DEFORMATIONS OF LEGENDRIAN CURVES

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Abstract. We construct versal and equimultiple versal deformations of the parametrization of a Legendrian curve.

1. Contact Geometry

Let $(X, \mathcal{O}_X)$ be a complex manifold of dimension 3. A differential form ω of degree 1 is said to be a *contact form* if $\omega \wedge d\omega$ never vanishes. Let ω be a contact form. By Darboux's theorem for contact forms there is locally a system of coordinates (x, y, p) such that $\omega = dy - pdx$. If ω is a contact form and f is a holomorphic function that never vanishes, $f\omega$ is also a contact form. We say that a locally free subsheaf $\mathcal{L}$ of Ω^1_X is a *contact structure* on X if $\mathcal{L}$ is locally generated by a contact form. If $\mathcal{L}$ is a contact structure on X the pair $(X, \mathcal{L})$ is called a *contact manifold*. Let $(X_1, \mathcal{L}_1)$ and $(X_2, \mathcal{L}_2)$ be contact manifolds. Let $\chi : X_1 \to X_2$ be a holomorphic map. We say that χ is a *contact transformation* if $\chi^*\omega$ is a local generator of $\mathcal{L}_1$ whenever ω is a local generator of $\mathcal{L}_2$.

Let $\theta = \xi dx + \eta dy$ denote the canonical 1-form of $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$. Let $\pi : \mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ be the *projective cotangent bundle* of $\mathbb{C}^2$, where $\pi(x, y; \xi : \eta) = (x, y)$. Let $U\,[V]$ be the open subset of $\mathbb{P}^*\mathbb{C}^2$ defined by $\eta \neq 0\,[\xi \neq 0]$. Then $\theta/\eta\,[\theta/\xi]$ defines a contact form $dy - pdx\,[dx - qdy]$ on $U\,[V]$, where $p = -\xi/\eta\,[q = -\eta/\xi]$. Moreover, $dy - pdx$ and $dx - qdy$ define a structure of contact manifold on $\mathbb{P}^*\mathbb{C}^2$.

If $\Phi(x, y) = (a(x, y), b(x, y))$ with $a, b \in \mathbb{C}\{x, y\}$ is an automorphism of $(\mathbb{C}^2, (0, 0))$, we associate to Φ the germ of contact transformation

$$\chi : (\mathbb{P}^*\mathbb{C}^2, (0, 0; 0 : 1)) \to \big(\mathbb{P}^*\mathbb{C}^2, (0, 0; -\partial_x b(0, 0) : \partial_x a(0, 0)\big)$$

defined by

$$\chi(x, y; \xi : \eta) = (a(x, y), b(x, y); \partial_y b\xi - \partial_x b\eta : -\partial_y a\xi + \partial_x a\eta)\,. \tag{1.1}$$

If $D\Phi_{(0,0)}$ leaves invariant $\{y = 0\}$, then $\partial_x b(0, 0) = 0$, $\partial_x a(0, 0) \neq 0$ and $\chi(0, 0; 0 : 1) = (0, 0; 0 : 1)$. Moreover,

$$\chi(x, y, p) = (a(x, y), b(x, y), (\partial_y bp + \partial_x b)/(\partial_y ap + \partial_x a))\,.$$

Let $(X, \mathcal{L})$ be a contact manifold. A curve L in X is called *Legendrian* if $\omega|_L = 0$ for each section ω of $\mathcal{L}$.

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Let Z be the germ at $(0,0)$ of an irreducible plane curve parametrized by

$$\varphi(t) = (x(t), y(t)). \tag{1.2}$$

We define the *conormal* of Z as the curve parametrized by

$$\psi(t) = (x(t), y(t); -y'(t) : x'(t)). \tag{1.3}$$

The conormal of Z is the germ of a Legendrian curve of $\mathbb{P}^*\mathbb{C}^2$.

We will denote the conormal of Z by $\mathbb{P}^*_Z\mathbb{C}^2$ and the parametrization (1.3) by $\mathcal{C}on\,\varphi$.

Assume that the tangent cone $C(Z)$ is defined by the equation $ax+by=0$, with $(a,b) \neq (0,0)$. Then $\mathbb{P}^*_Z\mathbb{C}^2$ is a germ of a Legendrian curve at $(0,0;a:b)$.

Let $f \in \mathbb{C}\{t\}$. We say the f has order k and write $ord\, f = k$ or $ord_t\, f = k$ if f/t^k is a unit of $\mathbb{C}\{t\}$.

Remark 1.1. Let Z be the plane curve parametrized by (1.2). Let $L = \mathbb{P}^*_Z\mathbb{C}^2$. Then:

(i) $C(Z) = \{y=0\}$ if and only if $ord\, y > ord\, x$. If $C(Z) = \{y=0\}$, L admits the parametrization
$$\psi(t) = (x(t), y(t), y'(t)/x'(t))$$
on the chart (x,y,p).

(ii) $C(Z) = \{y=0\}$ and $C(L) = \{x=y=0\}$ if and only if $ord\, x < ord\, y < 2ord\, x$.

(iii) $C(Z) = \{y=0\}$ and $\{x=y=0\} \not\subseteq C(L) \subset \{y=0\}$ if and only if $ord\, y \geq 2ord\, x$.

(iv) $C(L) = \{y=p=0\}$ if and only if $ord\, y > 2ord\, x$.

(v) $mult\, L \leq mult\, Z$. Moreover, $mult\, L = mult\, Z$ if and only if $ord\, y \geq 2ord\, x$.

If L is the germ of a Legendrian curve at $(0,0;a:b)$, $\pi(L)$ is a germ of a plane curve of $(\mathbb{C}^2,(0,0))$. Notice that all branches of $\pi(L)$ have the same tangent cone.

If Z is the germ of a plane curve with irreducible tangent cone, the union L of the conormal of the branches of Z is a germ of a Legendrian curve. We call L the *conormal* of Z.

If $C(Z)$ has several components, the union of the conormals of the branches of Z is a union of several germs of Legendrian curves.

If L is a germ of Legendrian curve, L is the conormal of $\pi(L)$.

Consider in the vector space $\mathbb{C}^2$, with coordinates x,p, the symplectic form $dp \wedge dx$. We associate to each symplectic linear automorphism

$$(p,x) \mapsto (\alpha p + \beta x, \gamma p + \delta x)$$

of $\mathbb{C}^2$ the contact transformation

$$(x,y,p) = (\gamma p + \delta x, y + \frac{1}{2}\alpha\gamma p^2 + \beta\gamma x p + \frac{1}{2}\beta\delta x^2, \alpha p + \beta x). \tag{1.4}$$

We call (1.4) a *paraboloidal contact transformation.*

In the case $\alpha = \delta = 0$ and $\gamma = -\beta = 1$ we get the so called *Legendre* transformation

$$\Psi(x, y, p) = (p, y - px, -x).$$

We say that a germ of a Legendrian curve L of $(\mathbb{P}^*\mathbb{C}^2, (0, 0; a : b))$ is in *generic position* if $C(L) \not\supset \pi^{-1}(0, 0)$.

Remark 1.2. Let L be the germ of a Legendrian curve on a contact manifold $(X, \mathcal{L})$ at a point o. By the Darboux's theorem for contact forms there is a germ of a contact transformation $\chi : (X, o) \to (U, (0, 0, 0))$, where $U = \{\eta \neq 0\}$ is the open subset of $\mathbb{P}^*\mathbb{C}^2$ considered above. Hence $C(\pi(\chi(L))) = \{y = 0\}$. Applying a paraboloidal transformation to $\chi(L)$ we can assume that $\chi(L)$ is in generic position. If $C(L)$ is irreducible, we can assume $C(\chi(L)) = \{y = p = 0\}$.

Following the above remark, from now on we will always assume that every Legendrian curve germ is embedded in $(\mathbb{C}^3_{(x,y,p)}, \omega)$, where $\omega = dy - pdx$.

Example 1.3.

(1) The plane curve $Z = \{y^2 - x^3 = 0\}$ admits a parametrization $\varphi(t) = (t^2, t^3)$. The conormal L of Z admits the parametrization $\psi(t) = (t^2, t^3, \frac{3}{2}t)$. Hence $C(L) = \pi^{-1}(0, 0)$ and L is not in generic position. If χ is the Legendre transformation, $C(\chi(L)) = \{y = p = 0\}$ and L is in generic position. Moreover, $\pi(\chi(L))$ is a smooth curve.

(2) The plane curve $Z = \{(y^2 - x^3)(y^2 - x^5) = 0\}$ admits a parametrization given by

$$\varphi_1(t_1) = ({t_1}^2, {t_1}^3), \quad \varphi_2(t_2) = ({t_2}^2, {t_2}^5).$$

The conormal L of Z admits the parametrization given by

$$\psi_1(t_1) = ({t_1}^2, {t_1}^3, \frac{3}{2}t_1), \quad \psi_2(t_2) = ({t_2}^2, {t_2}^5, \frac{5}{2}{t_2}^3).$$

Hence $C(L_1) = \pi^{-1}(0, 0)$ and L is not in generic position. If χ is the paraboloidal contact transformation

$$\chi : (x, y, p) \mapsto (x + p, y + \frac{1}{2}p^2, p),$$

then $\chi(L)$ has branches with parametrization given by

$$\chi(\psi_1)(t_1) = ({t_1}^2 + \frac{3}{2}t_1, {t_1}^3 + \frac{9}{8}{t_1}^2, \frac{3}{2}t_1),$$
$$\chi(\psi_2)(t_2) = ({t_2}^2 + \frac{5}{2}{t_2}^3, {t_2}^5 + \frac{25}{8}{t_2}^6, \frac{5}{2}{t_2}^3).$$

Then

$$C(\chi(L_1)) = \{y = p - x = 0\}, \quad C(\chi(L_2)) = \{y = p = 0\}$$

and L is in generic position.

2. Relative Contact Geometry

Set $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{z} = (z_1, \dots, z_m)$. Let I be an ideal of the ring $\mathbb{C}\{\mathbf{z}\}$. Let $\widetilde{I}$ be the ideal of $\mathbb{C}\{\mathbf{x}, \mathbf{z}\}$ generated by I.

Lemma 2.1. *(a) Let $f \in C\{\mathbf{x}, \mathbf{z}\}$, $f = \sum_\alpha a_\alpha \mathbf{x}^\alpha$ with $a_\alpha \in \mathbb{C}\{\mathbf{z}\}$. Then $f \in \widetilde{I}$ if and only if $a_\alpha \in I$ for each α.*

(b) If $f \in \widetilde{I}$, then $\partial_{x_i} f \in \widetilde{I}$ for $1 \leq i \leq n$.

(c) Let $a_1, \dots, a_{n-1} \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\}$. Let $b, \beta_0 \in \widetilde{I}$. Assume that $\partial_{x_n} \beta_0 = 0$. If β is the solution of the Cauchy problem

$$\partial_{x_n}\beta - \sum_{i=1}^{n-1} a_i \partial_{x_i}\beta = b, \qquad \beta - \beta_0 \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\}x_n, \tag{2.1}$$

then $\beta \in \widetilde{I}$.

Proof. There are $g_1, \dots, g_\ell \in \mathbb{C}\{\mathbf{z}\}$ such that $I = (g_1, \dots, g_\ell)$. If $a_\alpha \in I$ for each α, there are $h_{i,\alpha} \in \mathbb{C}\{\mathbf{z}\}$ such that $a_\alpha = \sum_{i=1}^\ell h_{i,\alpha} g_i$. Hence $f = \sum_{i=1}^\ell (\sum_\alpha h_{i,\alpha}\mathbf{x}^\alpha) g_i \in \widetilde{I}$.

If $f \in \widetilde{I}$, there are $H_i \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\}$ such that $f = \sum_{i=1}^\ell H_i g_i$. There are $b_{i,\alpha} \in \mathbb{C}\{\mathbf{z}\}$ such that $H_i = \sum_\alpha b_{i,\alpha}\mathbf{x}^\alpha$. Therefore $a_\alpha = \sum_{i=1}^\ell b_{i,\alpha} g_i \in I$.

We can perform a change of variables that rectifies the vector field $\partial_{x_n} - \sum_{i=1}^{n-1} a_i \partial_{x_i}$, reducing the Cauchy problem (2.1) to the Cauchy problem

$$\partial_{x_n}\beta = b, \qquad \beta - \beta_0 \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\}x_n.$$

Hence statements (b) and (c) follow from (a). ☐

Let J be an ideal of $\mathbb{C}\{\mathbf{z}\}$ contained in I. Let X, S and T be analytic spaces with local rings $\mathbb{C}\{\mathbf{x}\}, \mathbb{C}\{\mathbf{z}\}/I$ and $\mathbb{C}\{\mathbf{z}\}/J$. Hence $X \times S$ and $X \times T$ have local rings $\mathcal{O} := \mathbb{C}\{\mathbf{x}, \mathbf{z}\}/\widetilde{I}$ and $\widetilde{\mathcal{O}} := \mathbb{C}\{\mathbf{x}, \mathbf{z}\}/\widetilde{J}$. Let $\mathbf{a_1}, \dots, \mathbf{a_{n-1}}, \mathbf{b} \in \mathcal{O}$ and $\mathbf{g} \in \mathcal{O}/x_n\mathcal{O}$. Let $a_i, b \in \widetilde{\mathcal{O}}$ and $g \in \widetilde{\mathcal{O}}/x_n\widetilde{\mathcal{O}}$ be representatives of $\mathbf{a_i}, \mathbf{b}$ and $\mathbf{g}$. Consider the Cauchy problems

$$\partial_{x_n} f + \sum_{i=1}^{n-1} a_i \partial_{x_i} f = b, \qquad f + x_n\widetilde{\mathcal{O}} = g \tag{2.2}$$

and

$$\partial_{x_n} \mathbf{f} + \sum_{i=1}^{n-1} \mathbf{a_i} \partial_{x_i} \mathbf{f} = \mathbf{b}, \qquad \mathbf{f} + x_n\mathcal{O} = \mathbf{g}. \tag{2.3}$$

Theorem 2.2. *(a) There is one and only one solution of the Cauchy problem (2.2).*

(b) If f is a solution of (2.2), $\mathbf{f} = f + \widetilde{I}$ is a solution of (2.3).

(c) If $\mathbf{f}$ is a solution of (2.3) there is a representative f of $\mathbf{f}$ that is a solution of (2.2).

Proof. By Lemma 2.1, $\partial_{x_i}\widetilde{I} = \widetilde{I}$. Hence (b) holds.

Assume $J = (0)$. The existence and uniqueness of the solution of (2.2) is a special case of the classical Cauchy-Kowalevski Theorem. There is one and only one formal solution of (2.2). Its convergence follows from the majorant method.

The existence of a solution of (2.3) follows from (b).

Let $\mathbf{f_1}, \mathbf{f_2}$ be two solutions of (2.3). Let f_j be a representative of $\mathbf{f_j}$ for $j = 1, 2$. Then $\partial_{x_n}(f_2 - f_1) + \sum_{i=1}^{n-1} a_i \partial_{x_i}(f_2 - f_1) \in \widetilde{I}$ and $f_2 - f_1 + x_n\widetilde{\mathcal{O}} \in \widetilde{I} + x_n\widetilde{\mathcal{O}}$. By Lemma 2.1, $f_2 - f_1 \in \widetilde{I}$. Therefore $\mathbf{f_1} = \mathbf{f_2}$. This ends the proof of statement (a). Statement (c) follows from statements (a) and (b). □

Set $\Omega^1_{X|S} = \bigoplus_{i=1}^n \mathcal{O}dx_i$. We call the elements of $\Omega^1_{X|S}$ *germs of relative differential forms* on $X \times S$. The map $d : \mathcal{O} \to \Omega^1_{X|S}$ given by $df = \sum_{i=1}^n \partial x_i f dx_i$ is called the *relative differential* of f.

Assume that $dim\, X = 3$ and let $\mathcal{L}$ be a contact structure on X. Let $\rho : X \times S \to X$ be the first projection. Let ω be a generator of $\mathcal{L}$. We will denote by $\mathcal{L}_S$ the sub $\mathcal{O}$-module of $\Omega^1_{X|S}$ generated by $\rho^*\omega$. We call $\mathcal{L}_S$ a *relative contact structure* of $X \times S$. We call $(X \times S, \mathcal{L}_S)$ a relative contact manifold. We say that an isomorphism of analytic spaces

$$\chi : X \times S \to X \times S \tag{2.4}$$

is a *relative contact transformation* if $\chi(\mathbf{0}, s) = (\mathbf{0}, s)$, $\chi^*\omega \in \mathcal{L}_S$ for each $\omega \in \mathcal{L}_S$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{id_X} & X \\ \downarrow & & \downarrow \\ X \times S & \xrightarrow{\chi} & X \times S \\ \downarrow & & \downarrow \\ S & \xrightarrow{id_S} & S \end{array} \tag{2.5}$$

commutes.

The demand of the commutativeness of diagram (2.5) is a very restrictive condition but these are the only relative contact transformations we will need. We can and will assume that the local ring of X equals $\mathbb{C}\{x, y, p\}$ and that $\mathcal{L}$ is generated by $dy - pdx$.

Set $\mathcal{O} = \mathbb{C}\{x, y, p, \mathbf{z}\}/\widetilde{I}$ and $\widetilde{\mathcal{O}} = \mathbb{C}\{x, y, p, \mathbf{z}\}/\widetilde{J}$. Let $\mathfrak{m}_X$ be the maximal ideal of $\mathbb{C}\{x, y, p\}$. Let $\mathfrak{m}\,[\widetilde{\mathfrak{m}}]$ be the maximal ideal of $\mathbb{C}\{\mathbf{z}\}/I\,[\mathbb{C}\{\mathbf{z}\}/J]$. Let $\mathfrak{n}\,[\widetilde{\mathfrak{n}}]$ be the ideal of $\mathcal{O}\,[\widetilde{\mathcal{O}}]$ generated by $\mathfrak{m}_X\mathfrak{m}\,[\mathfrak{m}_X\widetilde{\mathfrak{m}}]$.

Remark 2.3. If (2.4) is a relative contact transformation, there are $\alpha, \beta, \gamma \in \mathfrak{n}$ such that $\partial_x \beta \in \mathfrak{n}$ and

$$\chi(x, y, p, \mathbf{z}) = (x + \alpha, y + \beta, p + \gamma, \mathbf{z}). \tag{2.6}$$

Theorem 2.4. *(a) Let $\chi : X \times S \to X \times S$ be a relative contact transformation. There is $\beta_0 \in \mathfrak{n}$ such that $\partial_p \beta_0 = 0$, $\partial_x \beta_0 \in \mathfrak{n}$, β is the solution of the Cauchy problem*

$$\left(1 + \frac{\partial \alpha}{\partial x} + p\frac{\partial \alpha}{\partial y}\right)\frac{\partial \beta}{\partial p} - p\frac{\partial \alpha}{\partial p}\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial p}\frac{\partial \beta}{\partial x} = p\frac{\partial \alpha}{\partial p}, \qquad \beta - \beta_0 \in p\mathcal{O} \tag{2.7}$$

and

$$\gamma = \left(1 + \frac{\partial \alpha}{\partial x} + p\frac{\partial \alpha}{\partial y}\right)^{-1}\left(\frac{\partial \beta}{\partial x} + p\left(\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x} - p\frac{\partial \alpha}{\partial y}\right)\right). \tag{2.8}$$

(b) Given $\alpha, \beta_0 \in \mathfrak{n}$ such that $\partial_p \beta_0 = 0$ and $\partial_x \beta_0 \in \mathfrak{n}$, there is a unique contact transformation χ verifying the conditions of statement (a). We will denote χ by χ_{α,β_0}.

(c) Given a relative contact transformation $\widetilde{\chi} : X \times T \to X \times T$ there is one and only one contact transformation $\chi : X \times S \to X \times S$ such that the diagram

$$\begin{array}{ccc} X \times S & \xrightarrow{\chi} & X \times S \\ \downarrow & & \downarrow \\ X \times T & \xrightarrow{\widetilde{\chi}} & X \times T \end{array} \tag{2.9}$$

commutes.

(d) Given $\alpha, \beta_0 \in \mathfrak{n}$ and $\widetilde{\alpha}, \widetilde{\beta}_0 \in \widetilde{\mathfrak{n}}$ such that $\partial_p \beta_0 = 0$, $\partial_p \widetilde{\beta}_0 = 0$, $\partial_x \beta_0 \in \mathfrak{n}$, $\partial_x \widetilde{\beta}_0 \in \widetilde{\mathfrak{n}}$ and $\widetilde{\alpha}, \widetilde{\beta}_0$ are representatives of α, β_0, set $\chi = \chi_{\alpha,\beta_0}$, $\widetilde{\chi} = \chi_{\widetilde{\alpha},\widetilde{\beta}_0}$. Then diagram (2.9) commutes.

Proof. Statements (a) and (b) are a relative version of Theorem 3.2 of [1]. In [1] we assume $S = \{0\}$. The proof works as long S is smooth. The proof in the singular case depends on the singular variant of the Cauchy-Kowalevski Theorem introduced in 2.2. Statement (c) follows from statement (b) of Theorem 2.2. Statement (d) follows from statement (c) of Theorem 2.2. □

Remark 2.5. (1) The inclusion $S \hookrightarrow T$ is said to be a *small extension* if the surjective map $\mathcal{O}_T \twoheadrightarrow \mathcal{O}_S$ has one dimensional kernel. If the kernel is generated by ε, we have that, as complex vector spaces, $\mathcal{O}_T = \mathcal{O}_S \oplus \varepsilon\mathbb{C}$. Every extension of Artinian local rings factors through small extensions.

Theorem 2.6. *Let $S \hookrightarrow T$ be a small extension such that*

$$\mathcal{O}_S \cong \mathbb{C}\{\mathbf{z}\},$$
$$\mathcal{O}_T \cong \mathbb{C}\{\mathbf{z}, \varepsilon\}/(\varepsilon^2, \varepsilon z_1, \ldots \varepsilon z_m) = \mathbb{C}\{\mathbf{z}\} \oplus \mathbb{C}\varepsilon.$$

Assume $\chi : X \times S \to X \times S$ is a relative contact transformation given at the ring level by

$$(x,y,p) \mapsto (H_1, H_2, H_3),$$

$\alpha, \beta_0 \in \mathfrak{m}_X$, such that $\partial_p \beta_0 = 0$ and $\beta_0 \in (x^2, y)$. Then, there are uniquely determined $\beta, \gamma \in \mathfrak{m}_X$ such that $\beta - \beta_0 \in p\mathcal{O}_X$ and $\widetilde{\chi}: \ X \times T \to X \times T$, given by

$$\widetilde{\chi}(x,y,p,\mathbf{z},\varepsilon) = (H_1 + \varepsilon\alpha, H_2 + \varepsilon\beta, H_3 + \varepsilon\gamma, \mathbf{z}, \varepsilon),$$

is a relative contact transformation extending χ (diagram (2.9) commutes). Moreover, the Cauchy problem (2.7) for $\widetilde{\chi}$ takes the simplified form

$$\frac{\partial \beta}{\partial p} = p\frac{\partial \alpha}{\partial p}, \qquad \beta - \beta_0 \in \mathbb{C}\{x,y,p\}p \tag{2.10}$$

and

$$\gamma = \frac{\partial \beta}{\partial x} + p(\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x}) - p^2 \frac{\partial \alpha}{\partial y}. \tag{2.11}$$

Proof. We have that $\widetilde{\chi}$ is a relative contact transformation if and only if there is $f := f' + \varepsilon f'' \in \mathcal{O}_T\{x,y,p\}$ with $f \notin (x,y,p)\mathcal{O}_T\{x,y,p\}$, $f' \in \mathcal{O}_S\{x,y,p\}$, $f'' \in \mathbb{C}\{x,y,p\} = \mathcal{O}_X$ such that

$$d(H_2 + \varepsilon\beta) - (H_3 + \varepsilon\gamma)d(H_1 + \varepsilon\alpha) = f(dy - pdx). \tag{2.12}$$

Since χ is a relative contact transformation we can suppose that

$$dH_2 - H_3 dH_1 = f'(dy - pdx).$$

Using the fact that $\varepsilon\mathfrak{m}_{\mathcal{O}_T}$ we see that (2.12) is equivalent to

$$\begin{cases} \frac{\partial \beta}{\partial p} = p\frac{\partial \alpha}{\partial p}, \\ \gamma = \frac{\partial \beta}{\partial x} + p(\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x}) - p^2\frac{\partial \alpha}{\partial y}, \\ f'' = \frac{\partial \beta}{\partial y} - p\frac{\partial \alpha}{\partial y}. \end{cases}$$

As $\beta - \beta_0 \in (p)\mathbb{C}\{x,y,p\}$ we have that β, and consequently γ, are completely determined by α and β_0. □

Remark 2.7. Set $\alpha = \sum_k \alpha_k p^k$, $\beta = \sum_k \beta_k p^k$, $\gamma = \sum_k \gamma_k p^k$, where $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}\{x,y\}$ for each $k \geq 0$ and $\beta_0 \in (x^2, y)$. Under the assumptions of Theorem 2.6,

(i) $\beta_k = \frac{k-1}{k}\alpha_{k-1}, \qquad k \geq 1$.

(ii) Moreover,

$$\gamma_0 = \frac{\partial \beta_0}{\partial x},\ \gamma_1 = \frac{\partial \beta_0}{\partial y} - \frac{\partial \alpha_0}{\partial x},\ \gamma_k = -\frac{1}{k}\frac{\partial \alpha_{k-1}}{\partial x} - \frac{1}{k-1}\frac{\partial \alpha_{k-2}}{\partial y}, \qquad k \geq 2.$$

Since,

$$\frac{\partial}{\partial y}\gamma_0 = \frac{\partial}{\partial x}(\frac{\partial \alpha_0}{\partial x} + \gamma_1),$$

β_0 is the solution of the Cauchy problem

$$\frac{\partial \beta_0}{\partial x} = \gamma_0, \quad \frac{\partial \beta_0}{\partial y} = \frac{\partial \alpha_0}{\partial x} + \gamma_1, \qquad \beta_0 \in (x^2, y).$$

3. Categories of Deformations

A category $\mathfrak{C}$ is called a *groupoid* if all morphisms of $\mathfrak{C}$ are isomorphisms.

Let $p : \mathfrak{F} \to \mathfrak{C}$ be a functor.

Let S be an object of $\mathfrak{C}$. We will denote by $\mathfrak{F}(S)$ the subcategory of $\mathfrak{F}$ given by the following conditions:

- Ψ is an object of $\mathfrak{F}(S)$ if $p(\Psi) = S$.
- χ is a morphism of $\mathfrak{F}(S)$ if $p(\chi) = id_S$.

Let $\chi\,[\Psi]$ be a morphism [an object] of $\mathfrak{F}$. Let $f\,[S]$ be a morphism [an object] of $\mathfrak{C}$. We say that $\chi\,[\Psi]$ is a morphism [an object] of $\mathfrak{F}$ over $f\,[S]$ if $p(\chi) = f\,[p(\Psi) = S]$.

A morphism $\chi' : \Psi' \to \Psi$ of $\mathfrak{F}$ over $f : S' \to S$ is called *cartesian* if for each morphism $\chi'' : \Psi'' \to \Psi$ of $\mathfrak{F}$ over f there is exactly one morphism $\chi : \Psi'' \to \Psi'$ over $id_{S'}$ such that $\chi' \circ \chi = \chi''$.

If the morphism $\chi' : \Psi' \to \Psi$ over f is cartesian, Ψ' is well defined up to a unique isomorphism. We will denote Ψ' by $f^*\Psi$ or $\Psi \times_S S'$.

We say that $\mathfrak{F}$ is a *fibered category* over $\mathfrak{C}$ if

1. For each morphism $f : S' \to S$ in $\mathfrak{C}$ and each object Ψ of $\mathfrak{F}$ over S there is a morphism $\chi' : \Psi' \to \Psi$ over f that is cartesian.
2. The composition of cartesian morphisms is cartesian.

A fibered groupoid is a fibered category such that $\mathfrak{F}(S)$ is a groupoid for each $S \in \mathfrak{C}$.

Lemma 3.1. *If $p : \mathfrak{F} \to \mathfrak{C}$ satisfies* (1) *and $\mathfrak{F}(S)$ is a groupoid for each object S of $\mathfrak{C}$, then $\mathfrak{F}$ is a fibered groupoid over $\mathfrak{C}$.*

Proof. Let $\chi : \Phi \to \Psi$ be an arbitrary morphism of $\mathfrak{F}$. It is enough to show that χ is cartesian. Set $f = p(\chi)$. Let $\chi' : \Phi' \to \Psi$ be another morphism over f. Let $f^*\Psi \to \Psi$ be a cartesian morphism over f. There are morphisms $\alpha : \Phi' \to f^*\Psi$, $\beta : \Phi \to f^*\Psi$ such that the solid diagram

$$\begin{array}{ccccc} f^*\Psi & \xleftarrow{\beta} & \Phi & \dashleftarrow & \Phi' \\ & \searrow & \downarrow \chi & \swarrow \chi' & \\ & & \Psi & & \end{array} \qquad (\alpha : \Phi' \to f^*\Psi) \tag{3.1}$$

commutes. Hence $\beta^{-1} \circ \alpha$ is the only morphism over f such that diagram (3.1) commutes.

□

Let $\mathfrak{A}n$ be the category of analytic complex space germs. Let 0 denote the complex vector space of dimension 0. Let $p : \mathfrak{F} \to \mathfrak{A}n$ be a fibered category.

Definition 3.2. Let T be an analytic complex space germ. Let $\psi\,[\Psi]$ be an object of $\mathfrak{F}(0)\,[\mathfrak{F}(T)]$. We say that Ψ is a *versal deformation* of ψ if given

- a closed embedding $f : T'' \hookrightarrow T'$,

- a morphism of complex analytic space germs $g : T'' \to T$,
- an object Ψ' of $\mathfrak{F}(T')$ such that $f^*\Psi' \cong g^*\Psi$,

there is a morphism of complex analytic space germs $h : T' \to T$ such that

$$h \circ f = g \qquad \text{and} \qquad h^*\Psi \cong \Psi'.$$

If Ψ is versal and for each Ψ' the tangent map $T(h) : T_{T'} \to T_T$ is determined by Ψ', Ψ is called a *semiuniversal deformation* of ψ.

Let T be a germ of a complex analytic space. Let A be the local ring of T and let $\mathfrak{m}$ be the maximal ideal of A. Let T_n be the complex analytic space with local ring $A/\mathfrak{m}^n$ for each positive integer n. The canonical morphisms

$$A \to A/\mathfrak{m}^n \qquad \text{and} \qquad A/\mathfrak{m}^n \to A/\mathfrak{m}^{n+1}$$

induce morphisms $\alpha_n : T_n \to T$ and $\beta_n : T_{n+1} \to T_n$.

A morphism $f : T'' \to T'$ induces morphisms $f_n : T''_n \to T'_n$ such that the diagram

$$\begin{array}{ccc}
T'' & \xrightarrow{f} & T' \\
\alpha''_n \big\uparrow & & \big\uparrow \alpha'_n \\
T''_n & \xrightarrow{f_n} & T'_n \\
\beta''_n \big\uparrow & & \big\uparrow \beta'_n \\
T''_{n+1} & \xrightarrow{f_{n+1}} & T'_{n+1}
\end{array}$$

commutes.

Definition 3.3. We will follow the terminology of Definition 3.2. Let $g_n = g \circ \alpha''_n$. We say that Ψ is a *formally versal deformation* of ψ if there are morphisms $h_n : T'_n \to T$ such that

$$h_n \circ f_n = g_n, \qquad h_n \circ \beta'_n = h_{n+1} \qquad \text{and} \qquad h_n^*\Psi \cong {\alpha'_n}^*\Psi'.$$

If Ψ is formally versal and for each Ψ' the tangent maps $T(h_n) : T_{T'_n} \to T_T$ are determined by ${\alpha'_n}^*\Psi'$, Ψ is called a *formally semiuniversal deformation* of ψ.

Theorem 3.4 ([4], Theorem 5.2)**.** *Let $\mathfrak{F} \to \mathfrak{C}$ be a fibered groupoid. Let $\psi \in \mathfrak{F}(0)$. If there is a versal deformation of ψ, every formally versal* [*semiuniversal*]*deformation of ψ is versal* [*semiuniversal*]*.*

Let Z be a curve of $\mathbb{C}^n$ with irreducible components $Z_1, \ldots, Z_r$. Set $\bar{\mathbb{C}} = \bigsqcup_{i=1}^r \bar{C}_i$ where each $\bar{C}_i$ is a copy of $\mathbb{C}$. Let φ_i be a parametrization of Z_i, $1 \leq i \leq r$. Let $\varphi : \bar{\mathbb{C}} \to \mathbb{C}^n$ be the map such that $\varphi|_{\bar{C}_i} = \varphi_i$, $1 \leq i \leq r$. We call φ the *parametrization* of Z.

Let T be an analytic space. A morphism of analytic spaces $\Phi : \bar{\mathbb{C}} \times T \to \mathbb{C}^n \times T$ is called a *deformation of* φ *over* T if the diagram

$$\begin{array}{ccc}
\bar{\mathbb{C}} & \xrightarrow{\varphi} & \mathbb{C}^n \\
\downarrow & & \downarrow \\
\bar{\mathbb{C}} \times T & \xrightarrow{\Phi} & \mathbb{C}^n \times T \\
\downarrow & & \downarrow \\
T & \xrightarrow{id_T} & T
\end{array}$$

commutes. The analytic space T is called de *base space* of the deformation.

We will denote by Φ_i the composition

$$\bar{C}_i \times T \hookrightarrow \bar{\mathbb{C}} \times T \xrightarrow{\Phi} \mathbb{C}^n \times T \to \mathbb{C}^n, \qquad 1 \leq i \leq r.$$

The maps Φ_i, $1 \leq i \leq r$, determine Φ.

Let Φ be a deformation of φ over T. Let $f : T' \to T$ be a morphism of analytic spaces. We will denote by $f^*\Phi$ the deformation of φ over T' given by

$$(f^*\Phi)_i = \Phi_i \circ (id_{\bar{C}_i} \times f).$$

We call $f^*\Phi$ the *pullback* of Φ by f.

Let $\Phi' : \bar{\mathbb{C}} \times T \to \mathbb{C}^n \times T$ be another deformation of φ over T. A morphism from Φ' into Φ is a pair (χ, ξ) where $\chi : \mathbb{C}^n \times T \to \mathbb{C}^n \times T$ and $\xi : \bar{\mathbb{C}} \times T \to \bar{\mathbb{C}} \times T$ are isomorphisms of analytic spaces such that the diagram

$$\begin{array}{ccccccc}
T & \longleftarrow & \bar{\mathbb{C}} \times T & \xrightarrow{\Phi} & \mathbb{C}^n \times T & \longrightarrow & T \\
\uparrow {\scriptstyle id_T} & \xi \nearrow & \uparrow & & \uparrow & \nwarrow \chi & \uparrow {\scriptstyle id_T} \\
 & & \bar{\mathbb{C}} & \xrightarrow{\varphi} & \mathbb{C}^n \times \{0\} & & \\
 & & \downarrow & & \downarrow & & \\
T & \longleftarrow & \bar{\mathbb{C}} \times T & \xrightarrow{\Phi'} & \mathbb{C}^n \times T & \longrightarrow & T
\end{array}$$

commutes.

Let Φ' be a deformation of φ over S and $f : S \to T$ a morphism of analytic spaces. A *morphism of* Φ' *into* Φ *over* f is a morphism from Φ' into $f^*\Phi$. There is a functor p that associates T to a deformation Ψ over T and f to a morphism of deformations over f.

Given $t \in T$ let Z_t be the curve parametrized by the composition

$$\bar{\mathbb{C}} \times \{t\} \hookrightarrow \bar{\mathbb{C}} \times T \xrightarrow{\Phi} \mathbb{C}^n \times T \to \mathbb{C}^n.$$

We call Z_t the *fiber of the deformation* Φ *at the point* t.

Let $\varphi : \bar{\mathbb{C}} \to \mathbb{C}^2$ be the parametrization of a plane curve Z. We will denote by $\mathcal{D}ef_\varphi\,[\mathcal{D}ef_\varphi^{em}]$ the *category of deformations [equimultiple deformations]* Φ *of (the parametrization* φ *of) the plane curve* Z.

Consider in $\mathbb{C}^3$ the contact structure given by the differential form $dy - pdx$. Let $\psi : \bar{\mathbb{C}} \to \mathbb{C}^3$ be the parametrization of a Legendrian curve L. We say that a deformation Ψ of ψ is a *Legendrian deformation of* ψ if all of its fibers are Legendrian. We say that (χ, ξ) is an isomorphism of Legendrian deformations if $\chi : X \times T \to X \times T$ is a relative contact transformation. We will denote by $\widehat{\mathcal{D}ef}_\psi\,[\widehat{\mathcal{D}ef}_\psi^{em}]$ the category of Legendrian [equimultiple Legendrian] deformations of ψ. All deformations are assumed to have trivial sections (see [3]).

Assume that $\psi = \mathcal{C}on\,\varphi$ parametrizes a germ of a Legendrian curve L, in generic position, in $(\mathbb{C}^3_{(x,y,p)}, \omega)$. If $\Phi \in \mathcal{D}ef_\varphi$ is given by

$$\Phi_i(t_i, \mathbf{s}) = (X_i(t_i, \mathbf{s}), Y_i(t_i, \mathbf{s}))\,, \qquad 1 \leq i \leq r, \tag{3.2}$$

such that $P_i(t_i, \mathbf{s}) := \partial_t Y_i(t_i, \mathbf{s}) / \partial_t X_i(t_i, \mathbf{s}) \in \mathbb{C}\{t_i, \mathbf{s}\}$ for $1 \leq i \leq r$, then

$$\Psi_i(t_i, \mathbf{s}) = (X_i(t_i, \mathbf{s}), Y_i(t_i, \mathbf{s}), P_i(t_i, \mathbf{s}))\,. \tag{3.3}$$

defines a deformation Ψ of ψ which we call *conormal of* Φ. Notice that in this case all fibers of Φ have the same tangent space $\{y = 0\}$. We will denote Ψ by $\mathcal{C}on\,\Phi$. If $\Psi \in \widehat{\mathcal{D}ef}_\psi$ is given by (3.3), we call *plane projection of* Ψ to the deformation Φ of φ given by (3.2). We will denote Φ by Ψ^π.

Let us consider the full subcategory $\overrightarrow{\mathcal{D}ef}_\varphi$ of the deformations $\Phi \in \mathcal{D}ef_\varphi^{em}$ such that all fibers of Φ have the same tangent space $\{y = 0\}$.

Remark 3.5. We see immediately that if $\Phi \in \overrightarrow{\mathcal{D}ef}_\varphi$ then $\mathcal{C}on\,\Phi$ exists. However, it should be noted that there are more deformations for which the conormal is defined:

Let Φ be the deformation of $\varphi = (t^3, t^{10})$ given by

$$X(t, s) = st + t^3; \quad Y(t, s) = \frac{5}{12} st^8 + t^{10}.$$

Then $\mathcal{C}on\,\Phi$ exists, but Φ is not equimultiple.

We define in this way the functors

$$\mathcal{C}on : \overrightarrow{\mathcal{D}ef}_\varphi \to \widehat{\mathcal{D}ef}_\psi, \qquad \pi : \widehat{\mathcal{D}ef}_\psi \to \mathcal{D}ef_\varphi.$$

Notice that the conormal of the plane projection of a Legendrian deformation always exists and we have that $\mathcal{C}on\,(\Psi^\pi) = \Psi$ for each $\Psi \in \widehat{\mathcal{D}ef}_\psi$ and $(\mathcal{C}on\,\Phi)^\pi = \Phi$ where $\Phi \in \overrightarrow{\mathcal{D}ef}_\varphi$.

Let us denote by $\overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi$ the subcategory of equimultiple deformations Φ of φ such that all fibres of Φ have fixed tangent space $\{y = 0\}$ with conormal in generic position. Then $\overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi \subset \overrightarrow{\mathcal{D}ef}_\varphi$ and if $\Phi \in \overrightarrow{\mathcal{D}ef}_\varphi$ is given by 3.2, then $\Phi \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi$ iff

$$\text{(3.4)} \qquad ord_{t_i} Y_i \geq 2 ord_{t_i} X_i, \qquad 1 \leq i \leq r.$$

Because we demand that Φ is equimultiple and all branches have tangent space $\{y = 0\}$, 3.4 is equivalent to

$$\text{(3.5)} \qquad ord_{t_i} Y_i \geq 2m_i, \qquad 1 \leq i \leq r,$$

where m_i is the multiplicity of the component Z_i of Z.

Lemma 3.6. *Under the assumptions above,*

$$\mathcal{C}on\,(\overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi) \subset \widehat{\mathcal{D}ef}_\psi^{em} \quad and \quad (\widehat{\mathcal{D}ef}_\psi^{em})^\pi \subset \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi.$$

Proof. Let m_i be the multiplicity of the component Z_i of Z. Let $Z_{i,s}[L_{i,s}]$ be the fiber of $\Phi[\Psi]$ (given by 3.2 [3.3]) at s . If $\Phi \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi$, $C(L_{i,s}) \not\supset \pi^{-1}(0,0)$ for each s, so $ord_{t_i} Y_i \geq 2ord_{t_i} X_i = 2m_i$. Hence $ord_{t_i} P_i \geq m_i$ and Ψ is equimultiple.

If $\Psi \in \widehat{\mathcal{D}ef}_\psi^{em}$, $ord_{t_i} P_i \geq ord_{t_i} X_i$ and we get that $C(L_{i,s}) \not\supset \pi^{-1}(0,0)$ for each s. Each component $L_{i,s}$ has multiplicity m_i for each s. Hence $mult\, Z_{i,s} \geq m_i$ for each s. Since multiplicity is semicontinuous, $mult\, Z_{i,s} = m_i$ for each s and Φ is equimultiple. □

Lemma 3.7. *If $\mathfrak{C}$ is one of the categories $\widehat{\mathcal{D}ef}_\psi$, $\widehat{\mathcal{D}ef}_\psi^{em}$, $p : \mathfrak{C} \to \mathfrak{A}n$ is a fibered groupoid.*

Proof. Let $f : S \to T$ be a morphism of $\mathfrak{A}n$. Let Ψ be a deformation over T. Then, $(\widetilde{\chi}, \widetilde{\xi}) : f^*\Psi \to \Psi$ is cartesian, with

$$\widetilde{\xi}(t_i, \mathbf{s}) = (t_i, \mathbf{s}), \qquad \widetilde{\chi}(x, y, p, \mathbf{s}) = (x, y, p, \mathbf{s}).$$

This is because if $(\chi, \xi) : \Psi' \to \Psi$ is a morphism over f, then by definition of morphism of deformations over different base spaces, (χ, ξ) is a morphism from Ψ' into $f^*\Psi$ over id_S. □

4. Equimultiple Versal Deformations

For Sophus Lie a contact transformation was a transformation that takes curves into curves, instead of points into points. We can recover the initial point of view. Given a plane curve Z at the origin, with tangent cone $\{y = 0\}$, and a contact transformation χ from a neighbourhood of $(0; dy)$ into itself, χ acts on Z in the following way: $\chi \cdot Z$ is the plane projection of the image by χ of the conormal of Z. We can define in a similar way the action of a relative contact transformation on a deformation of a plane curve Z, obtainning another deformation of Z.

We say that $\Phi \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi(T)$ is *trivial* (relative to the action of the group of relative contact transformations over T) if there is χ such that $\chi \cdot \Phi := \pi \circ \chi \circ \mathcal{C}on\, \Phi$ is the constant deformation of ϕ over T, given by

$$(t_i, \mathbf{s}) \mapsto \varphi_i(t_i), \qquad i = 1, \ldots, r.$$

Let Z be the germ of a plane curve parametrized by $\varphi : \bar{\mathbb{C}} \to \mathbb{C}^2$. In the following we will identify each ideal of $\mathcal{O}_Z$ with its image by $\varphi^* : \mathcal{O}_Z \to \mathcal{O}_{\bar{\mathbb{C}}}$. Hence

$$\mathcal{O}_Z = \mathbb{C}\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix} \right\} \subset \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\} = \mathcal{O}_{\bar{\mathbb{C}}}.$$

Set $\dot{\mathbf{x}} = [\dot{x}_1, \ldots, \dot{x}_r]^t$, where $\dot{x}_i$ is the derivative of x_i in order to t_i, $1 \leq i \leq r$. Let

$$\dot{\varphi} := \dot{\mathbf{x}}\frac{\partial}{\partial x} + \dot{\mathbf{y}}\frac{\partial}{\partial y}$$

be an element of the free $\mathcal{O}_{\bar{\mathbb{C}}}$-module

$$\mathcal{O}_{\bar{\mathbb{C}}}\frac{\partial}{\partial x} \oplus \mathcal{O}_{\bar{\mathbb{C}}}\frac{\partial}{\partial y}. \tag{4.1}$$

Notice that (4.1) has a structure of $\mathcal{O}_Z$-module induced by φ^*.

Let m_i be the multiplicity of Z_i, $1 \leq i \leq r$. Consider the $\mathcal{O}_{\bar{\mathbb{C}}}$-module

$$\left(\bigoplus_{i=1}^{r} t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^{r} t_i^{2m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\right). \tag{4.2}$$

Let $\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi}$ be the sub $\mathcal{O}_{\bar{\mathbb{C}}}$-module of (4.2) generated by

$$(a_1, \ldots, a_r)\left(\dot{\mathbf{x}}\frac{\partial}{\partial x} + \dot{\mathbf{y}}\frac{\partial}{\partial y}\right),$$

where $a_i \in t_i\mathbb{C}\{t_i\}$, $1 \leq i \leq r$. For $i = 1, \ldots, r$ set $p_i = \dot{y}_i/\dot{x}_i$. For each $k \geq 0$ set

$$\mathbf{p}^k = \left[p_1^k, \ldots, p_r^k\right]^t.$$

Let $\widehat{I}$ be the sub $\mathcal{O}_Z$-module of (4.2) generated by

$$\mathbf{p}^k\frac{\partial}{\partial x} + \frac{k}{k+1}\mathbf{p}^{k+1}\frac{\partial}{\partial y}, \qquad k \geq 1.$$

Set

$$\widehat{M}_\varphi = \frac{\left(\bigoplus_{i=1}^{r} t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^{r} t_i^{2m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\right)}{\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi} + (x,y)\frac{\partial}{\partial x} \oplus (x^2,y)\frac{\partial}{\partial y} + \widehat{I}}.$$

Given a category $\mathfrak{C}$ we will denote by $\underline{\mathfrak{C}}$ the set of isomorphism classes of elements of $\mathfrak{C}$.

Theorem 4.1. *Let ψ be the parametrization of a germ of a Legendrian curve L of a contact manifold X. Let $\chi : X \to \mathbb{C}^3$ be a contact transformation such that $\chi(L)$ is in generic position. Let φ be the plane projection of $\chi \circ \psi$. Then there is a canonical isomorphism*

$$\underline{\widehat{\mathcal{D}ef}}_{\psi}^{em}(T_\varepsilon) \xrightarrow{\sim} \widehat{M}_\varphi.$$

Proof. Let $\Psi \in \widehat{\mathcal{D}ef}_{\psi}^{em}(T_\varepsilon)$. By Lemma 3.6, Ψ is the conormal of its projection $\Phi \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_{\varphi}(T_\varepsilon)$. Moreover, Ψ is given by

$$\Psi_i(t_i,\varepsilon) = (x_i + \varepsilon a_i, y_i + \varepsilon b_i, p_i + \varepsilon c_i),$$

where $a_i, b_i, c_i \in \mathbb{C}\{t_i\}$, $ord\, a_i \geq m_i$, $ord\, b_i \geq 2m_i$, $i = 1,\dots,r$. The deformation Ψ is trivial if and only if Φ is trivial for the action of the relative contact transformations. Φ is trivial if and only if there are

$$\xi_i(t_i) = \widetilde{t}_i = t_i + \varepsilon h_i,$$
$$\chi(x,y,p,\varepsilon) = (x + \varepsilon\alpha, y + \varepsilon\beta, p + \varepsilon\gamma, \varepsilon),$$

such that χ is a relative contact transformation, ξ_i is an isomorphism, $\alpha,\beta,\gamma \in (x,y,p)\mathbb{C}\{x,y,p\}$, $h_i \in t_i\mathbb{C}\{t_i\}$, $1 \leq i \leq r$, and

$$x_i(t_i) + \varepsilon a_i(t_i) = x_i(\widetilde{t}_i) + \varepsilon\alpha(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i), p_i(\widetilde{t}_i)),$$
$$y_i(t_i) + \varepsilon b_i(t_i) = y_i(\widetilde{t}_i) + \varepsilon\beta(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i), p_i(\widetilde{t}_i)),$$

for $i = 1,\dots,r$. By Taylor's formula $x_i(\widetilde{t}_i) = x_i(t_i) + \varepsilon\dot{x}_i(t_i)h_i(t_i)$, $y_i(\widetilde{t}_i) = y_i(t_i) + \varepsilon\dot{y}_i(t_i)h_i(t_i)$ and

$$\varepsilon\alpha(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i), p_i(\widetilde{t}_i)) = \varepsilon\alpha(x_i(t_i), y_i(t_i), p_i(t_i)),$$
$$\varepsilon\beta(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i), p_i(\widetilde{t}_i)) = \varepsilon\beta(x_i(t_i), y_i(t_i), p_i(t_i)),$$

for $i = 1,\dots,r$. Hence Φ is trivialized by χ if and only if

$$a_i(t_i) = \dot{x}_i(t_i)h_i(t_i) + \alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \tag{4.3}$$
$$b_i(t_i) = \dot{y}_i(t_i)h_i(t_i) + \beta(x_i(t_i), y_i(t_i), p_i(t_i)), \tag{4.4}$$

for $i = 1,\dots,r$. By Remark 2.7 (i), (4.3) and (4.4) are equivalent to the condition

$$\mathbf{a}\frac{\partial}{\partial x} + \mathbf{b}\frac{\partial}{\partial y} \in \mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi} + (x,y)\frac{\partial}{\partial x} \oplus (x^2,y)\frac{\partial}{\partial y} + \widehat{I}.$$

□

Set

$$M_\varphi = \frac{\left(\bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\right)}{\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi} + (x,y)\frac{\partial}{\partial x} \oplus (x,y)\frac{\partial}{\partial y}},$$
$$\overset{\twoheadrightarrow}{M}_\varphi = \frac{\left(\bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^r t_i^{2m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\right)}{\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi} + (x,y)\frac{\partial}{\partial x} \oplus (x^2,y)\frac{\partial}{\partial y}}.$$

By Proposition 2.27 of [3],

$$\underline{\mathcal{D}ef}_{\varphi}^{em}(T_\varepsilon) \cong M_\varphi.$$

A similar argument shows that

$$\overset{\twoheadrightarrow}{\mathcal{D}ef}_{\varphi}(T_\varepsilon) \cong \overset{\twoheadrightarrow}{M}_\varphi.$$

We have linear maps

$$M_\varphi \overset{\imath}{\hookleftarrow} \overset{\twoheadrightarrow}{M}_\varphi \twoheadrightarrow \widehat{M}_\varphi. \tag{4.5}$$

Theorem 4.2 ([3], II Theorem 2.38 (3))**.** *Set $k = dim\, M_\varphi$. Let $\mathbf{a}^j, \mathbf{b}^j \in \bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}$, $1 \leq j \leq k$. If*

$$\mathbf{a}^j \frac{\partial}{\partial x} + \mathbf{b}^j \frac{\partial}{\partial y} = \begin{bmatrix} a_1^j \\ \vdots \\ a_r^j \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} b_1^j \\ \vdots \\ b_r^j \end{bmatrix} \frac{\partial}{\partial y}, \tag{4.6}$$

$1 \leq j \leq k$, represents a basis of M_φ, the deformation $\Phi : \bar{\mathbb{C}} \times \mathbb{C}^k \to \mathbb{C}^2 \times \mathbb{C}^k$ given by

$$X_i(t_i, \mathbf{s}) = x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i) s_j,\ Y_i(t_i, \mathbf{s}) = y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i) s_j, \tag{4.7}$$

$i = 1, \ldots, r$, is a semiuniversal deformation of φ in $\mathcal{D}ef_\varphi^{em}$.

Lemma 4.3. *Set $\overset{\twoheadrightarrow}{k} = dim\, \overset{\twoheadrightarrow}{M}_\varphi$. Let $\mathbf{a}^j \in \bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}$, $\mathbf{b}^j \in \bigoplus_{i=1}^r t_i^{2m_i}\mathbb{C}\{t_i\}$, $1 \leq j \leq \overset{\twoheadrightarrow}{k}$. If (4.6) represents a basis of $\overset{\twoheadrightarrow}{M}_\varphi$, the deformation $\overset{\twoheadrightarrow}{\Phi}$ given by (4.7), $1 \leq i \leq r$, is a semiuniversal deformation of φ in $\overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi$. Moreover, $\mathcal{C}on\, \overset{\twoheadrightarrow}{\Phi}$ is a versal deformation of ψ in $\widehat{\mathcal{D}ef}_\psi^{em}$.*

Proof. We will only show the completeness of $\overset{\twoheadrightarrow}{\Phi}$ and $\mathcal{C}on\, \overset{\twoheadrightarrow}{\Phi}$. Since the linear inclusion map $\imath$ referred in (4.5) is injective, the deformation $\overset{\twoheadrightarrow}{\Phi}$ is the restriction to $\overset{\twoheadrightarrow}{M}_\varphi$ of the deformation Φ introduced in Theorem 4.2. Let $\Phi_0 \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi(T)$. Since $\Phi_0 \in \mathcal{D}ef_\varphi^{em}(T)$, there is a morphism of analytic spaces $f : T \to M_\varphi$ such that $\Phi_0 \cong f^*\Phi$. Since $\Phi_0 \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi(T)$, $f(T) \subset \overset{\twoheadrightarrow}{M}_\varphi$. Hence $f^*\overset{\twoheadrightarrow}{\Phi} = f^*\Phi$.

If $\Psi \in \widehat{\mathcal{D}ef}_\psi^{em}(T)$, $\Psi^\pi \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi(T)$. Hence there is $f : T \to \overset{\twoheadrightarrow}{M}_\varphi$ such that $\Psi^\pi \cong f^*\overset{\twoheadrightarrow}{\Phi}$. Therefore $\Psi = \mathcal{C}on\, \Psi^\pi \cong \mathcal{C}on\, f^*\overset{\twoheadrightarrow}{\Phi} = f^*\mathcal{C}on\, \overset{\twoheadrightarrow}{\Phi}$. ∎

Theorem 4.4. *Let $\mathbf{a}^j \in \bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}$, $\mathbf{b}^j \in \bigoplus_{i=1}^r t_i^{2m_i}\mathbb{C}\{t_i\}$, $1 \leq j \leq \ell$. Assume that (4.6) represents a basis [a system of generators]of $\widehat{M}_\varphi$. Let Φ be the deformation given by (4.7), $1 \leq i \leq r$. Then $\mathcal{C}on\, \Phi$ is a semiuniversal [versal]deformation of ψ in $\widehat{\mathcal{D}ef}_\psi^{em}$.*

Proof. By Theorem 3.4 and Lemma 4.3 it is enough to show that $\mathcal{C}on\, \Phi$ is formally semiuniversal [versal].

Let $\imath : T' \hookrightarrow T$ be a small extension. Let $\Psi \in \widehat{\mathcal{D}ef}_\psi^{em}(T)$. Set $\Psi' = \imath^*\Psi$. Let $\eta' : T' \to \mathbb{C}^\ell$ be a morphism of complex analytic spaces. Assume that

(χ', ξ') define an isomorphism

$$\eta'^{*}\mathcal{C}on\,\Phi \cong \Psi'.$$

We need to find $\eta : T \to \mathbb{C}^{\ell}$ and χ, ξ such that $\eta' = \eta \circ \imath$ and χ, ξ define an isomorphism

$$\eta^{*}\mathcal{C}on\,\Phi \cong \Psi$$

that extends (χ', ξ'). Let $A\,[A']$ be the local ring of $T\,[T']$. Let δ be the generator of $Ker(A \twoheadrightarrow A')$. We can assume $A' \cong \mathbb{C}\{\mathbf{z}\}/I$, where $\mathbf{z} = (z_1, \ldots, z_m)$. Set

$$\widetilde{A}' = \mathbb{C}\{\mathbf{z}\} \quad \text{and} \quad \widetilde{A} = \mathbb{C}\{\mathbf{z}, \varepsilon\}/(\varepsilon^2, \varepsilon z_1, \ldots, \varepsilon z_m).$$

Let $\mathfrak{m}_A$ be the maximal ideal of A. Since $\mathfrak{m}_A \delta = 0$ and $\delta \in \mathfrak{m}_A$, there is a morphism of local analytic algebras from $\widetilde{A}$ onto A that takes ε into δ such that the diagram

$$\begin{array}{ccc} \widetilde{A} & \longrightarrow & \widetilde{A}' \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & A' \end{array} \tag{4.8}$$

commutes. Assume $\widetilde{T}\,[\widetilde{T}']$ has local ring $\widetilde{A}\,[\widetilde{A}']$. We also denote by $\imath$ the morphism $\widetilde{T}' \hookrightarrow \widetilde{T}$. We denote by κ the morphisms $T \hookrightarrow \widetilde{T}$ and $T' \hookrightarrow \widetilde{T}'$. Let $\widetilde{\Psi} \in \widehat{\mathcal{D}ef}_{\psi}^{\,em}(\widetilde{T})$ be a lifting of Ψ.

We fix a linear map $\sigma : A' \hookrightarrow \widetilde{A}'$ such that $\kappa^{*}\sigma = id_{A'}$. Set $\widetilde{\chi}' = \chi_{\sigma(\alpha),\sigma(\beta_0)}$, where $\chi' = \chi_{\alpha,\beta_0}$. Define $\widetilde{\eta}'$ by $\widetilde{\eta}'^{*}s_i = \sigma(\eta'^{*}s_i)$, $i = 1, \ldots, l$. Let $\widetilde{\xi}'$ be the lifting of ξ' determined by σ. Then

$$\widetilde{\Psi}' := \widetilde{\chi}'^{-1} \circ \widetilde{\eta}'^{*}\mathcal{C}on\,\Phi \circ \widetilde{\xi}'^{-1}$$

is a lifting of Ψ' and

$$\widetilde{\chi}' \circ \widetilde{\Psi}' \circ \widetilde{\xi}' = \widetilde{\eta}'^{*}\mathcal{C}on\,\Phi. \tag{4.9}$$

By Theorem 2.4 it is enough to find liftings $\widetilde{\chi}, \widetilde{\xi}, \widetilde{\eta}$ of $\widetilde{\chi}', \widetilde{\xi}', \widetilde{\eta}'$ such that

$$\widetilde{\chi} \cdot \widetilde{\Psi}^{\pi} \circ \widetilde{\xi} = \widetilde{\eta}^{*}\Phi$$

in order to prove the theorem.

Consider the following commutative diagram

$$
\begin{array}{ccccc}
\bar{\mathbb{C}} \times \widetilde{T}' & \hookrightarrow & \bar{\mathbb{C}} \times \widetilde{T} & \dashrightarrow & \bar{\mathbb{C}} \times \mathbb{C}^{\ell} \\
\downarrow{\scriptstyle \widetilde{\Psi}'} & & \downarrow{\scriptstyle \widetilde{\Psi}} & & \downarrow{\scriptstyle \mathcal{C}on\,\Phi} \\
\mathbb{C}^3 \times \widetilde{T}' & \hookrightarrow & \mathbb{C}^3 \times \widetilde{T} & \dashrightarrow & \mathbb{C}^3 \times \mathbb{C}^{\ell} \\
\downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} & & \downarrow \\
\widetilde{T}' & \hookrightarrow & \widetilde{T} & \overset{\widetilde{\eta}}{\dashrightarrow} & \mathbb{C}^{\ell}.
\end{array}
\qquad \widetilde{T}' \xrightarrow{\widetilde{\eta}'} \mathbb{C}^{\ell}
$$

If $\mathcal{C}on\,\Phi$ is given by

$$X_i(t_i, \mathbf{s}),\ Y_i(t_i, \mathbf{s}),\ P_i(t_i, \mathbf{s})\ \in \mathbb{C}\{\mathbf{s}, t_i\},$$

then $\widetilde{\eta}'^{*}\,\mathcal{C}on\,\Phi$ is given by

$$X_i(t_i, \widetilde{\eta}'(\mathbf{z})),\ Y_i(t_i, \widetilde{\eta}'(\mathbf{z})),\ P_i(t_i, \widetilde{\eta}'(\mathbf{z}))\ \in \widetilde{A}'\{t_i\} = \mathbb{C}\{\mathbf{z}, t_i\}$$

for $i = 1, \dots, r$. Suppose that $\widetilde{\Psi}'$ is given by

$$U_i'(t_i, \mathbf{z}),\ V_i'(t_i, \mathbf{z}),\ W_i'(t_i, \mathbf{z})\ \in \mathbb{C}\{\mathbf{z}, t_i\}.$$

Then, $\widetilde{\Psi}$ must be given by

$$U_i = U_i' + \varepsilon u_i,\ V_i = V_i' + \varepsilon v_i,\ W_i = W_i' + \varepsilon w_i\ \in \widetilde{A}\{t_i\} = \mathbb{C}\{\mathbf{z}, t_i\} \oplus \varepsilon\mathbb{C}\{t_i\}$$

with $u_i, v_i, w_i \in \mathbb{C}\{t_i\}$ and $i = 1, \dots, r$. By definition of deformation we have that, for each i,

$$(U_i, V_i, W_i) = (x_i(t_i), y_i(t_i), p_i(t_i))\ mod\ \mathfrak{m}_{\widetilde{A}}.$$

Suppose $\widetilde{\eta}' : \widetilde{T}' \to \mathbb{C}^{\ell}$ is given by $(\widetilde{\eta}_1', \dots, \widetilde{\eta}_{\ell}')$, with $\widetilde{\eta}_i' \in \mathbb{C}\{\mathbf{z}\}$. Then $\widetilde{\eta}$ must be given by $\widetilde{\eta} = \widetilde{\eta}' + \varepsilon\widetilde{\eta}^0$ for some $\widetilde{\eta}^0 = (\widetilde{\eta}_1^0, \dots, \widetilde{\eta}_{\ell}^0) \in \mathbb{C}^{\ell}$. Suppose that $\widetilde{\chi}' : \mathbb{C}^3 \times \widetilde{T}' \to \mathbb{C}^3 \times \widetilde{T}'$ is given at the ring level by

$$(x, y, p) \mapsto (H_1', H_2', H_3'),$$

such that $H' = id\ mod\ \mathfrak{m}_{\widetilde{A}'}$ with $H_i' \in (x, y, p)A'\{x, y, p\}$. Let the automorphism $\widetilde{\xi}' : \bar{\mathbb{C}} \times \widetilde{T}' \to \bar{\mathbb{C}} \times \widetilde{T}'$ be given at the ring level by

$$t_i \mapsto h_i'$$

such that $h' = id\ mod\ \mathfrak{m}_{\widetilde{A}'}$ with $h_i' \in (t_i)\mathbb{C}\{\mathbf{z}, t_i\}$.

Then, from 4.9 follows that

$$
\begin{aligned}
X_i(t_i, \widetilde{\eta}') &= H_1'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')), \\
Y_i(t_i, \widetilde{\eta}') &= H_2'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')), \\
P_i(t_i, \widetilde{\eta}') &= H_3'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')).
\end{aligned}
\tag{4.10}
$$

Now, $\widetilde{\eta}'$ must be extended to $\widetilde{\eta}$ such that the first two previous equations extend as well. That is, we must have

$$\begin{aligned}(4.11)\qquad X_i(t_i,\widetilde{\eta}) &= (H_1'+\varepsilon\alpha)(U_i(h_i'+\varepsilon h_i^0),V_i(h_i'+\varepsilon h_i^0),W_i(h_i'+\varepsilon h_i^0),\\ Y_i(t_i,\widetilde{\eta}) &= (H_2'+\varepsilon\beta)(U_i(h_i'+\varepsilon h_i^0),V_i(h_i'+\varepsilon h_i^0),W_i(h_i'+\varepsilon h_i^0).\end{aligned}$$

with $\alpha,\beta\in(x,y,p)\mathbb{C}\{x,y,p\}$, $h_i^0\in(t_i)\mathbb{C}\{t_i\}$ such that

$$(x,y,p)\mapsto(H_1'+\varepsilon\alpha,H_2'+\varepsilon\beta,H_3'+\varepsilon\gamma)$$

gives a relative contact transformation over $\widetilde{T}$ for some $\gamma\in(x,y,p)\mathbb{C}\{x,y,p\}$. The existence of this extended relative contact tranformation is guaranteed by Theorem 2.6. Moreover, again by Theorem 2.6 this extension depends only on the choices of α and β_0. So, we need only to find α, β_0, $\widetilde{\eta}^0$ and h_i^0 such that (4.11) holds. Using Taylor's formula and $\varepsilon^2=0$ we see that

$$\begin{aligned}X_i(t_i,\widetilde{\eta}'+\varepsilon\widetilde{\eta}^0) &= X_i(t_i,\widetilde{\eta}')+\varepsilon\sum_{j=1}^{\ell}\frac{\partial X_i}{\partial s_j}(t_i,\widetilde{\eta}')\widetilde{\eta}_j^0\\ (4.12)\qquad (\varepsilon\mathfrak{m}_{\widetilde{A}}=0)\qquad &= X_i(t_i,\widetilde{\eta}')+\varepsilon\sum_{j=1}^{\ell}\frac{\partial X_i}{\partial s_j}(t_i,0)\widetilde{\eta}_j^0,\\ Y_i(t_i,\widetilde{\eta}'+\varepsilon\widetilde{\eta}^0) &= Y_i(t_i,\widetilde{\eta}')+\varepsilon\sum_{j=1}^{\ell}\frac{\partial Y_i}{\partial s_j}(t_i,0)\widetilde{\eta}_j^0.\end{aligned}$$

Again by Taylor's formula and noticing that $\varepsilon\mathfrak{m}_{\widetilde{A}}=0$, $\varepsilon\mathfrak{m}_{\widetilde{A}'}=0$ in $\widetilde{A}$, $h'=id\ mod\ \mathfrak{m}_{\widetilde{A}'}$ and $(U_i,V_i)=(x_i(t_i),y_i(t_i))\ mod\ \mathfrak{m}_{\widetilde{A}}$ we see that

$$\begin{aligned}U_i(h_i'+\varepsilon h_i^0) &= U_i(h_i')+\varepsilon\dot{U}_i(h_i')h_i^0\\ (4.13)\qquad &= U_i'(h_i')+\varepsilon(\dot{x}_ih_i^0+u_i),\\ V_i(h_i'+\varepsilon h_i^0) &= V_i'(h_i')+\varepsilon(\dot{y}_ih_i^0+v_i).\end{aligned}$$

Now, $H'=id\ mod\ \mathfrak{m}_{\widetilde{A}'}$, so

$$\frac{\partial H_1'}{\partial x}=1\ mod\ \mathfrak{m}_{\widetilde{A}'},\qquad \frac{\partial H_1'}{\partial y},\frac{\partial H_1'}{\partial p}\in\mathfrak{m}_{\widetilde{A}'}\widetilde{A}'\{x,y,p\}.$$

In particular,

$$\varepsilon\frac{\partial H_1'}{\partial y}=\varepsilon\frac{\partial H_1'}{\partial p}=0.$$

By this and arguing as in (4.12) and (4.13) we see that

$$\begin{aligned}&(H_1'+\varepsilon\alpha)(U_i'(h_i')+\varepsilon(\dot{x}_ih_i^0+u_i),V_i'(h_i')+\varepsilon(\dot{y}_ih_i^0+v_i),W_i'(h_i')+\varepsilon(\dot{p}_ih_i^0+w_i))\\ &=H_1'(U_i'(h_i'),V_i'(h_i'),W_i'(h_i'))+\varepsilon(\alpha(U_i'(h_i'),V_i'(h_i'),W_i'(h_i'))+1(\dot{x}_ih_i^0+u_i))\\ &=H_1'(U_i'(h_i'),V_i'(h_i'),W_i'(h_i'))+\varepsilon(\alpha(x_i,y_i,p_i)+\dot{x}_ih_i^0+u_i),\\ &(H_2'+\varepsilon\beta)(U_i'(h_i')+\varepsilon(\dot{x}_ih_i^0+u_i),V_i'(h_i')+\varepsilon(\dot{y}_ih_i^0+v_i),W_i'(h_i')+\varepsilon(\dot{p}_ih_i^0+w_i))\\ &=H_2'(U_i'(h_i'),V_i'(h_i'),W_i'(h_i'))+\varepsilon(\beta(x_i,y_i,p_i)+\dot{y}_ih_i^0+v_i)\end{aligned}$$

Substituting this in (4.11) and using (4.10) and (4.12) we see that we have to find $\widetilde{\eta}^0 = (\widetilde{\eta}_1^0, \ldots, \widetilde{\eta}_\ell^0) \in \mathbb{C}^\ell$, h_i^0 such that

$$(4.14) \qquad (u_i(t_i), v_i(t_i)) = \sum_{j=1}^{\ell} \widetilde{\eta}_j^0 \left(\frac{\partial X_i}{\partial s_j}(t_i, 0), \frac{\partial Y_i}{\partial s_j}(t_i, 0) \right) -$$
$$-h_i^0(t_i)((\dot{x}_i(t_i), \dot{y}_i(t_i)) - (\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i))).$$

Note that, because of Remark 2.7 (i), $(\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i))) \in \widehat{I}$ for each i. Also note that $\widetilde{\Psi} \in \widehat{\mathcal{D}ef}_\psi^{em}(\widetilde{T})$ means that $u_i \in t_i^{m_i}\mathbb{C}\{t_i\}, v_i \in t_i^{2m_i}\mathbb{C}\{t_i\}$. Then, if the vectors

$$\left(\frac{\partial X_1}{\partial s_j}(t_1, 0), \ldots, \frac{\partial X_r}{\partial s_j}(t_r, 0) \right) \frac{\partial}{\partial x} + \left(\frac{\partial Y_1}{\partial s_j}(t_1, 0), \ldots, \frac{\partial Y_r}{\partial s_j}(t_r, 0) \right) \frac{\partial}{\partial y}$$
$$= (a_1^j(t_1), \ldots, a_r^j(t_r)) \frac{\partial}{\partial x} + (b_1^j(t_1), \ldots, b_r^j(t_r)) \frac{\partial}{\partial y}, \qquad j = 1, \ldots, \ell$$

form a basis of [generate] $\widehat{M_\varphi}$, we can solve (4.14) with unique $\widetilde{\eta}_1^0, \ldots, \widetilde{\eta}_\ell^0$ [respectively, solve] for all $i = 1, \ldots, r$. This implies that the conormal of Φ is a formally semiuniversal [respectively, versal] equimultiple deformation of ψ over $\mathbb{C}^\ell$. □

5. Versal Deformations

Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$. We will denote by $\int f dx_i$ the solution of the Cauchy problem

$$\frac{\partial g}{\partial x_i} = f, \qquad g \in (x_i).$$

Let ψ be a Legendrian curve with parametrization given by

$$(5.1) \qquad t_i \mapsto (x_i(t_i), y_i(t_i), p_i(t_i)) \qquad i = 1, \ldots, r.$$

We will call *fake plane projection* of (5.1) to the plane curve σ with parametrization given by

$$(5.2) \qquad t_i \mapsto (x_i(t_i), p_i(t_i)) \qquad i = 1, \ldots, r.$$

We will denote σ by ψ^{π_f}.

Given a plane curve σ with parametrization (5.2), we will cal *fake conormal* of σ to the Legendrian curve ψ with parametrization (5.1), where

$$y_i(t_i) = \int p_i(t_i) \dot{x}_i(t_i) dt_i.$$

We will denote ψ by $\mathcal{C}on_f\, \sigma$. Applying the construction above to each fibre of a deformation we obtain functors

$$\pi_f : \widehat{\mathcal{D}ef}_\psi \to \mathcal{D}ef_\sigma, \qquad \mathcal{C}on_f : \mathcal{D}ef_\sigma \to \widehat{\mathcal{D}ef}_\psi.$$

Notice that

$$(5.3) \qquad \mathcal{C}on_f\,(\Psi^{\pi_f}) = \Psi, \qquad (\mathcal{C}on_f\,(\Sigma))^{\pi_f} = \Sigma$$

for each $\Psi \in \widehat{\mathcal{D}ef}_\psi$ and each $\Sigma \in \mathcal{D}ef_\sigma$.

Let ψ be the parametrization of a Legendrian curve given by (5.1). Let σ be the fake plane projection of ψ. Set $\dot\sigma := \dot{\mathbf{x}}\frac{\partial}{\partial x} + \dot{\mathbf{p}}\frac{\partial}{\partial p}$. Let I^f be the linear subspace of

$$\mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial x} \oplus \mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial p} = \left(\bigoplus_{i=1}^r t_i\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^r t_i\mathbb{C}\{t_i\}\frac{\partial}{\partial p}\right)$$

generated by

$$\alpha_0\frac{\partial}{\partial x} - \left(\frac{\partial\alpha_0}{\partial x} + \frac{\partial\alpha_0}{\partial y}\mathbf{p}\right)\mathbf{p}\frac{\partial}{\partial p}, \qquad \left(\frac{\partial\beta_0}{\partial x} + \frac{\partial\beta_0}{\partial y}\mathbf{p}\right)\frac{\partial}{\partial p},$$

and

$$\alpha_k\mathbf{p}^k\frac{\partial}{\partial x} - \frac{1}{k+1}\left(\frac{\partial\alpha_k}{\partial x}\mathbf{p}^{k+1} + \frac{\partial\alpha_k}{\partial y}\mathbf{p}^{k+2}\right)\frac{\partial}{\partial p}, \quad k \geq 1,$$

where $\alpha_k \in (x,y), \beta_0 \in (x^2, y)$ for each $k \geq 0$. Set

$$M^f_\sigma = \frac{\mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial x} \oplus \mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial p}}{\mathfrak{m}_{\bar{\mathbb{C}}}\dot\sigma + I^f}.$$

Theorem 5.1. *Assuming the notations above,* $\underline{\widehat{\mathcal{D}ef}}_\psi(T_\varepsilon) \cong M^f_\sigma$.

Proof. Let $\Psi \in \widehat{\mathcal{D}ef}_\psi(T_\varepsilon)$ be given by

$$\Psi_i(t_i,\varepsilon) = (X_i, Y_i, P_i) = (x_i + \varepsilon a_i, y_i + \varepsilon b_i, p_i + \varepsilon c_i),$$

where $a_i, b_i, c_i \in \mathbb{C}\{t_i\}t_i$ and $Y_i = \int P_i\partial_{t_i}X_i dt_i$, $i = 1,\dots,r$. Hence

$$b_i = \int(\dot x_i c_i + \dot a_i p_i)dt_i, \qquad i = 1,\dots,r.$$

By (5.3) Ψ is trivial if and only if there an isomorphism $\xi : \bar{\mathbb{C}} \times T_\varepsilon \to \bar{\mathbb{C}} \times T_\varepsilon$ given by

$$t_i \to \widetilde{t_i} = t_i + \varepsilon h_i, \qquad h_i \in \mathbb{C}\{t_i\}t_i,\ i = 1,\dots,r,$$

and a relative contact transformation $\chi : \mathbb{C}^3 \times T_\varepsilon \to \mathbb{C}^3 \times T_\varepsilon$ given by

$$(x,y,p,\varepsilon) \mapsto (x + \varepsilon\alpha, y + \varepsilon\beta, p + \varepsilon\gamma, \varepsilon)$$

such that

$$X_i = x_i(\widetilde{t_i}) + \varepsilon\alpha(x_i(\widetilde{t_i}), y_i(\widetilde{t_i}), p_i(\widetilde{t_i})),$$
$$P_i = p_i(\widetilde{t_i}) + \varepsilon\gamma(x_i(\widetilde{t_i}), y_i(\widetilde{t_i}), p_i(\widetilde{t_i})),$$

$i = 1,\dots,r$. Following the argument of the proof of Theorem 4.1, Ψ^{π_f} is trivial if and only if

$$a_i(t_i) = \dot x_i(t_i)h_i(t_i) + \alpha(x_i(t_i), y_i(t_i), p_i(t_i)),$$
$$c_i(t_i) = \dot p_i(t_i)h_i(t_i) + \gamma(x_i(t_i), y_i(t_i), p_i(t_i)),$$

$i = 1,\dots,r$. The result follows from Remark 2.7 (ii). □

Lemma 5.2. *Let ψ be the parametrization of a Legendrian curve. Let Φ be the semiuniversal deformation in $\mathcal{D}ef_\sigma$ of the fake plane projection σ of ψ. Then $\mathcal{C}on_f\,\Phi$ is a versal deformation of ψ in $\widehat{\mathcal{D}ef}_\psi$.*

Proof. It follows the argument of Lemma 4.3. □

Theorem 5.3. *Let $\mathbf{a}^j, \mathbf{c}^j \in \mathfrak{m}_{\mathbb{C}}$ such that*

$$\mathbf{a}^j \frac{\partial}{\partial x} + \mathbf{c}^j \frac{\partial}{\partial p} = \begin{bmatrix} a_1^j \\ \vdots \\ a_r^j \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} c_1^j \\ \vdots \\ c_r^j \end{bmatrix} \frac{\partial}{\partial p}, \tag{5.4}$$

$1 \leq j \leq \ell$, represents a basis [a system of generators]of M_σ^f. Let $\Phi \in \mathcal{D}ef_\sigma$ be given by

$$X_i(t_i, \mathbf{s}) = x_i(t_i) + \sum_{j=1}^{\ell} a_i^j(t_i) s_j, \ P_i(t_i, \mathbf{s}) = p_i(t_i) + \sum_{j=1}^{\ell} c_i^j(t_i) s_j, \tag{5.5}$$

$i = 1, \ldots, r$. Then $\mathcal{C}on_f\,\Phi$ is a semiuniversal [versal]deformation of ψ in $\widehat{\mathcal{D}ef}_\psi$.

Proof. It follows the argument of Theorem 4.4, using Remark 2.7 (ii). □

6. Examples

Example 6.1. Let $\varphi(t) = (t^3, t^{10})$, $\psi(t) = (t^3, t^{10}, \frac{10}{3}t^7)$, $\sigma(t) = (t^3, \frac{10}{3}t^7)$. The deformations given by

- $X(t,\mathbf{s}) = t^3$, $\quad Y(t,\mathbf{s}) = s_1t^4 + s_2t^5 + s_3t^7 + s_4t^8 + t^{10} + s_5t^{11} + s_6t^{14}$;
- $X(t,\mathbf{s}) = s_1t + s_2t^2 + t^3$, $\quad Y(t,\mathbf{s}) = s_3t + s_4t^2 + s_5t^4 + s_6t^5 + s_7t^7 + s_8t^8 + t^{10} + s_9t^{11} + s_{10}t^{14}$;

are respectively

- an equimultiple semiuniversal deformation;
- a semiuniversal deformation

of φ. The conormal of the deformation given by

$$X(t,\mathbf{s}) = t^3, \qquad Y(t,\mathbf{s}) = s_1t^7 + s_2t^8 + t^{10} + s_3t^{11};$$

is an equimultiple semiuniversal deformation of ψ. The fake conormal of the deformation given by

$$X(t,\mathbf{s}) = s_1t + s_2t^2 + t^3, \quad P(t,\mathbf{s}) = s_3t + s_4t^2 + s_5t^4 + s_6t^5 + \frac{10}{3}t^7 + s_7t^8;$$

is a semiuniversal deformation of the fake conormal of σ. The conormal of the deformation given by

$$X(t,\mathbf{s}) = s_1t + s_2t^2 + t^3, \quad Y(t,\mathbf{s}) = \alpha_2t^2 + \alpha_3t^3 + \alpha_4t^4 + \alpha_5t^5 + \alpha_6t^6 + \alpha_7t^7 + \alpha_8t^8 + \alpha_9t^9 + \alpha_{10}t^{10} + \alpha_{11}t^{11};$$

with

$$\alpha_2 = \frac{s_1 s_3}{2}, \qquad \alpha_3 = \frac{s_1 s_4 + 2 s_2 s_3}{3}, \qquad \alpha_4 = \frac{3 s_3 + 2 s_2 s_4}{4},$$
$$\alpha_5 = \frac{3 s_4 + s_1 s_5}{5}, \qquad \alpha_6 = \frac{2 s_2 s_5 + s_1 s_6}{6}, \qquad \alpha_7 = \frac{3 s_5 + 2 s_2 s_6}{7},$$
$$\alpha_8 = \frac{10 s_1 + 9 s_6}{24}, \qquad \alpha_9 = \frac{3 s_1 s_7 + 20 s_2}{27}, \qquad \alpha_{10} = 1 + \frac{s_2 s_7}{5},$$
$$\alpha_{11} = \frac{3 s_7}{11},$$

is a semiuniversal deformation of ψ.

Example 6.2. Let $Z = \{(x, y) \in \mathbb{C}^2 : (y^2 - x^5)(y^2 - x^7) = 0\}$. Consider the parametrization φ of Z given by

$$x_1(t_1) = t_1^2,\ y_1(t_1) = t_1^5 \qquad x_2(t_2) = t_2^2,\ y_2(t_2) = t_2^7.$$

Let σ be the fake projection of the conormal of φ given by

$$x_1(t_1) = t_1^2,\ p_1(t_1) = \frac{5}{2} t_1^3 \qquad x_2(t_2) = t_2^2,\ p_2(t_2) = \frac{7}{2} t_2^5.$$

The deformations given by

- $X_1(t_1, \mathbf{s}) = t_1^2, \qquad Y_1(t_1, \mathbf{s}) = s_1 t_1^3 + t_1^5,$
  $X_2(t_2, \mathbf{s}) = t_2^2, \qquad Y_2(t_2, \mathbf{s}) = s_2 t_2^2 + s_3 t_2^3 + s_4 t_2^4 + s_5 t_2^5 + s_6 t_2^6 + t_2^7 + s_7 t_2^8 + s_8 t_2^{10} + s_9 t_2^{12};$
- $X_1(t_1, \mathbf{s}) = s_1 t_1 + t_1^2, \qquad Y_1(t_1, \mathbf{s}) = s_3 t_1 + s_4 t_1^3 + t_1^5,$
  $X_2(t_2, \mathbf{s}) = s_2 t_2 + t_2^2, \qquad Y_2(t_2, \mathbf{s}) = s_5 t_2 + s_6 t_2^2 + s_7 t_2^3 + s_8 t_2^4 + s_9 t_2^5 + s_{10} t_2^6 + t_2^7 + s_{11} t_2^8 + s_{12} t_2^{10} + s_{13} t_2^{12};$

are respectively

- an equimultiple semiuniversal deformation;
- a semiuniversal deformation

of φ. The conormal of the deformation given by

$$X_1(t_1, \mathbf{s}) = t_1^2, \qquad Y_1(t_1, \mathbf{s}) = t_1^5,$$
$$X_2(t_2, \mathbf{s}) = t_2^2, \qquad Y_2(t_2, \mathbf{s}) = s_1 t_2^4 + s_2 t_2^5 + s_3 t_2^6 + t_2^7 + s_4 t_2^8;$$

is an equimultiple semiuniversal deformation of the conormal of φ. The fake conormal of the deformation given by

$$X_1(t_1, \mathbf{s}) = s_1 t_1 + t_1^2, \quad P_1(t_1, \mathbf{s}) = s_3 t_1 + \frac{5}{2} t_1^3,$$
$$X_2(t_2, \mathbf{s}) = s_2 t_2 + t_2^2, \quad P_2(t_2, \mathbf{s}) = s_4 t_2 + s_5 t_2^2 + s_6 t_2^3 + s_7 t_2^4 + \frac{7}{2} t_2^5 + s_8 t_2^6;$$

is a semiuniversal deformation of the fake conormal of σ. The conormal of the deformation given by

$$\begin{aligned} X_1(t_1,\mathbf{s}) &= s_1t_1 + t_1^2, \quad Y_1(t_1,\mathbf{s}) = \alpha_2 t_1^2 + \alpha_3 t_1^3 + \alpha_4 t_1^4 + t_1^5, \\ X_2(t_2,\mathbf{s}) &= s_2t_2 + t_2^2, \quad Y_2(t_2,\mathbf{s}) = \beta_2 t_2^2 + \beta_3 t_2^3 + \beta_4 t_2^4 + \beta_5 t_2^5 + \beta_6 t_2^6 + \\ &\qquad\qquad + \beta_7 t_2^7 + \beta_8 t_2^8; \end{aligned}$$

with

$$\begin{aligned} \alpha_2 &= \frac{s_1s_3}{2}, & \alpha_3 &= \frac{2s_3}{3}, & \alpha_4 &= \frac{5s_1}{8}, \\ \beta_2 &= \frac{s_2s_4}{2}, & \beta_3 &= \frac{2s_4 + s_2s_5}{3}, & \beta_4 &= \frac{2s_5 + s_2s_6}{4}, \\ \beta_5 &= \frac{2s_6 + s_2s_7}{5}, & \beta_6 &= \frac{4s_7 + 7s_2}{12}, & \beta_7 &= 1 + \frac{s_2s_8}{7}, \\ \beta_8 &= \frac{2s_8}{8}, \end{aligned}$$

is a semiuniversal deformation of the conormal of φ.

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