On the conjecture of Kochar and Korwar

Nuria Torrado\textsuperscript{a,*}, Rosa E. Lillo\textsuperscript{b}, Michael P. Wiper\textsuperscript{b}

\textsuperscript{a} Universidad Carlos III de Madrid, Department of Statistics, Escuela Politécnica Superior, Campus de Leganés, Madrid, Spain
\textsuperscript{b} Universidad Carlos III de Madrid, Department of Statistics, Facultad de Ciencias Sociales y Jurídicas, Campus de Getafe, Madrid, Spain

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\textbf{A B S T R A C T}

In this article, we partially solve a conjecture by Kochar and Korwar (1996) \cite{9} in relation to the normalized spacings of the order statistics of a sample of independent exponential random variables with different scale parameters. In the case of a sample of size \(n=3\), they proved the ordering of the normalized spacings and conjectured that result holds for all \(n\). We prove this conjecture for \(n=4\) for both spacings and normalized spacings and generalize some results to \(n>4\).

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\textbf{1. Introduction}

Given a set of independent random variables, \(X_1, X_2, \ldots, X_n\), let the order statistics of these variables be \(X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}\). Then, the random variables

\[ D_i^n = X_{i:n} - X_{i-1:n} \quad \text{and} \quad D_{*i}^n = (n - i + 1) \left( X_{i:n} - X_{i-1:n} \right) \]

for \(i = 1, \ldots, n\), with \(X_{0:n} \equiv 0\), are called spacings and normalized spacings, respectively.

Spacings and their functions are important in statistics, in general, and in particular in the context of life testing and reliability models. It is well known that the exponential distribution shows no ageing over time and has constant failure rates, and spacings correspond to times elapsed between successive failures of components in a system, see e.g. \cite{1–3}.

Many authors have studied the stochastic properties of spacings of independent and identically distributed (i.i.d.) random variables, see \cite{4–6} for a review.

Spacings of non identically distributed variables (i.n.i.d.) have also been considered in the literature, see e.g. \cite{7}. In particular, Pledger and Proschan \cite{8} proved that, if the scale parameters of the exponential distributions are not all equal, then the \(i\)th normalized spacing is stochastically smaller than the \((i+1)\)th normalized spacing. Kochar and Korwar \cite{9} (hereafter K&K) conjectured that the successive normalized spacings are increasing in hazard rate ordering in the case when \(X_1, X_2, \ldots, X_n\) are independent exponential random variables with \(X_i\) having hazard rates \(\lambda_i\) for \(i = 1, \ldots, n\) and proved this conjecture for \(n = 3\). In a recent article, Hu et al. \cite{10} strengthened this last result from hazard rate ordering to likelihood ratio ordering in the case of spacings. They proved that, in general, the first spacing is smaller than the second spacing according to likelihood ratio ordering. Furthermore, if \(\lambda_i + \lambda_j \geq \lambda_k\) for all distinct \(i, j\) and \(k\), then \(D_{n-1:n} \prec_{lr} D_{n:n}\), (note that likelihood ratio ordering implies hazard rate ordering). Wen et al. \cite{11} established likelihood ratio ordering between consecutive spacings from a multiple-outlier exponential model and conjectured that this result can be strengthened to spacings from heterogeneous exponential variables. They also noticed that the result of K&K continues being a conjecture.

The purpose of this article is to investigate the hazard rate ordering of spacings and normalized spacings of heterogeneous exponential random variables. In particular, we prove the conjecture of K&K for \(n = 4\) and we show that the successive