Tilings on the butterfly lattice

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Abstract

Let $F$ be a simply connected figure constituted of cells of the butterfly lattice. We show that there exists a linear algorithm which says whether $F$ is tilable by the three tiles described in the paper. Moreover, we point out some specific geometrical properties, especially on the notion of flip.

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1. Introduction

We use flow theory in graphs to extend a result by Thurston on tiling [3] and a result by Chaboud [1] about the problem of tiling with two types of tile. In his paper, Thurston deals with some geometrical interpretations of tilings in terms of relevant three-dimensional surfaces. We show here that there exists a bijection between tilings and some specific tensions in a special graph (see Theorem 2.1). Moreover, we point out some specific geometrical properties, especially on the notion of flip. In this paper, we prove that if $F$ is a tilable B-figure, then the two following quantities are equal:

- the number of connected components in the flip-accessibility graph of the tilings of $F$
- the number of minimum tilings.

Precise definitions are given in what follows. However, we can already say that this theorem is related to the following classical result: All the domino tilings of a polyomino are accessible by flip.

The butterfly lattice is the subdivision of the plane into hexagons and equilateral triangles with the same side length and such that the sides are always shared by one hexagon and one triangle (see Fig. 1). A B-figure is a finite union of cells of the butterfly lattice. Now, let $G$ be...
the infinite graph on the nodes of the butterfly lattice, such that the edges join two nodes in a same triangle. In other words, $G$ is the 1-skeleton of the butterfly lattice. Let $F$ be a B-figure; we denote by $G_F$ the partial subgraph of $G$ constituted by the vertices and the edges of $G$ belonging (geometrically) to $F$.

The following important notion comes from Thurston [3]. The cellular orientation of a B-figure $F$ is an orientation of $G_F$ such that these edges (which become arcs) are oriented to travel along the boundary of the hexagons (resp. triangles) in the trigonometric (resp. clockwise) orientation. This oriented graph is denoted by $H_F$.

The union of one hexagon and two adjacent triangles is called a prototile (see Fig. 2). A tiling $Q$ of a B-figure $F$ is a partition of the cells of $F$ into prototiles. These prototiles are the tiles of the tiling $Q$. We say that a B-figure is tilable if it has at least one tiling (Fig. 3).

Let $F$ be a tilable B-figure and $Q$ a tiling of $F$. A tiling arc of $Q$ is a common arc of two tiles of $Q$ or a boundary arc.

2. Flows, flux and potentials

Let $G$ be an oriented graph; a flow on $G$ is a map $C$ from $E(G)$ into $\mathbb{C}$. A travel from a vertex $v$ to a vertex $v'$ is a chain linking $v$ to $v'$ provided with a way of travel. The flux of the flow $C$ on the travel $T$ is $\sum_{e \in T^+} C(e) - \sum_{e \in T^-} C(e)$ (where $T^+$ denotes the arcs travelled with respect to their orientation and $T^-$ the others). In simple words, we begin with 0 in $v$ and we follow $T$;
if we travel along an arc $e$ of $T$ in the good direction, we add $C(e)$, otherwise we subtract $C(e)$. The flux is the number that we obtain in $v'$. We denote by $F_T(C)$ the flux along $T$. We verify immediately that the flux is additive by concatenation of travels: $F_{T_1T_2}(C) = F_{T_1}(C) + F_{T_2}(C)$, where $T_1$ and $T_2$ are two travels such that the extremity of $T_1$ coincides with the origin of $T_2$ and where $T_1T_2$ is obtained by joining $T_1$ and $T_2$ end to end.

As for polyominoes (finite set of squares in an infinite chessboard), the boundary of a B-figure $F$ is the set of edges of $G$ which belong to a unique cell of $F$ and we denote it by $\partial F$. We say that a B-figure is full if its boundary is a topological circle. Let $F$ be a B-figure with a tiling $Q$; we are going to associate a flow $C_Q$ on $H_F$ to the tiling $Q$ as follows: $C_Q(e) = 1$ when $e$ is a tiling arc of $Q$ and $C_Q(e) = -2$ otherwise (see Fig. 4). For every cell $c$, let $T_c$ be the travel defined by turning around $c$ in the trigonometric way. We can notice that $F_{T_c}(C_Q) = 0$.

If for every closed travel (i.e. its origin and its extremity coincide) on the boundary of a B-figure $F$ we have $F_T(C) = 0$ where $C$ is a flow such that $C(e) = 1$ for all $e \in \partial F$, then we say that $F$ has a balanced boundary. In particular, we can notice that a full balanced B-figure has a balanced boundary. More specifically, if for all closed travel (not necessary on the boundary) we have $F_T(C) = 0$, then the flow $C$ is called a tension.

If $F$ is a B-figure with a balanced boundary and $Q$ is a tiling of $F$, we observe (because we can contract all closed paths by succession of contraction of cells and cycles on the boundary) that the flow $C_Q$ is a tension, and so, it derives from a potential $\varphi_Q$ on the vertices of $H_F$. In other words, there exists a potential $\varphi_Q$ defined by: for all $(x, y) \in V(H_F)^2$, $\varphi_Q(y) - \varphi_Q(x) = F_{T_{x,y}}(C_Q)$, where $T_{x,y}$ is a travel linking $x$ to $y$, and $\min_{x \in \partial F} \varphi_F(x) = 0$. The arbitrary condition $\min_{x \in \partial F} \varphi_F(x) = 0$ is necessary, otherwise $\varphi_Q$ is just defined up to a constant.

**Theorem 2.1.** Let $F$ be a B-figure with balanced boundary. There exists a one-to-one correspondence between its tilings and tensions $C$ on $H_F$ which satisfies:

- $\forall e \in \partial F, C_Q(e) > 0$.
- $\forall e \in F, C_Q(e) \in \{-2, 1\}$.

**Proof.** We have already shown that for each tiling of $F$, we can associate a tension which satisfies the hypotheses. Conversely, if we consider a tension which satisfies the hypotheses, we can notice that each hexagon of $F$ has exactly four positive arcs. They are the tiling arcs of a tiling of $F$. □

We say that $x$ is a (local) maximum vertex if for every adjacent vertex $v$ in $H_F$ we have $\varphi(x) \geq \varphi(v)$.

**Lemma 2.2.** Let $F$ be a B-figure with a balanced boundary (F is not necessarily a full B-figure). There exists a tiling $Q$ such that $\varphi_Q$ is only maximum on some vertices on the boundary of $F$. 
Lemma 2.4. Let $Q$ be a tiling such that, for every tiling $Q'$ of $F$, we have $\sum_{x \in V(H_F)} \varphi_Q(x) \leq \sum_{x \in V(H_F)} \varphi_{Q'}(x)$; we are going to prove that $Q$ satisfies the conditions of the lemma. Suppose that $x_m$ is a vertex which is not on the boundary and such that $\varphi_Q(x_m)$ is maximum. The vertices in
\[
\delta(x_m) = \{y; (x_m, y) \in E(H_F) \text{ or } (y, x_m) \in E(H_F)\}
\]
have a smaller potential than $x_m$, so the arcs in
\[
E^+(x_m) = \{(x_m, y); (x_m, y) \in E(H_F)\}
\]
are not tiling arcs of $Q$ and the arcs in $E^-(x_m) = \{(y, x_m); (y, x_m) \in E(H_F)\}$ are tiling arcs. Let $h_1, h_2, t_1, t_2$ be the two hexagons and the two triangles respectively which contain $x_m$. We can suppose that $h_1, t_1$ (resp. $h_2, t_2$) belong to the same tile $\{h_1, t_1, t_a\}$ (resp. $\{h_1, t_1, t_b\}$). Now, we can consider the tiling $Q'$ composed of the tiles of $Q$ but where we have replaced the tiles $\{h_1, t_1, t_a\}$ and $\{h_2, t_2, t_b\}$ by the tiles $\{h_1, t_2, t_a\}$ and $\{h_2, t_1, t_b\}$. Then $\sum_{x \in V(H_F)} \varphi_{Q'}(x) = \sum_{x \in V(H_F)} \varphi_Q(x) - 3$ because for every arc $e$ in $E^-(x_m)$ we have $C_Q(e) = -2$, for every arc $e$ in $E^+(x_m)$ we have $C_Q'(e) = 1$, and for all the other arcs $e$, $C_Q'(e) = C_Q(e)$ so $\varphi_{Q'}(x_m) = \varphi_Q(x_m) - 3$ and for all $x \in V(H_F) \setminus \{x_m\}$, $\varphi_{Q'}(x) = \varphi_Q(x)$. This is in contradiction with the hypotheses. 

Proof. Let $Q$ be a tiling such that, for every tiling $Q'$ of $F$, we have $\sum_{x \in V(H_F)} \varphi_{Q'}(x) \leq \sum_{x \in V(H_F)} \varphi_Q(x)$; we are going to prove that $Q$ satisfies the conditions of the lemma. Suppose that $x_m$ is a vertex which is not on the boundary and such that $\varphi_Q(x_m)$ is maximum. The vertices in
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Proof. Let $Q$ be a tiling such that, for every tiling $Q'$ of $F$, we have $\sum_{x \in V(H_F)} \varphi_{Q'}(x) \leq \sum_{x \in V(H_F)} \varphi_Q(x)$; we are going to prove that $Q$ satisfies the conditions of the lemma. Suppose that $x_m$ is a vertex which is not on the boundary and such that $\varphi_Q(x_m)$ is maximum. The vertices in
\[
\delta(x_m) = \{y; (x_m, y) \in E(H_F) \text{ or } (y, x_m) \in E(H_F)\}
\]
have a smaller potential than $x_m$, so the arcs in
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are not tiling arcs of $Q$ and the arcs in $E^-(x_m) = \{(y, x_m); (y, x_m) \in E(H_F)\}$ are tiling arcs. Let $h_1, h_2, t_1, t_2$ be the two hexagons and the two triangles respectively which contain $x_m$. We can suppose that $h_1, t_1$ (resp. $h_2, t_2$) belong to the same tile $\{h_1, t_1, t_a\}$ (resp. $\{h_1, t_1, t_b\}$). Now, we can consider the tiling $Q'$ composed of the tiles of $Q$ but where we have replaced the tiles $\{h_1, t_1, t_a\}$ and $\{h_2, t_2, t_b\}$ by the tiles $\{h_1, t_2, t_a\}$ and $\{h_2, t_1, t_b\}$. Then $\sum_{x \in V(H_F)} \varphi_{Q'}(x) = \sum_{x \in V(H_F)} \varphi_Q(x) - 3$ because for every arc $e$ in $E^-(x_m)$ we have $C_Q(e) = -2$, for every arc $e$ in $E^+(x_m)$ we have $C_Q'(e) = 1$, and for all the other arcs $e$, $C_Q'(e) = C_Q(e)$ so $\varphi_{Q'}(x_m) = \varphi_Q(x_m) - 3$ and for all $x \in V(H_F) \setminus \{x_m\}$, $\varphi_{Q'}(x) = \varphi_Q(x)$. This is in contradiction with the hypotheses. 

Lemma 2.3. Let $F$ be a tiling B-figure with a balanced boundary. For every tiling $Q$, there exists a unique tiling in $T \min_F$ obtained from $Q$ by a succession of flips.

Proof. We must prove that the tiling in $T \min_F$ obtained by flips around successive maxima does not depend on the successive choice of the maxima. But, in Lemma 2.2, we can observe that these two facts are equivalent:

- to make a flip around a maximum vertex $x_m$ (which transforms the tiling $Q$ into $Q'$)
- to transform $\varphi_Q(x_m)$ into $\varphi_{Q'}(x_m) = \varphi_Q(x_m) - 3$ and keep equal the other values.

So, if we have a choice between two maximum vertices $x_m$ and $x'_m$, it is clear that we have to make a flip on each of them and the result does not depend on the order of doing it.

Lemma 2.4. Let $F$ be a tiling B-figure with a balanced boundary. A tiling $Q$ in $T \min_F$ is totally determined by the values of $\varphi_Q$ on $\partial F$.

Proof. This is done by induction on the number of cells in $F$. We refer to the figure (Fig. 5). Let $v_1$ be an absolute maximum vertex (on the boundary) and set $\varphi_Q(v_1) = M$. Firstly, we can notice that $v_1$ is in the middle of a segment of length 2 belonging to the boundary. The arc in $E^+(v_1)$, denoted by $e_1 = (v_1, v_2)$, satisfies $C_Q(e_1) = -2$ (otherwise $\varphi_Q(v_1)$ will not be maximum). So, it does not belong to the boundary of $F$ and $\varphi_Q(v_2) = M - 2$. Now, we are going to separate the problem into four cases which depend on the position of the second negative arc of the unique black hexagon containing $v_1$. 

Fig. 5. Around a maximum.

- $(v_3, v_4)$ is the second negative arc of the black hexagon. We have two subcases. (a) The arc $(v_2, v_3)$ is on the boundary. (b) We have clearly that $\varphi_Q(v') = M$ or $\varphi_Q(v') = M - 3$. In fact, $\varphi_Q(v')$ is equal to $M$, otherwise $v_3$ is a (local) maximum and this is in contradiction with $Q \in T_{\min_F}$. Thus, $v'$ is on the boundary of $F$ and it is maximum.
- $(v_4, v_5)$ is the second negative arc of the black hexagon. In this case $\varphi_Q(v_4) = M$. So, $(v_3, v_4)$ is on the boundary of $F$.
- $(v_5, v_6)$ is not the second negative arc of the black hexagon. Indeed, in this case $\varphi_Q(v_5) = M + 1$. This is in contradiction with the maximality of $v_1$.
- $(v_2, v_3)$ is the second negative arc of the black hexagon.

Now, if $F$ is tilable, we can delete a tile of $Q$ which is totally determined. Indeed, we have three exclusive possibilities: if $v'$ (resp. $v_4$) is a maximum vertex on the boundary, then $(v_3, v_4)$ (resp. $(v_4, v_5)$) is a negative arc and we have determined the tile of $Q$ which covers $v_1$. Otherwise, $(v_2, v_3)$ is the second negative arc. By induction, the other tiles are also totally determined.

Theorem 2.5. Let $F$ be a tilable B-figure. The number of connected components in $A_F$ is equal to the cardinal of $T_{\min_F}$.

Proof. By Lemma 2.2, for every tiling $Q$ in a same connected component, the values of the potential $\varphi_Q$ on $\partial F$ do not depend on the tiling $Q$. Moreover, by Lemma 2.4, there is a unique $T \in T_{\min_F}$ compatible with these values.

Lemma 2.6. Let $F$ be a full tilable B-figure; there is a unique element in $T_{\min_F}$, in other words $A_F$ is connected.

Proof. Since $\partial F$ is a topological circle, the values of the potential $\varphi_Q$ on $\partial F$ are integrally determined by the cellular orientation of the arcs of $\partial F$ and do not depend on any tiling $Q$. So, there is a unique tiling in $T_{\min_F}$.

As a consequence, if $F$ is a full tilable B-figure, then for every vertex $x$ on the boundary of $F$, we simply denote by $\varphi(x)$ the common value of all the potentials $\varphi_Q$ in $x$. This restriction of $\varphi_Q$ on the boundary of $F$ is called – in the case of full polyominoes – height on the boundary by Fournier [2] and was introduced by Thurston [3].

The reader can notice that this potential on the boundary can be defined if $F$ has a balanced boundary. Since $\varphi$ can be calculated without knowing a tiling of $F$, for the full tilable B-figure, Lemma 2.4 obviously gives us, by induction, a linear time algorithm to build the unique minimum tiling (tiling in $T_{\min_F}$):
(1) We build (or complete) $\varphi$.
(2) We take the maximum vertices and then we deduce by Lemma 2.4 a tile of the minimum tiling.
(3) We delete this tile and then we go back to (1) with this new B-figure.

References