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The First Integral Method for Exact Solutions of Nonlinear Fractional Differential Equations

In this paper, we establish exact solutions for some nonlinear fractional differential equations (FDEs). The first integral method with help of the fractional complex transform (FCT) is used to obtain exact solutions for the time fractional modified Korteweg–de Vries (fmKdV) equation and the space–time fractional modified Benjamin–Bona–Mahony (fmBBM) equation. This method is efficient and powerful in solving kind of other nonlinear FDEs. [DOI: 10.1115/1.4028065]

Keywords: exact solutions, first integral method, fractional complex transform, time fmKdV equation, space–time fmBBM equation

1 Introduction

FDEs have recently proved to be valuable tools to the modeling of many physical phenomena, and have gained the focus of many studies due to their frequent appearance in various applications such as fluid flow, signal processing, control theory, systems identification, finance, fractional dynamics, and other areas. The FDEs are also used in modeling of many chemical processes mathematical biology and many other problems in physics and engineering. The FDEs have been investigated by many researchers [1–3].

In recent decades, a large amount of literature has been provided to construct the exact solutions of fractional ordinary differential equations and fractional partial differential equations of physical interest. Many powerful and efficient methods have been proposed to obtain approximate solutions and exact solutions of FDEs, such as the Adomian decomposition method [4,5], the variational iteration method [6,7], the homotopy analysis method [8,9], the homotopy perturbation method [10,11], the differential transformation method [12,13], the fractional subequation method [14–16], the first integral method [17], the exp-function method [18,19], the (G'/G)-expansion method [20,21], and so on. Based on these methods, a variety of FDEs have been investigated.

We note that as long as a different nonlinear fractional complex transformation form is taken for ξ, then a certain fractional partial differential equation can be turned into another ordinary differential equation of integer order, whose exact solutions are established based on Jumarie's modified Riemann–Liouville derivative.

The first integral method is a very powerful mathematical technique for finding exact solutions of partial and FDEs. It has been developed by Feng [22,23] and used successfully by many authors for finding exact solutions of partial differential equations (PDE) and FDEs in mathematical physics [24–28]. Using first integral method (see Ref. [17]), exact solutions of the fractional modified Korteweg–de Vries (fmKdV) equation and the space–time fmBBM equation were obtained.

In this article, we will suggest the first integral method, and utilize this method to solve the following two fractional nonlinear differential equations. This method finds exact solutions that are functions U of one variable that needs being determined, and this variable is searched among linear combinations of two powers of x and t: with its derivatives up to order $n-1$, such functions U form a vector that satisfies a differential system of first order, whose right hand side is made of polynomials of the unknowns. Searching for a polynomial that vanishes on system's trajectories help solving the problem.

We consider the following time fractional mKdV equation [29]:

$$D_t^{\alpha}u + u^2u_x + u_{xxx} = 0, \quad t>0, \quad 0<\alpha\le 1 \tag{1.1}$$

where α is a parameter describing the order of the fractional time-derivative.

We introduce the space–time fractional mBBM equation [30]

$$D_t^{\alpha}u + D_x^{\alpha}u - vu^2D_x^{\alpha}u + D_x^{3\alpha}u = 0 \tag{1.2}$$

where v is a nonzero positive constant. This equation was first derived to describe an approximation for surface long waves in nonlinear dispersive media. It can also characterize the hydromagnetic waves in cold plasma, acoustic waves in inharmonic crystals, and acoustic gravity waves in compressible fluids.

The present paper investigates for the applicability and effectiveness of the first integral method on fractional nonlinear partial differential equations.

The rest of this letter is organized as follows. In Sec. 2, we are given the modified Riemann–Liouville derivative and important properties and Sec. 3 we describe the first integral method and the FCT. In Secs. 4 and 5, to illustrate the validity and advantages of the method, we will apply it to the time fractional mKdV equation and the space–time fractional mBBM equation. In Sec. 6, some conclusions are given.

2 The Modified Riemann–Liouville Derivative

Jumarie proposed a modified Riemann–Liouville derivative. With this kind of fractional derivative and some useful formulas, we can convert FDEs into integer-order differential equations by variable transformation in Ref. [31].

In this section, we first give some properties and definitions of the modified Riemann–Liouville derivative which are used further in this paper.

Assume that f: $R \to R$, $x \to f(x)$ denote a continuous but not necessarily differentiable function. The Jumarie's modified Riemann–Liouville derivative of order α is defined by the expression

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$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1}[f(\xi)-f(0)]d\xi, & \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)]d\xi, & 0 < \alpha < 1 \\ (f^{(n)}(x))^{(\alpha-n)}, \quad n \le \alpha \le n+1, & n \ge 1 \end{cases} \tag{2.1}$$

A few properties of the modified Riemann–Liouville derivative were summarized and four famous formulas of them are

$$D_t^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0 \tag{2.2}$$

$$D_x^\alpha(u(x)v(x)) = v(x)D_x^\alpha u(x) + u(x)D_x^\alpha v(x) \tag{2.3}$$

$$D_x^\alpha f[u(x)] = f_u'(u)D_x^\alpha u(x) \tag{2.4}$$

$$D_x^\alpha f[u(x)] = D_u^\alpha f(u)(u'(x))^\alpha \tag{2.5}$$

which are direct consequences of the equality

$$d^\alpha x(t) = \Gamma(1+\alpha)dx(t) \tag{2.6}$$

which holds for nondifferentiable functions. In the above formulas (2.3)–(2.5), $u(x)$ is nondifferentiable function in Eqs. (2.3) and (2.4) and differentiable in Eq. (2.5). The function $v(x)$ is nondifferentiable, and $f(u)$ is differentiable in Eq. (2.4) and nondifferentiable in Eq. (2.5). That is why formulas (2.3)–(2.5) should be used carefully.

3 The First Integral Method and the Fractional Complex Transform

We consider the following nonlinear FDE of the type:

$$P(u, D_t^\alpha u, D_x^\beta u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\beta u, D_x^\beta D_x^\beta u, \ldots) = 0, \quad 0 < \alpha, \beta < 1 \tag{3.1}$$

where u is an unknown function and P is a polynomial of u and its partial fractional derivatives.

Step 1. The pioneer work of Li and He [32,33] introduced a FCT to convert FDEs into ordinary differential equations, so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. Using the FCT

$$\xi = \frac{\tau x^\beta}{\Gamma(1+\beta)} + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \tag{3.2}$$

where τ and λ are nonzero arbitrary constants and determined later. We can rewrite Eq. (3.1) in the following nonlinear ordinary differential equation:

$$Q(f(\xi), f'(\xi), f''(\xi), f'''(\xi), \ldots..) = 0 \tag{3.3}$$

where the prime denotes the derivation with respect to ξ. If possible, we should integrate Eq. (3.3) term by term one or more times.

Step 2. Suppose that the solution of ordinary differential equation (ODE) (3.3) can be written as follows:

$$u(x,t) = f(\xi) \tag{3.4}$$

Step 3. Now, we take a new independent variable

$$X(\xi) = f(\xi), \quad Y(\xi) = f_\xi(\xi) \tag{3.5}$$

which leads to a new system of

$$\begin{aligned} X_\xi(\xi) &= Y(\xi) \\ Y_\xi(\xi) &= H(X(\xi), Y(\xi)) \end{aligned} \tag{3.6}$$

Step 4. By the known theory of ordinary differential equations [34], if we can find the integrals to (3.6) under the same conditions, then the general solutions to (3.6) can be solved directly. With the help of the Division Theorem for two variables in complex domain C which is based on the Hilbert–Nullstellensatz Theorem [35], we obtain one first integral to (3.6) which can reduce (3.3) to a first order integrable ordinary differential equation. Then, an exact solution to (3.1) is obtained by solving this equation directly. Now, let us recall the Division Theorem:

Division Theorem. "*Suppose that P(w, z), Q(w, z) are polynomials in C(w, z) and P(w, z) is irreducible in C(w, z). If Q(w, z) vanishes at all zero points of P(w, z), then there exists a polynomial G(w, z) in C(w, z) such that*

$$Q[w,z] = P[w,z]G[w,z] \tag{3.7}$$

The fact that the real field $\mathbb{R}$ *is a subfield of the complex field* $\mathbb{C}$ *is well known. The extension of a given equation in* $\mathbb{R}$ *to an equation in* $\mathbb{C}$ *is always possible. If the extended equation has an algebraic curve solution in* $\mathbb{C}$*, then the intersection of the manifold of this solution and the real plane must be the algebraic curve solution of the original equation in* $\mathbb{R}$*. Thus, the Division Theorem stated in* $\mathbb{C}$ *can also be used in* $\mathbb{R}$ [36].

Feng and Roger [37], *pointed out, that the Division Theorem follows immediately from the Hilbert–Nullstellensatz Theorem* [35] *of commutative algebra.*"

Hilbert–nullstellensatz Theorem. *Let K be a field and L be an algebraic closure of K. Then:*

(1) *Every ideal* γ *of* $K[X_1, X_2, \ldots, X_n]$ *not containing 1 admits at least one zero in* L^n.
(2) *Let* $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$ *be two elements of* L^n *for the set of polynomials of* $K[X_1, X_2, \ldots, X_n]$ *zero at x to be identical with the set of polynomials of* $K[X_1, X_2, \ldots, X_n]$ *zero at y, it is necessary and sufficient that there exists a K–automorphism S of L such that* $y_i = S(x_i)$ *for* $1 \le i \le n$.
(3) *For an ideal* α *of* $K[X_1, X_2, \ldots, X_n]$ *to be maximal, it is necessary and sufficient that there exists an x in* L^n *such that* α *is the set of polynomials of* $K[X_1, X_2, \ldots, X_n]$ *zero at x.*
(4) *For a polynomial Q of* $K[X_1, X_2, \ldots, X_n]$ *to be zero on the set of zeros in* L^n *of an ideal* γ *of* $K[X_1, X_2, \ldots, X_n]$*, it is necessary and sufficient that there exists an integer* $m > 0$ *such that* $Q^m \in \gamma$.

4 The Time Fractional mKdV Equation

For our purpose, we introduce the following transformations:

$$u(x,t) = U(\xi), \quad \xi = cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \tag{4.1}$$

where λ and c are a constants.

Substituting Eq. (4.1) with Eq. (2.2) into Eq. (1.1), we can know that (1.1) reduced into an ODE

$$-\lambda U' + cU^2U' + c^3U''' = 0 \tag{4.2}$$

where "U'" $= dU/d\xi$. Since

$$U(\xi) = f(\xi) \tag{4.3}$$

then Eq. (4.2) can be written as

$$-\lambda f'(\xi) + cf^2(\xi)f'(\xi) + c^3f'''(\xi) = 0 \tag{4.4}$$

Integrating Eq. (4.4) once we obtain

$$\xi_0 - \lambda f(\xi) + \frac{c}{3}f^3(\xi) + c^3f''(\xi) = 0 \tag{4.5}$$

where ξ_0 is a integration constant.

Using Eqs. (3.5) and (3.6) we get

$$\dot{X}(\xi) = Y(\xi) \tag{4.6}$$

$$\dot{Y}(\xi) = \frac{3\lambda X(\xi) - 3\xi_0 - cX^3(\xi)}{3c^3} \tag{4.7}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (4.6), (4.7), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0 \tag{4.8}$$

where $a_i(X)$, $(i = 0, 1, \ldots, m)$ are polynomials of X and $a_m(X) \neq 0$. Equation (4.8) is called the first integral to (4.6)–(4.7), due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = [g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i \tag{4.9}$$

In this example, we take two different cases, assuming that $m = 1$ and $m = 2$ in Eq. (4.8).

Case I. Assume that $m = 1$, by equating the coefficients of Y^i $(i = 0, 1, 2)$ on both sides of Eq. (4.9), we get

$$\dot{a}_1(X) = h(X)a_1(X) \tag{4.10}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X) \tag{4.11}$$

$$a_1(X)\,\dot{Y} = g(X)a_0(X) = a_1(X)\left(\frac{3\lambda X - 3\xi_0 - cX^3}{3c^3}\right) \tag{4.12}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from Eq. (4.10) we infer that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find $a_0(X)$

$$a_0(X) = \frac{A_1}{2}X^2 + B_0X + A_0 \tag{4.13}$$

Substituting $a_0(X)$, $a_1(X)$, and $g(X)$ in Eq. (4.12) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = -\frac{i\lambda\sqrt{6}}{2c^2}, \qquad B_0 = 0, \qquad A_1 = \frac{i\sqrt{6}}{3c}, \qquad \xi_0 = 0 \tag{4.14}$$

Using Eq. (4.14) into Eq. (4.8), we obtain

$$Y(\xi) = \frac{i\lambda\sqrt{6}}{2c^2} - \frac{i}{\sqrt{6}c}X^2(\xi) \tag{4.15}$$

Combining Eq. (4.15) with Eq. (4.6), we obtain the exact solution to Eq. (4.5) as

$$X(\xi) = -i\sqrt{\frac{3\lambda}{c}}\tan\left(\frac{\sqrt{\lambda}(\xi + C_1)}{c\sqrt{2c}}\right) \tag{4.16}$$

where C_1 is integration constant. Thus the periodic wave solution to the time fractional fmKdV equation can be written as

$$u(x,t) = -i\sqrt{\frac{3\lambda}{c}}\tan\left(\frac{\sqrt{\lambda}\left(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + C_1\right)}{c\sqrt{2c}}\right) \tag{4.17}$$

Case II. Assume that $m = 2$, by equating the coefficients of Y^i $(i = 0, 1, 2, 3)$ on both sides of Eq. (4.9), we get

$$\dot{a}_2(X) = h(X)a_2(X) \tag{4.18}$$

$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X) \tag{4.19}$$

$$\dot{a}_0(X) = -2a_2(X)\left(\frac{3\lambda X - 3\xi_0 - cX^3}{3c^3}\right) + g(X)a_1(X) + h(X)a_0(X) \tag{4.20}$$

$$a_1(X)\,\dot{Y} = g(X)a_0(X) = a_1(X)\left(\frac{3\lambda X - 3\xi_0 - cX^3}{3c^3}\right) \tag{4.21}$$

Since $a_2(X)$ is a polynomial of X, then from Eq. (4.18) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For convenience, take $a_2(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find $a_1(X)$ and $a_0(X)$ as

$$a_1(X) = \frac{A_1}{2}X^2 + B_0X + A_0 \tag{4.22}$$

$$a_0(X) = \left(\frac{A_1^2}{8} + \frac{1}{6c^2}\right)X^4 + \left(\frac{A_1B_0}{2}\right)X^3 + \left(-\frac{\lambda}{c^3} + \frac{A_1A_0}{2} + \frac{B_0^2}{2}\right)X^2 + \left(\frac{2\xi_0}{c^3} + B_0A_0\right)X + e \tag{4.23}$$

Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$, and $g(X)$ in Eq. (4.21) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = -\frac{i\lambda\sqrt{6}}{c^2}, \qquad A_1 = \frac{2i}{c\sqrt{3}}, \qquad B_0 = 0, \qquad \xi_0 = 0, \qquad e = -\frac{3\lambda^2}{2c^4} \tag{4.24}$$

Using Eq. (4.24) into Eq. (4.8), we obtain

$$Y(\xi) = -\frac{i(-3\lambda + cX^2(\xi))}{c^2\sqrt{6}} \tag{4.25}$$

Combining Eq. (4.25) with Eq. (4.6), we obtain the exact solution to Eq. (4.5) as

$$X(\xi) = \sqrt{\frac{3\lambda}{c}}\tanh\left(\frac{\sqrt{-\lambda}(\xi + C_2)}{c\sqrt{2c}}\right) \tag{4.26}$$

where C_2 is integration constant. Thus the solitary wave solution to the time fractional fmKdV Eq. (4.2) can be written as

$$u(x,t) = \sqrt{\frac{3\lambda}{c}}\tanh\left(\frac{\sqrt{-\lambda}\left(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + C_2\right)}{c\sqrt{2c}}\right) \tag{4.27}$$

The established solutions have been checked with Maple by putting them back into the original Eq. (1.1). To the best of our knowledge, they have not obtained in literature.

5 The Space–Time Fractional mBBM Equation

First, we consider the following transformations:

$$u(x,t) = U(\xi) \tag{5.1}$$

$$\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} \tag{5.2}$$

where k and c are a nonzero constant.

Substituting Eq. (5.2) with Eq. (2.2) into Eq. (1.2), we can know that Eq. (1.2) reduced into an ODE

$$(c+k)U' - vkU^2U' + k^3U''' = 0 \tag{5.3}$$

where "U′" $= dU/d\xi$. Since $U(\xi) = f(\xi)$ then Eq. (5.3) once time integrating and setting the integration constant to zero we find

$$(c+k)f - vk\frac{f^3}{3} + k^3f'' = 0 \tag{5.4}$$

Using Eqs. (3.5) and (3.6) we get

$$\dot{X}(\xi) = Y(\xi) \tag{5.5}$$

$$\dot{Y}(\xi) = \frac{X(\xi)(-3c-3k+kvX^2(\xi))}{3k^3} \tag{5.6}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (5.5), (5.6), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0 \tag{5.7}$$

where $a_i(X)$, $(i = 0, 1, \ldots, m)$ are polynomials of X and $a_m(X) \neq 0$. Equation (5.7) is called the first integral to (5.5)–(5.6), due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = [g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i \tag{5.8}$$

In this example, we take two different cases, assuming that $m = 1$ and $m = 2$ in Eq. (5.7).

Case I. Assume that $m = 1$, by equating the coefficients of Y^i $(i = 0, 1, 2)$ on both sides of Eq. (5.8), we get

$$\dot{a_1}(X) = h(X)a_1(X) \tag{5.9}$$

$$\dot{a_0}(X) = g(X)a_1(X) + h(X)a_0(X) \tag{5.10}$$

$$a_1(X)\,\dot{Y} = g(X)a_0(X) = a_1(X)\left(\frac{X(-3c-3k+kvX^2)}{3k^3}\right) \tag{5.11}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from Eq. (5.9) we infer that $a_1(X)$ is constant and $h(X) = 0$. For convenience, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find $a_0(X)$:

$$a_0(X) = \frac{A_1}{2}X^2 + B_0X + A_0 \tag{5.12}$$

Substituting $a_0(X)$, $a_1(X)$, and $g(X)$ in Eq. (5.11) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = -\frac{(c+k)\sqrt{3}}{\sqrt{2v}k^2}, \qquad B_0 = 0, \qquad A_1 = \frac{\sqrt{2v}}{k\sqrt{3}} \tag{5.13}$$

Using Eq. (5.13) into Eq. (5.7), we obtain

$$Y(\xi) = \frac{(c+k)\sqrt{3}}{\sqrt{2v}k^2} - \frac{\sqrt{v}}{k\sqrt{6}}X^2(\xi) \tag{5.14}$$

Combining Eq. (5.14) with Eq. (5.5), we obtain the exact solution to (5.4) as

$$X(\xi) = \frac{\sqrt{3vkc+3vk^2}}{vk}\tanh\left(\frac{(\xi + C_1)\sqrt{2vkc+2vk^2}}{2k^2\sqrt{v}}\right) \tag{5.15}$$

where C_1 is integration constant. Thus the solitary wave solution to the space–time fractional fmBBM equation can be written as

$$u(x,t) = \frac{\sqrt{3vkc+3vk^2}}{vk}\tanh \times \left(\frac{\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} + C_1\right)\sqrt{2vkc+2vk^2}}{2k^2\sqrt{v}}\right) \tag{5.16}$$

Case II. Assume that $m = 2$, by equating the coefficients of Y^i $(i = 0, 1, 2, 3)$ on both sides of Eq. (5.8), we get

$$\dot{a_2}(X) = h(X)a_2(X) \tag{5.17}$$

$$\dot{a_1}(X) = g(X)a_2(X) + h(X)a_1(X) \tag{5.18}$$

$$\dot{a_0}(X) = -2a_2(X)\left(\frac{X(-3c-3k+kvX^2)}{3k^3}\right) + g(X)a_1(X) + h(X)a_0(X) \tag{5.19}$$

$$a_1(X)\,\dot{Y} = g(X)a_0(X) = a_1(X)\left(\frac{X(-3c-3k+kvX^2)}{3k^3}\right) \tag{5.20}$$

Since $a_2(X)$ is a polynomial of X, then from Eq. (5.17) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For convenience, take $a_2(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find $a_1(X)$ and $a_0(X)$ as

$$a_1(X) = \frac{A_1}{2}X^2 + B_0X + A_0 \tag{5.21}$$

$$a_0(X) = \left(\frac{A_1^2}{8} - \frac{v}{6k^2}\right)X^4 + \frac{A_1B_0X^3}{2} + \left(\frac{1}{k^2} + \frac{c}{k^3} + \frac{A_1A_0}{2} + \frac{B_0^2}{2}\right)X^2 + B_0A_0X + d \tag{5.22}$$

Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$, and $g(X)$ in Eq. (5.20) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = -\frac{(c+k)\sqrt{6}}{k^2\sqrt{v}}, \quad A_1 = \frac{2\sqrt{2v}}{k\sqrt{3}}, \quad B_0 = 0, \quad d = \frac{3(k^2+2ck+c^2)}{2vk^4} \tag{5.23}$$

Using Eq. (5.23) into Eq. (5.7), we obtain

$$Y(\xi) = -\frac{(-3k-3c+kvX^2(\xi))}{k^2\sqrt{6v}} \tag{5.24}$$

Combining Eq. (5.24) with Eq. (5.5), we obtain the exact solution to (5.4) as

$$X(\xi) = -i\frac{\sqrt{3vkc+3vk^2}}{vk}\tan\left(\frac{(\xi+C_2)\sqrt{2vkc+2vk^2}}{2k^2\sqrt{-v}}\right) \tag{5.25}$$

where C_2 is integration constant. Thus the periodic wave solution to the space–time fractional fmBBM equation can be written as

$$u(x,t) = -i\frac{\sqrt{3vkc+3vk^2}}{vk}\tan \times \left(\frac{\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{ct^\alpha}{\Gamma(1+\alpha)}+C_2\right)\sqrt{2vkc+2vk^2}}{2k^2\sqrt{-v}}\right) \tag{5.26}$$

6 Conclusion

We have proposed first integral method for fractional partial differential equations based on the sense of modified Riemann–Liouville derivative and fractional complex transformation, and applied it to the fmKdV equation and the fmBBM equation.

Among the process, the fractional complex transformation is very important, which ensures that a FDE can be turned into another differential equation of integer order, and then simplify the process of establishing exact solutions. From our results obtained in this paper, we conclude that the first integral method is powerful, effective, and convenient for nonlinear fractional PDEs. As one can see, this method has more general applications than the other methods, and can be applied to other fractional partial differential equations. We hope that the present solutions may be useful in further numerical analysis and these results are going to be very useful in further future research.

References

[1] Miller, K. S., and Ross, B., 1993, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York.
[2] Podlubny, I., 1999, *Fractional Differential Equations*, Academic, San Diego, CA.
[3] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J., 2006, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, Netherlands.
[4] El-Sayed, A. M. A., and Gaber, M., 2006, "The Adomian Decomposition Method for Solving Partial Differential Equations of Fractal Order in Finite Domains," Phys. Lett. A, **359**(3), pp. 175–182.
[5] Safari, M., Ganji, D. D., and Moslemi, M., 2009, "Application of He's Variational Iteration Method and Adomian's Decomposition Method to the Fractional KdV–Burgers–Kuramoto Equation," Comput. Math. Appl., **58**(11–12), pp. 2091–2097.
[6] Sweilam, N. H., Khader, M. M., and Al-Bar, R. F., 2007, "Numerical Studies for a Multi-Order Fractional Differential Equation," Phys. Lett. A, **371**(1–2), pp. 26–33.
[7] Inc, M., 2008, "The Approximate and Exact Solutions of the Space- and Time-Fractional Burgers Equations With Initial Conditions by Variational Iteration Method," J. Math. Anal. Appl., **345**(1), pp. 476–484.
[8] Song, L. N., and Zhang, H. Q., 2009, "Solving the Fractional BBM-Burgers Equation Using the Homotopy Analysis Method," Chaos, Solitons Fractals, **40**(4), pp. 1616–1622.
[9] Arafa, A. A. M., Rida, S. Z., and Mohamed, H., 2011, "Homotopy Analysis Method for Solving Biological Population Model," Commun. Theor. Phys., **56**(5), pp. 797–800.
[10] Gepreel, K. A., 2011, "The Homotopy Perturbation Method Applied to the Nonlinear Fractional Kolmogorov–Petrovskii–Piskunov Equations," Appl. Math. Lett., **24**(8), pp. 1428–1434.
[11] Gupta, P. K., and Singh, M., 2011, "Homotopy Perturbation Method for Fractional Fornberg–Whitham Equation," Comput. Math. Appl., **61**(2), pp. 50–254.
[12] Odibat, Z., and Momani, S., 2008, "Generalized Differential Transform Method for Linear Partial Differential Equations of Fractional Order," Appl. Math. Lett., **21**(2), pp. 194–199.
[13] Ertürk, V. S., Momani, S., and Odibat, Z., 2008, "Application of Generalized Differential Transform Method to Multi-Order Fractional Differential Equations," Commun. Nonlinear Sci. Numer. Simul., **13**(8), pp. 1642–1654.
[14] Zhang, S., and Zhang, H.-Q., 2011, "Fractional Sub-Equation Method and Its Applications to Nonlinear Fractional PDEs," Phys. Lett. A, **375**(7), pp. 1069–1073.
[15] Tong, B., He, Y., Wei, L., and Zhang, X., 2012, "A Generalized Fractional Sub-Equation Method for Fractional Differential Equations With Variable Coefficients," Phys. Lett. A, **376**(38–39), pp. 2588–2590.
[16] Guo, S., Mei, L., Li, Y., and Sun, Y., 2012, "The Improved Fractional Sub-Equation Method and Its Applications to the Space–Time Fractional Differential Equations in Fluid Mechanics," Phys. Lett. A, **376**(4), pp. 407–411.
[17] Lu, B., 2012, "The First Integral Method for Some Time Fractional Differential Equations," J. Math. Anal. Appl., **395**(2), pp. 684–693.
[18] Zhang, S., Zong, Q.-A., Liu, D., and Gao, Q., 2010, "A Generalized Exp-Function Method for Fractional Riccati Differential Equations," Commun. Fractional Calculus, **1**(1), pp. 48–51.
[19] Bekir, A., Güner, Ö., and Cevikel, A. C., 2013, "Fractional Complex Transform and Exp-Function Methods for Fractional Differential Equations," Abstr. Appl. Anal., **2013**, p. 426462.
[20] Zheng, B., 2012, "(G'/G)"-Expansion Method for Solving Fractional Partial Differential Equations in the Theory of Mathematical Physics," Commun. Theor. Phys., **58**(5), pp. 623–630.
[21] Gepreel, K. A., and Omran, S., 2012, "Exact Solutions for Nonlinear Partial Fractional Differential Equations," Chin. Phys. B, **21**(11), p. 110204.
[22] Feng, Z. S., 2002, "The First Integral Method to Study the Burgers-KdV Equation," J. Phys. A: Math. Gen., **35**(2), pp. 343–350.
[23] Feng, Z. S., and Wang, X. H., 2003, "The First Integral Method to the Two-Dimensional Burgers-KdV Equation," Phys. Lett. A, **308**(2–3), pp. 173–178.
[24] Raslan, K. R., 2008, "The First Integral Method for Solving Some Important Nonlinear Partial Differential Equations," Nonlinear Dyn., **53**(4), pp. 281–286.
[25] Abbasbandy, S., and Shirzadi, A., 2010, "The First Integral Method for Modified Benjamin–Bona–Mahony Equation," Commun. Nonlinear Sci. Numer. Simul., **15**(7), pp. 1759–1764.
[26] Tascan, F., and Bekir, A., 2009, "Travelling Wave Solutions of the Cahn–Allen Equation by Using First Integral Method," Appl. Math. Comput., **207**(1), pp. 279–282.
[27] Taghizadeh, N., and Mirzazadeh, M., 2011, "The First Integral Method to Some Complex Nonlinear Partial Differential Equations," J. Comput. Appl. Math., **235**(16), pp. 4871–4877.
[28] Deng, X., 2008, "Exact Peaked Wave Solution of CH-γ Equation by the First-Integral Method," Appl. Math. Comput., **206**(2), pp. 806–809.
[29] Kurulay, M., and Bayram, M., 2010, "Approximate Analytical Solution for the Fractional Modified KdV by Differential Transform Method," Commun. Nonlinear Sci. Numer. Simul., **15**, pp. 1777–1782.
[30] Alzaidy, J. F., 2013, "Fractional Sub-Equation Method and Its Applications to the Space–Time Fractional Differential Equations in Mathematical Physics," Br. J. Math. Comput. Sci., **3**(2), pp. 153–163.
[31] Jumarie, G., 2006, "Modified Riemann–Liouville Derivative and Fractional Taylor Series of Nondifferentiable Functions Further Results," Comput. Math. Appl., **51**(9–10), pp. 1367–1376.
[32] Li, Z. B., and He, J. H., 2010, "Fractional Complex Transform for Fractional Differential Equations," Math. Comput. Appl., **15**(5), pp. 970–973.
[33] Li, Z. B., and He, J. H., 2011, "Application of the Fractional Complex Transform to Fractional Differential Equations," Nonlinear Sci. Lett. A: Math. Phys. Mech., **2**(3), pp. 121–126, available at: http://works.bepress.com/ji_huan_he/52.
[34] Ding, T. R., and Li, C. Z., 1996, *Ordinary Differential Equations*, Peking University Press, Peking, China.
[35] Bourbaki, N., 1972, *Commutative Algebra*, Addison-Wesley, Paris, France.
[36] Feng, Z., and Wang, X., 2001, "Explicit Exact Solitary Wave Solutions for the Kundu Equation and the Derivative Schrödinger Equation," Phys. Scr., **64**(1), pp. 7–14.
[37] Feng, Z., and Roger, K., 2007, "Traveling Waves to a Burgers–Korteweg–de Vries-Type Equation With Higher-Order Nonlinearities," J. Math. Anal. Appl., **328**(2), pp. 1435–1450.