An algorithm to find two distance domination parameters in a graph

Gerd H. Fricke\textsuperscript{a}, Michael A. Henning\textsuperscript{b,*}, Ortrud R. Oellermann\textsuperscript{c},
Henda C. Swart\textsuperscript{d}

\textsuperscript{a} Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435, USA
\textsuperscript{b} Department of Mathematics and Applied Mathematics, University of Natal, Private Bag X01,
Pietermaritzburg 3209, South Africa
\textsuperscript{c} Department of Mathematics and Computer Science, Brandon University, Brandon, Mani., Canada
\textsuperscript{d} Department of Mathematics and Applied Mathematics, University of Natal, King George V Avenue,
Durban 4001, South Africa

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Abstract

Let $n > 1$ be an integer and let $G$ be a graph of order $p$. A set $\mathcal{D}$ of vertices of $G$ is a total $n$-dominating set of $G$ if every vertex of $V(G)$ is within distance $n$ from some vertex of $\mathcal{D}$ other than itself. The minimum cardinality among all total $n$-dominating sets of $G$ is called the total $n$-domination number and is denoted by $\gamma_n^t(G)$. A set $S$ of vertices of $G$ is $n$-independent if the distance (in $G$) between every pair of distinct vertices of $S$ is at least $n + 1$. The minimum cardinality among all maximal $n$-independent sets of $G$ is called the $n$-independence number of $G$ and is denoted by $i_n(G)$. In this paper, we present an algorithm for finding a total $n$-dominating set $\mathcal{D}$ and a maximal $n$-independent set $S$ in a connected graph with at least $p \geq 2n + 1$ vertices. It is shown that these sets $\mathcal{D}$ and $S$ satisfy the inequality $|S| + n|\mathcal{D}| \leq p$. Using this result, we conclude that if $G$ is a connected graph on $p \geq 2n + 1$ vertices, then $i_n(G) + n \cdot \gamma_n^t(G) \leq p$.

1. Introduction

In this paper, we shall use the terminology of [10]. Specifically, $p(G)$ denotes the number of vertices (order) of a graph $G$ with vertex set $V(G)$ and edge set $E(G)$. If $T$ is a rooted tree with root $r$ and $v$ is a vertex of $T$, then the level number of $v$, which we denote by $\ell(v)$, is the length of the unique $r-v$ path in $T$. The maximum of the level numbers of the vertices of $T$ is called the height of $T$ and is denoted by $h(T)$. If a vertex $v$ of $T$ is adjacent to $u$ and $l(u) > l(v)$, then $u$ is called a child of $v$. and $v$ is the parent of $u$. A vertex $w$ is a descendant of $v$ (and $v$ is an ancestor of $w$) if the level numbers of the vertices on the $v-w$ path are monotonically increasing. We will refer to an end-vertex of $T$ as a leaf.

\* Corresponding author.
Let $n \geq 1$ be an integer and let $G$ be a graph. A set $\mathcal{D}$ of vertices of $G$ is defined to be an $n$-dominating set (resp. total $n$-dominating set) of $G$ if every vertex in $V(G) - \mathcal{D}$ (resp. $V(G)$) is within distance $n$ from some vertex of $\mathcal{D}$ other than itself. The minimum cardinality among all total $n$-dominating sets of $G$ is called the total $n$-domination number of $G$ and is denoted by $\gamma'_n(G)$. A set $I$ of vertices of $G$ is defined to be $n$-independent in $G$ if every vertex of $I$ is at distance at least $n + 1$ from every other vertex of $I$ in $G$. Furthermore, $I$ is defined to be an $n$-independent dominating set of $G$ if $I$ is $n$-independent and $n$-dominating in $G$. The $n$-independent domination number $i_n(G)$ of $G$ is the minimum cardinality among all $n$-independent dominating sets of $G$. Hence, $1$-independent dominating sets of $G$ are independent dominating sets of $G$ and $i_1(G) = i(G)$.

Results on the concept of $n$-domination in graphs have been presented by, among others, Bascó and Tuza [2,3], Beineke and Henning [4], Bondy and Fan [5], Chang [6], Chang and Nemhauser [7–9], Fraisse [11], Fricke et al. [12,13], Hattingh and Henning [14,15], Henning et al. [16–20], Meir and Moon [21], Mo and Williams [22], Slater [23], Topp and Volkmann [24], and Xin He and Yesha [25].

2. Bounds relating $i_n(G)$ and $\gamma'_n(G)$

Allan et al. [1] established the following relationship between the independent domination number and total domination number of a graph.

**Theorem A.** If $G$ is a connected graph of order $p \geq 3$, then $i(G) + \gamma_n(G) \leq p$.

Henning et al. [18] extended this result for all trees of sufficiently large order.

**Theorem B.** For an integer $n \geq 2$, if $T$ is a tree of order $p \geq 2n + 1$, then

$$i_n(T) + n \cdot \gamma'_n(T) \leq p.$$ 

In this paper, we show that if $G$ is a connected graph on at least $p \geq 2n + 1$ vertices, then $i_n(G) + n \cdot \gamma'_n(G) \leq p$ for all integers $n \geq 1$. Note that this result is not an immediate consequence of Theorem B. For suppose $T$ is a spanning tree of a connected graph $G$. Then any total $n$-dominating set of $T$ is also a total $n$-dominating set of $G$, so $\gamma'_n(G) \leq \gamma'_n(T)$. However, an $n$-independent set of $T$ is not necessarily an $n$-independent set of $G$. For each positive integer $n$, Hedetniemi et al. [12] establish the existence of a connected graph $G$ every spanning tree $T$ of which satisfies $i_n(T) < i_n(G)$.

In what follows, let $n$ be a positive integer. First, we present an algorithm for finding a total $n$-dominating set in a connected graph of order at least $2n + 1$.

**Algorithm 1.** Given a connected graph $G$ on $p \geq 2n + 1$ vertices:

1. Find a rooted spanning tree $T$ of $G$ and let $r$ be the root of $T$.
2. If $h(T) \leq n$, then let $v$ be a vertex of $T$ different from $r$. Set $\mathcal{D} = \{v, r\}$, label $v$ and $r$ with the label $L(v) = L(r) = d$ (where $d$ indicates that $v$ and $r$ are...
n-dominated by a vertex of \( \mathcal{D} \) other than itself), and stop. The set \( \mathcal{D} \) is a minimum total \( n \)-dominating set of \( G \). If \( h(T) > n \), then continue.

3. Set \( i = 0 \), let \( T_0 = T \), \( \ell'_0 = h(T_0) \), \( D_0 = \emptyset \), and continue.

4. Select a leaf of \( T_i \) at level \( \ell_i \) and let \( v_{i+1} \) be the ancestor of the leaf at level \( \ell_i - n \).
   Let \( F_{i+1} \) be the subtree of \( T_i \) consisting of \( v_{i+1} \) and all its descendants. (Then \( F_{i+1} \)
   is a rooted tree with root \( v_{i+1} \) of height \( n \).)

5. Let \( \mathcal{D}_{i+1} = \mathcal{D}_i \cup \{ v_{i+1} \} \).
   
   5.1. Assign the label \( L(v) = d \) to all vertices of \( \mathcal{D}_{i+1} \) that are \( n \)-dominated by some
   other vertex of \( \mathcal{D}_{i+1} \) in \( G \).
   
   5.2. \( i \leftarrow i + 1 \) and continue.

6. If \( F_i = T_{i-1} \) and \( L(v_i) = d \), then go to Step 13.
   If \( F_i = T_{i-1} \) and \( v_i \) is unlabeled, then consider \( |V(F_i)| \).
   
   6.1. If \( |V(F_i)| \geq 2n + 1 \), then let \( v_{i+1} \) be a child of \( v_i \) and let \( F_{i+1} = F_i \).
   Also, let the grand family of \( F_i \) be defined by \( GF_i = F_i \). Go to Step 15.
   
   6.2. If \( |V(F_i)| < 2n + 1 \), then let \( v'_i \) be the vertex that immediately precedes \( v_i \)
   on the path \( v_{i-1} \cdots v_i \) path (of length \( n + 1 \)) in \( T \). Then \( v'_i \) \( n \)-dominates \( F_i \) and
   \( d_T(v'_i, v_{i-1}) = n \). Go to Step 14.

   Otherwise \( F_i \neq T_{i-1} \) and we let \( T_i = T_{i-1} - V(F_i) \). Let \( \ell_i = h(T_i) \) and continue.

7. If each vertex of \( \mathcal{D}_i \) is labeled (with the label \( d \)), then let \( \ell'_i = 0 \). Otherwise, let
   \( \ell'_i \) be the maximum level among all the unlabeled vertices of \( \mathcal{D}_i \). Continue.

8. If \( \ell_i \geq n \) and \( \ell_i > \ell'_i \), then go to Step 4; otherwise, continue.

9. If \( \ell_i \geq n \) and \( \ell_i \leq \ell'_i \), then go to Step 11; otherwise, continue.

10. If \( \ell'_i < n \), then go to Step 12.

11. Let \( v_t \) be the first unlabeled vertex of \( \mathcal{D}_i \) (so \( t \) is the smallest integer such that
    \( v_t \) is unlabeled). Then \( \ell(v_t) = \ell'_i \). Let \( v_{i+1} \) be that vertex at level \( \ell'_i - n \) that
    is an ancestor of \( v_t \). Let \( F_{i+1} \) be the subtree of \( T_i \) consisting of \( v_{i+1} \) and all its
    descendants. Then \( F_{i+1} \) is a rooted tree with root \( v_{i+1} \) of height \( n - 1 \) or \( n \). (Note
    that \( d_T(v_t, v_{i+1}) = n \).)

   11.1. If \( |V(F_{i+1})| \geq n + 1 \), then go to Step 5.

   11.2. If \( |V(F_{i+1})| = n \) (so \( F_{i+1} \) is a path), then form the grand family \( GF_i = \langle V(F_i) \cup V(F_{i+1}) \rangle_T \)
    induced by the vertices of \( F_i \) and \( F_{i+1} \) in \( T \), and return to Step 5.

12. If \( |V(F_i)| \geq n + 1 \), then let \( v_{i+1} \) be the root of \( T_i \) (so \( v_{i+1} = r \)) and let
    \( F_{i+1} = T_i \). (Since \( \ell_i < n \), we note that \( d(v_i, v_{i+1}) \leq n \).) Go to Step 15.

12.2. If \( |V(T_i)| \leq n \) and if there is no unlabeled vertex in \( \mathcal{D}_i \), then let \( F_{i+1} = T_i \)
    and go to Step 16. (Note that \( v_i \) \( n \)-dominates \( F_{i+1} \).)

12.3. If \( |V(T_i)| = n \) and if there is some unlabeled vertex in \( \mathcal{D}_i \), then let \( v_{i+1} \)
    be the root of \( T_i \) (so \( v_{i+1} = r \)) and let \( F_{i+1} = T_i \). Now form the grand family \( GF_i = \langle V(F_i) \cup V(F_{i+1}) \rangle_T \)
    induced by the vertices of \( F_i \) and \( F_{i+1} \) in \( T \) where \( t \) is the smallest integer such that \( v_t \) is unlabeled. (We note that
    \( d(v_t, v_{i+1}) \leq n \).) Go to Step 15.

12.4. If \( |V(T_i)| < n \) and if there is some unlabeled vertex in \( \mathcal{D}_i \), then consider
    \( |V(T_i)| - |V(F_i)| \).
12.4.1. If \(|V(T_i)| + |V(F_i)| \geq 2n + 1\), then let \(v_{i+1}\) be the child of \(v_i\) and let \(F_{i+1} = T_i\). Now form the grand family \(GF_i = \langle V(F_i) \cup V(F_{i+1}) \rangle_T\) induced by the vertices of \(F_i\) and \(F_{i+1}\) in \(T\), and go to Step 15.

12.4.2. If \(|V(T_i)| + |V(F_i)| < 2n + 1\), then set \(F_i := \langle V(T_i) \cup V(F_i) \rangle_T\). Further, let \(v'_i\) be the vertex that immediately precedes \(v_i\) on the \(v_{i-1} - v_i\) path (of length \(n + 1\)) in \(T\). Then \(v'_i\) \(n\)-dominates \(F_i\) and \(d_T(v'_i, v_{i-1}) = n\). Go to Step 14.

13. Let \(\mathcal{D} = \mathcal{D}_i\) and let \(F = \{F_1, F_2, ..., F_i\}\), and go to Step 17.

14. Let \(\mathcal{D}' = \mathcal{D}_i - \{v_i\}\). Let \(v_i \leftarrow v'_i\) and let \(\mathcal{D} = \mathcal{D}' \cup \{v_i\}\). Set \(F = \{F_1, F_2, ..., F_i\}\), and go to Step 17.

15. Let \(\mathcal{D} = \mathcal{D}_i \cup \{v_{i+1}\}\) and let \(F = \{F_1, F_2, ..., F_{i+1}\}\), and go to Step 17.

16. Let \(\mathcal{D} = \mathcal{D}_i\) and let \(F = \{F_1, F_2, ..., F_{i+1}\}\), and go to Step 17. (Note that \(|F| = |\mathcal{D}| + 1\).)

17. Label each \(v \in \mathcal{D}\) that is \(n\)-dominated by some other vertex of \(\mathcal{D}\) in \(G\) by \(L(v) = d\).

Let \(GF\) be the set of all grand families \(GF_j\). Output \(\mathcal{D}, F\) and \(GF\), and stop.

We now verify the validity of the algorithm.

**Theorem 1.** Algorithm 1 determines a total \(n\)-dominating set \(\mathcal{D}\) of a given connected graph on \(p \geq 2n + 1\) vertices.

**Proof.** It is evident that \(\mathcal{D}\) is an \(n\)-dominating set of \(G\). It remains to show that each member \(v\) in \(\mathcal{D}\) is \(n\)-dominated by some other vertex of \(\mathcal{D}\) in \(G\). It suffices to prove that at the completion of the algorithm, \(L(v) = d\) for all \(v \in \mathcal{D}\). If \(v \in \mathcal{D}\) belongs to some grand family \(GF_k\) for some \(k\), then it is evident that \(v\) is labeled. We now prove three claims.

**Claim 1.** In Step 6.2, the root \(v_i\) of \(F_i\) is the only unlabeled vertex in \(\mathcal{D}_i\).

**Proof.** Since the root \(v_i\) of \(F_i\) is unlabeled, it is evident that \(F_i = T_{i-1}\) was constructed in Step 4, so \(\ell(T_{i-1}) > \ell'_{i-1}\) (see Step 8). Furthermore, the root \(v_i\) of \(F_i\) is in fact the root \(r\) of \(T\), so \(\ell(T_{i-1}) = n\) and \(\ell(v_j) \geq n + 1\) for every \(j < i\) (for otherwise, \(d(v_i, v_j) \leq n\)). Hence, if \(\mathcal{D}_{i-1}\) contains an unlabeled vertex, then \(\ell'_{i-1} \geq n + 1 > \ell(T_{i-1})\), which produces a contradiction. Thus, each vertex of \(\mathcal{D}_{i-1}\) is labeled (so \(\ell'_{i-1} = 0\)).

**Claim 2.** In Step 12.4, the root \(v_i\) of \(F_i\) is the only unlabeled vertex of \(\mathcal{D}_i\).

**Proof.** Suppose \(v_j\) is unlabeled where \(j < i\). Then it is evident that every internal vertex of the \(v_j-v_i\) path belongs to \(T_i\). Since \(|V(T_i)| < n\), this path has length at most \(|V(T_i)| + 1 < n + 1\), so \(d(v_i, v_j) \leq n\). This contradicts the fact that \(v_j\) is unlabeled. Hence, \(v_i\) is labeled for each \(j < i\), so \(v_i\) is the only unlabeled vertex of \(\mathcal{D}_i\).

**Claim 3.** In Step 12.4.2, the \(v_{i-1}-v_i\) path in \(T\) does not contain \(r\), where \(v_i\) is the root of \(F_i\).
Proof. By Claim 2, $v_i$ is the only unlabeled vertex of $\mathcal{D}_i$. Since $v_i$ is unlabeled, it is evident that $F_i$ was constructed in Step 4, so $|V(F_i)| \geq n + 1$. We show that for each $j < i$, the $v_j-v_i$ path in $T$ does not contain $r$. If this is not the case, then let $j$ be the largest integer for which the $v_j-v_i$ path contains $r$. Then every internal vertex of the $v_j-v_i$ path belongs to $T_i$, so $d(v_i,v_j) \leq n$, contradicting the fact that $v_i$ is unlabeled. Hence, for each $j < i$, the $v_j-v_i$ path in $T$ does not contain $r$. In particular, the $v_{i-1}-v_i$ path in $T$ does not contain $r$. □

In the view of Claims 1 - 3, it is easily seen that each vertex $v$ of $\mathcal{D}$ is labeled at the completion of the algorithm. Hence, $\mathcal{D}$ is a total $n$-dominating set of $G$. □

Theorem 2. If $G$ is a connected graph on $p \geq 2n + 1$ vertices, then

$$i_n(G) + n\gamma^*_n(G) \leq p.$$  

Proof. Apply Algorithm 1 to the graph $G$. If $h(T) \leq n$, then $\mathcal{D} = \{v,r\}$ is a minimum total $n$-dominating set of $G$ and $S = \{r\}$ is a minimum $n$-independent dominating set of $G$, so $p = |V(G)| \geq 2n + 1 = n|\mathcal{D}| + |S| = i_n(G) + n\gamma^*_n(G)$. Hence, we may assume that $h(T) > n$, for otherwise there is nothing left to prove.

Let $\mathcal{D} = \{v_1,v_2,\ldots,v_m\}$ be the set $\mathcal{D}$ described by the algorithm where $v_m$ is the last vertex chosen by the algorithm. Then, by Theorem 1, $\mathcal{D}$ is a total $n$-dominating set of $G$, so $\gamma^*_n(G) \leq |\mathcal{D}|$. Let $GF$ be the set of all grand families $GF_k$ described by the algorithm, and let $S' = \{v_k \in \mathcal{D} | v_k \in GF_k$ for some $k\}$. Note, $S'$ contains exactly one vertex from each grand family. It is easily seen that $S'$ is an $n$-independent set in $G$. Let $S$ be any maximal $n$-independent set in $G$ that contains $S'$, so $i_n(G) \leq |S|$. For $j = 1,2,\ldots,[F]$, let $b_j = \beta_n(F_j)$, where the $n$-independent number $\beta_n(F_j)$ of $F_j$ is the maximum cardinality among the $n$-independent sets of vertices of $F_j$ in $G$.

For each $GF_k \in GF$, we have $S \cap V(GF_k) = \{v_k\}$ since $S' \subseteq S$. Thus, any grand family $GF_k$ in $GF$ contains two members of $\mathcal{D}$ and one member of $S$. Hence, since each grand family $GF_k$ has at least $2n + 1$ vertices, it follows that

$$|V(GF_k)| \geq 2n + 1 = |\mathcal{D} \cap V(GF_k)| \cdot n + |S \cap V(GF_k)|.$$

If $|F| = |\mathcal{D}| + 1$, then $|\mathcal{D} \cap V(F_{m+1})| = 0$ and $|S \cap V(F_{m+1})| \leq |V(F_{m+1})|$, so $|V(F_{m+1})| \geq |\mathcal{D} \cap V(F_{m+1})| \cdot n + |S \cap V(F_{m+1})|$. If $|F| = |\mathcal{D}|$, and if $F_j \in F$ is not labeled, then $|\mathcal{D} \cap V(F_j)| = \{v_j\}$ and $|S \cap V(F_j)| \leq b_j$. Furthermore, since $F_j$ is not labeled, $F_j$ was constructed in Steps 4, 6.2, 11.1, 12.1 or 12.4.2. In all cases, however, $|V(F_j)| \geq n + 1$ and any $n$-independent set of $F_j$ is of cardinality at most $|V(F_j)| - n$. Hence, if $|F| = |\mathcal{D}|$, and $F_j \in F$ is not labeled, then

$$|V(F_j)| \geq n + \beta_n(F_j)$$

$$= |\mathcal{D} \cap V(F_j)| \cdot n + \beta_n(F_j)$$

$$\geq |\mathcal{D} \cap V(F_j)| \cdot n + |S \cap V(F_j)|.$$
Hence,

\[ p = |V(G)| = \sum_{GF_k \in GF} |V(GF_k)| + \sum_{F_j \text{ unlabeled}} |V(F_j)| \]
\[ \geq \sum_{GF_k \in GF} \left[ (\emptyset \cap V(GF_k)) \cdot n + |S \cap V(GF_k)| \right] \]
\[ + \sum_{F_j \text{ unlabeled}} \left[ (\emptyset \cap V(F_j)) \cdot n + |S \cap V(F_j)| \right] \]
\[ = |\emptyset| \cdot n + |S| \geq n \cdot \gamma_n^1(G) + i_n(G), \]

so

\[ i_n(G) + n \cdot \gamma_n^1(G) \leq p. \]

That the bound in Theorem 2 is best possible may be seen by considering the graph \( G \) obtained from a star \( K(1,k), k \geq 1 \), by subdividing each edge \( 2n \) times. Then \( p(G) = (2n+1)k + 1, \gamma_n^1(G) = 2k \) and \( i_n(G) = k + 1 \), so that \( i_n(G) + n \cdot \gamma_n^1(G) = p(G) \).

References