Tail-Adaptive Location Rank Test for the Generalized Secant Hyperbolic Distribution

O. Y. Kravchuk & J. Hu

School of Land, Crop and Food Science, University of Queensland, Brisbane, Australia
Feinberg School of Medicine, Northwestern University, Chicago, Illinois, USA
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Inference

Tail-Adaptive Location Rank Test for the Generalized Secant Hyperbolic Distribution

O. Y. KRAVCHUK¹ AND J. HU²

¹School of Land, Crop and Food Science, University of Queensland, Brisbane, Australia
²Feinberg School of Medicine, Northwestern University, Chicago, Illinois, USA

The generalized secant hyperbolic distribution (GSHD) was recently introduced as a modeling tool in data analysis. The GSHD is a unimodal distribution that is completely specified by location, scale, and shape parameters. It has also been shown elsewhere that the rank procedures of location are regular, robust, and asymptotically fully efficient. In this article, we study certain tail weight measures for the GSHD and introduce a tail-adaptive rank procedure of location based on those tail weight measures. We investigate the properties of the new adaptive rank procedure and compare it to some conventional estimators.

Keywords Adaptive rank estimator; Generalized secant hyperbolic distribution; Location problem; Tail weight.

Mathematics Subject Classification 62G10.

1. Introduction

The generalized secant hyperbolic distribution (GSHD) was introduced in Vaughan (2002). The GSHD is a location-scale family of unimodal symmetric distributions with infinite support. A member of the distribution is completely specified by the location, \( \mu \in (-\infty, \infty) \), scale, \( \sigma > 0 \), and shape, \( t \in (-\pi, \infty) \), parameters. The density function is:

\[
f(x \mid \mu, \sigma, t) = \frac{b}{2\sigma} \left( \cosh \left( \frac{x - \mu}{\sigma} \right) + a \right)^{-1},
\]

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Address correspondence to O. Y. Kravchuk, School of Land, Crop, and Food Science, University of Queensland, Brisbane, Australia; E-mail: o.kravchuk@uq.edu.au
For simplicity of derivation and without loss of generality, in this article we operate with the GSHD centred around the origin: $\mu = 0$, and standardized to unit variance:

$$\sigma^2 = 3/(\pi^2 - t^2)$$

for $-\pi < t \leq 0$ and

$$\sigma^2 = 3/(\pi^2 + t^2)$$

for $t > 0$. In the text, we refer to a member of the standardized family as $GSHD(t)$.

Special members of the distribution are the logistic distribution at $t = 0$ and

$-$

the secant hyperbolic distribution at $t = -\pi/2$. The Cauchy distribution is a heavy-tailed limiting distribution of a scaled GSHD as $t$ approaches $-\pi$ (Kravchuk, 2006); the uniform distribution is a light-tailed limiting distribution as $t$ becomes infinitely large, $t \to \infty$ (Vaughan, 2002). The standardized GSHD also approximates, up to the fifth moment, the standardized central Student’s $t$-distribution with degrees of freedom larger than 4 at $-\pi < t < \pi$ (Vaughan, 2002).

There are several computationally attractive estimators of location for the GSHD. The modified maximum likelihood estimators of the location and scale parameters have been defined and studied by Vaughan (2002); the best linear unbiased estimators of the location and scale parameters have been investigated by Raqab and Ahsanullah (2004); and a fully efficient linear rank estimator of location has been introduced by Kravchuk (2006). The rank estimator of location is Pitman regular and robust to distributional misspecifications within wide intervals of the shape parameter $t$. It was suggested by Kravchuk (2006) that one may adaptively select a highly efficient rank procedure of location when a rough estimate of the shape parameter is available.

Many tail-adaptive procedures of location for symmetric distributions are based on the Hogg’s concept of adaptive tests (Hogg and Lenth, 1984): the underlying distribution is firstly classified by using either a tail weight, kurtosis, or peakedness estimator, and then an efficient test or estimator of location is selected from a set of available procedures. The Hogg’s concept may be naturally applied to the GSHD as its kurtosis is completely monotone specified by $t$ (Klein and Fischer, 2008):

$$\gamma_2 = \begin{cases} 
3 + \frac{6t^2 + \pi^2}{5\pi^2 - t^2}, & -\pi < t < 0, \\
3 - \frac{6t^2 - \pi^2}{5\pi^2 + t^2}, & t \geq 0.
\end{cases}$$

Adaptive location procedures for symmetric distributions are available in the literature for one-sample (Badahdah and Siddiqui, 1991; Hogg and Lenth, 1984), two-sample (Neuhauser et al., 2004), and regression (Büning, 1996) location problems. Various selector statistics of tail weight based on order statistics have been proposed for adaptive procedures. With such selectors, using rank tests and estimators within adaptive procedures is attractive as the statistical inference is not influenced by the selection process (Hájek et al., 1999). Unfortunately, the variance of the tail selectors commonly used in adaptive procedures is high, as shown, for example, in computational experiments in Schmid and Trede (2003). Therefore, their selection efficiency on moderate samples is low (Hogg and Lenth, 1984). However, the robustness of the location rank procedures of the GSHD
to shape (t) misspecifications (Kravchuk, 2006) overcomes this lack of precision in tail classification. In this article, we will introduce a tail-adaptive linear rank procedure of location that may be used for modeling data from symmetric unimodal distributions of various tails within the GSHD family.

This article is structured as follows. In Sec. 2, we discuss the Hogg’s (Hogg and Lenth, 1984) and Brys’s (Brys et al., 2006) tail weight and peakedness estimators based on order statistics in application to the GSHD. In Sec. 3, we introduce the consequent tail-adaptive rank procedure and develop a computational procedure for the exact adaptive confidence intervals of the location parameter of the GSHD. In Sec. 4, we compare the new adaptive procedures to conventional procedures based on order statistics. In Sec. 5, we conclude the article and suggest directions for further research.

2. Hogg’s and Brys’s Tail Weight Measures of the GSHD

Let us consider the tail measure, $T$, and the peakedness measure, $P$, that were initially proposed by Hogg and Lenth (1984) and have been recently studied in more detail by Schmid and Trede (2003):

$$T = \frac{X_{0.975} - X_{0.025}}{X_{0.975} - X_{0.125}}, \quad \text{and} \quad P = \frac{X_{0.875} - X_{0.125}}{X_{0.75} - X_{0.25}},$$

where $X_i$ are the corresponding distribution percentiles. For the GSHD, these measures are easy to express in terms of the tail parameter by using the following definition of its percentiles (Palmitesta and Provasi, 2004), $0 < p < 1$, standardized to unit variance:

$$X_p(t) = \frac{2}{c_2} \begin{cases} \tanh^{-1} \left( \cot \left( \frac{t}{2} \right) \tan \left( \frac{t}{2} (2p - 1) \right) \right), & -\pi < t < 0, \\ \tanh^{-1} (2p - 1), & t = 0, \\ \tanh^{-1} \left( \coth \left( \frac{t}{2} \right) \tanh \left( \frac{t}{2} (2p - 1) \right) \right), & t > 0, \end{cases}$$

where $c_2 = \sqrt{(\pi^2 - t^2)/3}$, $-\pi < t \leq 0$, and $c_2 = \sqrt{(\pi^2 + t^2)/3}$, $t > 0$.

The $T$ and $P$ measures of the GSHD are tabulated in Table 1 for certain values of $t$. In the limits, these measures converge to the corresponding tail and peakedness measures of the Cauchy and uniform distributions.

The normal distribution is not a member of the GSHD family. However, it is possible to approximate a normal distribution with a GSHD by selecting a value of $t$ close to $\pi$ (Vaughan, 2002). The goodness-of-fit attained depends on the actual value of $t$. At $t = 2.52$, the sample mean of the GSHD attains its maximal efficiency of 98.4% (Kravchuk, 2006). At $t = \pi$, the kurtosis of the GSHD is the same as that of the normal distribution (Vaughan, 2002). The minimal Kolmogorov-Smirnov distance of 0.0038 is achieved at $t = 2.8$. At $t = 2.605$, the Hogg’s tail measure of the GSHD is close to that of the normal distribution ($T = 1.7055$). The Kolmogorov-Smirnov distance between the standardized GSHD and the standard normal distribution at $t = 2.605$ is smaller than that at $t = \pi$ and $t = 2.52$ (0.0042 vs. 0.0072 and 0.0043, correspondingly). In this article, we assign GSHD(2.605) as the reference standard normal-tailed member of the distribution.
The Hogg’s or Brys’ tail classification procedures may be considered as being a special normality test. In this article, we are classifying the members of the GSHD into three categories: normal-tailed, heavy-tailed, and light-tailed, where the definition of tail weight is kurtosis-based. The normal-tailed members are those whose shape parameter is around \( t = \frac{2}{6} \). The good agreement between the standard normal and GSHD(2.605) allows us to use a tail classifier whose properties are well known under the normality assumption.

As can be seen in Table 1, both Hogg’s measures are good selectors for heavy-tailed, but not for light-tailed, distributions. The \( T \) measure has a higher gradient for positive values of \( t \), and we can envisage that its sample estimator has the better selection performance. We are not going to investigate the properties of the peakedness estimator, \( P \), any further in this article.

Following Schmid and Trede (2003), the estimator \( \hat{T} \) for the normal distribution is consistent and asymptotically normal with the variance \( n\sigma_T^2 = 3.0471 \). Schmid and Trede (2003) provided the tables of exact percentiles of the \( \hat{T} \) classifier for moderate and large samples (50 < \( n < 2000 \)) from the normal distribution and suggested that the approximate normal sampling distributions may be safely used only for very large samples, \( n > 2000 \).

The exact sampling distribution of the Hogg’s \( T \) classifier for GSHD(2.605) is close to the sampling distribution of the \( \hat{T} \) for the standard normal distribution. The common percentiles of the exact sampling distribution for GSHD(2.605) are presented in Table 2. For comparison, the corresponding exact percentiles of \( \hat{T} \) for the standard normal distribution at \( n = 200 \) are \( \hat{T}(0.01, 0.025, \ldots, 0.975, 0.99) = 1.455, 1.488, 1.518, 1.554, 1.869, 1.923, 1.972, 2.032 \) (Schmid and Trede, 2003).

In agreement with the observation by Schmid and Trede (2003), the convergence of the sampling distribution of the tail estimator \( \hat{T} \) to normal is slow. For not very large samples (\( n < 300 \)), the positive skewness is noticeable. However, the average converges to the expected value, 1.706, and the variance stabilizes, slowly, to around 3.2. As with many quantile-based summaries, the convergence of the \( \hat{T} \) estimator is not smooth.

By assigning the interval of normal tails of the GSHD as \( T \in [1.482, 2.195] \), we are able to distinguish between heavy-tailed and light-tailed members of the GSHD for samples of \( n > 150 \). This interval of \( T \) approximately corresponds to
Simulated percentiles and the estimated distributional parameters of $\hat{T}$ for various sample sizes for the standardized GSHD(2.605), 10,000 runs

<table>
<thead>
<tr>
<th>n</th>
<th>Percentiles</th>
<th>Median</th>
<th>Mean</th>
<th>nVar</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td></td>
<td>1.772</td>
<td>1.817</td>
<td>4.600</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>1.753</td>
<td>1.775</td>
<td>4.085</td>
</tr>
<tr>
<td>150</td>
<td></td>
<td>1.743</td>
<td>1.762</td>
<td>3.948</td>
</tr>
<tr>
<td>200</td>
<td></td>
<td>1.713</td>
<td>1.721</td>
<td>3.292</td>
</tr>
<tr>
<td>300</td>
<td></td>
<td>1.718</td>
<td>1.723</td>
<td>3.419</td>
</tr>
<tr>
<td>500</td>
<td></td>
<td>1.711</td>
<td>1.712</td>
<td>3.296</td>
</tr>
<tr>
<td>700</td>
<td></td>
<td>1.703</td>
<td>1.706</td>
<td>3.233</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$t \in [-\pi/2, 2\pi]$. However, such selection procedure has very low power even for large samples, for example, 30% for $t = -\pi/2$ at $n = 900$, and is thus not practical. Therefore, we have selected $T \in [1.535, 1.917]$ as the classification interval of normal tails; this interval corresponds to the 95% confidence interval of normal $\hat{T}$ at $n = 300$ (Schmid and Trede, 2003). Practically, this interval also matches well the exact 95% confidence interval of $\hat{T}$ at $t = 2.605$ and $n = 300$. This interval approximately describes the selection interval $t \in [-\pi/2, 3\pi/2]$ and provides a reasonably high power of selection, as can be seen in Table 3.

In summary, we suggest the following classification scheme for $n > 300$:

\[ \hat{T} \geq 1.917, \quad \text{heavy-tailed, i.e. } -\pi < t \leq -\pi/2, \]
\[ 1.535 < \hat{T} < 1.917, \quad \text{normal-tailed, i.e. } -\pi/2 < t \leq 3\pi/2, \] (2)
\[ \hat{T} \leq 1.535, \quad \text{light-tailed, i.e. } t > 3\pi/2. \]

The confidence of the selection increases rapidly; for example, the exact confidence of the procedure is 98% for $n = 400$. The selective performance of procedure (2) improves as either the sample size or the difference in tail weights between the normal and actual distributions increases. The lack of precision of selection at

<table>
<thead>
<tr>
<th>T classification, (2)</th>
<th>LQW classification, (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$t = -2$</td>
</tr>
<tr>
<td>----</td>
<td>---------</td>
</tr>
<tr>
<td>100</td>
<td>0.762</td>
</tr>
<tr>
<td>300</td>
<td>0.879</td>
</tr>
<tr>
<td>500</td>
<td>0.941</td>
</tr>
<tr>
<td>700</td>
<td>0.969</td>
</tr>
<tr>
<td>900</td>
<td>0.982</td>
</tr>
</tbody>
</table>
the classification boundaries may be overcome by using an adaptive procedure robust to minor misspecifications of \( t \), as will be shown in Sec. 3.

As we have discussed so far, the Hogg’s \( \hat{T} \) estimator of the tail weight is a suitable selector statistic for adaptive procedures that aim to distinguish between heavy-tailed, normal-tailed, and light-tailed members of the GSHD. This selector is consistent, and its exact and asymptotic distributions are known under the hypothesis of normality. The selector may be used in applications with samples of \( n > 200 \). One serious drawback of \( \hat{T} \) is that, by its definition, it has a low breakdown, 2.5%. In the following, we will discuss another tail weight estimator with a higher breakdown.

The following tail weight measure was recently introduced and discussed by Brys et al. (2006):

\[
LQW_p = -\frac{X_{(1-p)/2} + X_{p/2} - 2X_{0.25}}{X_{(1-p)/2} - X_{p/2}},
\]

and it was suggested there that the percentiles \( p = 0.125 \) and \( p = 0.25 \) should be used to preserve the robustness property of this measure. The sample estimator of the measure is based on the corresponding sample order statistics and is consistent and asymptotically normal. For the normal distribution, the corresponding asymptotic variances of the estimator are \( n/\sigma^2 = 2.23 \) for \( p = 0.125 \), and \( n/\sigma^2 = 3.71 \) for \( p = 0.25 \).

As we can see in Table 1, \( LQW_{0.125} \) has a steeper gradient and is a better selector than \( LQW_{0.25} \). The breakdown of the sample estimator of \( LQW_{0.125} \) is 6.25%, which is higher than that of the Hogg’s \( \hat{T} \) estimator. The selective ability of the measure is, however, worse than that of \( \hat{T} \). Based on the 95% approximate confidence interval of the \( LQW_{0.125} \) estimator for the normal distribution (Brys et al., 2006), we are not able to distinguish between a normal-tailed and the uniform distribution for small samples, \( n < 150 \). Even at \( n = 200 \), the selection interval of the normal-tailed members of the GSHD would be too wide: \( t \in (-2.5, 10.5) \). Only for very large samples, \( n = 800 \), the hyperbolic secant distribution can be classified as a heavy-tailed distribution. The \( LQW_{0.125} \) selector may thus be applied on large samples, \( n > 800 \), where its selection region, \( t \in [-\pi/2, 3\pi/2] \), is comparable to that of \( \hat{T} \). The power of the \( LQW \) selector is high for large samples, \( n > 800 \) (see Table 3). The following selection scheme is thus suggested for samples of \( n > 800 \):

\[
\begin{align*}
LQW_{0.125} &\geq 0.352, & \text{heavy-tailed, i.e. } -\pi < t \leq -\pi/2, \\
0.142 < LQW_{0.125} < 0.352 & \text{normal-tailed, i.e. } -\pi/2 < t \leq 3\pi/2, \\
LQW_{0.125} &\leq 0.142, & \text{light-tailed, i.e. } t > 3\pi/2.
\end{align*}
\]

In the following section, we will introduce a one-sample tail-adaptive rank procedure for the GSHD based on the \( T \) and \( LQW_{0.125} \) tail measures.

3. Adaptive Rank Procedure of the GSHD

The one-sample signed rank procedure optimal for the GSHD was introduced in Kravchuk (2006). The signed rank test and estimator are based on the linear signed
rank statistic:

\[ S(t, \mu) = \sum_{i=1}^{n} a(R_i^+, t) \text{sign}(x_i - \mu), \]  

(4)

where \( R_i^+ = \text{rank}(|x_i - \mu|) \), and the scores for the GSHD are:

\[
a(i, t) = \begin{cases} 
\frac{1}{\sin(t)} \sin(t/2n) \sin\left(\frac{(2i-1)t}{2n}\right), & -\pi/2 \leq t < 0, \\
\frac{2i-1}{2n}, & t = 0, \\
\frac{1}{\sinh(t)} \sinh(t/2n) \sinh\left(\frac{(2i-1)t}{2n}\right), & t > 0. 
\end{cases}
\]

The signed rank statistic is asymptotically normal with the variance of \( \sum_{i=1}^{n} (a(i, t))^2 \). A closed form of the variance is given in Kravchuk (2006). At \( t = 0 \), the test is equivalent to the one-sample Wilcoxon test and the corresponding rank estimator is the Hodges-Lehmann estimator.

Constructing an adaptive rank procedure based on selection procedures (2) or (3), we maintain the asymptotic relative efficiency (ARE) within the GSHD family at above 85%. This leads to the following selection of the test-values, \( t^* \), (Kravchuk, 2006):

\[
t^* = \begin{cases} 
-0.5\pi, & \text{heavy-tailed}, \\
0.8\pi, & \text{normal-tailed}, \\
1.5\pi, & \text{light-tailed}. 
\end{cases}
\]

The adaptive signed rank test is based on the following standardized scores governed by \( t^* \):

\[
a(i) = \begin{cases} 
\sqrt{\frac{2}{n}} \sin\left(\frac{2i-1}{4n}\frac{\pi}{4}\right), & \text{heavy-tailed}, \\
(0.3758)\sqrt{\frac{2}{n}} \sinh\left(\frac{2i-1}{n}\frac{0.4\pi}{4}\right), & \text{normal-tailed}, \\
(0.0552)\sqrt{\frac{2}{n}} \sinh\left(\frac{2i-1}{2n}\frac{1.5\pi}{2}\right), & \text{light-tailed}. 
\end{cases}
\]

The adaptive signed rank test statistic (4) with scores (6) is asymptotically standard normal. The adaptive signed rank estimator of location may be computed by using its linearized estimator, \( \hat{\mu} \), as introduced in Kraft and Van Eeden (1972) and discussed in application to the GSHD in Kravchuk (2006):

\[
\hat{\mu} = \hat{\mu}_1 + \frac{\hat{\sigma}}{\sqrt{I_\mu(t^*)n}} S(t^*, \mu_1),
\]

(7)
where $\hat{\mu}_t$ is an initial estimate of the location parameter, $\hat{\sigma}$ is an estimate of the scale parameter of the underlying distribution, $S$ is the adaptive signed rank statistic (4), and $I_\mu(t^*)$ is the location information number of standardized GSHD$(t^*)$.

$$I_\mu(t^*) = \begin{cases} 0.500, & \text{heavy-tailed,} \\ 0.188, & \text{normal-tailed,} \\ 0.106, & \text{light-tailed.} \end{cases}$$

The Hodges-Lehmann estimator may be used as an initial estimate of the location parameter, $\hat{\mu}_1$. The ratio of a truncated sample range to the corresponding distribution range, $(\hat{X}_{0.9} - \hat{X}_{0.1})/(X_{0.9} - X_{0.1})$, may be used as an initial estimate of the scale parameter (Mudholkar et al., 1997). To be consistent with using the Hodges-Lehmann estimator for location, we base the scale estimator on the percentiles of the logistic distribution:

$$\hat{\sigma} = (\hat{X}_{0.90} - \hat{X}_{0.10})/(2\sqrt{3}/\pi \tan^{-1}(40/9))$$. With these initial estimates, a simple numerical search around (7) converges quickly.

Computing the standard error of the location rank estimator is, however, more challenging. Although it is possible to derive the standard error computationally, for example, by bootstrapping, the standard error may not be well defined due to the discreteness of the rank estimator. We suggest computing the standard error of the Hodges-Lehmann estimator ($S_{HL}$) by using a quick algorithm suggested by Brown and Wang (2005) and consequently scaling this error up by using the asymptotic efficiency (AE) of the Hodges-Lehmann estimator within the GSH family: $S_{HL}/\sqrt{AE_\mu}(t = 0)$. The AE of the Hodges-Lehmann estimator is (Kravchuk, 2006):

$$AE_\mu(t = 0) = \begin{cases} 0.98, & t^* = -\pi/2, \\ 0.95, & t^* = 0.8\pi, \\ 0.78, & t^* = 3\pi/2, \\ 0.48, & t^* = 10. \end{cases}$$

Computational estimates of the exact standard error of the adaptive rank estimator for $n < 400$ are presented in Table 4 for various tail parameters of the standardized GSHD of unit variance. The values are given for each of the three members (5) of the adaptive rank estimator.

The adaptive location rank estimator based on scores (6) and shape classes (5) is consistent, robust, and regular (Kravchuk, 2006) with an asymptotic efficiency of at least 85% for $t \in [-0.9\pi, 14]$, for $n < 800$, and $t \in [-0.9\pi, 10.5]$, for $n > 800$. Moreover, the efficiency of the adaptive procedure for $t \to -\pi$ is still reasonable, >73%.

A practical alternative to this new efficient adaptive rank test would be an adaptive test based on order statistics, which is readily available in software packages and maintains good efficiency within the GSH family. In the following section, we discuss one of these common estimators in application to the GSHD.
The asymptotic variances of these location estimators are given by Mudholkar et al. (1997) for distributions with various tail exponents:

\[
\begin{align*}
\sigma^2 & = \frac{1}{250} \left[ (X_{0.90} - X_{0.90})^2 + (X_{0.05} - X_{0.05})^2 \right], \\
\end{align*}
\]

where \( \hat{X} \) stands for a sample percentile and \( X \) for a distribution percentile. The asymptotic variances of these location estimators \( \hat{\mu} \) are given by Mudholkar

Table 4

Exact standard errors (\( \times 10^2 \)) of the adaptive rank estimator (6) for a standardized GSHD\( (t^\ast) \) (10,000 runs)

<table>
<thead>
<tr>
<th>( t^\ast = 3\pi/2 )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>11.17</td>
<td>15.92</td>
<td>15.53</td>
<td>15.37</td>
<td>15.47</td>
<td>14.60</td>
<td>15.14</td>
<td>13.54</td>
<td>13.80</td>
<td>12.71</td>
</tr>
<tr>
<td>100</td>
<td>7.64</td>
<td>10.60</td>
<td>11.13</td>
<td>10.75</td>
<td>11.04</td>
<td>10.18</td>
<td>10.20</td>
<td>9.51</td>
<td>9.27</td>
<td>8.54</td>
</tr>
<tr>
<td>150</td>
<td>6.28</td>
<td>9.06</td>
<td>8.72</td>
<td>8.74</td>
<td>8.53</td>
<td>8.36</td>
<td>7.92</td>
<td>7.81</td>
<td>7.77</td>
<td>7.32</td>
</tr>
<tr>
<td>200</td>
<td>5.45</td>
<td>7.79</td>
<td>7.64</td>
<td>7.53</td>
<td>7.50</td>
<td>7.70</td>
<td>7.36</td>
<td>7.01</td>
<td>6.63</td>
<td>6.37</td>
</tr>
<tr>
<td>250</td>
<td>4.64</td>
<td>6.92</td>
<td>6.98</td>
<td>6.82</td>
<td>6.69</td>
<td>6.59</td>
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<td>6.20</td>
<td>6.05</td>
<td>5.55</td>
</tr>
<tr>
<td>300</td>
<td>4.23</td>
<td>6.32</td>
<td>6.21</td>
<td>6.15</td>
<td>6.04</td>
<td>5.96</td>
<td>5.73</td>
<td>5.64</td>
<td>5.36</td>
<td>5.01</td>
</tr>
<tr>
<td>350</td>
<td>3.93</td>
<td>5.58</td>
<td>5.74</td>
<td>5.74</td>
<td>5.68</td>
<td>5.62</td>
<td>5.40</td>
<td>5.00</td>
<td>4.92</td>
<td>4.60</td>
</tr>
</tbody>
</table>

\( t^\ast = 0.8\pi \)

<table>
<thead>
<tr>
<th>( \hat{\mu}, \hat{\sigma} ) =</th>
<th>( \text{heavy-tailed} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left( \hat{X}<em>{0.5} - \hat{X}</em>{0.25} \right) )</td>
<td>( \left( \hat{X}<em>{0.75} - \hat{X}</em>{0.25} \right) )</td>
</tr>
<tr>
<td>( \left( 0.3(\hat{X}<em>{0.25} + \hat{X}</em>{0.75}) + 0.4\hat{X}_{0.5} \right) )</td>
<td>( \left( \hat{X}<em>{0.90} - \hat{X}</em>{0.10} \right) )</td>
</tr>
<tr>
<td>( \left( 0.5(\hat{X}<em>{0.94} + \hat{X}</em>{0.06}) \right) )</td>
<td>( \left( \hat{X}<em>{0.94} - \hat{X}</em>{0.06} \right) )</td>
</tr>
</tbody>
</table>

\( \text{normal-tailed} \)

\( \text{light-tailed} \)

4. Adaptive Location Median-Gastwirth-Midrange Test in Application to the GSHD

The following adaptive location and scale estimators were proposed in (Mudholkar et al., 1997) for distributions with various tail exponents:

\[
(\hat{\mu}, \hat{\sigma}) = \begin{cases} 
\left( \hat{X}_{0.5}, \frac{\hat{X}_{0.75} - \hat{X}_{0.25}}{\hat{X}_{0.75} - \hat{X}_{0.25}} \right), & \text{heavy-tailed}, \\
\left( 0.3(\hat{X}_{0.25} + \hat{X}_{0.75}) + 0.4\hat{X}_{0.5}, \frac{\hat{X}_{0.90} - \hat{X}_{0.10}}{\hat{X}_{0.90} - \hat{X}_{0.10}} \right), & \text{normal-tailed}, \\
\left( 0.5(\hat{X}_{0.94} + \hat{X}_{0.06}), \frac{\hat{X}_{0.94} - \hat{X}_{0.06}}{\hat{X}_{0.94} - \hat{X}_{0.06}} \right), & \text{light-tailed},
\end{cases}
\]

where \( \hat{X} \) stands for a sample percentile and \( X \) for a distribution percentile.
et al. (1997) for the general case. The estimators can be easily computed for the GSHD by using its percentile function (1).

Vaughan (2002) noted that the sample midrange is asymptotically an MLE for the GSHD as $t \to \infty$ and Kravchuk (2006) showed that the sample median, is asymptotically more than 80% efficient at $-\pi < t < -\pi/2$. It may thus seem that the abovementioned estimator is efficient for the GSHD. However, the tail classification of that adaptive estimator is somewhat arbitrary and is well defined only for distributions with polynomial tails (Mudholkar et al., 1997). On the other hand, the GSHD behaves like a scaled Laplace distribution in tails (Vaughan, 2002). Therefore, it is not immediate that this adaptive test is efficient under the GSHD tail classification proposed in the current article. We computationally estimated the AREs of the Median-Gastwirth-Midrange estimators in comparison to the adaptive rank estimator (6), and the results are presented in Table 5.

In the case of the GSHD, we would argue that the sample median is not a good practical choice for $-\pi < t < -\pi/2$ as its efficiency does not exceed 87% and drops quickly below 80% for $t > -\pi/2$ (Kravchuk, 2006). On the other hand, the Gastwirth estimator can be used for the heavy-tailed members of the GSHD but should not replace the rank estimator for the normal-tailed group. The midrange’s efficiency relative to the corresponding rank estimator is consistently low for $-\pi < t < 6$. Although the midrange is reported to be optimal for the GSHD when

<table>
<thead>
<tr>
<th>Shape parameter $t$</th>
<th>−3</th>
<th>−2</th>
<th>$-\pi/2$</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>0.8π</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median ($t^* = -\pi/2$)</td>
<td>.97</td>
<td>.87</td>
<td>.81</td>
<td>.76</td>
<td>.76</td>
<td>.75</td>
<td>.67</td>
<td>.63</td>
<td>.59</td>
<td>.58</td>
<td>.53</td>
</tr>
<tr>
<td>Gastwirth ($t^* = -\pi/2$)</td>
<td>.98</td>
<td>.98</td>
<td>.93</td>
<td>.92</td>
<td>.92</td>
<td>.92</td>
<td>.88</td>
<td>.80</td>
<td>.80</td>
<td>.80</td>
<td>.72</td>
</tr>
<tr>
<td>Gastwirth ($t^* = 0.8\pi)$</td>
<td>1.86</td>
<td>1.19</td>
<td>1.10</td>
<td>1.02</td>
<td>.99</td>
<td>.94</td>
<td>.77</td>
<td>.61</td>
<td>.57</td>
<td>.54</td>
<td>.42</td>
</tr>
<tr>
<td>Midrange ($t^* = 0.8\pi$)</td>
<td>.02</td>
<td>.14</td>
<td>.18</td>
<td>.22</td>
<td>.24</td>
<td>.26</td>
<td>.34</td>
<td>.49</td>
<td>.58</td>
<td>.71</td>
<td>1.13</td>
</tr>
<tr>
<td>Midrange ($t^* = 3\pi/2$)</td>
<td>.04</td>
<td>.19</td>
<td>.24</td>
<td>.26</td>
<td>.29</td>
<td>.31</td>
<td>.38</td>
<td>.49</td>
<td>.51</td>
<td>.63</td>
<td>.84</td>
</tr>
</tbody>
</table>

Table 6

Exact standard errors ($\times 10^2$) of the Gastwirth estimator (6) for the standardized GSHD of unit variance (10,000 runs)

<table>
<thead>
<tr>
<th>−3</th>
<th>−2</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>3.97</td>
<td>8.51</td>
<td>9.79</td>
<td>10.08</td>
<td>10.25</td>
<td>10.92</td>
<td>11.70</td>
<td>12.21</td>
<td>12.61</td>
</tr>
<tr>
<td>200</td>
<td>2.82</td>
<td>6.04</td>
<td>6.85</td>
<td>7.09</td>
<td>7.18</td>
<td>7.63</td>
<td>8.15</td>
<td>8.70</td>
<td>8.98</td>
</tr>
<tr>
<td>250</td>
<td>2.46</td>
<td>5.45</td>
<td>6.10</td>
<td>6.38</td>
<td>6.58</td>
<td>6.94</td>
<td>7.38</td>
<td>7.75</td>
<td>8.15</td>
</tr>
<tr>
<td>300</td>
<td>2.29</td>
<td>4.94</td>
<td>5.56</td>
<td>5.81</td>
<td>5.95</td>
<td>6.28</td>
<td>6.74</td>
<td>7.04</td>
<td>7.42</td>
</tr>
<tr>
<td>350</td>
<td>2.11</td>
<td>4.52</td>
<td>5.21</td>
<td>5.36</td>
<td>5.50</td>
<td>5.79</td>
<td>6.24</td>
<td>6.51</td>
<td>6.89</td>
</tr>
</tbody>
</table>
$t \rightarrow \infty$ (Vaughan, 2002), using this estimator in practice is not efficient unless the shape parameter is very large, $t > 10$.

For completeness, simulated estimates of the standard errors of the Gastwirth estimator for the standardized GSHD of unit variance are provided in Table 6 for $n < 450$.

In summary, the tail classification of the Median-Gastwirth-Midrange adaptive test (Mudholkar et al., 1997) does not match the tail classification of the adaptive rank test introduced in this paper. The Gastwirth estimator is easy to compute, and this estimator is efficient for the GSHD at $-\pi < t < 0$. The median and midrange estimators should not be used for the GSHD.

5. Conclusion

We have introduced and studied a new tail-adaptive efficient signed rank location procedure for the generalized secant hyperbolic distribution. The procedure is based on the Hogg’s or Brys’s measures of the tail weight, and its asymptotic efficiency is maintained consistently above 85%. The test and estimation procedures are simple and should be useful to those using the GSHD for modelling and other purposes on large samples.

The article also contributes to the general research on the GSHD. In particular, the properties of the Hogg’s and Brys’s measures for the GSHD have been studied in detail, and we have also shown that the Gastwirth estimator is efficient for heavy-tailed members of the GSHD. Further research on the tail behavior of the GSHD is in progress.

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References


