)k. \quad \square$$  

The special cases of Theorem 4 when $T = \emptyset$ or $V$ were proved by Mader [1,7].

Corollary 5. If $G$ is a $T$-mixed $k$-connected graph that is minimal with respect to edge deletion and $|V| \geq 2$, then it has a node with degree $k$.

Proof. Assume on the contrary that $d(v_n) > k$, where $v_n$ is the last node of a legal ordering, then the edge $e = v_i v_n$ between the greatest indexed node $v_i$ ($i < n$) of legal ordering and $v_n$ does not belong to $F_k$, if the forests are constructed as in [8].

But $(V, F_1 \cup \cdots \cup F_k)$ is $T$-mixed $k$-connected if $G$ is $T$-mixed $k$-connected. This means that $G - e$ is also $T$-mixed $k$-connected, a contradiction. \quad \square

The special cases of Corollary 5 when $T = \emptyset$ or $T = V$ were proved by Lick [6] and Halin [5], respectively.

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References