Scattering of a spherical wave by a small sphere: An elementary solution

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Abstract: Wave scattering by objects that are small compared to the wavelength (Rayleigh scattering) is usually studied for plane incident waves. However, knowledge of the full Green’s function of the problem becomes necessary when the separation of scatterers from either an interface or each other is comparable to the scatterers’ dimensions. Here, an elementary analytic solution is derived for diffraction of a spherical sound wave by a small, soft sphere. The approximate solution is obtained from asymptotic expansions of an exact solution, holds everywhere outside the sphere, and reduces to classical results due to Kelvin and Rayleigh in appropriate special cases.

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1. Introduction

Diffraction of waves on objects with dimensions that are small compared to the wavelength, often referred to as Rayleigh scattering, is a classical problem of wave theory, which continues to attract considerable attention due to numerous applications in optics, acoustics, electromagnetics, fluid mechanics, and remote sensing. Rather general asymptotic results have been obtained for Rayleigh scattering of plane incident waves. However, many problems of current interest require knowledge of the full Green’s function in the presence of a scatterer, i.e., of the diffracted field in the case of an incident spherical wave. In acoustics, such problems include scattering from dense clouds of objects (e.g., from red blood cells in medical imaging or from fish schools in assessment of fish populations in the ocean) and radiation and scattering of sound by bodies located near interfaces (e.g., underground tunnels and pipelines, air bubbles in water, or buried mines). Even in the textbook case of an acoustically soft sphere no closed-form analytical solution is available for Rayleigh scattering of a spherical wave.

Following Rayleigh, the exact solution for scattering of a spherical wave by a soft sphere can be expressed as an infinite series in spherical harmonics. This solution is valid for arbitrary sound frequency and sphere radius. Kelvin obtained an elementary solution of the static problem. In this paper, through asymptotic expansion of the Rayleigh’s exact result, we derive an elementary analytic solution for diffraction of a spherical wave by a soft sphere, with the sphere radius being the only small parameter of the problem. The asymptotic solution holds everywhere outside the sphere and has a structure similar to the Kelvin’s solution. The present research was motivated by a study of the geophysical and biological implications of the anomalous transparency of gas-liquid and gas-solid interfaces to low-frequency sound, see Refs. 9–12, where it was found that diffraction effects lead to a dramatic increase in sound transmission into gas when a point source in a liquid or a solid approaches the interface.

2. Theory

Consider scattering of monochromatic (CW) waves of frequency \( \omega \) by an acoustically soft sphere of radius \( a \). The sphere is imbedded in a homogeneous fluid with sound
speed \(c\) and mass density \(\rho\). Introduce a spherical polar coordinate system \((r, \theta, \varphi)\) with an origin at the center of the sphere. Cartesian coordinates are related to the spherical coordinates by

\[
x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta.
\]

Let a spherical wave

\[
p_{\text{in}} = \exp[\text{i}kR(b)]/R(b), \quad R(z_0) = \sqrt{x^2 + y^2 + (z - z_0)^2}
\]

be incident on the sphere (Fig. 1). Here \(b > a\), \(k = \omega/c\) is the acoustic wave number, \(p\) is acoustic pressure, and \(R(z_0)\) is the distance between points \((x, y, z)\) and \((0, 0, z_0)\). The time dependence \(\exp(-\text{i} \omega t)\) of the acoustic field is assumed and suppressed. Outside the sphere, acoustic pressure \(p = p_{\text{in}} + p_{\text{sc}}\), where the pressure \(p_{\text{sc}}\) in the scattered wave satisfies the homogeneous Helmholtz equation \(\Delta p_{\text{sc}} + k^2 p_{\text{sc}} = 0\) at \(r > a\) and radiation conditions at infinity. The pressure on the surface of the soft sphere is constant (in particular, this is the case for pressure-release surfaces), and the boundary condition for the scattered wave is \(p_{\text{sc}} = -p_{\text{in}}\) at \(r = a\).

The incident wave and scattered waves can be represented \(^1\,^2\) as follows:

\[
p_{\text{i}} = -\text{i}k \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \theta) j_n^{(1)}(kr_>) j_n(kr<), \quad r_> = \max(r, b), \quad r_< = \min(r, b),
\]

\[
p_{\text{sc}} = -\text{i}k \sum_{n=0}^{\infty} (2n + 1) A_n P_n(\cos \theta) j_n^{(1)}(kb) h_n^{(1)}(kr), \quad r \geq a,
\]

where \(A_n = j_n(ka)/h_n^{(1)}(ka)\), \(P_n\) are Legendre polynomials, \(j_n\) are spherical Bessel functions, and \(h_n^{(1)}\) are spherical Hankel functions of the first kind \(^3\). The lowest-order Legendre polynomials are \(P_0(q) = 1, P_1(q) = q\); spherical Bessel and Hankel functions can be calculated using Rayleigh’s formulas \(^3\).

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Fig. 1. (Color online) Geometry of the problem.
\[ j_n(q) = q^n \left( -\frac{1}{q} \frac{d}{dq} \right)^n \frac{\sin q}{q}, \quad h_n^{(1)}(q) = -iq^n \left( -\frac{1}{q} \frac{d}{dq} \right)^n \frac{e^{iq}}{q}, \quad n = 0, 1, 2, \ldots \quad (4) \]

or expansions in powers of their argument

\[ j_n(q) = 2^n q^n \sum_{m=0}^{\infty} \frac{(n+m)!(\frac{-q^2}{m})^m}{m!(2n+2m+1)!}; \quad h_n^{(1)}(q) = i^n q^n \sum_{m=0}^{\infty} \frac{(n+m)!(\frac{-q^2}{m})^m}{m!(n-m)!}. \quad (5) \]

We are interested in the case of a small scatterer, where the radius of the sphere is small compared to the wavelength: \( ka \ll 1 \). This regime is known as Rayleigh scattering. From Eqs. (4) and (5), it follows that

\[ j_n(q) = \frac{2^n n! q^n}{(2n+1)!} \left[ 1 + O(q^2) \right], \quad h_n^{(1)}(q) = \frac{(2n)!}{2^n n! q^{n+1}} \left[ 1 + O(q^2) \right], \quad n = 1, 2, \ldots, \quad (6) \]

and, hence,

\[ j_0(q) = 1 - q^2/6 + O(q^4), \quad h_0^{(1)}(q) = e^{iq}/iq \quad (7) \]

The exact solution of the problem of scattering (diffraction) of a spherical wave by a soft sphere is given by an infinite series (3). In order to derive a closed-form, approximate solution to the problem for a small sphere, we first consider the case where the following two inequalities hold:

\[ kb \ll 1, \quad kr \ll 1. \quad (9) \]

The assumptions (9) mean that both the receiver and the source of the incident wave are located within a fraction of the wavelength from the scatterer’s center. Consider the series (2), which represents the incident spherical wave, in a low-frequency limit. Specifically, let \( kr \ll 1 \). Using Eq. (7) to evaluate the first term of the series and Eq. (6) to evaluate all other terms, we find

\[ p_n = \left[ \frac{e^{ikr}}{r} + \frac{1}{r} \sum_{n=1}^{\infty} P_n(\cos \theta) \left( \frac{r_\circ}{r} \right)^n \left[ 1 + O(k^2 r^2_\circ) \right] \right]. \quad (10) \]

Under conditions (9), we use Eqs. (6), (7), and (8) to evaluate the series (3) for the scattered wave and obtain

\[ p_\circ = -\frac{a}{b} \left[ \frac{e^{ikr(b-a)}}{r} + \frac{1}{r} \sum_{n=1}^{\infty} P_n(\cos \theta) \left( \frac{a^2}{br^2} \right)^n \left[ 1 + O(k^2 (b^2 + r^2)) \right] \right]. \quad (11) \]

Compare the right sides of Eqs. (10) and (11). The main terms, i.e., the terms of zero order in \( k \), in the developments of \( p_n \) and \( -bp_\circ/a \) will coincide, if \( r_\circ = r, r_\circ = a^2/b \). Expressing the infinite sum in Eq. (11) in terms of \( p_m \), we obtain

\[ p_\circ = -\frac{a}{b} \left\{ \frac{\exp[kR(a^2/b)]}{R(a^2/b)} + ik(b-a) \frac{\exp(ikr)}{r} \right\} \left[ 1 + O(k^2 (b^2 + r^2)) \right]. \quad (12) \]

Note that the quantity in braces in Eq. (12) is an exact solution to the Helmholtz equation for arbitrary \( a, b, \) and \( r \). Equation (12) is valid under condition (9) and can be viewed as an asymptotics of the scattered field in the regime, where \( k \to 0 \) while all the geometrical parameters—\( a, b, r, \) and \( \theta \)—are kept constant. The exponentials in
Eq. (12) are close to unity, but retaining either the first exponential or the first terms of its Taylor expansion is essential for capturing the subdominant term of the asymptotic expansion of \( p_{sc} \).

The stationary limit of Eq. (12),

\[
\lim_{k \to 0} p_{sc} = -\frac{a}{b} R \left( \frac{a^2}{b} \right),
\]

gives an exact solution to the boundary value problem for the Laplace equation \( \Delta p = 0 \) for arbitrary \( a, b, r, \) and \( \theta \). Mathematically, the stationary limit of the acoustic problem of scattering of a spherical wave by a soft sphere is equivalent to the electrostatic problem regarding the field due to a point charge in the presence of a grounded conducting sphere, with \( p \) giving the potential of the electric field. Equation (13) shows that, outside the sphere, the field of charges distributed on the conducting sphere coincides with the field due to a point charge—\( \bar{a} \) lb located at the point \( (0, 0, a^2/b) \) within the sphere. This is in agreement with the well-known direct solution of the electrostatic problem, which was first obtained by Kelvin \(^{11}\) by rather simple and elegant mathematical means. The point \( (0, 0, a^2/b) \) lies on the line connecting the center of the sphere with the point source (charge) and is known as the Kelvin’s inversion point.

Now, we return to the sound scattering problem and consider another case, where either the sound source or receiver (or both) are separated from the center of the scatterer by a distance that is large compared to its radius. Then

\[
\alpha \equiv \frac{a^2}{br} \ll 1.
\]

We no longer assume that inequalities (9) hold. No assumptions, other than \( ka \ll 1 \), are made about the wavelength. Under conditions \( ka \ll 1 \) and Eq. (14), the series (3) becomes rapidly converging because of a rapid decrease of the coefficients \( A_n \) (8) with \( n \) which cannot be compensated by the possible increase of the products \( h_n^0 (kb) h_n^0 (kr) \) with \( n \). Under the sum in Eq. (3), only the coefficients \( A_n \) (8) depend on \( a \). To calculate the scattered field with accuracy to the factor \( 1 + O(k^2 a^2 + z^2) \), it suffices to replace the series (3) by its first two terms. Using Eq. (8) for the coefficients of the series and Eq. (4) for spherical Hankel functions of zero and the first order, we find

\[
p_{sc} = -\frac{a}{b} e^{i k (b - a)} \left\{ \frac{\exp(ikr)}{r} - (1 - ikb) e^{ikb} \frac{a^2}{b} \frac{\partial}{\partial z} \left[ \frac{\exp(ikr)}{r} \right] \right\} \left[ 1 + O(k^2 a^2 + z^2) \right].
\]

Equation (15) also can be formally obtained from the two first terms of such an asymptotic expansion of the exact solution (3), (8) that \( a \to 0 \) while all other parameters of the problem: \( k, b, r, \) and \( \theta \), are kept constant. From Eq. (1), we have \( R(z_0) = 0 \) and

\[
\frac{\exp[ikR(z_0)]}{R(z_0)} = \frac{\exp(ikr)}{r} - z_0 \frac{\partial}{\partial z} \frac{\exp(ikr)}{r} + O \left( k^2 z_0^2 + \frac{z_0^2}{r^2} \right).
\]

Hence, within the accuracy of Eq. (15), the derivative in the right side of the latter can be replaced by a finite difference, leading to

\[
p_{sc} = -\frac{a e^{ikb}}{b} \left\{ (1 - ikb) \frac{\exp(ik (a^2/b))}{R(a^2/b)} + ik(b - a) \frac{\exp(ikr)}{r} \right\} \left[ 1 + O(k^2 a^2 + z^2) \right].
\]

Since \( ka \ll 1 \), the inequality (14) is met when either of the inequalities (9) is violated. Furthermore, inequalities (9) and (14) can be met simultaneously. Thus, the approximate solutions (12) and (17) have an overlapping domain of validity and together allow one to calculate \( p_{sc} \) at every point outside the scatterer. However, it is more convenient to have a single, uniformly valid expression for the scattered wave rather than two expressions with overlapping domains of validity. Let
\[ P_1 = -\frac{a}{bR(a^2/b)} \exp[ikR(a^2/b) + ik(b-a)]. \]  

By noting that, according to Eq. (1), \(|R(a^2/b) - r| \leq a^2/b = x^2\), it is easy to check that \(P_1\) differs from the local asymptotics (17) by the factor \(1 + O(x + k^2a^2)\) and from the local asymptotics (12) by the factor \(1 + (1 - a/b)/O(ka + k^2b^2 + k^2r^2)\). These factors are close to unity in the domains of validity of the respective local asymptotics. Hence, Eq. (18) provides a uniform asymptotics of the scattered field in the sense that \(P_1\) asymptotically approximates \(p_{sc}\) (3) for arbitrary \(k\), \(b\), \(r\), and \(\theta\) as long as the condition \(ka \ll 1\) is met.

Both local asymptotics (12) and (17) consist of fields of image sources at the center of the sphere and at the Kelvin’s inversion point and differ only by complex amplitudes of these sources. Equations (12) and (17) are subsumed by

\[ P_2 = -\frac{ae^{ik(b-a)}}{b(1 - ika)} \left[ (1 - ibk)e^{ikR(a^2/b)} + ik(b-a)e^{ikr} \right]. \]  

Comparison of Eq. (19) with Eqs. (12) and (17) shows that \(P_2\) reproduces the exact scattered field (3) with accuracy to the factor \(1 + O(k^2(b^2 + r^2))\), when condition (9) holds, and the factor \(1 + O(k^2a^2 + \phi^2)\), when condition (14) holds. Unlike \(P_1\), uniform asymptotics \(P_2\) maintains the second-order accuracy of the local asymptotics.

Uniform asymptotics \(P_1\) and \(P_2\) possess an important property that improves their accuracy at finite \(ka\). When a point monopole source approaches a pressure-release surface, acoustic field vanishes. Therefore, it is expected that \(p_{sc} = -p_{in}\) in the limit \(b \to a\). Comparison of Eq. (3) with Eq. (2) at \(b = a\) shows that this is indeed the case. According to Eqs. (18) and (19), the approximate solutions \(P_1\) and \(P_2\) have the same limit \(P_1 = P_2 = -p_{in}\) and, hence, become exact in the limit \(b \to a\). This result holds for the arbitrary sphere radius and acoustic frequency.

3. Numerical simulations

Accuracy of the asymptotics (18) and (19) at finite \(ka\) depends on dimensionless parameters \(kb\), \(kr\), and \(\theta\). We quantify the asymptotic errors by numerically evaluating the exact solution (3) using Mathematica®. Figure 2 illustrates that both asymptotics (18) and (19) are highly accurate at \(ka \ll 1\). As expected, asymptotic errors of \(P_2\) are proportional to \(k^2a^2\). The errors of \(P_1\) are proportional to \(ka\) when \(x = O(1)\). However, as \(x\) (14) increases, the accuracy of \(P_1\) improves and at \(x \gg 1\) becomes quadratic in \(ka\). \(P_2\) systematically provides higher accuracy than \(P_1\) for both the amplitude and phase of the scattered field (Figs. 2 and 3). Derived assuming \(ka \ll 1\), \(P_2\) (19) proves to be a good approximation to the exact solution for all possible \(ka\), when either \(kb \leq 1\) or \(kr \leq 1\), and for \(ka\) as large as \(\sim 0.75\), when \(kb > 1\) and \(kr > 1\) (Fig. 3).

4. Discussion

The classic problem\textsuperscript{1,2} of diffraction of a spherical acoustic wave on a small soft sphere allows an elementary solution with a clear physical meaning. This new solution is given by Eqs. (18), which is valid to the first order in \(ka\), and (19), which is valid to the second order in \(ka\). Unlike quasi-stationary solutions\textsuperscript{3} valid at \(kr \ll 1\), Eqs. (18) and (19) hold everywhere outside the scatterer and describe the transition of the scattered wave from the near to the far field of the scatterer. For fixed \(b\) and \(r\) and \(a \to 0\), the relative error of the uniform asymptotics is of the second order in \(a\), while their absolute error is of the third order in \(a\). \(P_1\) and \(P_2\) satisfy the Helmholtz equation and radiation conditions at infinity exactly, but generally meet the boundary conditions at \(r = a\) only approximately. Both uniform asymptotics reduce to the exact solution (13) when sound frequency tends to zero. When \(b \to a\), \(P_1\) and \(P_2\) have the same limit \(P_1 = P_2 = -p_{in}\) as the exact solution, at all frequencies and for arbitrary \(a\).
The physical meaning of the asymptotics $P_1$ is similar to the physical meaning of Kelvin’s solution to the static problem: to first order, the acoustic field scattered by a small, soft sphere coincides with the field radiated by a point image source located at the Kelvin’s inversion point $(0, 0, a^2/b)$ within the sphere. Note that the amplitude of the scattered wave is proportional to the radius of the sphere and remains finite in the limit $k \to 0$. According to the more accurate second-other asymptotics $P_2$ (19), the scattered wave consists of two spherical waves, which emanate from image sources located at the Kelvin inversion point $(0, 0, a^2/b)$ and the center of the sphere. The image sources have comparable magnitudes, unless $kb \ll 1$, in which case the source at the Kelvin inversion point dominates. When $kb \gg 1$, fields due to the image sources

![Fig. 2.](image1.jpg)

**Fig. 2.** (Color online) Frequency dependence of the accuracy of the asymptotics $P_1$ (dashed lines) and $P_2$ (solid lines) of the scattered wave field. Relative deviation $|1 - P_i/p_{exact}|$ of the asymptotics from the exact solution at $\theta = \pi/4$ is shown as a function of the non-dimensional radius $ka$ of the sphere for $b = r = 1.5a$ (1), $b = 1.5a$, $r = 10^3 a$ (2), $b = 10a$, $r = 1.5a$ (3), and $b = 10a$, $r = 10^3 a$ (4).

![Fig. 3.](image2.jpg)

**Fig. 3.** (Color online) Dependence of accuracy of the asymptotics $P_1$ (dashed lines) and $P_2$ (solid lines) of the scattered wave field on the size of the scatterer. Relative amplitude errors (a), (c), (e) and absolute phase errors (b), (d), (f) are shown as a function of the non-dimensional radius $ka$ of the sphere for $kb = 0.25$, $kr = 1$ (a), (b), $kb = 1$, $kr = 20$ (c), (d), $kb = 20$, $kr = 10$ (e), (f) and different values of the angle $\theta = 0$ (curves labeled 1), $\pi/3$ (2), $2\pi/3$ (3), and $\pi$ (4).
interfere destructively, so that the amplitude of the scattered wave is inversely proportional to $b$ just as the amplitude of the field incident on the sphere. The image sources coalesce when $b \to 1$.

Backscattering of spherical waves, where sound source and receiver are colocated and $R(a^2/b) = b - a^2/b$, was considered in Ref. 14. Inspection shows that Eqs. (18) and (19) agree with the results of Ref. 14 in this special case.

Diffraction of a plane wave is a particular case of spherical wave diffraction. A plane incident wave $\exp(-ikz) = \lim_{b \to 1} b \exp(-ikb)p_{in}$, according to Eq. (1). Solution $\tilde{p}_{sc}$ to the problem of scattering of the plane wave is obtained from $p_{sc}$ (3) as $\tilde{p}_{sc} = \lim_{b \to 1} b \exp(-ikb)p_{sc}$. The first- and second-order asymptotics of $\tilde{p}_{sc}$, $\tilde{P}_1 = \frac{a}{r} e^{ik(r-a)}$, $\tilde{P}_2 = \left[ 1 - \frac{ika^2(1 - ikr)}{r(1 - ika)} \cos \theta \right] \tilde{P}_1$, (20)

which are obtained by replacing $p_{sc}$ with $P_1$ (18) and $P_2$ (19), are in agreement with the classical results due to Rayleigh.1,2

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References and links