On the Number of Interference Alignment Solutions for the K-User MIMO Channel with Constant Coefficients

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Abstract

In this paper, we study the number of different interference alignment (IA) solutions that exists for the $K$-user multiple-input multiple-output (MIMO) interference channels with constant coefficients, when the alignment is performed via beamforming and without symbol extensions. When counting the number of IA solutions for a given scenario, the most interesting case arises when the number of equations in the polynomial system matches the number of variables and the system is feasible. In this situation, the number of IA solutions is finite and constant for any channel realization out of a zero-measure set and, as we prove in the paper, is given by an integral formula that can be numerically approximated using Monte Carlo integration methods. More precisely, the number of alignment solutions is the scaled average over a subset of the solution variety (formed by all triplets of channels, precoders and decoders satisfying the IA polynomial equations) of the determinant of a certain Hermitian matrix related to the geometry of the problem. Interestingly, while the value of this determinant at an arbitrary point can be used to check the feasibility of the IA problem, the average of the determinant (properly scaled) gives us the number of solutions. Our results can be applied to arbitrary interference MIMO networks, with any number of users, antennas and streams per user.

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Index Terms

Interference Alignment, MIMO Interference Channel, Polynomial Equations, Algebraic Geometry

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I. INTRODUCTION

Interference alignment (IA) has received a lot of attention in recent years as a key technique to achieve the maximum degrees of freedom (DoF) of wireless networks in the presence of interference. Originally proposed in [1], [2], the basic idea of IA consists of designing the transmitted signals in such a way that the interference at each receiver falls within a lower-dimensional subspace, therefore leaving a subspace free of interference for the desired signal [3]. This idea has been applied in different forms (e.g., ergodic interference alignment [4], signal space alignment [1], or signal scale alignment [5], [6]), and adapted to various wireless networks such as interference networks [1], X channels [2], downlink broadcast channels in cellular communications [7] and, more recently, to two-hop relay-aided networks in the form of interference neutralization [8].

In this paper we consider the linear IA problem (i.e., signal space alignment by means of linear beamforming) for the $K$-user multiple-input multiple-output (MIMO) interference channel with constant channel coefficients. Moreover, the MIMO channels are considered to be generic, without any particular structure, which happens for instance when the channel matrices have independent entries drawn from a continuous distribution. This setup has been the preferred option for recent experimental studies on IA [9], [10], [11].

The feasibility of linear IA for MIMO interference networks, which amounts to study the solvability of a set of polynomial equations, has been an active research topic during the last years [12], [13], [14], [15], [16]. Combining algebraic geometry tools with differential topology ones, it has been recently proved in [17], that an IA problem with any number of users, antennas and streams per user, is feasible iff the linear mapping given by the projection from the tangent space of $\mathcal{V}$ (the solution variety, whose elements are the triplets formed by the channels, decoders and precoders satisfying the IA equations) to the tangent space of $\mathcal{H}$ (the complex space of MIMO interference channels) at some element of $\mathcal{V}$ is surjective. Note that this implies in particular that the dimension of $\mathcal{V}$ must be larger than or equal to the dimension of $\mathcal{H}$. Exploiting this result, a general IA feasibility test with polynomial complexity has
also been proposed in [17], [18]. This test reduces to check whether the determinant of a given square Hermitian matrix is zero (meaning infeasible almost surely) or not (feasible).

In this paper we study the problem of how many different alignment solutions exist for a feasible IA problem. While the number of solutions is known for some particular cases, a general result is not available yet. For instance, it can be trivially shown that feasible systems for which the algebraic dimension of the solution variety is larger than that of the input space have an infinite number of alignment solutions. In plain words, these are MIMO interference networks for which the number of variables is larger than the number of equations of the polynomial system. These scenarios typically represent cases where not all available DoF are achieved (for instance, we might have more antennas than strictly needed to achieve a certain DoF tuple) and therefore they do not receive further consideration in this paper. Much more interesting and challenging is the case where the dimensions of \( \mathcal{V} \) and \( \mathcal{H} \) are exactly the same (identical number of variables and equations) \textit{and} the problem is feasible, because in this situation the number of IA solutions is finite and constant out of a zero measure set of \( \mathcal{H} \) as proved in [17]. Following the nomenclature recently introduced in [19] we refer to these systems as \textit{tightly feasible}, stressing the fact that removing a single antenna from the network turns the IA problem infeasible.

For tightly feasible single-beam (i.e., when all users wish to transmit \( d = 1 \) stream of data) MIMO networks, and elaborating on classic results from algebraic geometry, it was shown in [12] that the number of alignment solutions coincides with the mixed volume of the Newton polytopes that support each equation of the polynomial system. Although this solves theoretically the problem for single-beam networks, in practice the computation of the mixed volume of a set of IA equations using the available software tools [20] can be very demanding, therefore only a few cases have been solved so far. For single-beam networks, some upper bounds on the number of solutions using Bezout’s Theorem have also been proposed in [12], [21]. For multi-beam scenarios, however, the genericity of the polynomials system of equations is lost and it is not possible to resort to mixed volume calculations to find the number of solutions. Furthermore, the existing bounds in multi-beam cases are very loose.

The main contribution of this paper is an integral formula for the number of IA solutions for arbitrary, tightly feasible, networks. More specifically, we prove that while the feasibility problem is solved by checking the determinant of a certain Hermitian matrix, the number of IA solutions is given by the integral of the same determinant over a subset of the solution variety scaled by an appropriate constant. Although the integral, in general, is hard to compute analytically, it can be easily estimated using Monte Carlo integration. To speed up the convergence of the Monte Carlo integration method, we specialize the general integral formula for square symmetric multi-beam cases (i.e., equal number of transmit and
receive antennas and equal number of streams per user).

In addition to having a theoretical interest, the results proved in this work might also have some practical implications. For instance, to find scaling laws for the number of solutions with respect to the number of users could have interest to analyze the asymptotic performance of linear IA, as discussed in [21]. Also, for moderate-size networks for which the total number of solutions is not very high, the results of this paper also open the possibility to provide a systematic way to compute all (or practically all) interference alignment solutions for a channel realization. This idea is also briefly explored in the paper.

The rest of the paper is organized as follows. In Section II the system model and the IA feasibility problem are briefly reviewed, paying special attention to the feasibility test in [17]. The main results of the paper are presented in Section III where two integral formulas for the number of IA solutions are provided, one valid for arbitrary networks and the other for symmetric multi-beam scenarios (all users with the same number of antennas at both sides of the link and transmitting the same number of streams). Although these integrals, in general, cannot be computed in closed form, they can easily be estimated using Monte Carlo integration. A short review on Riemannian manifolds and other mathematical results that will also be used during the derivations as well as the proofs of the main theorems in Section III are relegated to appendices.

II. SYSTEM MODEL AND BACKGROUND MATERIAL

In this section we describe the system model considered in the paper, introduce the notation, define the main algebraic sets used throughout the paper, and briefly review the feasibility conditions of linear IA problems for arbitrary wireless networks.

A. Linear IA

We consider the $K$-user MIMO interference channel with transmitter $k$ having $M_k \geq 1$ antennas and receiver $k$ having $N_k \geq 1$ antennas. Each user $k$ wishes to send $d_k \geq 0$ streams or messages. We adhere to the notation used in [12] and denote this (fully connected) asymmetric interference channel as $\prod_{k=1}^{K} (M_k \times N_k, d_k) = (M_1 \times N_1, d_1) \cdots (M_K \times N_K, d_K)$. The symmetric case in which all users transmit $d$ streams and are equipped with $M$ transmit and $N$ receive antennas is denoted as $(M \times N, d)^K$. In the square symmetric case all users have the same number of antennas $M = N$.

The MIMO channel from transmitter $l$ to receiver $k$ is denoted as $H_{kl}$ and assumed to be flat-fading and constant over time. Each $H_{kl}$ is an $N_k \times M_l$ complex matrix with independent entries drawn from
a continuous distribution. We denote the set of users as \( \Upsilon = \{1, \ldots, K\} \) and the set of interfering links as \( \Phi = \{(k,l) \in \Upsilon \times \Upsilon : k \neq l\} \). Also, \( \sharp(\Phi) \) denotes the cardinality of \( \Phi \), that is the number of elements in the finite set \( \Phi \). In this paper we focus on fully connected interference channels and, consequently, \( \sharp(\Phi) = K(K-1) \).

User \( j \) encodes its message using an \( M_j \times d_j \) precoding matrix \( V_j \) and the received signal is given by

\[
y_j = H_{jj} V_j x_j + \sum_{i \neq j} H_{ji} V_i x_i + n_j, \quad 1 \leq j \leq K, \tag{1}
\]

where \( x_j \) is the \( d_j \times 1 \) transmitted signal and \( n_j \) is the zero mean unit variance circularly symmetric additive white Gaussian noise vector. The first term in (1) is the desired signal, while the second term represents the interference space. The receiver \( j \) applies a linear decoder \( U_j \) of dimensions \( N_j \times d_j \), i.e.,

\[
U_j^T y_j = U_j^T H_{jj} V_j x_j + \sum_{i \neq j} U_j^T H_{ji} V_i x_i + U_j^T n_j, \quad 1 \leq j \leq K, \tag{2}
\]

where superscript \( T \) denotes transpose.

The interference alignment (IA) problem is to find the decoders and precoders, \( V_j \) and \( U_j \), in such a way that the interfering signals at each receiver fall into a reduced-dimensional subspace and the receivers can then extract the projection of the desired signal that lies in the interference-free subspace. To this end it is required that the polynomial equations

\[
U_k^T H_{kl} V_l = 0, \quad (k,l) \in \Phi, \tag{3}
\]

are satisfied, while the signal subspace for each user must be linearly independent of the interference subspace and must have dimension \( d_k \), that is

\[
\text{rank}(U_k^T H_{kk} V_k) = d_k, \quad \forall \ k \in \Upsilon. \tag{4}
\]

\( B. \) Feasibility of IA: a brief review

The IA feasibility problem amounts to study the relationship between \( d_j, M_j, N_j, K \) such that the linear alignment problem is feasible. If the problem is feasible, the tuple \( (d_1, \ldots, d_K) \) defines the degrees of freedom (DoF) of the system, that is the maximum number of independent data streams that can be transmitted without interference in the channel. The IA feasibility problem and the closely related problem of finding the maximum DoF of a given network have attracted a lot of research over the last years. For instance, the DoF for the 2-user and, under some conditions, for the symmetric \( K \)-user MIMO interference channel have been found in [22] and [23], respectively. In this work we make the following assumptions:

\[
1 \leq d_k \leq N_k, \quad \forall \ k \in \Upsilon, \quad 1 \leq d_l \leq M_l, \quad \forall \ l \in \Upsilon, \tag{5}
\]
and
\[ d_k + d_l < N_k + M_l, \quad \forall (k, l) \in \Phi, \] (6)
which are necessary conditions for feasibility derived, respectively, for point-to-point MIMO links and for the 2-user MIMO channel.

The IA feasibility problem has also been deeply investigated in [12]–[16]. In the following we make a short review of the main feasibility result presented in [17], [18], which forms the starting point of this work.

We start by describing the three main algebraic sets involved in the feasibility problem.

- **Input space** formed by the MIMO matrices, which is formally defined as
  \[
  \mathcal{H} = \prod_{(k,l) \in \Phi} \mathcal{M}_{N_k \times M_l}(\mathbb{C})
  \]
  (7)
  where \( \prod \) holds for Cartesian product, and \( \mathcal{M}_{N_k \times M_l}(\mathbb{C}) \) is the set of \( N_k \times M_l \) complex matrices.
  Note that in [17], [18], we let \( \mathcal{H} \) be the product of projective spaces instead of the product of affine spaces. The use of affine spaces is more convenient for the purposes of root counting.

- **Output space** of precoders and decoders (i.e., the set where the possible outputs exist)
  \[
  \mathcal{S} = \left( \prod_{k \in \Upsilon} \mathbb{G}_{d_k, N_k} \right) \times \left( \prod_{l \in \Upsilon} \mathbb{G}_{d_l, M_l} \right),
  \]
  (8)
  where \( \mathbb{G}_{a,b} \) is the Grassmannian formed by the linear subspaces of (complex) dimension \( a \) in \( \mathbb{C}^b \).

- **The solution variety**, which is given by
  \[
  \mathcal{V} = \{(H,U,V) \in \mathcal{H} \times \mathcal{S} : (3) \text{ holds}\}
  \]
  (9)
  where \( H \) is the collection of all matrices \( H_{kl} \) and, similarly, \( U \) and \( V \) denote the set of \( U_k \) and \( V_l \), respectively. The set \( \mathcal{V} \) is given by certain polynomial equations, linear in each of the \( H_{kl}, U_k, V_l \) and therefore is an algebraic subvariety of the product space \( \mathcal{H} \times \mathcal{S} \). Let us remind here that the IA equations given by (3) hold or do not hold independently of the particular chosen affine representatives of \( U, V \).

Once the main algebraic sets have been defined, it is interesting to consider the following diagram

\[
\begin{array}{c}
\mathcal{V} \\
\pi_1 \swarrow \swarrow \pi_2 \\
\mathcal{H} & \mathcal{S}
\end{array}
\]
(10)
where the sets and the main projections involved in the feasibility problem are depicted. Note that, given 

\(H \in \mathcal{H}\), the set \(\pi_{1}^{-1}(H)\) is a copy of the set of \(U, V\) such that (3) holds, that is the solution set of the linear interference alignment problem. On the other hand, given \((U, V) \in \mathcal{S}\), the set \(\pi_{2}^{-1}(U, V)\) is a copy of the set of \(H \in \mathcal{H}\) such that (3) holds.

The feasibility question can then be restated as, is \(\pi_{1}^{-1}(H) \neq \emptyset\) for a generic \(H\)? The question was solved in [17], basically stating that the problem is feasible if and only if two conditions are fulfilled:

1) The algebraic dimension of \(\mathcal{V}\) must be larger than or equal to the dimension of \(\mathcal{H}\), i.e.,

\[
s = \left(\sum_{k \in \Upsilon} d_{k} (N_{k} + M_{k} - 2d_{k})\right) - \left(\sum_{(k,l) \in \Phi} d_{k}d_{l}\right) \geq 0.
\] (11)

In other words this condition means that, for the problem of polynomial equations to have a solution, the number of variables must be larger than or equal to the number of equations. This condition was already established in [12], hereby classifying interference channels as proper \((s \geq 0)\) or improper \((s < 0)\). More recently, in [13] it was rigorously proved that improper systems are always infeasible.

2) For some element \((H, U, V) \in \mathcal{V}\), the linear mapping

\[
\theta : \left(\prod_{k \in \Upsilon} \mathcal{M}_{N_{k} \times d_{k}}(\mathbb{C}) \right) \times \left(\prod_{l \in \Upsilon} \mathcal{M}_{M_{l} \times d_{l}}(\mathbb{C})\right) \rightarrow \prod_{(k,l) \in \Phi} \mathcal{M}_{d_{k} \times d_{l}}(\mathbb{C})
\]

\[
\left\{ \{\hat{U}_{k}\}_{k \in \Upsilon}, \{\hat{V}_{l}\}_{l \in \Upsilon} \right\} \mapsto \left\{ \hat{U}_{k}^{T} H_{k,l} \hat{V}_{l} + U_{k}^{T} H_{k,l} \hat{V}_{l} \right\}_{(k,l) \in \Phi}
\] (12)

is surjective, i.e., it has maximal rank equal to \(\sum_{(k,l) \in \Phi} d_{k}d_{l}\). This condition amounts to saying that the projection from the tangent plane at an arbitrary point of the solution variety to the tangent plane of the input space must be surjective: that is, one tangent plane must cover the other. Moreover, in this case, the mapping (12) is surjective for almost every \((H, U, V) \in \mathcal{V}\).

III. THE NUMBER OF SOLUTIONS OF FEASIBLE IA PROBLEMS

A. Preliminaries

As it was shown in [17], [18], the surjectivity of the mapping \(\theta\) in (12) can easily checked by a polynomial-complexity test that can be applied to arbitrary \(K\)-user MIMO interference networks. The test basically consists of two main steps: i) to find an arbitrary point in the solution variety and ii) to check the rank of a matrix constructed from that point. To find an arbitrary point in the solution variety, in [17] we generated a set of random precoders and decoders, and then solved a linear underdetermined problem to get a set of channel matrices satisfying the IA equations (3): this was called inverse IA problem in [17].
In this paper we choose an even simpler (trivial) solution satisfying the IA equations. Specifically, we take structured matrices given by

$$H_{kl} = \begin{pmatrix} 0_{d_k \times d_i} & A_{kl} \\ B_{kl} & C_{kl} \end{pmatrix},$$

with precoders and decoders given by

$$V_l = \begin{pmatrix} I_{d_l} \\ 0_{(M_l-d_i) \times d_i} \end{pmatrix}, \quad U_k = \begin{pmatrix} I_{d_k} \\ 0_{(N_k-d_k) \times d_k} \end{pmatrix},$$

which trivially satisfy $U_k^T H_{kl} V_l = 0$ and therefore belong to the solution variety. We claim that essentially all the useful information about $V$ can be obtained from the subset of $V$ consisting on triples $(H_{kl}, U_k, V_l)$ of the form (13) and (14). The reason is that given any other element $(\tilde{H}_{kl}, \tilde{U}_k, \tilde{V}_l) \in V$, one can easily find sets of orthogonal matrices $P_k$ and $Q_l$ satisfying

$$U_k = P_k \tilde{U}_k, \quad V_l = Q_l \tilde{V}_l,$$

and

$$\tilde{U}_k^T \tilde{H}_{kl} \tilde{V}_l = U_k^T (P_k^*)^T \tilde{H}_{kl} Q_l^* V_l = 0,$$

where the superscript $*$ denotes Hermitian. That is, the transformed channels $H_{kl} = (P_k^*)^T \tilde{H}_{kl} Q_l^*$ have the form (13), and the transformed precoders $V_l$ and decoders $U_k$ have the form (14). Thus, we have just described an isometry which sends $(\tilde{H}_{kl}, \tilde{U}_k, \tilde{V}_l)$ to $(H_{kl}, U_k, V_l)$. The situation is thus similar to that of a torus: every point can be sent to some predefined vertical circle through a rotation, thus the torus is essentially understood by “moving” a circumference and keeping track of the visited places. The same way, $V$ can be thought of as moving the set of triples of the form (13) and (14), and keeping track of the visited places. Technically, $V$ is the orbit of the set of triples of the form (13) and (14) under the isometric action of a product of unitary groups.

In summary, the main idea is that, for the purpose of checking feasibility or counting solutions, we can replace the set of arbitrary complex matrices $\mathcal{H}$ by the set of structured matrices

$$\mathcal{H}_I = \prod_{k \neq l} \begin{pmatrix} 0_{d_k \times d_i} & A_{kl} \\ B_{kl} & C_{kl} \end{pmatrix} \equiv \pi_2^{-1} \left( \left( I_{d_k} \right)_{k \in \Upsilon}, \left( I_{d_l} \right)_{l \in \Upsilon} \right).$$

The mapping $\theta$ in (12) has a more simple form for triples of the form (13) and (14), and can be replaced by a new mapping $\Psi$ defined as

$$\Psi : \left( \prod_{k \in \Upsilon} \mathcal{M}_{(N_k-d_k) \times d_k}(\mathbb{C}) \times \prod_{l \in \Upsilon} \mathcal{M}_{(M_l-d_l) \times d_i}(\mathbb{C}) \right) \ni (\{\tilde{U}_k\}_{k \in \Upsilon}, \{\tilde{V}_l\}_{l \in \Upsilon}) \mapsto \prod_{(k,l) \in \Phi} \mathcal{M}_{d_k \times d_i}(\mathbb{C}) \ni (\tilde{U}_k^T B_{kl} + A_{kl})_{k,l},$$

(15)
We will be interested in the function \( \det(\Psi \Psi^*) \), which depends on the channel realization \( H \) only through the blocks \( A_{kl} \) and \( B_{kl} \). The vectorization of the mapping (15) reveals that \( \Psi \) is composed of two main kinds of blocks, \( \Psi_{kl}^{(A)} \) and \( \Psi_{kl}^{(B)} \), i.e.

\[
\text{vec}(U_k^T B_{kl} + A_{kl} V_l) = (A_{kl} \otimes I_{d_k}) K_{(N_k-d_k),d_k} \text{vec}(U_k) + (I_{d_l} \otimes B_{kl}^T) \text{vec}(V_l),
\]

where \( \otimes \) denotes Kronecker product and \( K_{m,n} \) is the \( mn \times mn \) commutation matrix which is defined as the matrix that transforms the vectorized form of an \( m \times n \) matrix into the vectorized form of its transpose. Block \( \Psi_{kl}^{(B)} \) has dimensions \( d_l d_k \times d_l (M_l - d_l) \), whereas block \( \Psi_{kl}^{(A)} \) is \( d_l d_k \times d_k (N_k - d_k) \).

For a given tuple \((k,l)\), \( \Psi_{kl}^{(B)} \) and \( \Psi_{kl}^{(A)} \) are placed in the row partition that corresponds to the interfering link indicated by the tuple \((k,l)\). \( \Psi_{kl}^{(B)} \) is placed in the \( l+K \)-th column partition, whereas \( \Psi_{kl}^{(A)} \) occupies the \( k \)-th column partition. The rest of blocks are occupied by null matrices. The dimensions of \( \Psi \) are therefore \( \sum_{k \neq l} d_k d_l \times \sum_{j=1}^{K} (M_j + N_j - 2d_j) d_j \). In the particular case of \( s = 0 \), \( \Psi \) is a square matrix of size \( \sum_{k \neq l} d_k d_l \).

Notice that \( \Psi \) has the same structure as the incidence matrix of the network connectivity graph. Taking the 3-user interference channel as an example, \( \Psi \) is constructed as follows

\[
\begin{bmatrix}
\Psi_{12}^{(A)} & 0 & 0 & 0 & \Psi_{12}^{(B)} & 0 \\
\Psi_{13}^{(A)} & 0 & 0 & 0 & 0 & \Psi_{13}^{(B)} \\
0 & \Psi_{21}^{(A)} & 0 & \Psi_{21}^{(B)} & 0 & 0 \\
0 & \Psi_{23}^{(A)} & 0 & 0 & 0 & \Psi_{23}^{(B)} \\
0 & 0 & \Psi_{31}^{(A)} & \Psi_{31}^{(B)} & 0 & 0 \\
0 & 0 & \Psi_{32}^{(A)} & 0 & \Psi_{32}^{(B)} & 0
\end{bmatrix}
\]

where the blocks \( \Psi_{kl}^{(B)} \) and \( \Psi_{kl}^{(A)} \) are given by (16).

**B. Main results**

We use the following notation: given a Riemannian manifold \( X \) with total finite volume denoted as \( \text{Vol}(X) \) (the volume of the manifolds used in this paper are reviewed in Appendix A), let

\[
\int_{x \in X} f(x) \, dx = \frac{1}{\text{Vol}(X)} \int_{x \in X} f(x) \, dx
\]

be the average value of a integrable (or measurable and nonnegative) function \( f : X \to \mathbb{R} \). Fix \( d_j, M_j, N_j \) and \( \Phi \) satisfying (5) and (6) and let \( s \geq 0 \) be defined as in (11). The main results of the paper are Theorems 1, 2 and 3 below, which give integral expressions for the number of IA solutions when \( s = 0 \).
and the system is tightly feasible: this number is denoted as \( \sharp(\pi_1^{-1}(H_0)) \), which is the same for all channel realizations out of some zero-measure set.

**Theorem 1:** Assume that \( s = 0 \), and let \( \mathcal{H}_\epsilon \subseteq \mathcal{H} \) be any open set such that the following holds: if \( H = (H_{kl}) \in \mathcal{H}_\epsilon \) and \( P_k, Q_k, 1 \leq k \leq K \) are unitary matrices of respective sizes \( N_k, M_k \), then

\[
(P_k^T H_{kl} Q_l) \in \mathcal{H}_\epsilon.
\]

(We may just say that \( \mathcal{H}_\epsilon \) is invariant under unitary transformations). Then, for every \( H_0 \in \mathcal{H} \) out of some zero-measure set, we have:

\[
\sharp(\pi_1^{-1}(H_0)) = C \int_{H \in \mathcal{H} \cap \mathcal{H}_\epsilon} \det(\Psi \Psi^*) \, dH,
\]

where

\[
C = \frac{Vol(S)}{Vol(\mathcal{H}_\epsilon)},
\]

with \( S \) being the output space (Cartesian product of Grassmannians) in Eq. (8).

**Proof:** See Appendix B.

If we take \( \mathcal{H}_\epsilon \) to be the set

\[
\{(H_{kl}) : \|H_{kl}\|_F \in (1 - \epsilon, 1 + \epsilon)\}
\]

and we let \( \epsilon \rightarrow 0 \) we get:

**Theorem 2:** For a tightly feasible \( (s = 0) \) fully connected interference channel, and for every \( H_0 \in \mathcal{H} \) out of some zero-measure set, we have:

\[
\sharp(\pi_1^{-1}(H_0)) = C \int_{H \in \mathcal{H}, \|H_{kl}\|_F = 1} \det(\Psi \Psi^*) \, dH,
\]

where

\[
C = \prod_{(k,l) \in \Phi} \left( \frac{\Gamma(NKM_l)}{\Gamma(2) \cdot \Gamma(N_k - d_k)} \right) \times \prod_{k \in \Upsilon} \left( \frac{\Gamma(2) \cdot \Gamma(d_k) \cdot \Gamma(N_k - d_k)}{\Gamma(2) \cdot \Gamma(N_k)} \right) \times \prod_{l \in \Upsilon} \left( \frac{\Gamma(2) \cdot \Gamma(M_l - d_l)}{\Gamma(2) \cdot \Gamma(M_l)} \right)
\]

**Proof:** See Appendix C.

**Remark 1:** As proved in [17] (see also [18]), if the system is infeasible then \( \det(\Psi \Psi^*) = 0 \) for every choice of \( H, U, V \) and hence Theorem 1 still holds. Moreover, if the system is feasible and \( s > 0 \) then there is a continuous of solutions for almost every \( H_{kl} \) and hence it is meaningless to count them (the
value of the integrals in our theorems is not related to the number of solutions in that case). Note also that the equality of Theorem 1 holds for every unitarily invariant open set \( \mathcal{H}_e \), which in particular implies that the right-hand side of \( (17) \) has the same value for all such \( \mathcal{H}_e \) (recall that we proved in [17] that almost all channel realizations in \( \mathcal{H} \) have the same number of solutions).

Theorem 2 can be used to approximate the number of solutions of a given MIMO system using Montecarlo integration (see Section 11-C below). However, the convergence of the integral is quite slow in general. In the square symmetric case when all the \( d_k \) and all the \( N_k \) and \( M_k \) are equal \( \forall k \) and greater than \( 2d \), that is \( N = M \geq 2d \), which holds automatically when \( s = 0 \) and \( K \geq 3 \); we can write another integral which has faster convergence in practice:

**Theorem 3:** Let us consider a tightly feasible \( (s = 0) \) square interference channel \( (N_k = M_k = N \) and \( d_k = d, \forall k) \). Assuming additionally that \( K \geq 3 \), then for every \( H_0 \in \mathcal{H} \) out of some zero-measure set, we have:

\[
\sharp \pi_1^{-1}(H_0) = \left( \frac{2^{d^2} V(\mathcal{U}_{N-d})^2}{V(\mathcal{U}_N) V(\mathcal{U}_{N-2d})} \right)^{\sharp(\Phi)} V(\mathcal{S}) \int_{(A_{kl},B_{kl}) \in \mathcal{U}_{(N-d) \times d}} \det(\Psi^{\Psi}) dH,
\]

where \( \Psi \) is again defined by (15) and the input space of MIMO channels where we have to integrate are now

\[
H_{kl} = \begin{pmatrix}
0_{d \times d} & A_{kl} \\
B_{kl} & 0_{(N-d) \times (N-d)}
\end{pmatrix},
\]

whose blocks, \( A_{kl} \) and \( B_{kl} \), are matrices in the complex Stiefel manifold, denoted as \( \mathcal{U}_{(N-d) \times d} \), and formed by all orthonormal \( d \)-dimensional vectors in \( \mathbb{C}^{(N-d)} \). On the other hand, \( \mathcal{U}_a \) denotes the unitary group of dimension \( a \), whose volume can be found in Appendix [A].

**Remark 2:** If the problem is fully connected, the value of the constant preceding the integral in Theorem 3 is (using that \( 2N - dK - d = 0 \) if \( s = 0 \)):

\[
C = \left( \frac{2^{d^2} V(\mathcal{U}_{N-d})^2}{V(\mathcal{U}_N) V(\mathcal{U}_{N-2d})} \right)^{K(K-1)} V(\mathcal{S}) = \left( \frac{\Gamma(N-d+1) \cdots \Gamma(N)}{\Gamma(N-2d+1) \cdots \Gamma(N-d)} \right)^{K(K-1)} \left( \frac{\Gamma(2) \cdots \Gamma(d)}{\Gamma(N-d+1) \cdots \Gamma(N)} \right)^{2K}
\]

Additionally, if \( N = 2d \) (which implies \( K = 3 \)) then this constant is exactly equal to 1.

**Proof:** See Appendix [D].

In the next section we discuss how the results in Theorems 1 and 2 can be used to get approximations to the number of IA solutions for a given interference network.
C. Estimating the number of solutions by Monte Carlo integration

The integrals in Theorems 2 and 3 are too difficult to be computed analytically, but one can certainly try to compute them approximately using Monte Carlo integration. Our main reference here is [24, Sec. 5]. The Crude Monte Carlo method for computing the average

\[ \int_{x \in X} f(x) \, dx \]

of a function \( f \) defined on a finite-volume manifold \( X \) consists just in choosing many points at random, say \( x_1, \ldots, x_n \) for \( n \gg 1 \), uniformly distributed in \( X \), and approximating

\[ \int_{x \in X} f(x) \, dx \approx E_n = \frac{1}{n} \sum_{j=1}^{n} f(x_j). \] (18)

The most reasonable way to implement this in a computer program is to write down an iteration that computes \( E_1, E_2, E_3, \ldots \). The unique point to be decided is how many such \( x_j \) we must choose to get a reasonable approximation of the integral. A usual tool for measuring that is the standard deviation, that can be approximated by

\[ \Sigma_n = \left( \frac{1}{n-1} \sum_{j=1}^{n} (f(x_j) - E_n)^2 \right)^{1/2}. \] (19)

If we stop the iteration when

\[ \frac{\Sigma_n}{E_n} < \varepsilon, \]

then, with a probability of 0.95 on the set of random sequences of \( n \) terms, the relative error satisfies

\[ \left| \frac{\int_{x \in X} f(x) \, dx - E_n}{E_n} \right| \leq 2\varepsilon. \]

For example, if we stop the iteration when

\[ \frac{\Sigma_n}{E_n} < 0.05 \]

then we can expect to be making an error of about 10 percent in our calculation of \( \int_{x \in X} f(x) \, dx \).

The whole procedure for a general system is illustrated in Algorithm 1 which follows Theorem 2. Its particularization to square systems is shown in Algorithm 2 and follows Theorem 3.

D. The single-beam case

Although the results of Theorems 1, 2 and 3 are general and can be applied to arbitrary systems, for the particular case of single-beam MIMO networks (\( d_k = 1, k \in \Upsilon \)) it is possible to develop specific, much more efficient, techniques to count the exact number of alignment solutions. This subsection is devoted
Algorithm 1: Computing the number of IA solutions for general scenarios $\prod_{k=1}^{K} (M_k \times N_k, d_k)$.

**Input:** Relative error, $\varepsilon$; number of antennas, $\{M_l\}$ and $\{N_k\}$; streams, $d$; and users, $K$.

**Output:** Approximate number of IA solutions, $E_n$.

$n = 1$

repeat

Generate a set of i.i.d. matrices $\{A_{kl}\}$, $\{B_{kl}\}$ and $\{C_{kl}\}$.

Build channel matrices $\{H_{kl}\}$ according to (13).

Normalize every channel matrix $H_{kl}$ such that $\|H_{kl}\|_F = 1$.

Build matrix $\Psi$ according to (16).

Compute $D_n = C \det(\Psi \Psi^*)$ where $C$ is taken from Theorem 2

Calculate $E_n$ and $\Sigma_n$ according to (18) and (19), respectively, where $f(x_j)$ is now $D_j$.

$n = n + 1$.

until $\frac{\Sigma_n}{E_n} < \varepsilon$

Algorithm 2: Computing the number of IA solutions for symmetric square scenarios $(N \times N, d)^K$.

**Input:** Relative error, $\varepsilon$; number of antennas, $N$; streams, $d$; and users, $K$.

**Output:** Approximate number of IA solutions, $E_n$.

$n = 1$

repeat

Generate a set of i.i.d. $(N - d) \times d$ matrices $\{A_{kl}^*\}$ and $\{B_{kl}\}$ in the Stiefel manifold.

Build matrix $\Psi$ according to (16).

Compute $D_n = C \det(\Psi \Psi^*)$ where $C$ is taken from Theorem 3

Calculate $E_n$ and $\Sigma_n$ according to (18) and (19), respectively, where $f(x_j)$ is now $D_j$.

$n = n + 1$.

until $\frac{\Sigma_n}{E_n} < \varepsilon$

to this particular case. First, we should mention that, from a theoretical point of view, the single-beam case was solved in [12], where it was shown that the number of IA solutions for single-beam feasible systems coincides with the mixed volume of the Newton polytopes that support each equation of the
However, from a practical point of view, the computation of the mixed volume of a set of bilinear equations using the available software tools \[20\] can be very demanding. In consequence, the exact number of IA solutions is only known for some particular cases \[12, 21\].

The main idea that allows us to count efficiently the number of IA solution for single-beam MIMO networks is that, as first discussed in \[25\], for single-beam MIMO networks the mixed volume does not change if we consider rank-one MIMO channels instead of full-rank channels. The proof of this fact is straightforward by taking into account that the same monomials are present in both systems of equations and, thus, the Newton polytopes that support each equation are identical in both cases. Therefore, for the purpose of counting the number of alignment solutions in single-beam feasible systems, we can simplify our problem by considering rank-one channels without loss of generality. Assuming rank-one MIMO channels, \( H_{kl} = f_{kl}g_{kl}^* \), the set of alignment equations \((3)\) can be rewritten as

\[
\begin{align*}
    u_k^T f_{kl} g_{kl} v_l &= 0, \quad (k, l) \in \Phi, \\
    L(u_k) &\cap L(v_l)
\end{align*}
\]

where now \( v_l \) and \( u_k \) are column vectors representing the precoders and decoders for the particular case of \( d_k = 1 \ \forall k \). We notice that there are \( K(K-1) \) equations, each one being the product of two linear factors \( L(u_k) \) and \( L(v_l) \) in the entries of \( u_k \) and \( v_l \), respectively, as indicated in \((20)\). Finding a solution to this system reduces to choose from every equation exactly one factor and force it to be zero, i.e., either \( u_k^T f_{kl} = 0 \) or \( g_{kl}^* v_l = 0 \). Now the question is how many different solutions exist for such a system.

As a first approach, one may think that the total number of solutions would be \( 2^{K(K-1)} \) since we have 2 different choices for each of the \( K(K-1) \) different equations. Actually, this number matches the number of solutions given by the classical Bezout’s Theorem, which states that the number of solutions of a system of polynomials is upper bounded by the product of the degrees of those polynomials. This number, however, is just an upper bound (typically very loose) on the number of solutions because the equations are coupled through \( u_k \) and \( v_l \).

A tighter bound would be obtained by considering that we can design \( v_l \) (of size \( M_l \times 1 \)) to lie in, at most, \( M_l - 1 \) non-intersecting nullspaces. In other words, for a given \( l \), \( L(v_l) = g_{kl}^* v_l = 0 \) can be satisfied for a maximum of \( M_l - 1 \) values of \( k \). This observation would allow us to upper bound the number of solutions by \((K(K-1)) \sum_{l} (M_l-1)\), or equivalently, \((K(K-1)) \sum_{l} (N_k-1)\). Although this bound is much tighter than \( 2^{K(K-1)} \) for small values of \( K \), they are asymptotically equivalent since the rate of growth with \( K \) of the latter is also exponential.

\(^1\)This is not true though for multibeam cases because in this case the genericity of the system of equations is lost.
Due to the fact that all the equations in the system are strongly coupled and they cannot be solved independently, the combinatorics of finding the exact number of solutions is much more complicated than this last approach and forces us to design a counting routine. In order to explain how this computational routine works, we will use the \((2 \times 3, 1)(3 \times 2, 1)(2 \times 4, 1)(2 \times 2, 1)\) system as an example. The proposed routine proceeds as follows:

1) We start from a \(K \times K\) table. Each cell in the table corresponds to a link of the interference channel. Cells in the main diagonal represent direct links and they are ruled out since they do not play any role in the IA problem. All other cells correspond to interfering links. The table for the \((2 \times 3, 1)(3 \times 2, 1)(2 \times 4, 1)(2 \times 2, 1)\) system (or any 4-user system) would be as follows.

```
  1 2 3 4
Receiver (k)
  1 2 3 4
Transmitter (l)
```

2) We will now fill the cells according to some rules. The value in the cell \((k, l)\) indicates how the equation corresponding to the \((k, l)\) link has been satisfied. If it has been satisfied by forcing \(L(v_l) = 0\) it will contain a one. Otherwise, it will contain a zero, meaning that it has been satisfied by setting \(L(u_k) = 0\). We recall that given \(l\), \(L(v_l) = 0\) can be satisfied for a maximum of \(M_l - 1\) values of \(k\) and given \(k\), \(L(u_k) = 0\) can be satisfied for a maximum of \(N_k - 1\) values of \(l\). We also recall that when \(s = 0\), \(\sum_l(M_l - 1) + \sum_k(N_k - 1) = K(K - 1)\). Thus, for any valid solution, the \(l\)-th column of the table must contain exactly \(M_l - 1\) ones whereas the \(k\)-th row must contain exactly \(N_k - 1\) zeros. On the other hand, all cells must contain either a zero or a one.

As an example, for the \((2 \times 3, 1)(3 \times 2, 1)(2 \times 4, 1)(2 \times 2, 1)\) system we just have to find how many \(4 \times 4\) matrices exist with exactly \((1, 2, 1, 1)\) ones in columns 1, \ldots, 4 and \((2, 1, 3, 1)\) zeros in rows 1, \ldots, 4, respectively (not counting those in the main diagonal). It can be seen that there are only two possibilities which are shown in Figure 1.

3) The approach to fill the table for an arbitrary single-beam network is a recursive tree search approach, commonly known as backtracking procedure \cite{26} which is widely used to solve combinatorial enumeration problems. We first start with an all-zeros table and try to build up our solution cell by cell, filling it with ones, starting from the upper left corner; first right, then bottom. We can keep track of the approaches that we explored so far by maintaining a backtracking tree whose root
is the all-zeros board and where each level corresponds to the number of ones we have placed so far. Figure 2 shows the backtracking tree for our example system which was constructed according to Algorithm 3.

In general this procedure is much more efficient than resorting to general software packages to compute the mixed volume since it exploits the specific structure of the IA bilinear equations. As a matter of fact, our enumeration method for the $(3 \times 5,1)$ scenario takes less than 6 minutes to count the 357435 solutions, whereas state-of-the-art software for computing mixed volumes could take several days to obtain the same solution without additional information regarding the structure of the equations.

1) Connections with other combinatorial and graph theory problems: For the particular case of symmetric $(M \times N,1)^K$ scenarios, the IA solution counting problem can be restated as several well-studied combinatorial and graph theory problems. According to Step 2 above, every column of valid solution contains $M - 1$ ones whereas every row has $N - 1$ zeros. We recall that $s = 0$ or, equivalently, $M + N = K + 1$, which allows us rephrase Step 2 in a much simpler way. That is, the number of distinct IA solutions matches the number of $K \times K$ binary matrices with exactly $M - 1$ ones per row and column which are not in the main diagonal.

When this matrix is seen as the adjacency matrix of a graph (or the biadjacency matrix of a bipartite graph) some connections to graph theory problems arise. Most of these problems have been of historical interest and hence a lot of research has been done on them. It is natural, then, to find out that the number of solutions for some scenarios have already been computed in this field. We mention a few of them in the following.

![Diagram](image_url)  

Fig. 1. Representation of the two valid solutions for the system $(2 \times 3,1)(3 \times 2,1)(2 \times 4,1)(2 \times 2,1)$. 
Fig. 2. Backtracking tree for the system \((2 \times 3, 1)(3 \times 2, 1)(2 \times 4, 1)(2 \times 2, 1)\). The two tables at the lower level correspond to the two valid solutions.

- The number of solutions for \((2 \times (K-1), 1)^K\) scenarios is given by the number of derangements (permutations of \(K\) elements with no fixed points), rencontres numbers or subfactorial. It is also the number of labeled 1-regular digraphs with \(K\) nodes. Interestingly, as found in [27], [28, p.195], they are equal to

\[
\text{round} \left( \frac{K!}{e} \right).
\]

- The number of solutions for \((3 \times (K-2), 1)^K\) systems matches the number of labeled 2-regular digraphs with \(K\) nodes. In this case, a closed-form expression is also available [27]:

\[
\sum_{k=0}^{K} \sum_{s=0}^{K-k} \sum_{j=0}^{K-k} \frac{(-1)^{k+j-s}K!(K-k)!(2K-k-2j-s)!}{s!(k-s)!(K-k-j)!2j12^{2K-2k-j}}.
\]

- In general, for \((M \times (K-M+1), 1)^K\) scenarios, closed-form solutions do not exist and most of them have not even been studied. It is clear that this problem matches that of counting the number

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Algorithm 3: Backtracking procedure for counting the number of IA solutions for arbitrary single-beam scenarios \( \prod_{k=1}^{K} (M_k \times N_k, 1) \).

**Input:** Number of antennas, \( \{M_k\} \) and \( \{N_k\} \); and users, \( K \).

**Output:** Number of solutions, \( S \).

\[
S = 0 \quad // \text{No solutions found yet}
\]
\[
table = 0 \quad // \text{Empty table to fill with 1s}
\]
\[
row = 0, \ col = 0 \quad // \text{Row and column indexes}
\]

\( S = \text{backtrack} (table, row, col, S) \)

```latex
function S = backtrack (table, row, col, S)

if table is a valid solution then
    S = S + 1 \quad // Valid solution found
else
    foreach (row, col) in get_candidates (table, row, col) do
        table(row, col) = 1 \quad // Fill the cell with a 1
        backtrack (table, row, col, S) \quad // Recursive call
        table(row, col) = 0 \quad // Remove the 1
    return S

function ((crow_1, ccol_1),..., (crow_N, ccol_N)) = get_candidates (table, row, col)
return list of candidate cells to store the next 1
```

of labeled \((M - 1)\)-regular digraphs with \( K \) nodes but, as far as we know, no closed-form solution has been found yet. Further details can be found in Section IV.

IV. Numerical experiments

In this section we present some results obtained by means of the integral formulae in Theorem 2 (for arbitrary interference channels) and Theorem 3 (for square symmetric interference channels). We first evaluate the accuracy provided by the approximation of the integrals by Monte Carlo methods. To this end, we focus initially on single-beam systems, for which the procedure described in Section III-D allows us to efficiently obtain the exact number of IA solutions for a given scenario. The true number of solutions can thus be used as a benchmark to assess the accuracy of the approximation.
Tables I and II show the number of solutions given by the exact and the approximate procedures, respectively. To simplify the analysis, we have considered \((M \times (K - M + 1), 1)^K\) symmetric single-beam networks for increasing values of \(M\) and \(K\). As shown in Section III-D1, counting IA solutions for this scenario is equivalent to the well-studied graph theory problem of counting labeled \((M - 1)\)-regular digraphs with \(K\) nodes. Thus, additional terms and further information can be retrieved from integer sequences databases such as [27] from its corresponding A-number given in the last row of Table I. Percentages in Table II represent the upper bound for the relative error, \(2\varepsilon \cdot 100\), obtained in each scenario (see Section III-C).

<table>
<thead>
<tr>
<th>(M = 2)</th>
<th>(M = 3)</th>
<th>(M = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2 \times (K - 1), 1)^K)</td>
<td>((3 \times (K - 2), 1)^K)</td>
<td>((4 \times (K - 3), 1)^K)</td>
</tr>
<tr>
<td>(K = 2)</td>
<td>(1)</td>
<td>–</td>
</tr>
<tr>
<td>(K = 3)</td>
<td>(2)</td>
<td>(1)</td>
</tr>
<tr>
<td>(K = 4)</td>
<td>(9)</td>
<td>(9)</td>
</tr>
<tr>
<td>(K = 5)</td>
<td>(44)</td>
<td>(216)</td>
</tr>
<tr>
<td>(K = 6)</td>
<td>(265)</td>
<td>(7570)</td>
</tr>
<tr>
<td>(K = 7)</td>
<td>(1854)</td>
<td>(357435)</td>
</tr>
<tr>
<td>(K = 8)</td>
<td>(14833)</td>
<td>(22040361)</td>
</tr>
<tr>
<td>(K &gt; 8)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(K &gt; 8)</td>
<td>A000166</td>
<td>A007107</td>
</tr>
</tbody>
</table>

**TABLE I**

**Exact number of IA solutions for several symmetric single-beam scenarios, \((M \times (K - M + 1), 1)^K\).**

Figure 3 compares the results in Tables I and II in a logarithmic scale, to illustrate how the number of solutions increase with \(K\) for different values of \(M\). The exact number of solutions are also depicted to further stress the accuracy of the approximation.

Now we move to multi-beam scenarios, for which the exact number of solutions is only known for a few scenarios. Table III shows the results obtained for some instances of the \((M \times (2K - M + 2), 2)^K\) network. These results have been obtained using the integral formula in Theorem 2, except the square cases \((M = N)\), for which we used the expression in Theorem 3. For instance, we can mention that the system \((5 \times 5, 2)^4\) has, with a high confidence level, about 3700 different solutions. As it can be observed, the estimate of the integral formula in Theorem 3 converges much faster than that of Theorem 2.
TABLE II
APPROXIMATE NUMBER OF IA SOLUTIONS FOR SEVERAL SYMMETRIC SINGLE-BEAM SCENARIOS, \((M \times (K - M + 1), 1)^K\).

<table>
<thead>
<tr>
<th>Number of users, K</th>
<th>(M = 2)</th>
<th>(M = 3)</th>
<th>(M = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K = 2)</td>
<td>1</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>(K = 3)</td>
<td>(\approx 2) (1.0 %)</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>(K = 4)</td>
<td>(\approx 9) (1.6 %)</td>
<td>(\approx 9) (1.6 %)</td>
<td>1</td>
</tr>
<tr>
<td>(K = 5)</td>
<td>(\approx 44) (2.6 %)</td>
<td>(\approx 216) (1.5 %)</td>
<td>(\approx 44) (2.6 %)</td>
</tr>
<tr>
<td>(K = 6)</td>
<td>(\approx 266) (3.3 %)</td>
<td>(\approx 7291) (5.5 %)</td>
<td>(\approx 7291) (5.5 %)</td>
</tr>
<tr>
<td>(K = 7)</td>
<td>(\approx 1868) (9.6 %)</td>
<td>(\approx 361762) (8.7 %)</td>
<td>(\approx 1936679) (7.0 %)</td>
</tr>
<tr>
<td>(K = 8)</td>
<td>(\approx 13144) (20.6 %)</td>
<td>(\approx 22419610) (11.3 %)</td>
<td>(\approx 739668504) (14.1 %)</td>
</tr>
</tbody>
</table>

allowing us to get smaller relative errors. For the sake of completeness, Table IV shows the approximate number of solutions for some additional square symmetric multi-beam scenarios. For some of them the exact number of solutions was already known, as indicated in the table. For others (those indicated as

Fig. 3. Comparison of exact and approximate values for some single-beam scenarios, \((M \times (K - M + 1), 1)^K\).
N/A in the table) the exact number of solutions was unknown.

\[
M = 3 \quad (3 \times (2K - 1), 2)^K \\
M = 4 \quad (4 \times (2K - 2), 2)^K \\
M = 5 \quad (5 \times (2K - 3), 2)^K \\
M = 6 \quad (6 \times (2K - 4), 2)^K
\]

<table>
<thead>
<tr>
<th>(K)</th>
<th>(d)</th>
<th>Scenario</th>
<th>Exact</th>
<th>Ref.</th>
<th>Approximate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>((2 \times 2, 1)^3)</td>
<td>2</td>
<td>1</td>
<td>(\approx 2) (0.9 %)</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>((4 \times 4, 2)^3)</td>
<td>6</td>
<td>1</td>
<td>(\approx 6) (0.9 %)</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>((6 \times 6, 3)^3)</td>
<td>20</td>
<td>1</td>
<td>(\approx 20) (1.4 %)</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>((5 \times 5, 2)^3)</td>
<td>N/A</td>
<td>N/A</td>
<td>(\approx 3700) (0.1 %)</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>((10 \times 10, 4)^4)</td>
<td>N/A</td>
<td>N/A</td>
<td>(\approx 13887464893004) (6.8 %)</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>((3 \times 3, 1)^5)</td>
<td>216</td>
<td>21</td>
<td>(\approx 216) (0.6 %)</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>((6 \times 6, 2)^5)</td>
<td>N/A</td>
<td>N/A</td>
<td>(\approx 387724347) (0.7 %)</td>
</tr>
</tbody>
</table>

**TABLE III**

Approximate number of IA solutions for several symmetric 2-beam scenarios, \((M \times (2K - M + 2), 2)^K\).

Although these results have mainly a theoretical interest, they might also have some important practical implications. For instance, knowing the rate of increase of the number of solutions with \(K\) could have interest to analyze the asymptotic performance of linear IA, as discussed in [21]. Also, for moderate-size networks for which the total number of solutions is not very high, the results of this paper also open the possibility to provide a systematic way to compute all (or practically all) interference alignment solutions for a channel realization. Although all IA solutions are asymptotically equivalent, their sum-rate performance in low or moderate SNRs behavior may differ significantly [21], [29]. The main idea here is that if we are able to obtain all or almost all IA solutions for a particular channel realization, we can get all or almost all IA solutions for any other channel realization by using a homotopy-continuation...
based method such as that described in [25]. This idea is illustrated in Figure 4 which shows in grey the sum-rate curves of 973 different solutions for the \((4 \times 6, 2)^4\) network. The maximum sum-rate solution is plotted in a thicker solid line, while the average sum-rate of all solutions is represented with a dashed line. The relative performance improvement provided by the maximum sum-rate solution over the average is always above 10 % for SNR values below 40 dB, and is more than 20 % for SNR=20 dB. We note that this improvement is comparable to the one provided by sum-rate optimization algorithms which take into account additional information in the optimization procedure such as direct channels and noise variance.

![Fig. 4. Comparison of the sum rate achieved by approximately different solutions for the system \((4 \times 6, 2)^4\).](image)

V. CONCLUSION

In this paper we have provided two integral formulae to compute the finite number of IA solutions in tightly feasible problems, including multi-beam \((d_k > 1)\) networks. The first one can be applied to arbitrary \(K\)-user channels, whereas the second one solves the symmetric square case. Both integrals can be estimated by means of Monte Carlo methods. For single-beam networks, it is possible to obtain the exact number of solution resorting to more classic results on algebraic geometry.
APPENDIX A
MATHEMATICAL PRELIMINARIES

To facilitate reading, in this section we recall the mathematical results used in this paper. Firstly, we provide a short review on mappings between Riemannian manifolds and the main mathematical result used to derive the number of IA solutions, which is the Coarea formula. Secondly, we review the volume of the complex Stiefel and Grassmanian manifolds and the volume of the unitary group, which are also used throughout the paper.

A. Riemannian manifolds and the Coarea formula

The following result is immediate from [30, Th. 9.23].

**Theorem 4:** Let $X$ be a compact, embedded, (real) codimension $c$ submanifold of the Riemannian manifold $Y$. Then, for sufficiently small $\epsilon > 0$,

$$\text{Vol}(y \in Y : d(y, X) < \epsilon) = \text{Vol}(X)\text{Vol}(r \in \mathbb{R}^c : \|r\| \leq 1)\epsilon^c + O(\epsilon^{c+1}).$$

Here, $\text{Vol}(X)$ is the volume of $X$ w.r.t. its natural Riemannian structure inherited from that of $Y$. One of our main tools is the so-called Coarea Formula. The most general version we know may be found in [31], but for our purposes a smooth version as used in [32, p. 241] or [33] suffices. We first need a definition.

**Definition A.1:** Let $X$ and $Y$ be Riemannian manifolds, and let $\varphi : X \to Y$ be a $C^1$ surjective map. Let $k = \dim(Y)$ be the real dimension of $Y$. For every point $x \in X$ such that the differential mapping $D\varphi(x)$ is surjective, let $v^1_x, \ldots, v^k_x$ be an orthogonal basis of $\text{Ker}(D\varphi(x))^\perp$. Then, we define the Normal Jacobian of $\varphi$ at $x$, $\text{NJ}_\varphi(x)$, as the volume in the tangent space $T_{\varphi(x)}Y$ of the parallelepiped spanned by $D\varphi(x)(v^1_x), \ldots, D\varphi(x)(v^k_x)$. In the case that $D\varphi(x)$ is not surjective, we define $\text{NJ}_\varphi(x) = 0$.

**Theorem 5 (Coarea formula):** Let $X, Y$ be two Riemannian manifolds of respective dimensions $k_1 \geq k_2$. Let $\varphi : X \to Y$ be a $C^\infty$ surjective map, such that the differential mapping $D\varphi(x)$ is surjective for almost all $x \in X$. Let $\psi : X \to \mathbb{R}$ be an integrable mapping. Then, the following equality holds:

$$\int_{x \in X} \psi(x)\text{NJ}_\varphi(x) \, dX = \int_{y \in Y} \int_{x \in \varphi^{-1}(y)} \psi(x) \, dx \, dy. \quad (21)$$

Note that from the Preimage Theorem and Sard’s Theorem (see [34, Ch. 1]), the set $\varphi^{-1}(y)$ is a manifold of dimension equal to $\dim(X) - \dim(Y)$ for almost every $y \in Y$. Thus, the inner integral of (21) is well defined as an integral in a manifold. Moreover, if $\dim(X) = \dim(Y)$ then $\varphi^{-1}(y)$ is a finite set for almost every $y$, and then the inner integral is just a sum with $x \in \varphi^{-1}(y)$.
The following result, which follows from the Coarea formula, is [32, p. 243, Th. 5].

**Theorem 6:** Let $X, Y$ and $V \subseteq X \times Y$ be smooth Riemannian manifolds, with $\dim(V) = \dim(X)$ and $Y$ compact. Assume that $\pi_2 : V \to Y$ is regular (i.e. $D\pi_2$ is everywhere surjective) and that $D\pi_1(x, y)$ is surjective for every $(x, y) \in V$ out of some zero measure set. Then, for every open set $U \subseteq X$ contained in some compact set $K \subseteq X$,

$$
\int_{x \in U} \det(\pi_1^{-1}(x)) \, dx = \int_{y \in Y} \int_{x \in U : (x, y) \in V} DET(x, y)^{-1} \, dx \, dy,
$$

where $DET(x, y) = \det(DG_{x,y}(x)DG_{x,y}(x)^*)$ and $G_{x,y}$ is the (locally defined) implicit function of $\pi_1$ near $x = \pi_1(x, y)$. That is, close to $(x, y)$ the sets $V$ and $\{(x, G_{x,y}(x))\}$ coincide.

**Corollary 1:** In addition to the hypotheses of Theorem 6, assume that there exists $y_0 \in Y$ such that for every $y \in Y$ there exists an isometry $\varphi_y : Y \to Y$ with $\varphi_y(y) = y_0$ and an associated isometry $\chi_y : X \to X$ such that $\chi_y(U) = U$ and $(\chi_y \times \varphi_y)(V) = V$. Then,

$$
\int_{x \in U} \det(\pi_1^{-1}(x)) \, dx = Vol(Y) \int_{x \in U : (x, y_0) \in V} DET(x, y_0)^{-1} \, dx.
$$

**Proof:** Let $y \in Y$ and let $\varphi_y$, $\chi_y$ as in the hypotheses. Then, consider the mapping

$$
\chi_y \mid_{\{x \in U : (x, y) \in V\}} : \{x \in U : (x, y) \in V\} \to \{x \in U : (x, y) \in V\}
$$

$$
\begin{array}{ccc}
\chi_y & \mid & V \\
\mid & \downarrow & \mid \\
\pi_1 & \downarrow & \pi_1 \\
\mid & \downarrow & \mid \\
X & \to & X \\
\end{array}
$$

which is the restriction of an isometry, hence an isometry. Let $G_x$ be the local inverse of $\pi_1$ close to $x \in X$. The change of variables formula then implies:

$$
\int_{x \in U : (x, y) \in V} DET(x, y)^{-1} \, dx = \int_{x \in U : (x, y_0) \in V} DET(\chi_y^{-1}(x), y_0)^{-1} \, dx.
$$

Note that the following diagram is commutative:

$$
\begin{array}{ccc}
V \cap \pi_1^{-1}(U) & \overset{\chi_y^{-1} \times \varphi_y^{-1}}{\longrightarrow} & V \cap \pi_1^{-1}(U) \\
\pi_1 \downarrow & \overset{\chi_y^{-1}}{\longrightarrow} & \downarrow \pi_1 \\
X & \overset{\chi_y^{-1}}{\longrightarrow} & X \\
\end{array}
$$

Thus, the mapping $(\chi_y^{-1} \times \varphi_y^{-1}) \circ G_{x,y_0} \circ \chi_y$ is a local inverse of $\pi_1$ near $(\chi_y^{-1}(x), y)$, that is

$$
G_{\chi_y^{-1}(x), y} = (\chi_y^{-1} \times \varphi_y^{-1}) \circ G_{x,y_0} \circ \chi_y,
$$

and the composition rule for the derivative gives:

$$
DG_{\chi_y^{-1}(x), y}(\chi_y^{-1}(x)) = D(\chi_y^{-1} \times \varphi_y^{-1})(G_{x,y_0}(x))DG_{x,y_0}(x)D\chi_y^{-1}(y)(\chi_y(x)).
$$

Now, $\chi_y$, $\varphi_y$ and $\chi_y \times \varphi_y$ are isometries of their respective spaces. Thus, we conclude:

$$
\det(DG_{\chi_y^{-1}(x), y}(\chi_y^{-1}(x)))DG_{\chi_y^{-1}(x), y}(\chi_y^{-1}(x))^t = \det(DG_{x,y_0}(x))DG_{x,y_0}(x)^t,
$$

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that is $\text{DET}(\chi^{-1}(x), y) = \text{DET}(x, y_0)$. Then, \eqref{eq:23} reads

$$\int_{x \in U : (x,y) \in V} \text{DET}(x, y)^{-1} \, dx = \int_{x \in U : (x,y) \in V} \text{DET}(x, y_0)^{-1} \, dx.$$ 

That is, the inner integral in the right-hand side term \eqref{eq:22} is constant. The corollary follows.

\section*{B. The volume of classical spaces}

Some helpful formulas are collected here:

\begin{equation}
\text{Vol}(S(C^a)) = \text{Vol}(S(R^{2a})) = \frac{2\pi^a}{\Gamma(a)} \tag{24}
\end{equation}

is the volume of the complex sphere of dimension $a$.

\begin{equation}
\text{Vol}(U_a) = \frac{(2\pi)^{a(a+1)/2}}{\Gamma(1) \cdots \Gamma(a)} \tag{25}
\end{equation}

is the volume of the unitary group of dimension $a$. Note that, as pointed out in \cite{35} p. 54] there are other conventions for the volume of unitary groups. Our choice here is the only one possible for Theorem 4 to hold: the volume of $U_a$ is the one corresponding to its Riemannian metric inherited from the natural Frobenius metric in $M_a(C)$.

We finally recall the volume of the complex Grassmannian. Let $1 \leq a \leq b$; then,

\begin{equation}
\text{Vol}(G_{a,b}) = \pi^{a(b-a)} \frac{\Gamma(2) \cdots \Gamma(2) \cdot \Gamma(2) \cdots \Gamma(b-a)}{\Gamma(2) \cdots \Gamma(b)} \tag{26}
\end{equation}

\section*{Appendix B}

\section*{Proof of Theorem 1}

We will apply Corollary 1 to the double fibration given by \eqref{eq:10}. In the notations of Corollary 1 we consider $X = \mathcal{H}$, $Y = \mathcal{S}$, $V$ the solution variety and

$$y_0 = \left( \begin{pmatrix} \text{Id} \\ 0_{N_k-d_k} \end{pmatrix}, \begin{pmatrix} \text{Id} \\ 0_{M_k-d_k} \end{pmatrix} \right) \in \mathcal{S}.$$ 

Given any other element $y = (U_k, V_k) \in \mathcal{S}$, let $P_k$ and $Q_k$ be unitary matrices of respective sizes $N_k$ and $M_k$ such that

$$U_k = P_k \begin{pmatrix} \text{Id} \\ 0_{N_k-d_k} \end{pmatrix}, \quad V_k = Q_k \begin{pmatrix} \text{Id} \\ 0_{M_k-d_k} \end{pmatrix}.$$ 

Then consider the mapping

$$\varphi_y(U_k, V_k) = (P_k^* \tilde{U}_k, Q_k^* \tilde{V}_k),$$
which is an isometry of \( S \) and satisfies \( \varphi_y(y) = y_0 \) as demanded by Corollary 1. We moreover have the associated mapping \( \chi_y : \mathcal{H} \to \mathcal{H} \) given by

\[
\chi_y((H_{kl})_{(k,l)\in \Phi}) = (P_k^TH_{kl}Q_l)_{(k,l)\in \Phi}
\]

which is an isometry of \( \mathcal{H} \). Moreover, \( \chi_y(\mathcal{H}_e) = \mathcal{H}_e \) and \( \chi_y \times \varphi_y(\mathcal{V}) = \mathcal{V} \). We can thus apply Corollary 1 which yields

\[
\int_{H \in \mathcal{H}_e} \sharp(\pi_1^{-1}(x)) \, dx = \text{vol}(\mathcal{S}) \int_{H \in \mathcal{H}_I \cap \mathcal{H}_e} \det(DG(H)DG(H)^*)^{-1} \, dH,
\]

(27)

where \( \mathcal{H}_e \) is any open subset of \( \mathcal{H} \) and \( G \) is the local inverse of \( \pi_1 \) close to \( H \) at \( (H, y_0) \). We now compute \( \det(DG(H)DG(H)^*)^{-1} \). From the definition of \( G \) we have

\[
DG(H)\dot{H} = -\Psi_H^{-1}(\dot{R}_{kl}).
\]

A straight–forward computation shows that:

\[
DG(H)^*(\dot{U}, \dot{V}) = (-U_k^T\Psi_H^*\dot{U} + \dot{V})_{k \neq l}.
\]

Thus, writing \( \Psi = \Psi_H \), we have:

\[
DG(H)DG(H)^*(\dot{U}, \dot{V}) = \Psi^{-1}\Psi^{-*}(\dot{U}, \dot{V}).
\]

Therefore, \( (DG(H)DG(H)^*)^{-1} = \Psi^*\Psi \) and

\[
\det(DG(H)DG(H)^*)^{-1} = \det(\Psi^*\Psi) = \det(\Psi^*\Psi).
\]

From this last equality and (27) we have:

\[
\int_{H \in \mathcal{H}_e} \sharp(\pi_1^{-1}(H)) \, dH = \text{Vol}(\mathcal{S}) \int_{H \in \mathcal{H}_I \cap \mathcal{H}_e} \det(\Psi^*\Psi) \, dH.
\]

Theorem 1 follows dividing both sides of this equation by \( \text{Vol}(\mathcal{H}_e) \) and using the fact that for every choice of \( H \) out of a zero measure set, the number of elements in \( \pi_1^{-1}(H) \) is constant (see [17, Th. 2]).
APPENDIX C

PROOF OF THEOREM

Let $\mathcal{H}$ be the product for $(k,l) \in \Phi$ of the sets

$$\{ H_{kl} : d(H_{kl}, \{ R \in \mathcal{M}_{N_k \times M_l}(\mathbb{C}) : \| R \|_F = 1 \}) < \epsilon \}.$$  

From Theorem 4, each of these sets have volume equal to

$$2Vol(\{ R \in \mathcal{M}_{N_k \times M_l}(\mathbb{C}) : \| R \|_F = 1 \})\epsilon + O(\epsilon^2) = \frac{4\pi^{N_k M_l}}{\Gamma(N_k M_l)} \epsilon + O(\epsilon^2)$$

Thus, using (24),

$$Vol(\mathcal{H}) = \prod_{(k,l) \in \Phi} \frac{4\pi^{N_k M_l}}{\Gamma(N_k M_l)} \epsilon^2(\Phi) + O(\epsilon^2(\Phi) + 1).$$

On the other hand, consider the smooth mapping

$$f : \mathcal{H}_I \rightarrow \mathcal{H}_I \cap \prod_{(k,l) \in \Phi} \{ H_{kl} : \| H_{kl} \|_F = 1 \}$$

and apply Theorem 5 to get

$$\int_{H \in \mathcal{H}_I \cap \mathcal{H}_*} \det(\Psi_H\Psi_H^*) dH = \int_{H \in \mathcal{H}_I \cap \prod_{(k,l) \in \Phi} \{ H_{kl} : \| H_{kl} \|_F = 1 \}} \int_{t \in [-\epsilon, \epsilon]} \det(\Psi_H\Psi_H^*) NJf(H) d\vec{t} dH,$$

where $\hat{H}_{kl} = H_{kl}(1 + t_{kl})$. Note that the function inside the inner integral is smooth and hence for any $H \in \mathcal{H}_I \cap \prod_{(k,l) \in \Phi} \{ H_{kl} : \| H_{kl} \|_F = 1 \} \epsilon$ we have

$$\det(\Psi_H\Psi_H^*) NJf(H) = \det(\Psi_H\Psi_H^*) NJf(H) + O(\epsilon).$$

We have thus proved (using $\approx$ for equalities up to $O(\epsilon)$):

$$\int_{H \in \mathcal{H}_I \cap \mathcal{H}_*} \det(\Psi_H\Psi_H^*) dH \approx \int_{H \in \mathcal{H}_I \cap \prod_{(k,l) \in \Phi} \{ H_{kl} : \| H_{kl} \|_F = 1 \}} (2\epsilon)^2(\Phi) \det(\Psi_H\Psi_H^*) NJf(H) dH,$$

It is very easy to see that $NJf(H) = 1$ if $H = (H_{kl})$ with $\| H_{kl} \|_F = 1$. Thus, we have

$$\int_{H \in \mathcal{H}_I \cap \mathcal{H}_*} \det(\Psi_H^*\Psi_H) dH \approx (2\epsilon)^2(\Phi) \int_{H \in \mathcal{H}_I \cap \prod_{(k,l) \in \Phi} \{ H_{kl} : \| H_{kl} \|_F = 1 \}} \det(\Psi_H^*\Psi_H) dH.$$

From Theorem 1 and taking limits we then have that for almost every $H_0 \in \mathcal{H}$,

$$\#(\pi^{-1}_1(H_0)) = C \int_{H \in \mathcal{H}_I \cap \prod_{(k,l) \in \Phi} \{ H_{kl} : \| H_{kl} \|_F = 1 \}} \det(\Psi^*) dH,$$

where

$$C = \frac{\frac{\gamma(\Phi) Vol(H \in \mathcal{H}_I \cap \prod_{(k,l) \in \Phi} \{ H_{kl} : \| H_{kl} \|_F = 1 \})}{\prod_{(k,l) \in \Phi} \frac{4\pi^{N_k M_l}}{\Gamma(N_k M_l)} Vol(S)}}.$$
Now, $\mathcal{H}_I \cap \prod_{(k,l) \in \Phi} \{H_{kl} : \|H_{kl}\|_F = 1\}$ is a product of spheres and thus from (24)

$$\text{Vol} \left( \mathcal{H}_I \cap \prod_{(k,l) \in \Phi} \{H_{kl} : \|H_{kl}\|_F = 1\} \right) = \prod_{(k,l) \in \Phi} \frac{2\pi^{N_k M_l - d_k d_l}}{\Gamma \left( N_k M_l - d_k d_l \right)}.$$ 

Finally, $S = (\prod_{k \in \Upsilon} \mathbb{C}_{d_k, N_k}) \times (\prod_{l \in \Upsilon} \mathbb{C}_{d_l, M_l})$ is a product of complex Grassmannians, and its volume is thus the product of the respective volumes, given in (26). That is,

$$\text{Vol}(S) = \left( \prod_{k \in \Upsilon} \pi^{d_k (N_k - d_k)} \frac{\Gamma(2) \cdot \Gamma(d_k) \cdot \Gamma(2) \cdot \cdots \Gamma(N_k - d_k)}{\Gamma(2) \cdot \cdots \cdot \Gamma(N_k)} \right) \times \left( \prod_{l \in \Upsilon} \pi^{d_l (M_l - d_l)} \frac{\Gamma(2) \cdot \cdots \cdot \Gamma(d_l) \cdot \Gamma(2) \cdot \cdots \cdot \Gamma(M_l - d_l)}{\Gamma(2) \cdot \cdots \cdot \Gamma(M_l)} \right).$$

Putting these computations together, we get the value of $C$ claimed in Theorem 2.

**APPENDIX D**

**PROOF OF THEOREM 3**

The proof of this theorem is quite long and nontrivial. We will apply Theorem 1 to the sets

$$\mathcal{H}_\epsilon = \{(H_{kl}) : d(H_{kl}, \mathcal{U}_{N_k}) \leq \epsilon, (k, l) \in \Phi\}. \quad (29)$$

Then, because (17) holds for every $\epsilon$, one can take limits and conclude that for almost every $H_0 \in \mathcal{H}$,

$$\#(\pi^{-1}_1(H_0)) = \lim_{\epsilon \to 0} \frac{\text{Vol}(\mathcal{H}_I \cap \mathcal{H}_\epsilon) \text{Vol}(S)}{\text{Vol}(\mathcal{H}_\epsilon)} \int_{H \in \mathcal{H}_I \cap \mathcal{H}_\epsilon} \det(\Psi \Psi^*) \, dH. \quad (30)$$

The claim of Theorem 3 will follow from the (difficult) computation of that limit. We organize the proof in several subsections.

### A. Unitary matrices with some zeros

In this section we study the set of unitary matrices of size $N \geq 2d$ which have a principal $d \times d$ minor equal to 0, and the set of closeby matrices. For simplicity of the exposition, the notations of this section are inspired in, but different from, the notations of the rest of the paper. Let

$$\mathcal{T} = \mathcal{T}_{N,d} = \left\{ H = \begin{pmatrix} 0_{d \times d} & A \\ B & C \end{pmatrix} \right\} \subseteq \mathcal{M}_{N \times N}(\mathbb{C}).$$

Note that $\mathcal{T}$ is a vector space of complex dimension $N^2 - d^2$. Our three main results are:

**Proposition 1:** The set $\mathcal{U}_N \cap \mathcal{T}$ is a manifold of codimension $N^2$ inside $\mathcal{T}$. Moreover,

$$\text{Vol}(\mathcal{U}_N \cap \mathcal{T}) = \frac{\text{Vol}(\mathcal{U}_{N-d})^2}{\text{Vol}(\mathcal{U}_{N-2d})}.$$
Proposition 2: The following equality holds:
\[
\lim_{\epsilon \to 0} \frac{Vol(H \in T : d(H, U_N) \leq \epsilon)}{\epsilon N^2} = 2^{d^2} Vol(U_N) Vol(x \in \mathbb{R}^N : \|x\| \leq 1).
\]

Proposition 3: Let \( \Psi : T \to \mathbb{R} \) be a smooth mapping defined in the \( T \) and such that \( \Psi(H) \) depends only on the \( A \) and \( B \) part of \( H \), but not on the part \( C \). Denote \( \Psi(H) = \Psi(A, B) \). Then,
\[
\lim_{\epsilon \to 0} \frac{\int_{H \in T \in T : d(H, U_N) \leq \epsilon} \Psi(H) dH}{Vol(U_N) Vol(U_N^2)} = \frac{2^{d^2} Vol(U_{N-d})^2}{Vol(U_N) Vol(U_{N-2d})} \int_{(A^*, B) \in U_{N-d} \times d} \Psi(A, B) d(A, B).
\]

1) Proof of Proposition 3: Let
\[
\xi : U_{N-d}^2 \to U_N \cap T
\]
\[
(U, V) \mapsto \begin{pmatrix} I_d & 0 \\ 0 & U \end{pmatrix} J \begin{pmatrix} I_d & 0 \\ 0 & V^* \end{pmatrix}
\]
where
\[
J = \begin{pmatrix} 0 & I_d & 0 \\ I_d & 0 & 0 \\ 0 & 0 & I_{N-2d} \end{pmatrix}.
\]

We claim that \( \xi \) is surjective. Indeed, let
\[
H = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix} \in U_N \cap T.
\]
From \( HH^* = I_N \) we have that \( A \) satisfies \( AA^* = I_d \), i.e. the rows of \( A \) can be completed to form a unitary basis of \( \mathbb{C}^{N-d} \). Namely, there exists \( V \in U_{N-d} \) such that \( A = (I_d 0)V \). Similarly, there exists \( U \in U_{N-d} \) such that \( B = U(I_d^* 0) \). Then,
\[
H = \begin{pmatrix} I_d & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} 0 & I_d & 0 \\ I_d & R_1 & R_2 \\ 0 & R_3 & R_4 \end{pmatrix} \begin{pmatrix} I_d & 0 \\ 0 & V \end{pmatrix},
\]
where \( R \) satisfies \( URV = C \). Now, this implies that the matrix
\[
\begin{pmatrix} 0 & I_d & 0 \\ I_d & R_1 & R_2 \\ 0 & R_3 & R_4 \end{pmatrix}
\]
is unitary, which forces \( R_1 = 0, R_2 = 0, R_3 = 0 \) and \( R_4 \) unitary. That is
\[
H = \left( \begin{array}{ccc}
I_d & 0 & 0 \\
0 & U & 0 \\
0 & 0 & R_4
\end{array} \right) \left( \begin{array}{ccc}
I_d & 0 \\
I_d & 0 \\
0 & V
\end{array} \right) = \left( \begin{array}{ccc}
I_d & 0 \\
0 & U \\
0 & 0 & R_4
\end{array} \right) J \left( \begin{array}{ccc}
I_d & 0 \\
0 & I_d \\
0 & 0 & R_4
\end{array} \right) \left( \begin{array}{ccc}
I_d & 0 \\
0 & U \\
0 & 0 & V
\end{array} \right),
\]
that is
\[
H = \xi \left( U, V^* \left( \begin{array}{ccc}
I_d & 0 \\
0 & R_4
\end{array} \right) \right),
\]
and the surjectivity of \( \xi \) is proved. Moreover, this describes \( U_N \cap T \) as the orbit of \( J \) under the action in \( T \) given by
\[
((U, V), X) \mapsto \left( \begin{array}{ccc}
I_d & 0 \\
0 & U \\
0 & 0 & V
\end{array} \right) X \left( \begin{array}{ccc}
I_d & 0 \\
0 & U \\
0 & 0 & V
\end{array} \right) = \left( \begin{array}{ccc}
I_d & 0 \\
0 & U \\
0 & 0 & V
\end{array} \right)
\]
Then, \( U_N \cap T \) is a smooth manifold diffeomorphic to the quotient space
\[
\mathcal{U}_{N-d}/S_J,
\]
where \( S_J \) is the stabilizer of \( J \). Now, \( (U, V) \in S_J \) if and only if
\[
\left( \begin{array}{ccc}
I_d & 0 & 0 \\
0 & U_1 & U_2 \\
0 & U_3 & U_4
\end{array} \right) \left( \begin{array}{ccc}
I_d & 0 & 0 \\
0 & V_1^* & V_3^* \\
0 & V_2^* & V_4^*
\end{array} \right) \left( \begin{array}{ccc}
I_d & 0 & 0 \\
0 & I_{N-2d} \\
0 & 0 & I_{N-2d}
\end{array} \right) = \left( \begin{array}{ccc}
0 & I_d & 0 \\
0 & 0 & U_4
\end{array} \right),
\]
which implies \( U_1 = I_d, U_2 = 0, U_3 = 0, V_1 = I_d, V_2 = 0, V_3 = 0 \) and \( U_4 = V_4 \). Thus,
\[
S_J = \left\{ \left( \begin{array}{ccc}
I_d & 0 & 0 \\
0 & U_4
\end{array} \right), \left( \begin{array}{ccc}
I_d & 0 \\
0 & U_4
\end{array} \right) : U_4 \in \mathcal{U}_{N-2d} \right\}.
\]
Then,
\[
\dim(U_N \cap T) = \dim(U_{N-d}/S_J) = 2 \dim(U_{N-d})^2 - \dim(S_J) = 2(N-d)^2 - (N-2d)^2 = N^2 - 2d^2.
\]
On the other hand, \( \dim(T) = 2N^2 - 2d^2 \) and thus
\[
\text{codim}_T(U_N \cap T) = 2N^2 - 2d^2 - (N^2 - 2d^2) = N^2,
\]
as claimed. We now apply the Coarea formula to \( \xi \) to compute the volume of \( U_N \cap T \). Note that by unitary invariance the Normal Jacobian of \( \xi \) is constant, and so is \( Vol(\xi^{-1}(H)) \). We can easily compute
\[
Vol(\xi^{-1}(H)) = \frac{Vol(\xi^{-1}(J))}{Vol(S_J)} = \sqrt{2} \frac{(N-2d)^2}{32} Vol(U_{N-2d}).
\]
For the Normal Jacobian of $\xi$, note that
\[
D\xi(I_{N-d}, I_{N-d})(\dot{U}, \dot{V}) = \begin{pmatrix} 0 & \dot{V}_1^* & -\dot{V}_2 \\ \dot{U}_1 & 0 & \dot{U}_2 \\ -\dot{U}_2^* & \dot{V}_2^* & \dot{U}_4 + \dot{V}_4^* \end{pmatrix}.
\]

Thus, $D\xi(I_{N-d}, I_{N-d})$ preserves the orthogonality of the natural basis of $T_U \mathcal{U}_{N-d} \times T_V \mathcal{U}_{N-d}$ but for the elements such that $\dot{U}_4 \neq 0$ or $\dot{V}_4 \neq 0$. We then conclude that $NJ(\xi)(I_{N-d}, I_{N-d}) = NJ(\eta)$ where

$$
\eta : \{M \in \mathcal{M}_{N-2d} : M + M^* = 0\}^2 \to \{M \in \mathcal{M}_{N-2d} : M + M^* = 0\}
$$

$$(\dot{U}_4, \dot{V}_4) \mapsto \dot{U}_4 + \dot{V}_4^*.$$  

It is a routine task to see that $\eta^*(L) = (L, L^*)$ which implies $\eta\eta^*(L) = 2L$, that is

$$\det(\eta\eta^*) = 2^{2\dim(\{M \in \mathcal{M}_{N-2d} : M + M^* = 0\})} = 2^{(N-2d)^2}.$$  

Hence, $NJ(\eta) = \sqrt{\det(\eta\eta^*)} = \sqrt{2^{(N-2d)^2}}$. As we have pointed out above, the value of the Normal Jacobian of $\xi$ is constant. Thus, for every $U, V$,

$$NJ(\xi)(U, V) = NJ(\eta) = \sqrt{2^{(N-2d)^2}}.$$  

The Coarea formula applied to $\xi$ then yields:

$$Vol(\mathcal{U}_{N-d}) = \int_{(U, V) \in \mathcal{U}_{N-d}} 1 d(U, V) = \int_{H \in \mathcal{U}_{N} \cap \mathcal{T}} \frac{Vol(\xi^{-1}(H))}{NJ(\xi)} dH = Vol(\mathcal{U}_{N} \cap \mathcal{T}) Vol(\mathcal{U}_{N-2d}).$$  

The value of $Vol(\mathcal{U}_{N} \cap \mathcal{T})$ is thus as claimed in Proposition 1.

2) Some notations: Given a matrix of the form

$$H = \begin{pmatrix} 0 & \sigma & 0 \\ \alpha & C_1 & C_2 \\ 0 & C_3 & C_4 \end{pmatrix}, \quad (33)$$

($\alpha$ and $\sigma$ are $d \times d$ diagonal matrices with real positive ordered entries) we denote by $\tilde{H}$ the associated matrix

$$\tilde{H} = \begin{pmatrix} \alpha & C_1 & C_2 \\ 0 & \sigma & 0 \\ 0 & U_0^* C_3 & U_0^* C_4 \end{pmatrix},$$

where $U_0$ is some unitary matrix which minimizes the distance from $C_4$ to $\mathcal{U}_{N-2d}$. Note that

$$\tilde{H} = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & U_0^* \end{pmatrix} H.$$
and hence
\[ d(H,\mathcal{U}_N) = d(\tilde{H},\mathcal{U}_N). \]

We also let
\[ T_1(H) = \|\alpha - I_d\|^2 + \|\sigma - I_d\|^2 + \|C_4 - U_0\|^2 + \frac{\|C_1\|^2 + \|C_2\|^2 + \|C_3\|^2}{2}, \]
\[ T_2(H) = \|\alpha - I_d\|^2 + \|\sigma - I_d\|^2 + \|C_4 - U_0\|^2 + \|C_1\|^2 + \frac{\|C_2\|^2 + \|C_3\|^2}{2}. \]

Note that
\[ T_2(H) \geq T_1(H) \geq \frac{\|\tilde{H} - I_N\|^2}{2} \geq \frac{d(\tilde{H},\mathcal{U}_N)^2}{2} = \frac{d(H,\mathcal{U}_N)^2}{2}. \quad (34) \]

3) Approximate distance to \( \mathcal{U}_N \) and \( \mathcal{U}_N \cap \mathcal{T} \): In this section we prove that for small values,
\[ d(H,\mathcal{U}_N) \approx T_1(H)^{1/2}, \quad d(H,\mathcal{U}_N \cap \mathcal{T}) \approx T_2(H)^{1/2}. \]

More precisely:

**Proposition 4:** For sufficiently small \( \epsilon > 0 \), if \( d(H,\mathcal{U}_N) \leq \epsilon \) then,
\[ |d(H,\mathcal{U}_N) - T_1(H)^{1/2}| \leq O(\epsilon^2), \]
\[ |d(H,\mathcal{U}_N \cap \mathcal{T}) - T_2(H)^{1/2}| \leq O(\epsilon^2). \]

Here, we are writing \( O(\epsilon^2) \) for some function of the form \( c(d)\epsilon^2 \).

Before proving Proposition 4 we state the following intermediate result.

**Lemma 1:** There is an \( \epsilon_0 > 0 \) such that \( \|\tilde{H} - I_N\| \leq \epsilon < \epsilon_0 \) implies:
\[ T_1(H)^{1/2} - 9\epsilon^2 \leq d(H,\mathcal{U}_N) \leq T_1(H)^{1/2} + 9\epsilon^2, \]
\[ T_2(H)^{1/2} - 30\epsilon^2 \leq d(H,\mathcal{U}_N \cap \mathcal{T}) \leq T_2(H)^{1/2} + 30\epsilon^2. \]

**Proof:** We will use the concept of normal coordinates (see for example [30, p. 14]). Consider the exponential mapping in \( \mathcal{U}_N \), which is given by the matrix exponential
\[ T_1 \mathcal{U}_N = \{ R \in \mathcal{M}_N(\mathbb{C}) : R + R^* = 0 \} \rightarrow \mathcal{U}_N \]
\[ R \mapsto e^R = I + R + \sum_{k \geq 2} \frac{R^k}{k!}, \]
which is an isometry from a neighborhood of \( 0 \in T_1 \mathcal{U}_N \) to a neighborhood of \( I \in \mathcal{U}_N \) and defines the normal coordinates. Thus, for sufficiently small \( \epsilon_1 > 0 \) there exists \( \epsilon_0 > 0 \) such that if \( U \in \mathcal{U}_N \), \( \|U - I\| < \epsilon_0 \) then there exists a skew–symmetric matrix \( R \) such that
\[ U = e^R, \quad \|R\| = d_{\mathcal{U}_N}(U, I), \quad \|R\| \leq \epsilon_1. \]
Let $R \in \mathcal{M}_N(\mathbb{C})$ be a skew–Hermitian matrix such that
\[
\|\tilde{H} - e^R\| = d(\tilde{H}, \mathcal{U}_N) = \delta \leq \epsilon, \quad \|R\| = d_{\mathcal{U}_N}(e^R, I), \quad \|R\| \leq \epsilon_1.
\]
Let $S = \sum_{k \geq 2} R^k / k!$. Then, $e^R = I + R + S$ and
\[
\|S\| \leq \sum_{k \geq 2} \frac{\|R\|^k}{k!} \leq 2\|R\|^2.
\]
If we denote $a = \|e^R - I\| = \|R + S\|$ and $b = d_{\mathcal{U}_N}(e^R, I) = \|R\|$, we have proved that
\[
b - 2b^2 \leq a \leq b + 2b^2.
\]
Using that $b < 1/2$ and some arithmetic, this implies
\[
a + 6a^2 \geq b, \quad \text{that is} \quad \|R\| \leq \|e^R - I\| + 6\|e^R - I\|^2.
\]
Now,
\[
\|e^R - I\| \leq \|e^R - \tilde{H}\| + \|\tilde{H} - I\| \leq 2\epsilon,
\]
which implies
\[
\|R\| \leq 2\epsilon + 24\epsilon^2 \leq 3\epsilon.
\]
In particular, $\|S\| \leq 9\epsilon^2$. We conclude:
\[
d(H, \mathcal{U}_N) = d(\tilde{H}, \mathcal{U}_N) = \|\tilde{H} - e^R\| \geq \|\tilde{H} - (I + R)\| - \|S\| \geq \|\tilde{H} - (I + R)\| - 9\epsilon^2.
\]
We now solve the following elementary minimization problem:
\[
\min_{R: R + R^* = 0} \|\tilde{H} - (I + R)\|.
\]
Let
\[
R = \begin{pmatrix} R_1 & R_2 & R_3 \\ -R_2^* & R_5 & R_6 \\ -R_3^* & -R_6^* & R_9 \end{pmatrix}, \quad R_1 + R_1^* = 0, R_5 + R_5^* = 0, R_9 + R_9^* = 0.
\]
Then, $\|\tilde{H} - (I + R)\|$ is minimized when $R_1 = 0, R_5 = 0, R_9 = 0$ and
\[
R_2 = \text{argmin}(\|C_1 - R_2\|^2 + \|R_2\|^2)
\]
\[
R_3 = \text{argmin}(\|C_2 - R_3\|^2 + \|R_3\|^2)
\]
\[
R_6 = \text{argmin}(\|U_0^* C_3 - R_6^*\|^2 + \|R_6\|^2).
\]
It is easily seen that the solutions to these problems are:

\[ R_2 = \frac{C_1}{2} \rightarrow \|C_1 - R_2\|^2 + \|R_2\|^2 = \frac{\|C_1\|^2}{2}, \]
\[ R_3 = \frac{C_2}{2} \rightarrow \|C_2 - R_3\|^2 + \|R_3\|^2 = \frac{\|C_2\|^2}{2} \]
\[ R_6 = \frac{C_3^*U_0}{2} \rightarrow \|U_0^*C_3 - R_6^*\|^2 + \|R_6\|^2 = \frac{\|C_3\|^2}{2}. \]

We have then proved

\[ \min_{R:R+R^* = 0} \|\tilde{H} - (I + R)\| = T_1(H)^{1/2}, \]

and the minimum is reached at

\[ R = \begin{pmatrix} 0 & C_1/2 & C_2/2 \\ -C_1^*/2 & 0 & C_3^*U_0/2 \\ -C_2^*/2 & -U_0C_3^*/2 & 0 \end{pmatrix} \]  \hspace{1cm} (35) \]

Hence,

\[ d(H, U_N) \geq T_1(H)^{1/2} - 9\epsilon^2, \]

and the first lower bound claimed in the lemma follows. For the upper bound let \( R \) be defined by (35) and note that (following a similar reasoning to the one above)

\[ d(H, U_N) = d(\tilde{H}, U_N) \leq \|\tilde{H} - e^R\| \leq \|\tilde{H} - (I + R)\| + \sum_{k \geq 2} \frac{\|R\|^k}{k!} = T_1(H)^{1/2} + \sum_{k \geq 2} \frac{\left(\frac{\|C_1\|^2 + \|C_2\|^2 + \|C_3\|^2}{2}\right)^{k/2}}{k!} \]

Now, \( \|\tilde{H} - I_N\| \leq \epsilon \) in particular implies \( \|C_1\|^2 + \|C_2\|^2 + \|C_3\|^2 \leq \epsilon^2 \) and then we have

\[ d(H, U_N) \leq T_1(H)^{1/2} + \sum_{k \geq 2} \frac{\left(\frac{\epsilon^2}{2}\right)^{k/2}}{k!} \leq T_1(H)^{1/2} + 2\epsilon^2, \]

as wanted. Now, for the second claim of the lemma, the same argument is used but now \( R \) is such that \( e^R \) minimizes \( \|\tilde{H} - e^R\| \) and

\[ e^R = \begin{pmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{pmatrix}. \]

Now, from the equality

\[ I + R = e^R - S, \]

and arguing as above we have that

\[ \|R_2\| \leq \|S\| \leq 9\epsilon^2, \] which implies \( \|R - \tilde{R}\| = \sqrt{2}\|R_2\| \leq 20\epsilon^2, \]
where we denote by $\tilde{R}$ the matrix resulting from letting $R_2 = 0$. Thus,

$$||\tilde{H} - e^R|| - ||\tilde{H} - (I + \tilde{R})|| \leq ||\tilde{H} - I - R|| - ||\tilde{H} - I - \tilde{R}|| + ||S|| \leq ||\tilde{R} - R|| + 9e^2 \leq 30e^2.$$

We have then proved

$$d(H, U_N \cap T) - \min_{R: R + R^* = 0, R_2 = 0} ||\tilde{H} - (I + R)|| \leq ||\tilde{R} - R|| + 9e^2 \leq 30e^2,$$

and as before we can easily see that the minimum is reached when $R_1 = 0$, $R_2 = 0$, $R_5 = 0$, $R_9 = 0$, $R_3 = C_2/2$ and $R_6 = C_3^* U_0/2$ which proves that

$$\min_{R: R + R^* = 0, R_2 = 0} ||\tilde{H} - (I + R)|| = T_2(H).$$

The lemma is now proved.

**Proof of Proposition 4**

Let $E$ be a matrix such that $||E|| \leq \epsilon$ and $H = U + E$ for some unitary matrix $U$. Then,

$$||HH^* - I|| = ||UU^* + UE^* + EE^* + EE^* - I|| = ||UE^* + UE^* + EE^*|| \leq 2\epsilon + \epsilon^2 \leq 4\epsilon.$$

On the other hand,

$$HH^* - I = \begin{pmatrix} \sigma^2 - I & \sigma C_1^* \\ C_1 \sigma & X \\ C_3 \sigma & X \end{pmatrix},$$

where the entries $X$ are terms which we do not need to compute. In particular, we have $||C_1 \sigma|| \leq 4\epsilon$ and

$$||\sigma^2 - I|| \leq 4\epsilon,$$

which implies $||\sigma^{-2}|| = ||\sigma^{-2} - I + I|| \leq \sqrt{d} + 4\epsilon$ and hence

$$||C_1|| = ||C_1 \sigma \sigma^{-1}|| \leq ||C_1 \sigma|| ||\sigma^{-1}|| \leq 4\epsilon \sqrt{\sqrt{d} + 4\epsilon} \leq 4\sqrt{d\epsilon}.$$

A similar argument works for $C_3$ as well, and using a symmetric argument for $H^* H$ we get the same bound for $C_2$ and an equivalent bound for $\alpha$ to that of (36). Summarizing these bounds, we have:

$$||C_1||^2 + ||C_2||^2 + ||C_3||^2 \leq 48de^2$$

(37)

Moreover, we also have

$$||C_4 C_4^* - I|| \leq ||C_3 C_3^*|| + ||C_3 C_3^* + C_4 C_4^* - I|| \leq 16d\epsilon^2 + 4\epsilon \leq 20de,$$

which implies

$$\sum_{j=0}^{N-d} (\beta_j^2 - 1)^2 = ||C_4 C_4^* - I||^2 \leq 400d^2 \epsilon^2.$$
where the $\beta_j$ are the singular values of $C_4$. In particular,
\[
\|U_0^*C_4 - I_{N-d}\|^2 = d(C_4, U_{N-d})^2 = \sum_{j=1}^{N-d} (\beta_j - 1)^2 \leq \sum_{j=1}^{N-d} (\beta_j - 1)^2 (\beta_j + 1)^2 = \sum_{j=1}^{N-d} (\beta_j^2 - 1)^2 \leq 400d^2 \epsilon^2,
\]
and we conclude that
\[
\|U_0^*C_4 - I_{N-d}\| \leq 20d \epsilon. \tag{38}
\]
Using (36), (37) and (38) above we get:
\[
\|\tilde{H} - I_N\|^2 = \|\tilde{\sigma} - I_d\|^2 + \|\tilde{\alpha} - I_d\|^2 + \|C_4\|^2 + \|C_2\|^2 + \|C_3\|^2 + \|U_0^*C_4 - I_{N-d}\|^2 \leq C(d) \epsilon^2,
\]
where $C(d)$ depends only on $d$. Let $\epsilon$ be small enough for $C(d)\epsilon$ to satisfy the hypotheses of Lemma [1].

The Proposition [4] follows from applying that lemma.

4) How the sets of closeby matrices to $U_N$ and $U_N \cap T$ compare: Our main result in this section is the following.

**Proposition 5:** Let $\alpha > 1$. For sufficiently small $\epsilon > 0$, we have:
\[
2^d V ol \left( H \in T : d(H, U_N \cap T) \leq \frac{\epsilon}{\alpha} \right) \leq
V ol(H \in T : d(H, U_N) \leq \epsilon) \leq
2^d V ol(H \in T : d(H, U_N \cap T) \leq \alpha \epsilon)
\]

Before the proof we state two technical lemmas.

**Lemma 2:** Let $\sigma, \alpha$ be as in (33). Then,
\[
V ol \left( C : T_1 \left( \begin{array}{ccc} 0 & \sigma & 0 \\ \alpha & C_1 & C_2 \\ 0 & C_3 & C_4 \end{array} \right) \leq \epsilon \right) = 2^d V ol \left( C : T_2 \left( \begin{array}{ccc} 0 & \sigma & 0 \\ \alpha & C_1 & C_2 \\ 0 & C_3 & C_4 \end{array} \right) \leq \epsilon \right).
\]

**Proof:** Let
\[
S_i(C) = T_i \left( \begin{array}{cc} 0 & A \\ B & C \end{array} \right), \quad i = 1, 2,
\]
where $A = (\sigma 0)$ and $B^T = (\alpha 0)$. The claim of the lemma is that
\[
V ol(C : S_1(C) \leq \epsilon) = 2^d V ol(C : S_2(C) \leq \epsilon).
\]
Indeed, consider the mapping
\[
\varphi \left( \begin{array}{cc} C_1 & C_2 \\ C_3 & C_4 \end{array} \right) = \left( \begin{array}{cc} \sqrt{2} C_1 & C_2 \\ C_3 & C_4 \end{array} \right),
\]
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which has Jacobian equal to $\sqrt{2^{2d^2}} = 2^d$. The change of variables theorem yields:

$$2^d Vol(C : S_1(\varphi(C)) \leq \epsilon) = Vol(\varphi(C) : S_1(\varphi(C)) \leq \epsilon) = Vol(C : S_1(C) \leq \epsilon).$$

The lemma follows from the fact that $S_1(\varphi(C)) = S_2(C)$. \hfill \blacksquare

Lemma 3: Let $\alpha > 1$ and let $A, B$ be complex matrices of respective sizes $d \times (N - d)$ and $(N - d) \times d$. Then, for sufficiently small $\epsilon > 0$ we have

$$2^d Vol\left(C : d\left(\begin{array}{c}
0 \\
B \\
C
\end{array}\right), \mathcal{U}_N \cap \mathcal{T}\right) \leq \frac{\epsilon}{\alpha}.$$  

$$Vol\left(C : d\left(\begin{array}{c}
0 \\
A \\
B \\
C
\end{array}\right), \mathcal{U}_N\right) \leq \epsilon.$$  

$$2^d Vol\left(C : d\left(\begin{array}{c}
0 \\
A \\
B \\
C
\end{array}\right), \mathcal{U}_N \cap \mathcal{T}\right) \leq \alpha \epsilon.$$  

Proof: Let $U_A, V_A, U_B, V_B$ be such that

$$A = U_A(\sigma 0)V_A^*, \quad B = U_B\left(\begin{array}{c}
\alpha \\
0
\end{array}\right)V_B^*$$

are singular value decompositions of $A$ and $B$ respectively. Then,

$$Vol\left(C : d\left(\begin{array}{c}
0 \\
A \\
B \\
C
\end{array}\right), \mathcal{U}_N\right) \leq \epsilon = Vol\left(C : d\left(\begin{array}{c}
U_A^* \quad 0 \\
0 \quad U_B^*
\end{array}\right)\left(\begin{array}{c}
0 \\
A \\
B \\
C
\end{array}\right), \mathcal{U}_N\right) \leq \epsilon = Vol\left(C : d\left(\begin{array}{c}
0 \\
(\sigma 0) \\
0 \\
C
\end{array}\right), \mathcal{U}_N\right) \leq \epsilon,$$

where the last inequality follows from unitary invariance of the volume. Let $H$ be as in (33). From Proposition 4 we conclude:

$$Vol\left(C : d(H, \mathcal{U}_N) \leq \epsilon\right) \leq Vol(C : T_1(H)^{1/2} \leq \epsilon + c(d)\epsilon^2) = Vol(C : T_1(H) \leq (\epsilon + c(d)\epsilon^2)^2) = \text{Lemmal} 2^d Vol(C : T_2(H) \leq (\epsilon + c(d)\epsilon^2)^2).$$

From (34), for sufficiently small $\epsilon > 0$, $T_2(H) \leq (\epsilon + c(d)\epsilon^2)^2$ implies $d(H, \mathcal{U}_N)$ is as small as wanted. Hence, from Proposition 4 for sufficiently small $\epsilon > 0$ we have

$$Vol(C : T_2(H) \leq (\epsilon + c(d)\epsilon^2)^2) = Vol(C : T_2(H)^{1/2} \leq \epsilon + c(d)\epsilon^2) \leq Vol\left(C : d(H, \mathcal{U}_N \cap \mathcal{T}) \leq \epsilon + 2c(d)\epsilon^2\right).$$
In particular, for every $\alpha > 1$ and for sufficiently small $\epsilon > 0$ we have proved that

$$\text{Vol} (C : d(H, U_N) \leq \epsilon) \leq 2^d \text{Vol} (C : d(H, U_N \cap T) \leq \alpha \epsilon).$$

This proves the upper bound of the lemma. The lower bound is proved with a symmetric argument, using the opposite inequalities of Proposition 4.

**Proof of Proposition 5** Let $\alpha > 1$. From Fubini’s Theorem,

$$\text{Vol}(H \in T : d(H, U_N) \leq \epsilon) = \int_{A \in M_{d \times (N-d)}(C), B \in M_{(N-d) \times d}(C)} \text{Vol}(C : d(H, U_N) \leq \epsilon) d(A, B).$$

From Lemma 3, for sufficiently small $\epsilon > 0$ this is at most

$$\int_{A \in M_{d \times (N-d)}(C), B \in M_{(N-d) \times d}(C)} 2^d \text{Vol}(C : d(H, U_N \cap T) \leq \alpha \epsilon) d(A, B).$$

Again from Fubini’s Theorem, this last equals

$$2^d \text{Vol}(H : d(H, U_N \cap T) \leq \alpha \epsilon),$$

proving the upper bound of the proposition. The lower bound follows from a symmetrical argument.

5) **Proof of Proposition 2** Let $\alpha > 1$. From Proposition 5, we have

$$\lim_{\epsilon \to 0} \frac{\text{Vol}(H \in T : d(H, U_N) \leq \epsilon)}{\epsilon^{N^2}} \leq 2^d \lim_{\epsilon \to 0} \frac{\text{Vol}(H \in T : d(H, U_N \cap T) \leq \alpha \epsilon)}{\epsilon^{N^2}}.$$

Note that $N^2$ is the (real) codimension of $U_N \cap T$ inside $T$. Thus, from Theorem 4

$$\lim_{\epsilon \to 0} \frac{\text{Vol}(H \in T : d(H, U_N \cap T) \leq \alpha \epsilon)}{\epsilon^{N^2}} = \text{Vol}(U_N \cap T) \alpha N^2 \text{Vol}(x \in \mathbb{R}^{N^2} : \|x\| \leq 1).$$

We have thus proved that for every $\alpha > 1$ we have

$$\lim_{\epsilon \to 0} \frac{\text{Vol}(H \in T : d(H, U_N) \leq \epsilon)}{\epsilon^{N^2}} \leq 2^d \text{Vol}(U_N \cap T) \alpha N^2 \text{Vol}(x \in \mathbb{R}^{N^2} : \|x\| \leq 1).$$

This implies:

$$\lim_{\epsilon \to 0} \frac{\text{Vol}(H \in T : d(H, U_N) \leq \epsilon)}{\epsilon^{N^2}} \leq 2^d \text{Vol}(U_N \cap T) \text{Vol}(x \in \mathbb{R}^{N^2} : \|x\| \leq 1).$$

The reverse inequality is proved the same way using the other inequality of Proposition 5.
6) Integrals of functions of the subset of matrices in $\mathcal{T}$ which are close to $U_N$: We are now close to the proof of Proposition [3] but we still need some preparation. We state two lemmas.

**Lemma 4:** Let $\Psi : \mathcal{T} \to [0, \infty)$ be a smooth mapping. Then,
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon N^2} \int_{H \in \mathcal{T} : d(H, U_N) \leq \epsilon} \Psi(H) \, dH = 2^d \int_{U \cap \mathcal{T}} \text{Vol}(x \in \mathbb{R}^{N^2} : \|x\| \leq 1) \int_{U \cup U \cap \mathcal{T}} \Psi(U) \, dU
\]

**Proof:** For sufficiently small $\epsilon > 0$, given $H \in \mathcal{T}$ such that $d(H, U_N) < \epsilon$, there is a unique $U \in U \cap \mathcal{T}$ such that the distance $d(H, U \cap \mathcal{T})$ is minimized (see for example [30, p. 32]). Let $\pi(H)$ be such $U$. Moreover, $\pi$ is a smooth mapping. We thus have
\[
\int_{H \in \mathcal{T} : d(H, U_N) \leq \epsilon} \Psi(H) \, dH = \int_{U \in U \cap \mathcal{T}} \int_{H \in \mathcal{T} : d(H, U_N) \leq \epsilon, \pi(H) = U} NJ \pi(H) \Psi(H) \, dH \, dU.
\]

Now, $\Psi$ is smooth and hence $\Psi(H) = \Psi(U) + \tilde{\Psi}(H)$ where $|\tilde{\Psi}(H)| \leq O(\epsilon)$. We thus have
\[
\int_{H \in \mathcal{T} : d(H, U_N) \leq \epsilon} \Psi(H) \, dH = \int_{U \in U \cap \mathcal{T}} \Psi(U) \int_{H \in \mathcal{T} : d(H, U_N) \leq \epsilon, \pi(H) = U} NJ \pi(H) \, dH \, dU + O(\epsilon).
\]

The integral inside this last expression is unitary invariant and thus its value is a constant $c_\epsilon$. Moreover, the same argument applied to $\Psi \equiv 1$ yields
\[
\text{Vol}(H \in \mathcal{T} : d(H, U_N) \leq \epsilon) = \int_{U \in U \cap \mathcal{T}} c_\epsilon \, dU.
\]

That is,
\[
c_\epsilon = \frac{\text{Vol}(H \in \mathcal{T} : d(H, U_N) \leq \epsilon)}{\text{Vol}(U \cap \mathcal{T})}.
\]

We have then proved
\[
\int_{H \in \mathcal{T} : d(H, U_N) \leq \epsilon} \Psi(H) \, dH = \frac{\text{Vol}(H \in \mathcal{T} : d(H, U_N) \leq \epsilon)}{\text{Vol}(U \cap \mathcal{T})} \int_{U \in U \cap \mathcal{T}} \Psi(U) \, dU = \int_{U \in U \cap \mathcal{T}} \Psi(U) \, dU,
\]

The lemma follows from Proposition [2].

**Lemma 5:** Let $\Psi$ be a smooth mapping. Then,
\[
\lim_{\epsilon \to 0} \frac{\int_{H \in \mathcal{T} : d(H, U_N) \leq \epsilon} \Psi(H) \, dH}{\text{Vol}(U \cap \mathcal{T})} = \frac{2^d \text{Vol}(U \cap \mathcal{T})}{\text{Vol}(U_N)} \int_{U \cup U \cap \mathcal{T}} \Psi(U) \, dU
\]

**Proof:** From Theorem [4] and using that the codimension of $U_N$ in $\mathcal{M}_N(\mathbb{C})$ is $N^2$ we know that
\[
\text{Vol}(H \in \mathcal{M}_N(\mathbb{C}) : d(H, U_N) \leq \epsilon) = \text{Vol}(U_N) \epsilon^{N^2} \text{Vol}(x \in \mathbb{R}^{N^2} : \|x\| \leq 1) (1 + O(\epsilon)),
\]

where $\lim_{\epsilon \to 0} O(\epsilon) = 0$. The lemma now follows from Lemma [4].
7) proof of Proposition [3] This result is almost immediate from Lemma [5] and Proposition [1]. Let \( \xi \) be the mapping defined in (31). We have computed the Normal Jacobian of \( \xi \) and the volume of the preimage of \( \xi \) in Section D-A1 From Theorem 5,

\[
\int_{(U,V) \in U_d^2} N_{\xi} \left( \Psi(U,V) \right) \, d(U,V) = \int_{H \in U_d \cap T} \Psi(H) \frac{Vol(\xi^{-1}(H))}{NJ_\xi} \, dH = Vol(U_{N-2d}) \int_{H \in U_d \cap T} \Psi(H) \, dH.
\]

Hence, as \( \Psi \) does not depend on \( C \), and writing \( \Psi(H) = \Psi(A,B) \) (note the abuse of notation),

\[
\int_{H \in U_d \cap T} \Psi(H) \, dH = \frac{1}{Vol(U_{N-2d})} \int_{(U,V) \in U_d^2} \Psi \left( (Id_0)V^*, U \left( \frac{Id}{0} \right) \right) \, d(U,V).
\]

Normalizing we get

\[
\int_{H \in U_d \cap T} \Psi(H) \, dH = \int_{(U,V) \in U_d^2} \Psi \left( (Id_0)V^*, U \left( \frac{Id}{0} \right) \right) \, d(U,V).
\]

Now, generating at random unitary matrices \( U,V \) and then taking \( (Id_0)V^*, U \left( \frac{Id}{0} \right) \) is the same as generating at random two elements in \( S \text{tiefel}_{(N-d)\times d} \). The proposition is proved.

B. Proof of Theorem [3]

Recall that we have defined \( \mathcal{H}_\epsilon \) in (29), and we want to compute the limit (30):

\[
\lim_{\epsilon \to 0} \frac{Vol(\mathcal{H}_I \cap \mathcal{H}_\epsilon) Vol(S)}{Vol(\mathcal{H}_\epsilon)} \int_{H \in \mathcal{H}_I \cap \mathcal{H}_\epsilon} \det(\Psi^*) \, dH = \lim_{\epsilon \to 0} \frac{Vol(S)}{Vol(\mathcal{H}_\epsilon)} \int_{H \in \mathcal{H}_I \cap \mathcal{H}_\epsilon} \det(\Psi^*) \, dH.
\]

Now, we use Fubini’s theorem to convert the last integral into an iterated integral

\[
\int_{H(k_1, l_1) \in T, d(H(k_1, l_1), S) < \epsilon} \cdots \int_{H(k_1, l_r) \in T, d(H(k_r, l_r), S) < \epsilon} \det(\Psi^*) \, dH(k_1, l_1) \cdots dH(k_r, l_r),
\]

where \( (k_1, l_1), \ldots, (k_r, l_r) \) are the elements of the connectivity set \( \Phi \), ordered with respect to some (irrelevant) criterion. From Proposition 3 the last inner integral satisfies:

\[
\int_{H(k_r, l_r) \in T, d(H(k_r, l_r), S) < \epsilon} \det(\Psi^*) \, dH(k_r, l_r) = O(\epsilon^*) + Vol(H \in \mathcal{M}_N(\mathbb{C}) : d(H, U_N) \leq \epsilon) \times \frac{2^d Vol(U_{N-2d})^2}{Vol(U_N Vol(U_{N-2d})} \int_{(A,B) \in U_d \times U_d} \det(\Psi^*) \, d(A,B),
\]

where \( \Psi \) is computed for

\[
H(k_r, l_r) = \begin{pmatrix} 0_{d \times d} & A \\ B & 0_{(N-d) \times (N-d)} \end{pmatrix}.
\]

Here, \( O(\epsilon^*) \) is an expression such that

\[
\lim_{\epsilon \to 0} \frac{O(\epsilon^*)}{Vol(H \in \mathcal{M}_N(\mathbb{C}) : d(H, U_N) \leq \epsilon)} = 0.
\]
By repeating the procedure and using Fubini’s theorem again to convert the iterated integral into a unique multiple integral, we conclude:

\[
\int_{H \in H \cap H_a} \det(\Psi \Psi^*) \, dH = O(\epsilon^*) + Vol(H \in \mathcal{M}_N(\mathbb{C}) : d(H, \mathcal{U}_N) \leq \epsilon)^2(\Phi) \times \\
\left( \frac{2^d Vol(\mathcal{U}_{N-d})^2}{Vol(\mathcal{U}_N)Vol(\mathcal{U}_{N-2d})} \right)^2(\Phi) \int_{(A_{kl}, B_{kl}) \in U_{N-d} \times (k, l) \in \Phi} \det(\Psi \Psi^*) \, d(A_{kl}, B_{kl}),
\]

where \( \Psi \) is computed for

\[
H_{kl} = \begin{pmatrix}
0_{d \times d} & A_{kl} \\
B_{kl} & 0_{(N-d) \times (N-d)}
\end{pmatrix}.
\]

Here, \( O(\epsilon^*) \) is an expression such that

\[
\lim_{\epsilon \to 0} \frac{O(\epsilon^*)}{Vol(H \in \mathcal{M}_N(\mathbb{C}) : d(H, \mathcal{U}_N) \leq \epsilon)^2(\Phi)} = 0.
\]

On the other hand, also from Fubini’s theorem we have

\[
Vol(H_n) = Vol(H \in \mathcal{M}_N(\mathbb{C}) : d(H, \mathcal{U}_N) \leq \epsilon)^2(\Phi).
\]

The claim of the Theorem \([3]\) follows.

REFERENCES


  2616–2626, 2011.


