A fractional order SEIR model with vertical transmission

Nuri Özlü, Elif Demirci

Ankara University, Faculty of Sciences, Department of Mathematics, Besevler, 06100, Ankara, Turkey

1. Introduction

Mathematical modeling in epidemiology provides understanding of the mechanisms that influence the spread of a disease and it suggests control strategies [1]. One of the early triumphs of mathematical epidemiology [2] was a formulation to predict the behaviour of a disease. In this model, the total population is assumed to be constant and divided into three classes namely suspended, infectious and recovered. Over the years, more complex models have been derived. For some diseases it is found that for a period of time, a part of the infectious class does not show the symptoms. For modelling such diseases SEIR models are used [3].

Although a large amount of work has been done in modelling the dynamics of epidemiological diseases, it has been restricted to integer order (delay) differential equations. In recent years, it has turned out that many phenomena in different fields can be described very successfully by the models using fractional order differential equations [4–7].

In this paper, we first introduce a fractional order SEIR model with vertical transmission in a non-constant population. We show that the model introduced in this paper has nonnegative solutions. We give a detailed analysis for the asymptotic stability of disease free and positive fixed points. Numerical simulations are also presented to verify the obtained results.

2. Model derivation

We first give the definitions of fractional order integral and derivative [8]. For the fractional order differentiation, we will use the Caputo’s definition due to its convenience for initial conditions of the differential equations.

**Definition 1.** The fractional integral of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is defined by

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1}f(\tau)d\tau$$

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and the Caputo fractional order derivative of order \( \alpha \in (n - 1, n) \) of \( f(t) \) is defined by

\[
D^\alpha f(t) = I^{n-\alpha} D^n f(t)
\]

with \( n-1 \) being the integer part of \( \alpha \) and \( D = d/dt \).

Here and elsewhere \( I^\alpha \) denotes the Gamma function.

Note that under natural conditions on the function \( f(t) \), for \( \alpha \to n \) the Caputo derivative becomes a conventional derivative [8].

Many infectious diseases in nature have both horizontal and vertical transmission routes. These include such human diseases as Rubella, Herpes Simplex, Hepatitis B, Chagas, and the HIV/AIDS. Horizontal transmission of diseases among human and animals, occurs through physical contact with hosts or through disease vectors like mosquitos, flies, etc. Vertical transmission is the transmission of an infection from parent to child during the perinatal period.

The model that we study in this paper is a fractional order SEIR epidemic model with vertical transmission. The total host population, \( N(t) \), is partitioned into four classes namely susceptible, exposed, infectious and recovered with densities denoted, respectively, by \( S(t), E(t), I(t) \) and \( R(t) \). Let \( \Delta \) denote the number of recruits per unit of time. The natural death rate is assumed to be a non-negative constant \( d \). The horizontal transmission of the disease is assumed to take place with direct contact between infectious and susceptible hosts with a transmission rate \( r > 0 \). For the vertical transmission of the disease, we assume that the offspring from exposed and infectious classes are born into an exposed class with probabilities of \( p \) and \( q \), respectively.

These assumptions lead to the following system of differential equations of order \( \alpha \) with \( \beta, \gamma > 0 \) being the rate that exposed individuals become infectious and recovery rate, respectively, and \( \theta \geq 0 \) being the infection related death rate:

\[
\begin{align*}
D^\alpha S &= \Delta - \frac{p \Delta E}{N} - \frac{q \Delta I}{N} - \frac{SI}{N} - dS \\
D^\alpha E &= \frac{p \Delta E}{N} + \frac{q \Delta I}{N} + \frac{SI}{N} - dE - \beta E \\
D^\alpha I &= \beta E - dI - \theta I - \gamma I \\
D^\alpha R &= \gamma I - dR
\end{align*}
\]

\[S(0) = S_0, \quad E(0) = E_0, \quad I(0) = I_0, \quad R(0) = R_0\]  
(1)

where \( 0 < \alpha \leq 1 \), \( N = S + E + I + R \), \( (S, E, I, R) \in \mathbb{R}^4_+ \). The reason for considering a fractional order system instead of its integer order counterpart is that fractional order differential equations are generalizations of integer order differential equations. Also using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modelling real life phenomena. We should note that system (1) can be reduced to an integer order system by setting \( \alpha = 1 \).

Adding up the equations given in (1), we have

\[
D^\alpha N(t) = \Delta - dN - \theta I.
\]  
(3)

3. Non-negative solutions

Denote \( \mathbb{R}^4_+ = \{ X \in \mathbb{R}^4 : X \geq 0 \} \) and let \( X(t) = (S(t), E(t), I(t), R(t))^T \). For the proof of the theorem about the non-negative solutions we need the following lemma [9].

**Lemma 1** \( (\text{Generalized Mean Value Theorem [9]}). \) Let \( f(x) \in C[a, b] \) and \( D^\alpha f(x) \in C(a, b) \) for \( 0 < \alpha \leq 1 \), then we have

\[
f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\xi)(x - a)^\alpha \]

with \( 0 \leq \xi \leq x, \forall x \in (a, b) \).

**Remark 1.** Suppose \( f(x) \in C[0, b] \) and \( D^\alpha f(x) \in C(0, b) \) for \( 0 < \alpha \leq 1 \). It is clear from the Lemma 1 that if \( D^\alpha f(x) \geq 0, \forall x \in (0, b) \), then the function \( f \) is nondecreasing and if \( D^\alpha f(x) \leq 0, \forall x \in (0, b) \), then the function \( f \) is nonincreasing for all \( x \in [0, b] \).

**Theorem 1.** There is a unique solution for the initial value problem given with (1)–(2) and the solution remains in \( \mathbb{R}^4_+ \).

**Proof.** The existence and uniqueness of the solution of (1)–(2) in \( (0, \infty) \) can be obtained from Theorem 3.1 and Remark 3.2 in [10]. We need to show that the domain \( \mathbb{R}^4_+ \) is positively invariant. Since
\[ D^s S |_{S=0} = \Delta - p\Delta E N - q\Delta I N = \frac{\Delta}{N} (N - pE - qI) \geq 0, \]
\[ D^s E |_{E=0} = q\Delta I N + r\frac{IS}{N} \geq 0, \]
\[ D^s I |_{I=0} = \beta E \geq 0, \]
\[ D^s R |_{R=0} = \gamma I \geq 0. \]
on each hyperplane bounding the nonnegative orthant, the vector field points into \( \mathbb{R}^4_+ \). \hfill \Box

**Remark 2.** It is clear that \( N(t) \) also remains nonnegative. For convenience, we will use the first three equations of (1) and the Eq. (3) for the stability analysis.

### 4. Equilibrium points and stability

Consider the system of differential equations of order \( \alpha, \ 0 < \alpha \leq 1, \)
\[ D^\alpha S = \Delta - p\Delta E N - q\Delta I N = \frac{\Delta}{N} (N - pE - qI) \geq 0, \]
\[ D^\alpha E = q\Delta I N + r\frac{IS}{N} \geq 0, \]
\[ D^\alpha I = \beta E \geq 0, \]
\[ D^\alpha R = \gamma I \geq 0. \]

To evaluate the equilibrium points of (4), let
\[
\begin{cases}
D^\alpha S = 0 \\
D^\alpha E = 0 \\
D^\alpha I = 0 \\
D^\alpha N = 0.
\end{cases}
\]

Then the equilibrium points are \( F_0 = \left( \frac{\Delta}{d}, \ 0, \ 0, \ \frac{\Delta}{d} \right) \) and \( F_1 = (\bar{S}, \bar{E}, \bar{I}, \bar{N}) \) where
\[
\bar{I} = \frac{\Delta((d+\theta+\gamma)(pd-d-\beta)+\beta(qd+r))}{(d+\beta)(d+\theta+\gamma)(r-\theta)},
\]
\[
\bar{S} = \frac{\Delta \beta - (d+\beta)(d+\theta+\gamma)\bar{I}}{d\beta},
\]
\[
\bar{E} = \frac{(d+\theta+\gamma)\bar{I}}{\beta},
\]
\[
\bar{N} = \frac{\Delta - \theta \bar{I}}{d}.
\]
The Jacobian matrix \( J(F_0) \) for the system given in (4) evaluated at the disease free equilibrium is as follows:
\[
J(F_0) = \begin{pmatrix}
-d & -pd & -qd - r & 0 \\
0 & pd - d - \beta & qd + r & 0 \\
0 & \beta & -d - \theta - \gamma & 0 \\
0 & 0 & -\theta & -d
\end{pmatrix}.
\]

**Theorem 2.** The disease free equilibrium of system (4) is asymptotically stable if \( \frac{(qd+r)\beta}{(d+\theta+\gamma)(pd-d-\beta)} < 1. \)

**Proof.** The disease free equilibrium is asymptotically stable if all of the eigenvalues, \( \lambda_i, \ i = 1, 2, 3, 4, \) of \( J(F_0) \) satisfy the following condition [11,12]:
\[
| \arg \lambda_i | > \frac{\pi}{2}. \tag{5}
\]
These eigenvalues can be determined by solving the characteristic equation
\[
det(J(F_0) - \lambda I) = 0
\]
which leads to the equation
\[
(d + \lambda)^2(\lambda^2 + \lambda(A + B) + AB - C) = 0
\]
Fig. 1. $S(t)$ for $\alpha = 1, 0.95, 0.9$.

where

\[
A = d + \theta + \gamma \\
B = d - pd + \beta \\
C = (qd + r)\beta.
\]

The roots of the characteristic equation are

\[
\lambda_{1,2} = -d, \\
\lambda_{3,4} = \frac{- (A + B) \pm \sqrt{(A + B)^2 - 4(AB - C)}}{2}.
\]

Since $p$ is a probability, it is clear that $A + B > 0$. If $AB > C$, then all of the eigenvalues, $\lambda_i$, $i = 1, 2, 3, 4$ satisfy the condition given by (5).

The value $\frac{(qd+r)\beta}{(d+\theta+\gamma)(d-p\beta)}$ is known as basic reproduction number denoted by $R_0$, which is the number of secondary cases that one case would produce in a completely susceptible population. The biological interpretation of $R_0$ is that if $R_0$ is less than one, then the infection dies out, if it exceeds one, then the infection persists.

We now discuss the asymptotic stability of the endemic (positive) equilibrium of the system given by (4). The Jacobian matrix $J(F_1)$ evaluated at the endemic equilibrium is given as:

\[
J(F_1) = \begin{pmatrix}
-p\Delta & -q\Delta & -rS & p\Delta + q\Delta I + rSI \\
-N & -N & -N & N^2 \\
N & N & N & -N^2 \\
0 & 0 & 0 & -d
\end{pmatrix}.
\]

The characteristic equation of the linearized system is in the form

\[
P(\lambda) = -(d + \lambda)R(\lambda) = 0
\]

with $R(\lambda)$ being $(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3)$, where

\[
A_1 = 3d + \theta + \gamma + \beta + \frac{1}{N}(r - p\Delta), \\
A_2 = d(3d + 2\theta + 2\gamma) + \beta(2d + \theta + \gamma)\left(\frac{1}{N}\right)(1 + (\rho\beta - \rho^2)) - \Delta(p\theta + \gamma + 2d) + \beta q) \\
A_3 = (d + \theta + \gamma)\left(\frac{rI}{N} + d\right) - \beta d\left(q\Delta + r\tilde{S}\right) - \beta \frac{\theta}{N^2}(p\Delta \tilde{E} + q\Delta I + r\tilde{SI}).
\]

Let $D(f)$ denote the discriminant of a polynomial $f$. If $f(x) = x^3 + a_1x^2 + a_2x + a_3$ then

\[
D(f) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_2a_1^3 - 4a_3^2 - 27a_1^2.
\]
Using the proposition given in [14], we have the following result.

**Corollary 1.** The positive equilibrium point \( F_1 \) of the system (4) is asymptotically stable if one of the following conditions holds for polynomial \( R \) which is given as in (6) and coefficients \( A_1, A_2, A_3 \) which are given as in (7).

(i) \( D(R) > 0, A_1 > 0, A_3 > 0 \) and \( A_1 A_2 > A_3 \).
(ii) \( D(R) < 0, A_1 \geq 0, A_2 \geq 0, A_3 > 0 \) and \( \alpha < \frac{2}{3} \).
(iii) \( D(R) < 0, A_1 < 0, A_2 < 0 \) and \( \alpha > \frac{2}{3} \).

5. Numerical methods and simulations

For the numerical solutions of a system of fractional differential equations, using an Adams-type predictor corrector method is appropriate [13]. For the parameters

<table>
<thead>
<tr>
<th>( A )</th>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( d )</th>
<th>( \beta )</th>
<th>( \theta )</th>
<th>( \gamma )</th>
</tr>
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<td>0.221176</td>
<td>0.8</td>
<td>0.95</td>
<td>0.05</td>
<td>0.008</td>
<td>0.05</td>
<td>0.002</td>
<td>0.003</td>
</tr>
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</table>

with the initial conditions

\[ S(0) = 140, \quad E(0) = 0.01, \quad I(0) = 0.02, \quad N(0) = 141 \]

which are realistic, there exists a positive fixed point, \( (2.19794, 3.51022, 13.5009, 24.2718) \). The approximate solutions \( S(t), E(t), I(t) \) and \( N(t) \) are displayed in **Figs. 1–4**, respectively. In each figure three different values of \( \alpha, \alpha = 1, 0.95, 0.90 \), are considered.

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Fig. 4. $N(t)$ for $\alpha = 1, 0.95, 0.9$. 

References