The Dual Complexity Space as the Dual of a Normed Cone

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Abstract

In [15] M. Schellekens introduced the complexity (quasi-metric) space as a part of the research in Theoretical Computer Science and Topology, with applications to the complexity analysis of algorithms. Later on, S. Romaguera and M. Schellekens ([13]) introduced the so-called dual complexity (quasi-metric) space and established several quasi-metric properties of the complexity space via the analysis of the dual. These authors also proved in [14] that actually the dual complexity space \( C^* \) can be modeled as a norm-weightable cone whose induced quasi-metric is Smyth complete. This fact suggests the existence of deep connections between a general theory of (dual) complexity spaces and Asymmetric Functional Analysis. These connections have been recently explored in [3], [4] and [8]. In particular, it was proved in [3] that the so-called dual \( p \)-complexity space \( C^*_p \), with \( 1 \leq p < \infty \), is isometrically isomorphic to the positive cone of the classical Banach space \( l_p \). The space \( C^*_1 \) is exactly the dual complexity space, and thus it is isometrically isomorphic to the positive cone of the Banach space \( l_1 \) of all absolutely summable real sequences. Here, we continue the analysis of the structure of the dual complexity space \( C^* \). We show that it is the dual space of the positive cone of the Banach space \( c_0 \) of all real sequences converging to zero, and that its dual space is the positive cone of the Banach space \( l_{\infty} \) of all bounded real sequences. Furthermore, the dual space of \( C^*_p \), \( 1 < p < \infty \), is \( C^*_q \) where \( 1/p + 1/q = 1 \). These results extend to this setting well-known theorems of the classical theory of Functional Analysis.

Keywords: Complexity

1 Introduction and preliminaries

Throughout this paper the letters \( \mathbb{R}^+ \), \( \mathbb{N} \) and \( \omega \) will denote the set of nonnegative real numbers, the set of natural numbers and the set of nonnegative integers numbers, respectively.

Recall that a monoid is a semigroup \((X,+)\) with neutral element 0.

According to [9] and [16], a cone (on \( \mathbb{R}^+ \)) is a triple \((X,+,\cdot)\) such that \((X,+)\) is an Abelian monoid, and \( \cdot \) is a function from \( \mathbb{R}^+ \times X \) to \( X \) such that for all \( x,y \in X \) and \( r,s \in \mathbb{R}^+ \): (i) \( r \cdot (s \cdot x) = (rs) \cdot x \); (ii) \( r \cdot (x+y) = (r \cdot x) + (r \cdot y) \); (iii) \( (r+s) \cdot x = (r \cdot x) + (s \cdot x) \); (iv) \( 1 \cdot x = x \).

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A cone \((X, +, \cdot)\) is called cancellative if for all \(x, y, z \in X\), \(z + x = z + y\) implies \(x = y\).

Obviously, every linear space \((X, +, \cdot)\) can be considered as a cancellative cone when we restrict the operation \(\cdot\) to \(\mathbb{R}^+ \times X\).

We will say that a subset \(A\) of a linear space \(X\) is a cone of \(X\) if \(A\) is a cone for the restriction to \(A\) of the operations of linear space on \(X\).

A quasi-norm on a cone \((X, +, \cdot)\) is a function \(q : X \to \mathbb{R}^+\) such that for all \(x, y \in X\) and \(r \in \mathbb{R}^+\):

(i) \(x = 0\) if and only if there is \(-x \in X\) and \(q(x) = q(-x) = 0\); (ii) \(q(r \cdot x) = rq(x)\); (iii) \(q(x + y) \leq q(x) + q(y)\).

If the quasi-norm \(q\) satisfies: (i’) \(q(x) = 0\) if and only if \(x = 0\), then \(q\) is called a norm on the cone \((X, +, \cdot)\).

A (quasi-)normed cone is a pair \((X, q)\) where \(X\) is a cone and \(q\) is (quasi-)norm on \(X\). When the cone \(X\) is, in fact, a linear space the pair \((X, q)\) will be called (quasi-)normed linear space (asymmetric normed linear space in [3], [4] and [5]).

A cone \(B\) of a linear space \(X\) is called pointed if \(B \cap -B = 0\) (see, for example, [6]).

Let us recall that a linear function from a cone \((X, +, \cdot)\) to a cone \((Y, +, \cdot)\) is a function \(f : X \to Y\) such that \(f(\alpha \cdot x + \beta \cdot y) = \alpha \cdot f(x) + \beta \cdot f(y)\) for all \(x, y \in X\) and \(\alpha, \beta \in \mathbb{R}^+\).

Let us recall that a quasi-metric on a (nonempty) set \(X\) is a nonnegative real-valued function \(d\) on \(X \times X\) such that for all \(x, y, z \in X\) : (i) \(d(x, y) = d(y, x) = 0\) if and only if \(x = y\); (ii) \(d(x, z) \leq d(x, y) + d(y, z)\).


A quasi-metric space is a pair \((X, d)\) such that \(X\) is a nonempty set and \(d\) is a quasi-metric on \(X\).

Each quasi-metric \(d\) on a set \(X\) induces a topology \(\tau(d)\) on \(X\) which has as a base the family of open \(d\)-balls \(\{B_d(x, r) : x \in X, r > 0\}\), where \(B_d(x, r) = \{y \in X : d(x, y) < r\}\) for all \(x \in X\) and \(r > 0\).

Observe that if \(d\) is a quasi-metric, then \(\tau(d)\) is a \(T_0\) topology.

Given a a quasi-normed linear space \((X, q)\) it is easily seen that the quasi-norm \(q\) induces, in a natural way, a quasi-metric \(d_q\) on \(X\), defined by

\[d_q(x, y) = q(y - x)\]

for all \(x, y \in X\) (see, for example, [3]).

Note that, in general, we can not induce a quasi-metric on a cone \(Y\) of \(X\) from the restriction \(q|_Y\) of \(q\) to \(Y\) by using the above technique. This follows from the fact that the difference \(y - x\) of two elements \(x, y\) of \(Y\) does not necessarily belong to \(Y\), and so that it is not possible to compute the distance form \(x\) to \(y\) as \(d_q|_Y(x, y) = q|_Y(y - x)\). For this reason, in the following, when we rely on a cone \(Y\) of a linear space \(X\) we will always consider the quasi-metric structure induced on \(Y\) by the restriction of the quasi-metric \(d_q\), defined on \(X\), to the cone \(Y\), i.e. \((Y, (d_q)|_Y)\). The restriction of \(d_q\) to \(Y\) will also be denoted by \(d_q\) if no confusion
arises.

The following well-known example will be useful later on. For each \( x, y \in \mathbb{R} \) let \( u(x) = x \lor 0 \). It is clear that \( u \) is a quasi-norm on \( \mathbb{R} \) which induces the so-called upper quasi-metric on \( \mathbb{R} \), that is \( d_u(x, y) = (y - x) \lor 0 \).

In the last years quasi-normed cones and other related “nonsymmetric” structures from topological algebra and functional analysis, have been successfully applied to several problems in theoretical computer science and approximation theory, respectively (see Sections 11 and 12 of [11], and also [13], [3], [5], [9], [12], [14], [16], etc.). In particular, in 1995, it was introduced by Schellekens the complexity (quasi-metric) space as a part of the development of a topological foundation for the complexity analysis of programs and algorithms ([15]). The complexity space consists of the pair \( (\mathcal{C}, d_{\mathcal{C}}) \), where

\[
\mathcal{C} = \{ f : \omega \to (0, \infty) : \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \},
\]

and \( d_{\mathcal{C}} \) is the quasi-metric on \( \mathcal{C} \) given by

\[
d_{\mathcal{C}}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left[ \frac{1}{g(n)} - \frac{1}{f(n)} \right] \lor 0,
\]

for all \( f, g \in \mathcal{C} \).

Later, Romaguera and Schellekens ([13]) introduced the so-called dual (quasi-metric) complexity space and studied several quasi-metric properties of the original complexity space which are interesting from a computational point of view, via the analysis of its dual. The dual complexity space is the pair \( (\mathcal{C}^*, d_{\mathcal{C}^*}) \), with

\[
\mathcal{C}^* = \{ f \in (\mathbb{R}^+)^\omega : \sum_{n=0}^{\infty} 2^{-n} f(n) < \infty \},
\]

and \( d_{\mathcal{C}^*} \) the quasi-metric on \( \mathcal{C}^* \) given by

\[
d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \{(g(n) - f(n)) \lor 0\}
\]

for all \( f, g \in \mathcal{C}^* \).

Actually \( \mathcal{C}^* \) can become a normed cone if we endowed it with the nonnegative real valued function \( q_{\mathcal{C}^*} \) given by \( q_{\mathcal{C}^*}(f) = \sum_{n=0}^{\infty} 2^{-n} u(f(n)) \) (see [14]). Observe that the quasi-metric \( d_{\mathcal{C}^*} \) can be obtained from the norm \( q_{\mathcal{C}^*} \) as follows:

\[
d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} u(g(n) - f(n))
\]

for all \( f, g \in \mathcal{C}^* \).

Recently, in [3], García-Raffi, Romaguera and Sánchez-Pérez have extended the notion of the dual complexity space to the so-called dual \( p \)-complexity space \( (\mathcal{C}_{p}^*, q_p) \),
with \( p > 1 \), in order to include the study, from a dual point of view, of some kind of exponential time algorithms. Among other results, they proved that the dual \( p \)-complexity space is isometrically isomorphic to the positive cone of the classical Banach space \( l_p \) endowed with the asymmetric norm \( || \cdot ||_p \) (see Section 2).

In this paper we obtain analogous results to the classical theorems on duality for normed linear spaces (compare \([10]\) and \([7]\)), extending them to the (quasi-metric) complexity setting. Specifically we show that the dual complexity space \((C^*, q_{C^*})\) is the “dual space” of the positive cone of the Banach space \( c_0 \) of all real sequences converging to zero. Moreover, we analyze its dual, as a normed cone, and we prove that the mentioned dual is exactly the positive cone of the Banach space \( l_\infty \) of all bounded real sequences. Finally, we observe that the dual space of \( C^*_p \), with \( p > 1 \), is the \( q \)-complexity space \( C^*_q \), with \( p \) and \( q \) “conjugated” in the classical sense, i.e. \( \frac{1}{p} + \frac{1}{q} = 1 \).

2 Representing the dual complexity space as a dual normed cone

In our context, given a cone \( A \) of a quasi-normed linear space \((X, q)\), we define

\[
A^* = \{ f : (A, \tau(d_q)) \to (\mathbb{R}^+, \tau(d_u)) : f \text{ is linear and continuous} \}.
\]

Obviously \( A^* \) is a cone.

A very useful fact is given by the next result.

**Lemma 2.1** Let \((X, q)\) be a quasi-normed linear space, let \( A \) be a cone of \( X \) and let \( f \in A^* \). Then, there is \( M > 0 \) such that \( f(x) \leq Mq(x) \) for all \( x \in A \).

**Proof.** Given \( f \in A^* \), there exists \( \delta > 0 \) such that \( f(B_{d_q}(0, \delta)) \subseteq [0, 1] \). Put \( M = 2/\delta \). Fix \( x \in A \). If \( q(x) = 0 \), then \( f(x) = 0 \) (indeed, if \( f(x) > 0 \), we have \( q(x/f(x)) = 0 \) but \( f(x/f(x)) = 1 \), a contradiction). If \( q(x) > 0 \), then \( x/Mq(x) \in B_{d_q}(0, \delta) \) and thus \( f(x/Mq(x)) < 1 \). We conclude that \( f(x) \leq Mq(x) \) for all \( x \in A \).

Now, consider the set \( A^* - A^* := \{ f - g : f, g \in A^* \} \). Clearly \( A^* - A^* \) equipped with the usual pointwise operations becomes a linear space and, thus, \( A^* \) is a cone of \( A^* - A^* \).

Next, for each \( f, g \in A^* \) put

\[
q^*(f - g) = \sup \{ f(x) - g(x) : q(x) \leq 1, x \in A \}.
\]

Since \( f \in A^* \), by Lemma 2.1 there exists \( M > 0 \) such that

\[
q^*(f - g) \leq q^*(f) \leq M.
\]

So \( q^* \) is well-defined on \( A^* - A^* \). It is routine to show that \((A^* - A^*, q^*)\) is a quasi-normed linear space. Notice that, \( q^*|_{A^*} \) is, in fact, a norm on \( A^* \).
In the following we will refer to \((A^*, q^*)\) as the dual normed cone of \((A, q)\), or simply the dual cone of \((A, q)\).

Of course (see Section 1), \(q^*\) induces a quasi-metric \(d_{q^*}\) on \(A^*\) such that for each pair \(f, g \in A^*\),

\[
d_{q^*}(f, g) = q^*(g - f) = \sup\{g(x) - f(x) : q(x) \leq 1, x \in A\}.
\]

Observe that \(q^*(f) = d_{q^*}(0, f)\) for all \(f \in A^*\).

The next notion will play a crucial role in our context.

**Definition 2.2** Let \((X, q)\) and \((Y, p)\) be two quasi-normed linear spaces and let \(A\) and \(B\) be two cones of \(X\) and \(Y\), respectively. We say that \((A, q)\) is isometrically isomorphic to \((B, p)\) if there is a linear map \(\varphi\) from \(A\) onto \(B\) such that \((d_p)_{|B}(\varphi(x), \varphi(y)) = (d_q)_{|A}(x, y)\) for all \(x, y \in A\).

Note that it immediately follows from the preceding definition that \(\varphi\) is one-to-one and that \(p(\varphi(x)) = q(x)\) for all \(x \in A\). Thus we restate in this way the classical notion of isometric isomorphism between (quasi-)normed linear spaces.

As we indicated in Section 1, the notion of the dual complexity space was extended to the \(p\)-dual case in [3]. To this end, it was introduced the quasi-normed linear space \((B_p^*, q_p)\), where \(B_p^*\) is the linear space given by

\[
B_p^* = \{ f \in \mathbb{R}^\omega : \sum_{n=0}^{\infty} (2^{-n}|f(n)|)^p < \infty \}
\]

and \(q_p\) denote the nonnegative real valued function defined on \(B_p^*\) by

\[
q_p(f) = \left( \sum_{n=0}^{\infty} (2^{-n}u(f(n))^p \right)^{1/p}.
\]

Thus, the normed cone \((C_p^*, q_p)\) is called the dual \(p\)-complexity space where

\[
C_p^* = \{ f \in (\mathbb{R}^+)^\omega : \sum_{n=0}^{\infty} (2^{-n}f(n))^p < \infty \}
\]

and the restriction of \(q_p\) to \(C_p^*\) is also denoted by \(q_p\). Note that \(C_p^*\) is a pointed cone of \(B_p^*\).

Among other results, it was shown in [3] that such a cone is isometrically isomorphic to the normed cone \((l_p^+, ||\cdot||_p)\), where \(l_p^+\) is the positive cone of \(l_p\) (i.e. \(l_p^+ = \{ x \in l_p : x \geq 0 \}\)), \((l_p, ||\cdot||_p)\) is the well-known Banach space of all infinite sequences \(x := (x_n)_n\) of real numbers such that \(||x||_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p} < \infty\), and the quasi-norm \(||\cdot||_p\) is defined on \(l_p\) by \(||x||_p = (\sum_{n=1}^{\infty} u(x_n)^p)^{1/p}\). As a particular case of above construction can be retrieved the so-called dual complexity space of...
[13] when one takes \( p = 1 \). As a consequence we obtain that \((C^*, q_{C^*})\) is isometrically isomorphic to the normed cone \((l_1^+, || \cdot ||_1)\).

Next we prove that the dual complexity space is also isometrically isomorphic to the dual cone of a distinguished normed cone, namely \( c_0^+ \). To this end, consider the linear space \( c_0 \) of all infinitie sequences \( x := (x_n) \) of real numbers such that \( \lim_{n \to \infty} x_n = 0 \), endowed with the norm \( || \cdot ||_0 \) given by \( ||x||_0 := \sup_{n \in \mathbb{N}} |x_n| \) for all \( x \in c_0 \). Define a nonnegative real valued function \( q_0 \) on \( c_0 \) by \( q_0(x) = \sup_{n \in \mathbb{N}} |x_n| \). Clearly \( q_0 \) is a quasi-norm and hence the pair \((c_0, q_0)\) is a quasi-normed linear space.

In the following, as usual, the Euclidean norm on \( \mathbb{R} \) will be denoted by \( |\cdot| \).

**Remark 2.3** Denote by \((c_0^+, || \cdot ||_0^*)\) the classical (normed) dual space of \((c_0, || \cdot ||_0)\). In the context of normed linear spaces is well-known that the Banach space \((l_1, || \cdot ||_1)\) is isometrically isomorphic to \((c_0^+, || \cdot ||_0^*)\), by means of the mapping \( \Phi : l_1 \to c_0^+ \) given by \( \Phi(f)(x) = \sum_{k=1}^\infty x_k f_k \), for \( f := (f_k)_k \in l_1 \) and \( x := (x_k)_k \in c_0 \). In fact \( \Phi \) is linear, bijective and preserves the norms (see [10] for more details).

Now, if we denote by \( c_0^+ \) the positive cone of \( c_0 \) we have the following.

**Theorem 2.4** The dual complexity space \((C^*, q_{C^*})\) is isometrically isomorphic to the dual cone of the normed cone \((c_0^+, q_0)\).

**Proof.** Note that it is suffices to show that \((l_1^+, || \cdot ||_1)\) is isometrically isomorphic to the dual cone of \((c_0^+, q_0)\). To this end, for each \( f := (f_k)_k \in l_1^+ \) put

\[
\Psi(f)(x) = \sum_{k=1}^\infty x_k f_k,
\]

whenever \( x := (x_k)_k \in c_0^+ \). Note that, actually, \( \Psi(f) \) is the restriction to \( c_0^+ \) of the mapping \( \Phi(f) \) defined in Remark 2.3. Hence, since \( l_1^+ \) is a subset of \( l_1 \), \( \Psi(f)(x) < \infty \) for all \( x \in c_0^+ \). Therefore \( \Psi(f) \) is a nonnegative real valued function on \( c_0^+ \).

Next we prove that \( \Psi(f) \in (c_0^+)^* \).

Clearly \( \Psi(f) \) is linear on \( c_0^+ \). Moreover it is upper semicontinuous on \((c_0^+, \tau_{q_0})\). Indeed, suppose that \( (x(n))_n \) is a sequence in \( c_0^+ \) such that \( q_0(x(n) - x) \to 0 \) for some \( x := (x_k)_k \in c_0^+ \). Then, given \( \varepsilon > 0 \) we have

\[
\Psi(f)(x(n)) - \Psi(f)(x) = \sum_{k=1}^\infty x(n)_k f_k - \sum_{k=1}^\infty x_k f_k = \sum_{k=1}^\infty (x(n)_k - x_k) f_k \\
\leq \sum_{k=1}^\infty q_0(x(n) - x) f_k < \varepsilon \sum_{k=1}^\infty f_k = \varepsilon ||f||_{1^+},
\]

eventually. So \( \Psi(f) \) is upper semicontinuous on \((c_0^+, \tau_{q_0})\), and, consequently \( \Psi(f) \in (c_0^+)^* \).

Thus, we have constructed a mapping \( \Psi : l_1^+ \to (c_0^+)^* \), such that \( \Psi(f)(x) = \sum_{k=1}^\infty x_k f_k \) for all \( f \in l_1^+ \) and \( x \in (c_0^+)^* \). As in the classical case (compare Remark 2.3), \( \Psi \) is linear.

Now let \( f, g \in l_1^+ \) and \( x \in c_0^+ \). Then
\[ \Psi(g)(x) - \Psi(f)(x) = \sum_{k=1}^{\infty} x_k(g_k - f_k) \leq q_0(x) \sum_{k=1}^{\infty} \{ (g_k - f_k) \vee 0 \} = q_0(x) \| g - f \|_+ . \]

So
\[ d_{q_0^+}(\Psi(f), \Psi(g)) = \sup\{ \Psi(g)(x) - \Psi(f)(x) : q_0(x) \leq 1 \} \leq \| g - f \|_+ = d_{\| \cdot \|_+}(f, g) . \]

On the other hand, if for each \( n \in \mathbb{N} \) we define \( x(n) = \sum_{k=1}^{n} \chi_k e_k \), where \( \chi_k = 1 \)

if \( g_k > f_k \) and \( \chi_k = 0 \) if \( g_k \leq f_k \), and \( e_k = (0, \ldots, 0, 1, 0, \ldots) \), we clearly obtain that \( x(n) \in c_0^+ \) and \( q_0(x(n)) \leq 1 \). Hence

\[ \Psi(g)(x(n)) - \Psi(f)(x(n)) = \sum_{k=1}^{n} \chi_k(g_k - f_k) = \sum_{k=1}^{n} \{ (g_k - f_k) \vee 0 \} . \]

Therefore
\[ d_{q_0^+}(\Psi(f), \Psi(g)) \geq \sup\{ \Psi(g)(x(n)) - \Psi(f)(x(n)) : n \in \mathbb{N} \} = \sum_{k=1}^{\infty} \{ (g_k - f_k) \vee 0 \} = d_{\| \cdot \|_+}(f, g) . \]

We conclude that \( d_{q_0^+}(\Psi(f), \Psi(g)) = d_{\| \cdot \|_+}(f, g) \), for all \( f, g \in l_1^+ \).

It remains to show that \( \Psi \) is onto.

Given \( F \in (c_0^+)^* \), for each \( x \in c_0^+ \) we can write \( x = \sum_{k=1}^{\infty} x_k e_k \) and it is obvious that \( \| x - \sum_{k=1}^{n} x_k e_k \|_0 = \sup_{k>n} | x_k | \rightarrow 0 \) as \( n \rightarrow \infty \). Since \( x - \sum_{k=1}^{n} x_k e_k \in c_0^+ \), then \( 0 \leq F(x - \sum_{k=1}^{n} x_k e_k) = F(x) - F(\sum_{k=1}^{n} x_k e_k) \). By Lemma 2.1 there is \( M > 0 \) such that

\[ | F(x) - F(\sum_{k=1}^{n} x_k e_k) | \leq M q_0(x - \sum_{k=1}^{n} x_k e_k) = M \| x - \sum_{k=1}^{n} x_k e_k \|_0 . \]

Therefore \( F(x) = \sum_{k=1}^{\infty} x_k F(e_k) \).

Let \( \tilde{f} = (\tilde{f}_n)_n \) such that \( \tilde{f}_n = F(e_n) \) for all \( n \in \mathbb{N} \). We show that \( \tilde{f} \in l_1^+ \). Indeed, let \( x(n) = \sum_{k=1}^{n} \chi_k e_k \) where \( \chi_k = 1 \) if \( F(e_k) \neq 0 \) and \( \chi_k = 0 \) if \( F(e_k) = 0 \). It is clear that \( x(n) \in c_0^+ \) and \( q_0(x(n)) \leq 1 \). Thus \( \sum_{k=1}^{n} \tilde{f}_n = \sum_{k=1}^{n} \chi_k F(e_k) = F(x(n)) \) for all \( n \in \mathbb{N} \). Whence, by Lemma 2.1, we obtain that \( \sum_{k=1}^{n} \tilde{f}_n \leq M q_0(x(n)) \leq M \) for all \( n \in \mathbb{N} \). Consequently \( \tilde{f} \in l_1^+ \). Finally, it is easily seen that \( \Psi(\tilde{f}) = F \).

Next we give the description of the dual cone of the (normed) complexity space \((C^*, q_{C^*})\). To this end, let \( l_\infty \) be the linear space of all infinitie sequences \( x := (x_n)_n \) of real numbers such that \( \| x \|_0 < \infty \). It is clear that \( l_\infty \) endowed with the quasi-norm \( q_0 \) has a quasi-normed linear structure. Let us denote by \( l_\infty^+ \) the positive cone of \( l_\infty \).
In order to help the reader we proceed, as is the case of the Theorem 2.4, recalling briefly in the below remark the classical version of the our next result (Theorem 2.6).

**Remark 2.5** It is well known that the classical dual space \((l_1^+, || \cdot ||_1^+)\) of the Banach space \((l_1, || \cdot ||_1)\) is isometrically isomorphic to \((l_\infty, || \cdot ||_0)\) ([7], [10]) by means of the mapping \(\Upsilon : l_1^+ \to l_\infty\) given by \(\Upsilon(f) = (f(e_n))_n\) for each \(f \in l_1^+\). In fact \(\Upsilon\) is a linear, bijective and preserves the norms.

**Theorem 2.6** The dual cone \(((l_1^+)^*)^+ \cap || \cdot ||_{+1}^+\) of the normed cone \((l_1^+, || \cdot ||_1^+)\) is isometrically isomorphic to the normed cone \((l_\infty^+, q_0)\).

**Proof.** We only give a sketch of the proof because of it is very similar to the proof of the above theorem. First, for \(F \in (l_1^+)^*\), put \(\Gamma(F)\) as the sequence \((F(e_k))_k\).

Clearly \(\sup_{k \in \mathbb{N}} F(e_k) \leq K q_0(e_k) \leq K\) for some \(K > 0\) since \(F \in (l_1^+)^*\) and by Lemma 2.1. So \(\Gamma(F) \in l_\infty^+\). Hence we can define the mapping \(\Gamma : (l_1^+)^* \to l_\infty^+\) as \(\Gamma(F) = (f(e_k))_k\).

Note that \(\Gamma\) is the restriction of \(\Upsilon\) defined in Remark 2.5.

Since \(\Upsilon\) is linear we immediately obtain that \(\Gamma\) is linear.

Next we show \(\Gamma\) is, in fact, an isometry. Given \(\varepsilon > 0\), there is \(k_0 \in \mathbb{N}\) such that

\[
d_{q_0}(\Gamma(F), \Gamma(G)) = q_0(\Gamma(G) - \Gamma(F)) = \sup_{k \in \mathbb{N}} (G(e_k) - F(e_k)) \geq 0
\]

\[
< (G(e_{k_0}) - F(e_{k_0})) \geq 0 + \varepsilon
\]

\[
\leq \sup\{G(x) - F(x) : ||x||_{+1} \leq 1\} + \varepsilon = d_{|| \cdot ||_{+1}^+}(F, G) + \varepsilon.
\]

Since the above inequality is true for all \(\varepsilon > 0\) we have that \(d_{q_0}(\Gamma(F), \Gamma(G)) \leq d_{|| \cdot ||_{+1}^+}(F, G)\) for all \(F, G \in (l_1^+)^*\).

On the other hand, since \(l_1^+\) is a subset of \(l_1\) we can write each \(x = \sum_{k=1}^{\infty} x_k e_k\) and obviously \(||x - \sum_{k=1}^{n} x_k e_k||_1 = \sum_{k=n+1}^{\infty} ||x_k|| \to 0\) as \(n \to \infty\). Furthermore, for each \(F \in (l_1^+)^*\) we have that \(0 \leq F(x - \sum_{k=1}^{n} x_k e_k) = F(x) - F(\sum_{k=1}^{n} x_k e_k)\) because \(x - \sum_{k=1}^{n} x_k e_k \in l_1^+\). Hence, from Lemma 2.1, we deduce that there is \(K > 0\) such that

\[
||F(x) - F(\sum_{k=1}^{n} x_k e_k)|| \leq K ||x - \sum_{k=1}^{n} x_k e_k||_{+1} = K ||x - \sum_{k=1}^{n} x_k e_k||_1.
\]

Therefore \(F(x) = \sum_{k=1}^{\infty} x_k F(e_k)\) for all \(x \in l_1^+\).

Thus, given \(\varepsilon > 0\), there is \(y \in l_1^+\) such that

\[
d_{|| \cdot ||_{+1}^+}(F, G) = \sup\{G(x) - F(x) : ||x||_{+1} \leq 1\} < G(y) - F(y) + \varepsilon
\]

\[
= \left(\sum_{k=1}^{\infty} y_k G(e_k) - \sum_{k=1}^{\infty} y_k F(e_k)\right) + \varepsilon
\]

\[
\leq \left(\sum_{k=1}^{\infty} y_k ((G(e_k) - F(e_k)) \vee 0)\right) + \varepsilon
\]

\[
\leq q_0(\Gamma(G) - \Gamma(F)) ||y||_{+1} + \varepsilon \leq q_0(\Gamma(G) - \Gamma(F)) + \varepsilon
\]

\[
= d_{q_0}(\Gamma(F), \Gamma(G)) + \varepsilon.
\]
Since this is true for all $\varepsilon > 0$ we have that $d_{||\cdot||^*_{+1}}(F, G) \leq d_{q_0}(\Gamma(F), \Gamma(G))$ for all $F, G \in (l^+_1)^*$. So that $d_{||\cdot||^*_{+1}}(F, G) = d_{q_0}(\Gamma(F), \Gamma(G))$.

Now we have to prove that $\Gamma$ is onto.

Let $x \in l^+_\infty$. Then following the technique of the classical proof we take $\widetilde{F}$ as $\widetilde{F}(f) = \sum_{k=1}^{\infty} x_k f_k$. Hence since $f \in l^+_1$ and $x \in l^+_\infty$, $\widetilde{F}(f)$ is a nonnegative real number. Consequently we can define the function $\widetilde{F} : l^+_1 \rightarrow \mathbb{R}^+$ as $\widetilde{F}(f) = \sum_{k=1}^{\infty} x_k f_k$. Clearly $\widetilde{F}$ is linear. Moreover if $(z^{(n)})_n$ is a convergent sequence to an element $z$ in $(l^+_1, ||\cdot||_{+1})$ we have that

$$\widetilde{F}(z) - \widetilde{F}((z^{(n)}) = \sum_{k=1}^{\infty} x_k z_k - \sum_{k=1}^{\infty} x_k z_k^{(n)}$$

$$= \sum_{k=1}^{\infty} (z_k - z_k^{(n)}) x_k \leq q_0(x) \sum_{k=1}^{\infty} \left( |z_k - z_k^{(n)}| \vee 0 \right)$$

$$= q_0(x) ||z - (z^{(n)})||_{+1}.$$

Therefore $\widetilde{F}(z^{(n)}) \rightarrow \widetilde{F}(z)$ as $n \rightarrow \infty$ in $(\mathbb{R}^+, u)$. So $\widetilde{F} \in (l^+_1)^*$. Finally it is clear that $\Gamma(\widetilde{F}) = x$. □

**Corollary 2.7** The dual cone of the dual complexity space $(C^*, q_{C^*})$ is isometrically isomorphic to the normed cone $(l^+_{\infty}, q_0)$.

The following lemma, whose proof can be found in [1], is crucial to obtain the dual space of the Banach space $(l_p, ||\cdot||_p)$, for $p > 1$.

**Lemma 2.8** (Hölder’s Inequality). Suppose that $p > 1$, and let $q$ be the number such that $\frac{1}{p} + \frac{1}{q} = 1$. If $(a_k)_k$, $(b_k)_k$ are two sequences of nonnegative real numbers, then

$$\sum_{k=1}^{\infty} a_k b_k \leq \left( \sum_{k=1}^{\infty} a_k^p \right)^{1/p} \left( \sum_{k=1}^{\infty} b_k^q \right)^{1/q}.$$

**Remark 2.9** ([7]). If $p > 1$, then the dual space $(l^*_p, ||\cdot||^*_p)$ is isometrically isomorphic to the Banach space $(l_q, ||\cdot||_q)$, where $\frac{1}{p} + \frac{1}{q} = 1$. The isometry is given as in Remark 2.5.

An appropriate modification of the proof of the above theorems jointly with Hölder’s inequality permit us to state the following.

**Theorem 2.10** For each $p \in (1, \infty)$ the dual $p$-complexity space $(C^*_p, q_p)$ is isometrically isomorphic to the dual cone of the normed cone $(l^+_{\infty}, ||\cdot||_{+q})$, where $\frac{1}{p} + \frac{1}{q} = 1$.

**References**


